The Singularity Theorems of General Relativity in Low Regularity

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Overview

Long-term project on

Lorentzian geometry and general relativity

with metrics of low regularity

Contents

1. Intro: The basic setup of general relativity
2. Why low regularity: Physics & analysis vs. geometry
3. $\mathcal{C}^{0,1}$-metrics and below:
   completeness of impulsive gravitational waves
4. $\mathcal{C}^{1,1}$-metrics: causality theory and
   the Penrose and Hawking singularity theorems

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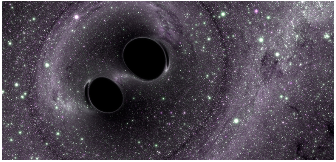
The basic physical setup of General Relativity

- Albert Einstein's theory of
  space, time and gravitation
  created 100 years ago
- current description of
  gravitation & universe at large



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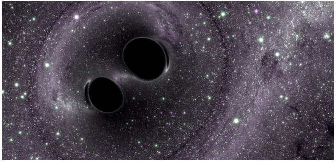

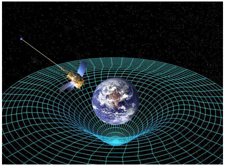

- **geometric theory**
  due to Galileo's principle of equivalence:
  all bodies fall the same in a gravitational field
  $\rightsquigarrow$ gravitational field as property of the surrounding space
- Gravitational field influences how we measure lengths and angles
  $\rightsquigarrow$ **curvature of space and time**

The basic mathematical setup of GR, 1

Lorentzian geometry (basic geometric setup)

- smooth 4-dimensional spacetime manifold M
- smooth **spacetime metric g**:
  symmetric, non-degenerate scalar product in any tangent space

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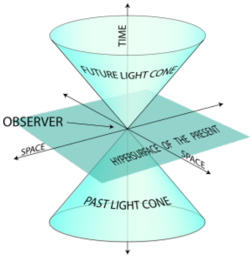
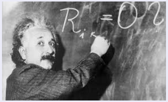


- lightcone in any T_pM:
  timelike, null (causal), spacelike vectors
- Particles travel on timelike curves c
  light travels on null curves
- chronological/causal future $I^+(p)$ / $J^+(p)$
  $\rightsquigarrow$ causality theory
- Free-falling particles/photons move on geodesics: $\ddot{\gamma} = 0$

The basic mathematical setup of GR, 2

Field equations (basic physical/analytical setup)

$$\underbrace{\mathbf{R}_{ij}[\mathbf{g}] - \frac{1}{2}\mathbf{R}[\mathbf{g}]\,\mathbf{g}_{ij} + \Lambda\,\mathbf{g}_{ij}}_{\text{spacetime curvature}} = \underbrace{8\pi\mathbf{T}_{ij}}_{\text{matter}}$$



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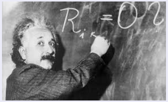

- Ricci-tensor $\mathbf{R}_{ij}$, scalar curvature $\mathbf{R}$ built from **Riemann tensor**

$$R_{XY}Y = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$$

- locally: $R^m{}_{ikp} = \partial_k \Gamma^m{}_{ip} - \partial_p \Gamma^m{}_{ik} + \Gamma^a{}_{ip}\Gamma^m{}_{ak} - \Gamma^a{}_{ik}\Gamma^m{}_{ap}$
  and Christoffel symbols $\Gamma^i_{jk} = \mathbf{g}^{il}\Gamma_{ljk} = \frac{1}{2}\mathbf{g}^{il}(\partial_k \mathbf{g}_{lj} + \partial_j \mathbf{g}_{kl} - \partial_l \mathbf{g} jk)$

$$\Longrightarrow \mathbf{R}_{ij}, \mathbf{R} \sim \partial^2 \mathbf{g} + (\partial \mathbf{g})^2$$

- coupled system of 10 quasi-linear PDEs of 2$^{\text{nd}}$ order for $\mathbf{g}$

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Why Low Regularity?

