Semi-Riemannian Metrics of Low Differentiability

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PDEs and Function Spaces
Imperial College, London
December 3, 2008

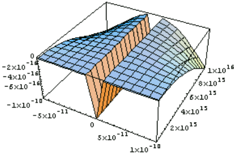
Basic geometry for General Relativity in low regularity situations

1. The problem
   - concentrated sources of the gravitational field
     curvature from a metric of low regularity
   - analytical demand: low regularity vs.
     geometrical demand: nonlinearities
2. Different approaches
   (0) the classical C^∞-setting still fine down to $C^{1,1}$-metrics
   (1) linear distributional geometry (tensor distributions)
   restricted “maximal” distributional setting using Sobolev spaces
   (2) nonlinear distributional geometry (Colombeau algebras)
   unrestricted generalised setting
3. The question
   - Compatibility: in the range where (1) and (2) work, do they agree?

Outline

Why is it needed?

- General Relativity
  - space-time (M, g) Einstein equations $G_{ab}[g] = \kappa\, T_{ab}$
  - problem: many reasonable space-times are singular e.g.:



impulsive gravitational waves,
shell-crossing singularities,
cosmic strings, ...

  - often the metric is below $C^{1,1}$
    T_{ab} “concentrated” but locally integrable
  - singularity theorems:
    incompleteness not just due to failure of smoothness
- Singular Yang-Mills Theory
  - fractionally charged instantons
    $\rightsquigarrow$ singular connections in fibre bundles

Maximal distributional setting for GR

- distributional metric [Marsden, 68],[Parker, 79]

$$g \in \mathcal{D}'^{0}_{2}(M) \cong \mathcal{D}'(M) \otimes_{\mathcal{C}^\infty} \mathcal{T}^0_2(M) \cong L_{\mathcal{C}^\infty}(\mathfrak{X}(M), \mathfrak{X}(M); \mathcal{D}'(M))$$

symmetric and nondegenerate, i.e., $g(X, Y) = 0 \ \forall Y \Rightarrow X = 0$.

$\rightsquigarrow$ no way to define, inverse, curvature, ...

- maximal “reasonable” setting: Geroch-Traschen class

$$g \in \left(H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}\right)^0_2(M)$$

(gt-setting) [Geroch&Traschen, 87], [LeFloch&Mardare, 07]

Pro’s: allows to define curvature $\mathrm{Riem}[g]$, $\mathrm{Ric}[g]$, $\mathrm{R}[g]$ in distributions
consistent limits $\rightsquigarrow$ valid modelling

Con’s: Bianchi identities fail $\rightsquigarrow$ energy conservation ?
$\dim(\mathrm{supp}(\mathrm{Riem}[g])) \geq 3 \rightsquigarrow$ thin shells yes, but strings no!

Colombeau Algebras

Algebras of generalised functions in the sense of J.F. Colombeau [Colombeau 1984, 1985] are differential algebras

- that contain the vector space of distributions and
- display maximal consistency with classical analysis (in the light of L. Schwartz' impossibility result).
  In particular the construction preserves
  - the product of $\mathcal{C}^\infty$-functions
  - (Lie) derivatives of distributions.

Main ideas of the construction are

- regularisation of distributions by nets of $\mathcal{C}^\infty$-functions
- asymptotic estimates in terms of a regularisation parameter
  (quotient construction)

The (special) algebra on manifolds

- scalars: $\mathcal{G}(M) := \mathcal{E}_M(M)/\mathcal{N}(M)$

$$\mathcal{E}_M(M) := \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^{\infty}{}^{(0,1]} : \forall K\, \forall P\, \exists l : \sup_{x\in K} |Pu_\varepsilon(x)| = O(\varepsilon^{-l})\}$$

$$\mathcal{N}(M) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(M) : \forall K \quad \forall m : \sup_{x\in K} | \quad u_\varepsilon(x)| = O(\varepsilon^{m})\}$$

  notation: $\mathcal{G} \ni u = [(u_\varepsilon)_\varepsilon]$
  fine sheaf of differential algebras w.r.t. $L_X u := [(L_X u_\varepsilon)_\varepsilon]$

