The Singularity Theorems in Low Regularity

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Overview

Long-term project on

Lorentzian geometry and general relativity

with metrics of low regularity

jointly with

- ‘theoretical branch’ (Vienna & U.K.):
  Melanie Graf, James Grant, Günther Hörmann, Mike Kunzinger, Clemens Sämann, James Vickers
- ‘exact solutions branch’ (Vienna & Prague):
  Jiří Podolský, Clemens Sämann, Robert Švarc

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Remarks on low Regularity

Why low regularity?

1. Physics: Realistic matter models $\rightsquigarrow$ $\mathbf{g} \in \mathcal{C}^{1,1}$ (derivs. loc. Lip.)
2. Analysis: ivp $\mathbf{g} \in H^{5/2}(M), \mathcal{C}^1(\Sigma)$, recent big improvements

The challenge

Physics and Analysis	vs.	**Lorentzian geometry**
want/need low regularity		needs high regularity

Regularity matters [Hartman&Wintner, 1951]

$$\mathbf{g}_{ij}(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 - |x|^\lambda \end{pmatrix} \quad 1 < \lambda < 2, \; \mathbf{g} \in \mathcal{C}^{1,\lambda-1}$$

- minimising curves not unique, even locally
- geodesics that are non-minimising between any of its points

The Lorentzian character matters

Riemannian case (added regularity of geodesics)

- $\mathbf{g} \in \mathcal{C}^0 \implies$ shortest (Lipschitz) curves exist [Hilbert, 1899]
- $\mathbf{g} \in \mathcal{C}^{0,\alpha} \implies$ all shortest curves are $\mathcal{C}^{1,\beta}$ with $\beta = \frac{\alpha}{2-\alpha}$ (optimal) [Calabi, Hartman, 70], [Lytchak, Yaman, 06]
  in particular $\alpha = 1 = \beta$ and $\ddot{\gamma} = 0$ a.e.

Lorentzian case

- no length structure $\rightsquigarrow$ use geodesic equation.
- $\mathbf{g} \in \mathcal{C}^{0,1} \implies$ geodesics in the sense of Fillipov are $\mathcal{C}^{1,1}$ [S., 2014]
- Is there an analogue of the Lytchak-Yaman result?
- causality goes severely wrong,
  light cones "bubble up" below $\mathcal{C}^{0,1}$ [CG,12]

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Regularity for the singularity theorems of GR

Pattern singularity theorem **[Senovilla, 98]**

In a $\mathcal{C}^2$-spacetime the following are incompatible

(i) Energy condition
(ii) Causality condition
(iii) Initial or boundary condition
(iv) Causal geodesic completeness

Theorem allows (i)–(iv) and $g \in \mathcal{C}^{1,1}$. But $\mathcal{C}^{1,1}$-spacetimes

- are physically reasonable models
- are not *really* singular (curvature bounded)
- still allow unique solutions of geodesic eq. $\rightsquigarrow$ formulation sensible

Moreover below $\mathcal{C}^{1,1}$ we have

- unbded curv., non-unique geos, no convexity $\rightsquigarrow$ 'really singular'

Hence $\mathcal{C}^{1,1}$ is the natural regularity class for singularity theorems!

The classical Theorems

Theorem **[Hawking, 1967]**

A $\mathcal{C}^2$ -spacetime is future timelike geodesically incomplete, if

(i) $\mathrm{Ric}\,(X,X) \geq 0$ for every timelike vector X

(ii) There exists a compact space-like hypersurface S in M

(iii) The unit normals to S are everywhere converging

Theorem **[Penrose, 1965]**

A $\mathcal{C}^2$ -spacetime is future null geodesically incomplete, if

(i) $\mathrm{Ric}\,(X,X) \geq 0$ for every null vector X

(ii) There exists a non-compact Cauchy hypersurface S in M

(iii) There exists a trapped surface $\mathcal{T}$
(cp. achronal spacelike 2-srf. w. past-pt. timelike mean curvature)

The $\mathcal{C}^{1,1}$-Theorems

Theorem **[Kunzinger, S., Stojković, Vickers, 2015]**

A $\mathcal{C}^{1,1}$-spacetime is future timelike geodesically incomplete, if

(i) $\mathrm{Ric}\,(X,X) \geq 0$ for every smooth timelike local vector field X

(ii) There exists a compact space-like hypersurface S in M

(iii) The unit normals to S are everywhere converging

Theorem **[Kunzinger, S., Vickers, 2015]**

A $\mathcal{C}^{1,1}$-spacetime is future null geodesically incomplete, if

(i) $\mathrm{Ric}\,(X,X) \geq 0$ for every Lip-cont. local null vector field X

(ii) There exists a non-compact Cauchy hypersurface S in M

(iii) There exists a trapped surface $\mathcal{T}$
(cp. achronal spacelike 2-srf. w. past-pt. timelike mean curvature)

Obstacles in the $\mathcal{C}^{1,1}$-case

- No appropriate version of calculus of variations available (second variation, maximizing curves, focal points, index form, ...)
- $\mathcal{C}^2$-causality theory rests on local equivalence with Minkowski space. This requires good properties of exponential map.