(1) Realistic matter—Physics

- want discontinuous matter configurations $\rightsquigarrow$ $\mathbf{T} \notin \mathcal{C}^0 \implies \mathbf{g} \notin \mathcal{C}^2$
- finite jumps in $\mathbf{T} \rightsquigarrow \mathbf{g} \in \mathcal{C}^{1,1}$ (derivatives locally Lipschitz)
- more extreme situations (impulsive waves): $\mathbf{g}$ piecew. $\mathcal{C}^3$, globally $\mathcal{C}^0$

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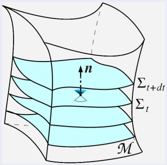
(2) Initial value problem—Analysis

Local existence and uniqueness Thms.
for Einstein eqs. in terms of Sobolev spaces

- classical [CB,HKM]: $\mathbf{g} \in H^{5/2} \implies \mathcal{C}^1(\Sigma)$
- recent big improvements [K,R,M,S]: $\mathbf{g} \in \mathcal{C}^0(\Sigma)$



Regularity matters

Riemannian counterexample [Hartman&Wintner, 51]

$$\mathbf{g}_{ij}(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1-|x|^\lambda \end{pmatrix} \qquad \text{on } (-1,1)\times\mathbb{R}\subseteq\mathbb{R}^2$$

- $\lambda \in (1,2) \implies \mathbf{g} \in \mathcal{C}^{1,\lambda-1}$ Hölder, slightly below $\mathcal{C}^{1,1}$
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BUT

- shortest curves from (0,0) to (0,y) are two symmetric arcs

  $\rightsquigarrow$ minimising curves not unique, even locally

- the y-axis is a geodesic which is

  non-minimising between any of its points

The Lorentzian character matters

Riemannian case (added regularity of geodesics)

- $\mathbf{g} \in \mathcal{C}^0 \implies$ shortest (Lipschitz) curves exist [Hilbert, 1899]
- $\mathbf{g} \in \mathcal{C}^{0,\alpha} \implies$ all shortest curves are $\mathcal{C}^{1,\beta}$ with $\beta = \frac{\alpha}{2-\alpha}$ (optimal) [Calabi, Hartman, 70], [Lytchak, Yaman, 06]
  in particular $\alpha = 1 = \beta$ and $\ddot{\gamma} = 0$ a.e.
- $\mathbf{g} \in \mathcal{C}^1 \implies$ all shortes curves satisfy $\ddot{\gamma} = 0$ and $\gamma \in \mathcal{C}^2$

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Lorentzian case

- no length structure $\rightsquigarrow$ use geodesic equation.
- $\mathbf{g} \in \mathcal{C}^{0,1} \implies$ geodesics in the sense of Fillipov are $\mathcal{C}^{1,1}$ [S., 2014]
- counterexample [Kunzinger, Sämann, very recent!]
  $\mathbf{g} \in \mathcal{C}^{0,\frac{1}{2}}$ where
  - no longest curve is $\mathcal{C}^1$, and
  - $\exists$ longest curve which is not even piecewise $\mathcal{C}^1$

GR and low regularity

The challenge

Physics and Analysis	vs.	**Lorentzian geometry**
want/need low regularity		needs high regularity to maintain standard results

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Lorentzian geometry and regularity

- classically $\mathbf{g} \in \mathcal{C}^\infty$, for all practical purposes $\mathbf{g} \in \mathcal{C}^2$
- things go wrong below $\mathcal{C}^2$
  - convexity goes wrong for $\mathbf{g} \in \mathcal{C}^{1,\alpha}$ $(\alpha < 1)$ [HW, 51]
  - causality goes wrong, light cones "bubble up" for $\mathbf{g} \in \mathcal{C}^0$ [CG, 12]

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Impulsive gravitational waves

- model short but strong pulses of gravitational radiation propagating in Minkowski or (anti-)de Sitter universes
- relevant models of ultrarelativistic particle