- tensor fields: $\mathcal{G}^r_s(M) := \mathcal{E}_M{}^r_s(M)/\mathcal{N}^r_s(M)$

$$\mathcal{G}^r_s(M) \cong \mathcal{G}(M) \otimes_{\mathcal{G}} \mathcal{T}^r_s(M) \cong L_{\mathcal{C}^\infty(M)}(\Omega^1(M)^r, \mathfrak{X}(M)^s; \mathcal{G}(M))$$
$$\cong L_{\mathcal{G}(M)}(\mathcal{G}^0_1(M)^r, \mathcal{G}^1_0(M)^s; \mathcal{G}(M))$$

  fine sheaf of finitely generated and projective $\mathcal{G}(M)$-modules

- Embeddings: $\exists$ injective sheaf morphisms (basically convolution)

$$\iota : \mathcal{T}^r_s(\_) \hookrightarrow \mathcal{D}'^r_s(\_) \hookrightarrow \mathcal{G}^r_s(\_).$$

Generalised setting for GR

- generalised metric: (technicalities on the index skipped)
  $g \in \mathcal{G}^0_2(M)$ symmetric and $\det(g)$ invertible in $\mathcal{G}$, i.e.,

$$\forall K \text{ comp. } \exists m : \inf_{p \in K} |\det(g_\varepsilon(p))| \geq \varepsilon^m \qquad (N_\varepsilon)$$

  - for all generalised points $g(\tilde{x})$ is nondegenerate as map $\tilde{\mathbb{R}}^n \times \tilde{\mathbb{R}}^n \to \tilde{\mathbb{R}}$ (pointwise generalised nondegeneracy)
  - locally there exists a representative g_ε consisting of smooth metrics and $\det(g)$ invertible in $\mathcal{G}$ (idea of smoothing)
  - g induces an isomorphism $\mathcal{G}^1_0(M) \ni X \mapsto X^\flat := g(X, \,.\,) \in \mathcal{G}^0_1(M)$
  - $\exists!$ generalised Levi-Civita connection for g
- generalised curvature $\mathrm{Riem}[g], \mathrm{Ric}[g], \mathrm{R}[g]$.
  defined via usual coordinate formulae for fixed ε
- basic $\mathcal{C}^2$-compatibility: $g_\varepsilon \to g$ in $\mathcal{C}^2$, g a vacuum solution of Einstein's equation $\Rightarrow \mathrm{Ric}[g_\varepsilon] \to 0$ in $\mathcal{D}'^1_3$.

Compatibility

- $g \in \left(H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}\right)^0_2(M)$ two ways to calculate the curvature
  - (i) gt-setting: coordinate formulae in $\mathcal{D}'$ resp. $W^{m,p}_{\mathrm{loc}}$ $\rightsquigarrow \mathrm{Riem}[g] \in {\mathcal{D}'}^1_3$
  - (ii) $\mathcal{G}$-setting: embed g via convolution with a mollifier usual formulae for fixed ε $\rightsquigarrow \mathrm{Riem}[g_\varepsilon] \in \mathcal{G}^1_3$
- Do we get the same answer?

$$\begin{array}{ccc} H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}} \ni g & \xrightarrow{*\rho_\varepsilon} & [g_\varepsilon] \in \mathcal{G} \\ \text{gt-setting} \downarrow & & \downarrow \mathcal{G}\text{-setting} \\ \mathrm{Riem}[g] & \xleftarrow{\lim_{\varepsilon\to 0}} & \mathrm{Riem}[g_\varepsilon] \end{array}$$

Embeddings and association

- scalars on $\Omega \subseteq \mathbb{R}^n$ open: $u \in \mathcal{E}'(\Omega)$

$$\begin{array}{c} u_\varepsilon := u * \rho_\varepsilon \\ \iota(u) := [(u_\varepsilon)_\varepsilon] \end{array} \quad \text{with} \quad \begin{array}{l} \rho \in \mathcal{S}(\mathbb{R}^n),\ \int \rho = 1,\ \rho_\varepsilon := \frac{1}{\varepsilon^n} \rho\left(\frac{\cdot}{\varepsilon}\right) \\ \int x^\alpha \rho(x) dx = 0\ \forall |\alpha| \geq 1 \end{array}$$