$\rightsquigarrow$ big parts of causality theory have to be redone

- Ricci tensors is only L^∞

$\rightsquigarrow$ problems with energy conditions

strategy:

- Proof that the exponential map is a bi-Lipschitz homeo
- Re-build causality theory for $\mathcal{C}^{1,1}$-metrics regularisation adapted to causal structure replacing calculus of var.
- use surrogate energy condition

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The exponential map in low regularity

Optimal regularity

- $\mathbf{g} \in \mathcal{C}^{1,1} \Rightarrow exp_p$ local homeo [Whitehead, 1932]
- $\mathbf{g} \in \mathcal{C}^{1,1} \Rightarrow exp_p$ local bi-Lipschitz homeo

- [KSS,14]: regularisation & comparison methods [LeFloch&Chen,08]
- [Minguzzi,15]: Picard-Lindelöf approx. & Lip. inv. funct. thm.

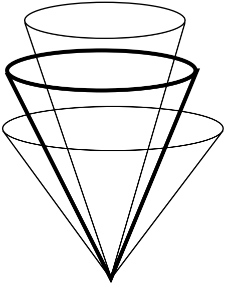
Consequences in $\mathcal{C}^{1,1}$

- convexity is okay (remember [HW, 51]!)
- Gauss lemma holds

$\rightsquigarrow$ bulk of causality theory remains true [CG,12, KSSV,14, Ming.,15]

Chrusciel-Grant regularization of the metric

Regularisation adapted to the causal structure [CG,12], [KSSV, 14]



Sandwich null cones of $\mathbf{g}$ between null cones of two approximating families of smooth metrics so that

$$\check{\mathbf{g}}_\varepsilon \prec \mathbf{g} \prec \hat{\mathbf{g}}_\varepsilon.$$

- applies to continuous metrics
- local convolution plus small shift

Properties of the approximations for $\mathbf{g} \in \mathcal{C}^{1,1}$

(i) $\check{\mathbf{g}}_\varepsilon$, $\hat{\mathbf{g}}_\varepsilon \to \mathbf{g}$ locally in C^1

(ii) $D^2\check{\mathbf{g}}_\varepsilon$, $D^2\hat{\mathbf{g}}_\varepsilon$ locally uniformly bded. in ε, but $\mathrm{Ric}[\mathbf{g}_\varepsilon] \not\to \mathrm{Ric}[\mathbf{g}]$

Surrogate energy condition (Hawking case)

Lemma **[KSSV, 15]**

Let $(M, \mathbf{g})$ be a $\mathcal{C}^{1,1}$-spacetime satisfying the energy condition

$$\mathrm{Ric}\,[\mathbf{g}]\,(X, X) \geq 0 \quad \text{for all timelike local } \mathcal{C}^\infty\text{-vector fields } X.$$

Then for all $K \subset\subset M \;\; \forall C > 0 \;\; \forall \delta > 0 \;\; \forall \kappa < 0 \;\; \forall \varepsilon$ small

$$\mathrm{Ric}\,[\check{\mathbf{g}}_\varepsilon](X, X) > -\delta \quad \forall X \in TM|_K : \; \check{\mathbf{g}}_\varepsilon(X, X) \leq \kappa, \; \|X\|_h \leq C.$$

Proof.

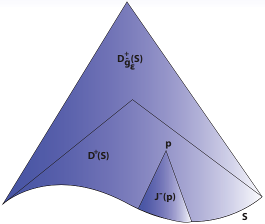
- $\check{g}_\varepsilon - g * \rho_\varepsilon \to 0$ in $\mathcal{C}^2 \rightsquigarrow$ only consider $g_\varepsilon := g * \rho_\varepsilon$
- $R_{jk} = R^i_{jki} = \partial_{x^i}\Gamma^i_{kj} - \partial_{x^k}\Gamma^i_{ij} + \Gamma^i_{im}\Gamma^m_{kj} - \Gamma^i_{km}\Gamma^m_{ij}$
- Blue terms$|_\varepsilon$ converge uniformly
- For red terms use variant of Friedrich's Lemma:

$$\rho_\varepsilon \geq 0 \implies \big(\mathrm{Ric}[\mathbf{g}](X, X)\big) * \rho_\varepsilon \geq 0$$

$$\big(\mathrm{Ric}[\mathbf{g}](X, X)\big) * \rho_\varepsilon - \mathrm{Ric}[\mathbf{g}_\varepsilon](X, X) \to 0 \text{ unif.}$$