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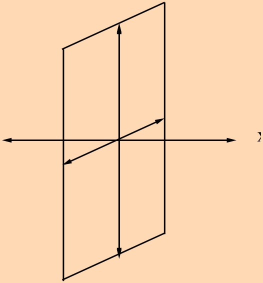
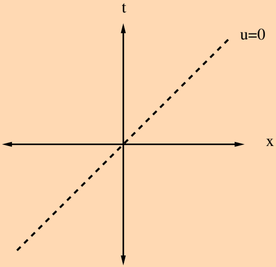


ultrarelativistic particle



spacetime diagr. ($\Lambda = 0$)

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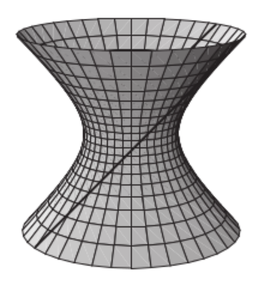
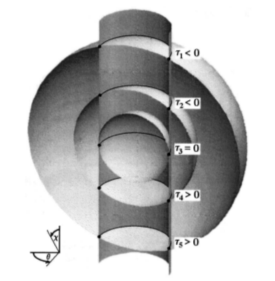


de Sitter universe



propagating wave

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Metric e.g. in the non-expanding case (coords $(U, V, Z, \bar{Z})$)

$$ds^2 = \frac{2\,|dZ + U_+(H_{,Z\bar{Z}}dZ + H_{,\bar{Z}\bar{Z}}d\bar{Z})|^2 - 2\,dUdV}{[\,1 + \frac{1}{6}\Lambda(Z\bar{Z} - UV - U_+(H - ZH_{,Z} - \bar{Z}H_{,\bar{Z}}))\,]^2}$$

Geodesics: regularity, matching, completeness

$\mathcal{C}^1$-matching of the geodesics in impulsive grav. waves

- Physicists match geodesics of background across wave-surface.
- Only possible if geodesics — are $\mathcal{C}^1$ across the wave-surface
  — are unique

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Uniquenes criteria for Filippov solutions

plus explicit use of the respective geometry of the solutions.

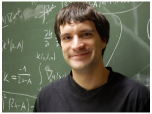
Completeness results

Prague Relativity Group

Diana Vienna

Jiří Podolský

Robert Švarc



Clemens Sämann

Alexander Lecke

- $\mathcal{C}^{0,1}$, $\Lambda = 0$, non-exp. [Lecke, S., Švarc, 14]
- $\mathcal{C}^{0,1}$, $\Lambda \neq 0$, non-exp. [Podolský, Sämann, S., Švarc, 15]
- $\mathcal{D}'$, $\Lambda \neq 0$, non-exp. [Sämann, S., Lecke, Podolský, 16]
- $\mathcal{C}^{0,1}$, $\Lambda \neq 0$, expanding [Podolský, Sämann, S., Švarc, 16]
- $\mathcal{D}'$, general non-flat wave-surface [Sämann, S., 12, 15]
- $\mathcal{D}'$, gyratons [Podolský, S., Švarc, 14] [Podolský, Sämann, S., Švarc, 16]

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Singularities in GR

- singularities occur in exact solutions; high degree of symmetries
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Theorem (Pattern singularity theorem [Senovilla, 97])

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(i) *Energy condition*
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- (iii) initial condition $\rightsquigarrow$ causal geodesics start focussing
- (i) energy condition $\rightsquigarrow$ focussing goes on $\rightsquigarrow$ focal point
- (ii) causality condition $\rightsquigarrow$ no focal points
- way out: one causal geodesic has to be incomplete, i.e., $\neg$ (iv)

The classical theorems

Theorem ([Penrose, 1965] Gravitational collapse)

A $\mathcal{C}^2$-spacetime is future null geodesically incomplete, if

(i) $Ric(X, X) \geq 0$ *for every null vector* X

(ii) *There exists a non-compact Cauchy hypersurface* S *in* M

(iii) *There exists a trapped surface*
(cp. achronal spacelike 2-srf. w. past-pt. timelike mean curvature)

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Theorem ([Hawking, 1967] Big Bang)

A $\mathcal{C}^2$-spacetime is future timelike geodesically incomplete, if

(i) $Ric(X, X) \geq 0$ *for every timelike vector* X

(ii) *There exists a compact space-like hypersurface* S *in* M

(iii) *The unit normals to* S *are everywhere converging,* $\theta := -tr\,\mathbf{K} < 0$.