- $u \in \mathcal{D}'(\Omega)$: sheaf theoretic construction, or
  set $u_\varepsilon = u * \psi_\varepsilon, \quad \psi_\varepsilon(x) = \chi\left(\frac{x}{\sqrt{\varepsilon}}\right) \rho_\varepsilon(x), \quad \chi$ a cut-off
- ψ_ε is a strict δ-net (moderate, asymptotic vanishing moments)

  (i) $\mathrm{supp}(\psi_\varepsilon) \to \{0\}$ $(\varepsilon \to 0)$ (ii) $\int \psi_\varepsilon \to 1$ $(\varepsilon \to 0)$
  (iii) $\|\psi_\varepsilon\|_{L^1} \leq C$ for all ε (small)

- $g \in \left(H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}\right)^0_2(M)$: $g^\varepsilon_{ij} := g_{ij} * \psi_\varepsilon$, $\rightsquigarrow$ metric g_ε, $\iota(g) = [(g_\varepsilon)_\varepsilon]$
- association: $\mathcal{G} \ni u \approx v \in \mathcal{D}' :\Leftrightarrow \int u_\varepsilon \omega \to \langle v, \omega \rangle$

On the gt-class of metrics

- $H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ is an algebra
- $f \in H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ invertible $:\Leftrightarrow$ loc. uniformly bounded away from 0,

$$\forall K \text{ compact } \exists C: \ |f(x)| \geq \ C > 0 \text{ a.e. on } K$$

then f^{-1} is again loc. unif. bded away from 0

Definition (Nondegenerate gt-metrics [LeFM07], [SV08])

A gt-regular metric is a section $g \in \left(H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}\right)^0_2(M)$, which is a Semi-Riemannian metric almost everywhere. It is called nondegenerate, if

$$\forall K \text{ compact } \exists C: \ |\det g(x)| \geq \ C > 0 \text{ a.e. on } K. \qquad (N)$$

$\Rightarrow \ g^{-1} \in \left(H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}\right)^0_2(M)$ and nondegenerate, i.e.,

$\det(g^{-1})$ loc. unif. bded away from 0

Smoothing gt-metrics

Basic properties of smoothing (ψ_ε a strict δ-net)

- $f \in L^1_{\mathrm{loc}} \Rightarrow f_\varepsilon = f * \psi_\varepsilon \in \mathcal{C}^\infty(\Omega_{\psi_\varepsilon})$
- $f \in W^{m,p}_{\mathrm{loc}} \;\Rightarrow\; f_\varepsilon := f * \psi_\varepsilon \to f$ in $W^{m,p}_{\mathrm{loc}}$ for all m, $1 \le p < \infty$
- $f, h \in H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}} \;\Rightarrow\; f_\varepsilon h_\varepsilon \to fh$ in $H^1_{\mathrm{loc}} \cap L^p_{\mathrm{loc}}$ for all $p < \infty$

Lemma (Stability of the determinant)

Let g be nondegenerate, gt-regular, then

$$\det(g_\varepsilon) \;\to\; \det g \quad \textit{in} \quad H^1_{\mathrm{loc}} \cap L^p_{\mathrm{loc}} \;\; \textit{for all } p < \infty.$$

- But (N) for g does not imply (N_ε) for g_ε and $m = 0$, (N^0_ε)!

g nondegenerate gt-regular metric $\not\Rightarrow$ g_ε generalised metric

Preserving nondegeneracy (1)

problem (1): preserving positivity for scalars

- want: $0 \leq f \in H^1_{\text{loc}} \cap L^\infty_{\text{loc}}$ & loc. unif. bounded away from 0

  $\Rightarrow \ \forall K$ compact $\exists C, \varepsilon_0 : \ f_\varepsilon(x) \geq C > 0 \quad \forall x \in K, \ \varepsilon \leq \varepsilon_0 \qquad (N'_\varepsilon)$

  Then $1/f_\varepsilon$ smooth, locally uniformly bounded net, and

  $1/f_\varepsilon \to 1/f$ in $H^1_{\text{loc}} \cap L^p_{\text{loc}}$ for all $p < \infty$.