The $\mathcal{C}^{1,1}$-proof (Hawking case)

- $D^+(S) \subseteq D^+_{\check{\mathbf{g}}_\varepsilon}(S)$:



- Limiting argument $\Rightarrow \exists$ maximising $\mathbf{g}$-geodesic γ for all $p \in D^+(S)$ and $\gamma = \lim \gamma_{\check{\mathbf{g}}_\varepsilon}$ in $\mathcal{C}^1$
- Surrogate energy condition for $\check{\mathbf{g}}_\varepsilon$ and Raychaudhury equation
  $\Rightarrow D^+(S)$ relatively compact
  otherwise $\exists$ $\check{\mathbf{g}}_\varepsilon$-focal pt. too early
  $\Rightarrow H^+(S) \subseteq \overline{D^+(S)}$ compact
- Derive a contradiction as in the $\mathcal{C}^\infty$-case using $\mathcal{C}^{1,1}$-causality

Surrogate energy condition (Penrose case)

Lemma **[KSV, 15]**

Let $(M, \mathbf{g})$ be a $\mathcal{C}^{1,1}$-spacetime satisfying the energy condition

$$\mathrm{Ric}\,[\mathbf{g}]\,(X, X) \geq 0 \quad \text{for every local Lip. null vector field } X.$$

Then for all $K \subset\subset M$ $\forall C > 0$ $\forall \delta > 0$ $\exists \eta > 0$ s.t. we have

$$\mathrm{Ric}\,[\hat{\mathbf{g}}_\varepsilon](X, X) > -\delta$$

for all $p \in K$ and all $X \in T_pM$ with $\|X\|_h \leq C$ which are close to a $\mathbf{g}$-null vector in the sense that

$$\exists Y_0 \in TM|_K \quad \mathbf{g}\text{-null}, \quad \|Y_0\|_h \leq C, \quad d_h(X, Y_0) \leq \eta.$$

The $\mathcal{C}^{1,1}$-proof (Penrose case)

- Choose $\hat{\mathbf{g}}_\varepsilon$ globally hyperbolic (stability [NM,11], [S,15])
- Surrogate energy condition is strong enough to guarantee that

$$E^+_\varepsilon(\mathcal{T}) = J^+_\varepsilon(\mathcal{T}) \setminus I^+_\varepsilon(\mathcal{T}) \quad \text{is relatively compact}$$

  in case of null geodesic completeness
- $\hat{\mathbf{g}}_\varepsilon$ globally hyperbolic $\Rightarrow$

$$E^+_\varepsilon(\mathcal{T}) = \partial J^+_\varepsilon(\mathcal{T}) \text{ is a } \hat{\mathbf{g}}_\varepsilon\text{-achronal, compact } \mathcal{C}^0\text{-hypersrf.}$$

- $\mathbf{g} < \hat{\mathbf{g}}_\varepsilon \Rightarrow E^+_\varepsilon(\mathcal{T})$ is $\mathbf{g}$-achronal
- derive usual (topological) contradiction

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Outlook

Related results, comparison geometry

- volume estimates for nullcones [Grant, 2011]
- volume comparison with (warped product) model spacetimes
  $\rightsquigarrow$ new $\mathcal{C}^\infty$-proof of Hawking's theorem [Grant, Treude, 2013]
- comparison geometry proof of $\mathcal{C}^{1,1}$-Hawking theorem [Graf, 2016]
- rigidity results for singularity thms [Graf, 2016]

Current research

- Comparison approach to Penrose singularity theorem
  Evolve trapped surface along null geodesics, quantify area, should also give new proof in $C^{1,1}$.
- Mid term goal: ▸ Hawking & Penrose singularity theorem in $C^{1,1}$:
  Will require completely new methods.

Some related Literature

[CG,12] P.T. Chruściel, J.D.E. Grant, *On Lorentzian causality with continuous metrics.* CQG 29 (2012)

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[S,14] R. Steinbauer, *Every Lipschitz metric has C^1-geodesics.* CQG 31, 057001 (2014)

Thank you for your attention!

Lemma **[Hawking and Penrose, 1967]**

In a causally complete $\mathcal{C}^2$-spacetime, the following cannot all hold:

1. Every inextendible causal geodesic has a pair of conjugate points
2. M contains no closed timelike curves and
3. there is a future or past trapped achronal set S

Theorem

A $\mathcal{C}^2$-spacetime M is causally incomplete if Einstein's eqs. hold and

1. M contains no closed timelike curves
2. M satisfies an energy condition
3. *Genericity*: nontrivial curvature at some pt. of any causal geodesic
4. M contains either
   - a trapped surface
   - some p s.t. convergence of all null geodesics changes sign in the past
   - a compact spacelike hypersurface Back

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