Hawking's Thm: proof strategy ($\mathcal{C}^2$-case)

- **Analysis**: θ evolves along the normal geodesic congruence of S by *Raychaudhury's* equation

$$\theta' + \frac{\theta^2}{3} + \mathrm{Ric}(\dot\gamma, \dot\gamma) + \mathrm{tr}(\sigma^2) = 0$$

  - (i) $\implies \theta' + (1/3)\theta^2 \leq 0 \implies (\theta^{-1})' \geq 1/3$
  - (iii) $\implies \theta(0) < 0 \implies \theta \to \infty$ in finite time $\implies$ focal point
- **Causality theory**: $\exists$ longest curves in the Cauchy development $\implies$ no focal points in the Cauchy development
- **completeness** $\implies \overline{D^+(S)} \subseteq exp([0,T]\cdot \mathbf{n}_S)\ldots$ compact $\implies$ horizon $H^+(M)$ compact, $\rightsquigarrow$ 2 possibilities

(1) $H^+(M) = \emptyset$. Then $I^+(S) \subseteq D^+(S) \implies$ timlike incomplete $\lightning$

(2) $H^+(M) \neq \emptyset$ compact $\implies p \mapsto d(S,p)$ has min on $H^+(S)$
But from every point in $H^+(M)$ there starts a past null generator γ (inextendible past directed null geodesic contained in $H^+(S)$) and $p \mapsto d(S,p)$ strictly decreasing along $\gamma \implies$ unbounded $\lightning$

Why go to $\mathcal{C}^{1,1}$?

Recall:

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Theorem allows (i)–(iv) and $g \in \mathcal{C}^{1,1}$. But $\mathcal{C}^{1,1}$-spacetimes

- are physically reasonable models
- are not *really* singular (curvature bounded)
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- unbounded curvature, non-unique geodesics $\rightsquigarrow$ 'really singular'

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Hence $\mathcal{C}^{1,1}$ is the natural regularity class for singularity theorems!

Obstacles in the $\mathcal{C}^{1,1}$-case

- No appropriate version of calculus of variations available (second variation, maximizing curves, focal points, index form, ...)
- $\mathcal{C}^2$-causality theory rests on local equivalence with Minkowski space. This requires good properties of exponential map.

$\rightsquigarrow$ big parts of causality theory have to be redone

- Ricci tensors is only L^∞

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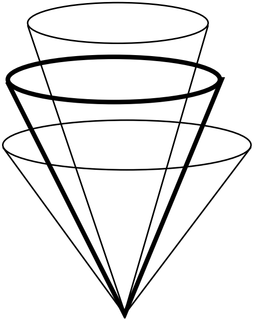
strategy:

- Employ regularisation adapted to causal structure
- Avoid calculus of variations
- Re-build causality theory for $\mathcal{C}^{1,1}$-metrics

Chrusciel-Grant regularization of the metric

Regularisations of the metric adapted to the causal structure

[Chrusciel, Grant, 12], [KSSV, 14]



$\mathbf{g} \prec \mathbf{h} \;:\Leftrightarrow$
$\mathbf{g}(X,X) \leq 0 \Rightarrow \mathbf{h}(X,X) < 0$

$g \in \mathcal{C}^0$: $\forall \varepsilon > 0 \;\; \exists \check{\mathbf{g}}_\varepsilon, \hat{\mathbf{g}}_\varepsilon \in \mathcal{C}^\infty$:

$$\check{\mathbf{g}}_\varepsilon \prec \mathbf{g} \prec \hat{\mathbf{g}}_\varepsilon,$$

$$d_h(\check{\mathbf{g}}_\varepsilon, \mathbf{g}) + d_h(\hat{\mathbf{g}}_\varepsilon, \mathbf{g}) < \varepsilon$$