- not true if $\psi_\varepsilon \not\geq 0$, ρ with vanishing moments $\Rightarrow \rho \not\geq 0 \Rightarrow \psi_\varepsilon \not\geq 0$

Lemma (Existence of admissible mollifiers)

There exist moderate strict delta nets ρ_ε with

(i) $\text{supp}(\rho_\varepsilon) \subseteq B_\varepsilon(0)$ (ii) $\int \rho_\varepsilon(x)\,dx = 1$

(iii) $\forall j \in \mathbb{N}\ \exists \varepsilon_0 : \ \int x^\alpha \rho_\varepsilon(x)\,dx = 0$ *for all* $1 \leq |\alpha| \leq j$ *and all* $\varepsilon \leq \varepsilon_0$

(iv) $\forall \eta > 0\ \exists \varepsilon_0 : \ \int |\rho_\varepsilon(x)|\,dx \leq 1 + \eta$ *for all* $\varepsilon \leq \varepsilon_0$.

Convolution with ρ_ε provides an embedding ι_ρ into $\mathcal{G}$ with (N'_ε).

Preserving nondegeneracy (2)

problem (2): preserving nondegeneracy for metrics

- want: $\forall K$ cp. $\exists C, \varepsilon_0 : |\det(g_\varepsilon)| \geq C_K > 0 \,\forall x \in K,\ \varepsilon \leq \varepsilon_0 \quad (N^0_\varepsilon)$

Definition (Stability condition)

Let g be a gt-regular metric and $\lambda_i, \ldots, \lambda_n$ its eigenvalues.

(i) For any compact K we set $\mu_K := \min\limits_{1\leq i\leq n} \operatorname*{esinf}\limits_{x\in K} |\lambda^i(x)|$.

(ii) We call g stable if on K there is A^K continuous, such that

$$\max_{i,j} \operatorname*{essup}_{x\in K} |g_{ij}(x) - A^K_{ij}(x)| \leq C < \frac{\mu_K}{2n}.$$

Lemma (Nondegeneracy of smoothed gt-regular metrics)

Let g be a nondegenerate, stable, and gt-regular metric.
Let g_ε be a smoothing of g with an admissible mollifier $(\rho_\varepsilon)_\varepsilon$.
Then (N^0_ε) holds, and the embedding $\iota_\rho(g)$ is a gen. metric.

Stability results

Lemma (Stability of the inverse and Christoffel symbols)

Let g be a nondegenerate, stable, and gt-regular metric. Let g_ε be a smoothing of g with an admissible mollifier $(\rho_\varepsilon)_\varepsilon$.

(i) *The inverse of the smoothing $(g_\varepsilon)^{-1}$ is a smooth and locally uniformly bounded net (on rel. cp. sets for ε small), and*

$$(g_\varepsilon)^{-1} \to g^{-1} \text{ in } H^1_{\mathrm{loc}} \cap L^p_{\mathrm{loc}} \text{ for all } p < \infty.$$

In particular, for any embedding we have that $(\iota_\rho(g))^{-1} \approx g^{-1}$.

(ii) *The Christoffel symbols of the smoothing $\Gamma_{ijk}[g_\varepsilon]$, $\Gamma^i_{jk}[g_\varepsilon]$ are smooth and L^2_{loc}-bounded nets (on rel. cp. sets for ε small), and*

$$\Gamma_{ijk}[g_\varepsilon] \to \Gamma_{ijk} \text{ and } \Gamma^i_{jk}[g_\varepsilon] \to \Gamma^i_{jk} \text{ in } L^2_{\mathrm{loc}}$$

In particular, for any embedding $\Gamma_{ijk}[\iota_\rho(g)] \approx \Gamma_{ijk}[g]$ and $\Gamma^i_{jk}[\iota_\rho(g)] \approx \Gamma^i_{jk}[g]$.