$\mathbf{g} \in \mathcal{C}^{1,1}$, $\mathbf{g}_\varepsilon$ one of the above,

(i) $\mathbf{g}_\varepsilon \to \mathbf{g}$ in $\mathcal{C}^1$

(ii) $D^2\mathbf{g}_\varepsilon$ loc. unif. bded. in ε

(iii) but $\mathrm{Ric}[\mathbf{g}_\varepsilon] \not\to \mathrm{Ric}[\mathbf{g}]$

The exponential map in low regularity

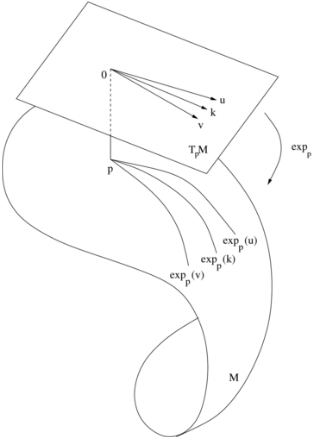
- $exp_p: \ T_pM \ni v \mapsto \gamma_v(1) \in M$, where γ_v is the (unique) geodesic starting at p in direction of v
- $\mathbf{g} \in \mathcal{C}^2 \ \Rightarrow exp_p$ local diffeo
- $\mathbf{g} \in \mathcal{C}^{1,1} \Rightarrow exp_p$ loc. homeo [W,32]

Optimal regularity

$\mathbf{g} \in \mathcal{C}^{1,1} \Rightarrow exp_p$ bi-Lipschitz homeo

- [KSS,14]: regularisation & comparison geometry
- [Minguzzi,15]: refined ODE methods

$\rightsquigarrow$ bulk of causality theory remains true in $\mathcal{C}^{1,1}$ [CG,12, KSSV,14, Ming.,15]



Surrogate energy condition

Lemma (Regularising Ricci-curvature [KSSV, 15])

Let $(M, \mathbf{g})$ be a $\mathcal{C}^{1,1}$-spacetime satisfying the energy condition

$$Ric[\mathbf{g}](X,X) \geq 0 \quad \textit{for every timelike Lipschitz vector field } X.$$

Then for all $K \subset\subset M$ $\forall C > 0$ $\forall \delta > 0$ $\forall \kappa < 0$ $\forall \varepsilon$ small

$$Ric[\check{\mathbf{g}}_\varepsilon](X,X) > -\delta \quad \forall X \in TM|_K : \; \check{\mathbf{g}}_\varepsilon(X,X) \leq \kappa, \; \|X\|_h \leq C.$$

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Proof.

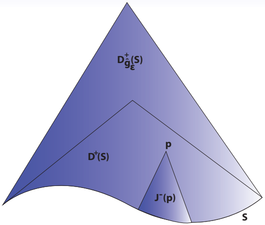
- $\check{g}_\varepsilon - g * \rho_\varepsilon \to 0$ in $\mathcal{C}^2 \rightsquigarrow$ only consider $g_\varepsilon := g * \rho_\varepsilon$
- $R_{jk} = R^i_{jki} = \partial_{x^i}\Gamma^i_{kj} - \partial_{x^k}\Gamma^i_{ij} + \Gamma^i_{im}\Gamma^m_{kj} - \Gamma^i_{km}\Gamma^m_{ij}$
- Blue terms$|_\varepsilon$ converge uniformly
- For red terms use variant of Friedrich's Lemma:

$$\rho_\varepsilon \geq 0 \implies \big(\mathrm{Ric}[\mathbf{g}](X, X)\big) * \rho_\varepsilon \geq 0$$

$$\big(\mathrm{Ric}[\mathbf{g}](X, X)\big) * \rho_\varepsilon - \mathrm{Ric}[\mathbf{g}_\varepsilon](X, X) \to 0 \text{ uniformly}$$

The $\mathcal{C}^{1,1}$-proof

- $D^+(S) \subseteq D^+_{\check{\mathbf{g}}_\varepsilon}(S)$:



- Limiting argument $\Rightarrow \exists$ maximising $\mathbf{g}$-geodesic γ for all $p \in D^+(S)$ and $\gamma = \lim \gamma_{\check{\mathbf{g}}_\varepsilon}$ in $\mathcal{C}^1$
- Surrogate energy condition for $\check{\mathbf{g}}_\varepsilon$ and Raychaudhury equation
  $\Rightarrow D^+(S)$ relatively compact
  otherwise $\exists$ $\check{\mathbf{g}}_\varepsilon$-focal pt. too early
  $\Rightarrow H^+(S) \subseteq \overline{D^+(S)}$ compact
- Derive a contradiction as in the $\mathcal{C}^\infty$-case using $\mathcal{C}^{1,1}$-causality

The $\mathcal{C}^{1,1}$-theorems

Theorem ([Hawking, 1967] Big Bang)

A $\mathcal{C}^2$ -spacetime is future timelike geodesically incomplete, if

(i) *$Ric(X, X) \geq 0$ for every timelike vector X*

(ii) *There exists a compact space-like hypersurface S in M*

(iii) *The unit normals to S are everywhere converging*

The $\mathcal{C}^{1,1}$-theorems

Theorem ([Kunzinger, S., Stojković, Vickers, 2015])

A $\mathcal{C}^{1,1}$-spacetime is future timelike geodesically incomplete, if

(i) $Ric(X, X) \geq 0$ *for every smooth timelike local vector field* X

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Theorem ([Penrose, 1965] Gravitational collapse)

A $\mathcal{C}^2$ -spacetime is future null geodesically incomplete, if

(i) $Ric(X, X) \geq 0$ *for every null vector* X

(ii) *There exists a non-compact Cauchy hypersurface* S *in* M

(iii) *There exists a trapped surface*
(cp. achronal spacelike 2-srf. w. past-pt. timelike mean curvature)

The $\mathcal{C}^{1,1}$-theorems

Theorem ([Kunzinger, S., Stojković, Vickers, 2015])

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Theorem ([Kunzinger, S., Vickers, 2015])

A $\mathcal{C}^{1,1}$-spacetime is future null geodesically incomplete, if

(i) $Ric(X,X) \geq 0$ *for every Lip-cont. local null vector field* X

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(cp. achronal spacelike 2-srf. w. past-pt. timelike mean curvature)

The Hawking & Penrose Singularity Theorem

Lemma [Hawking and Penrose, 1967]

In a causally complete $\mathcal{C}^2$-spacetime, the following cannot all hold:

1. Every inextendible causal geodesic has a pair of conjugate points
2. M contains no closed timelike curves and
3. there is a future or past trapped achronal set S

Theorem

A $\mathcal{C}^2$-spacetime M is causally incomplete if Einstein's eqs. hold

1. M contains no closed timelike curves
2. M satisfies an energy condition
3. *Genericity*: nontrivial curvature at some pt. of any causal geodesic
4. M contains either
   - a trapped surface
   - some p s.t. convergence of all null geodesics changes sign in the past
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Outlook

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Lorentzian comparison geometry

- volume comparison with (warped product) model spacetimes
  $\rightsquigarrow$ comparison geometry proof of Hawking's theorem in $\mathcal{C}^\infty$
  [Grant, Treude, 2013]
- comparison geometry proof of $\mathcal{C}^{1,1}$-Hawking theorem [Graf, 2016]
- volume estimates for nullcones [Grant, 2011]
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Synthetic geometry

- Metric geometry: Length spaces, Alexandrov spaces (synthetic curvature bounds), but only few analogs for Lorentzian setting
- Hope: synthetic description of conjugate points and genericity condition

Thank you for your attention!

Some related Literature

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