Compatibility results (1)

Theorem (Compatibility of the gt- with the $\mathcal{G}$-setting)

Let g be a nondegenerate, stable, and gt-regular metric, and denote its Riemann tensor by $\mathrm{Riem}[g]$.
Let g_ε be a smoothing of g with an admissible mollifier $(\rho_\varepsilon)_\varepsilon$.
Then we have for the Riemann tensor $\mathrm{Riem}[g_\varepsilon]$ of g_ε

$$\mathrm{Riem}[g_\varepsilon] \to \mathrm{Riem}[g] \text{ in } \mathcal{D}'^1_3.$$

Hence for any embedding ι_ρ we have $\mathrm{Riem}[\iota_\rho(g)] \approx \mathrm{Riem}[g]$.

$$\begin{array}{ccc} \underset{\text{nondeg., stable}}{H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}} \ni g & \xrightarrow{*\iota_\rho \text{ admissible}} & [g_\varepsilon] \in \mathcal{G} \\ \text{gt-setting} \downarrow & & \downarrow \mathcal{G}\text{-setting} \\ \mathrm{Riem}[g] & \xleftarrow{\approx} & \mathrm{Riem}[g_\varepsilon] \end{array}$$

Compatibility results (2)

for other curvature quantities in the gt-setting use following trick

$$\begin{aligned}\frac{1}{2}g^{rs}R^i_{jkl} &= g^{rs}(\partial_{[l}\Gamma^i_{k]j} + g^{rs}\Gamma^i_{m[l}\Gamma^m_{k]j}) \\ &= \partial_{[l}(g^{rs}\Gamma^i_{k]j}) - (\partial_{[l}g^{rs})\Gamma^i_{k]j} + g^{rs}\Gamma^i_{m[l}\Gamma^m_{k]j},\end{aligned}$$

analogously for any product of the form $\otimes_m g \otimes_l g^{-1} \otimes \mathrm{Riem}[g]$

Corollary (Compatibility for curvature quantities)

Let g be a nondegenerate, stable, and gt-regular metric. Let g_ε be a smoothing of g with an admissible mollifier $(\rho_\varepsilon)_\varepsilon$. Then we have ($m, l \in \mathbb{N}$)

$$\otimes_m g_\varepsilon \otimes_l g_\varepsilon^{-1} \otimes \mathrm{Riem}[g_\varepsilon] \to \otimes_m g \otimes_l g^{-1} \otimes \mathrm{Riem}[g] \text{ in } \mathcal{D}'^{1+2l}_{3+2m}.$$

In particular, we have for any embedding ι_ρ

$$\mathrm{Ric}[\iota_\rho(g)] \approx \mathrm{Ric}[g],\ R[\iota_\rho(g)] \approx R[g],\ W[\iota_\rho(g)] \approx W[g].$$

Discussion

Relation to older stability results: $(g_n)_n$ gt-regular sequence

- [LeFloch&Mardare, 07]
  $g_n \to g$ in H^1_{loc}, $g_n^{-1} \to g^{-1}$ in $L^\infty_{\mathrm{loc}} \Rightarrow \mathrm{Riem}[g_n] \to \mathrm{Riem}[g]$, in $\mathcal{D}'^1_3$.
  for smoothings via convolution $g_n^{-1} \not\to g^{-1}$ in L^∞_{loc}.
- [Geroch&Traschen, 87]
  $g_n \to g$ in H^1_{loc}, $g_n^{-1} \to g^{-1}$ in L^2_{loc}, g_n, g_n^{-1} bded in L^∞_{loc} $(*)$
  $$\Rightarrow \mathrm{Riem}[g_n] \to \mathrm{Riem}[g] \text{ in } \mathcal{D}'^1_3.$$

Existence of approximating sequences with $(*)$

- [Geroch&Traschen, 87]
  Only for continuous g, open for general g.
- Positive answer for general g by the above Theorem.

Further prospects

- Jump conditions along singular hypersurfaces in the spirit of [LeFloch&Mardare, 07], [Lichnerowicz, 55-79] in the generalised setting plus compatibility.

  Applications to gravitational shock waves.
- Regularity of generalised solutions to wave equations in singular space-times ([Grant, Mayerhofer, S., 08]).
- Compatibility for connections in fibre bundles ([Kunzinger, Vickers, S., 05]).

Some References

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Conference Announcement

GF2009

International Conference on Generalized Functions

August 31 – September 4, 2009
Vienna, Austria
http://www.mat.univie.ac.at/˜gf2009