Lorentzian Geometry

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Preface

These are lecture notes accompanying a course on Lorentzian geometry at the Faculty of Mathematics at Vienna University. They are based mainly on the notes [1] of Christian Bär and the standard text book [4] by Barrett O'Neill. The prerequisites for following the course are a solid working knowledge in analysis on manifolds and some basics of Riemannian geometry, as provided by [2, 3].

We are greatly indebted to Artemis Kaliger for creating these beautiful lecture notes!

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PRELIMINARIES

1.1 Curvature

- Let M be a SRMF with Levi-Civita connection ∇. (Riemannian) curvature tensor is defined as

$$R : \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$$

$$R(Z, X, Y) = R(X, Y)Z = R_{XY}Z \coloneqq \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z.$$

- It is a $(1, 3)$-tensor field locally given by

$$R_{\partial_k \partial_l}(\partial_j) = R^i_{jkl}\partial_i.$$

Recall ([3], 1.3.8) that

$$\nabla_{\partial_i}\partial_j \eqqcolon \Gamma^k_{ij}\partial_k$$

and, explicitly ([3], 1.3.9),

$$\Gamma^i_{jk} = g^{il}\Gamma_{ljk} = \frac{1}{2}g^{il}(g_{lj,k} + g_{kl,j} - g_{jk,l})$$

are Christoffel symbols.
For components of the curvature tensor in terms of Christoffel symbols we have ([3], 3.1.1):

$$R^i_{jkl} = \partial_l\Gamma^i_{kj} - \partial_k\Gamma^i_{lj} + \Gamma^i_{lm}\Gamma^m_{kj} - \Gamma^i_{km}\Gamma^m_{lj}.$$

- Since R is a tensor field, one may also insert individual tangent vectors into its slots. Upon inserting individual tangent vectors one obtains the curvature operator of $p \in M$ for $x, y \in T_pM$ via

$$R_{xy} : T_pM \to T_pM$$
$$z \mapsto R_{xy}z.$$

It has the following symmetries for all $x, y, v, w \in T_pM$ ([3], 3.2.1):

1. antisymmetry: $R_{xy} = -R_{yx}$
2. skew-adjointness: $\langle R_{xy}v, w\rangle = -\langle R_{xy}w, v\rangle$
3. 1^{st} Bianchi identity: $R_{xy}z + R_{yz}x + R_{zx}y = 0$
4. pair symmetry: $\langle R_{xy}v, w\rangle = \langle R_{vw}x, y\rangle$

as well as the 2^{nd} Bianchi identity:

$$(\nabla_z R)(x,y) + (\nabla_x R)(y,z) + (\nabla_y R)(z,x) = 0.$$

Definition 1.1.1. Every 2-dimensional subspace Π of T_pM is called a tangent plane to M at p.

For $v, w \in \Pi$ let

$$Q(v,w) = \langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2 = \det\begin{pmatrix}\langle v,v\rangle & \langle v,w\rangle \\ \langle v,w\rangle & \langle w,w\rangle\end{pmatrix}.$$

From [3], 1.1.3, we know that

$$\Pi \text{ non-degenerate } (:\iff g|_\Pi \text{ non-degenerate}) \iff Q(v,w) \neq 0 \text{ for any basis } \{v,w\} \text{ of } \Pi.$$

Moreover, using ONB (exists by [3], 1.1.12), we have

$$\begin{aligned} Q(v,w) > 0 &\iff g|_\Pi \text{ is definite and} \\ Q(v,w) < 0 &\iff g|_\Pi \text{ is indefinite.} \end{aligned}$$

It is easy to calculate $Q(v,w)$ for $v = e_1$ and $w = e_2$, where $\{e_1, e_2\}$ is an ONB.
On the other hand, for $v = ax + by$ and $w = cx + dy$ by an easy calculation we get

$$Q(v,w) = \underbrace{(ad-bc)^2}_{>0} Q(x,y),$$

which shows that the sign of Q is independent of basis.

Lemma 1.1.2 (Sectional Curvature)**.** Let Π be a non-degenerate tangent plane of $p \in M$, where M is any SRMF. The sectional curvature of Π

$$K(v,w) := \frac{\langle R_{vw}v, w\rangle}{Q(v,w)}$$

is independent of the choice of basis $\{v,w\}$ of Π (and so one can also denote it by $K(\Pi)$).

Proof. Let $\{x,y\}$ be another basis of Π. Then there are $a,b,c,d \in \mathbb{R}$ $(ad - bc \neq 0)$ such that $v = ax + by$ and $w = cx + dy$.

$$\begin{aligned} \langle R_{vw}v, w\rangle &= \langle R_{(ax+by)(cx+dy)}(ax+by), cx+dy\rangle \\ &\overset{R_{xx}=0}{=} (ad-bc)\langle R_{xy}(ax+by), cx+dy\rangle \\ &\overset{[3],\, 3.1.2}{=} (ad-bc)^2\langle R_{xy}x, y\rangle \end{aligned}$$

Now $Q(v,w) = (ad-bc)^2 Q(x,y)$ proves the claim. ☐

Lemma 1.1.3 (Approximation Lemma)**.** Let V be a vector space with scalar product (with possibly nontrivial signature, see 1.1.4 in [3]) and let $v, w \in V$. Then there exist $\overline{v}, \overline{w}$ arbitrarily near v, w such that $\overline{v}$, $\overline{w}$ span a non-degenerate plane.

Proof. Without loss of generality we may assume that v and w are linearly independent and that they span a degenerate plane. If $\langle v,v\rangle = 0$ (i.e. if v is a null vector) choose x with scalar product $\langle x,v\rangle \neq 0$ (note that such an x exists since $\langle\cdot,\cdot\rangle$ is non-degenerate). If v is not a null vector (i.e. if it is timelike or spacelike) choose x with a different causal character from that of v. In both cases we have $Q(v,x) < 0$:

$$\text{if } \langle v,v\rangle = 0 \implies Q(v,x) = -\langle v,x\rangle^2 < 0 \text{ and}$$

$$\text{if } \langle v,v\rangle \neq 0 \implies Q(v,x) = \langle v,v\rangle\langle x,x\rangle - \langle v,x\rangle^2 \leq \underbrace{\langle v,v\rangle\langle x,x\rangle}_{\text{different signs}} < 0.$$

It suffices to show that v and $w + \delta x$ span a non-degenerate plane for δ small.

$$Q(v, w+\delta x) \overset{(\star)}{=} \underbrace{Q(v,w)}_{\substack{=0,\\ \text{span}(v,w) \text{ is degenerate}}} + 2b\delta + \delta^2 \underbrace{Q(v,x)}_{<0} \text{ for some } b \in \mathbb{R}$$

and so there are two cases, namely, either

$b = 0$ and $Q(v, w+\delta x) < 0$ or

$b \neq 0$ and for δ small $Q(v, w+\delta x) \neq 0$.

Finally, we prove $(\star)$:

$$\begin{aligned} Q(v,w+\delta x) &= \langle v,v\rangle(\langle w,w\rangle + 2\delta\langle w,x\rangle + \delta^2\langle x,x\rangle) - (\langle v,w\rangle + \delta\langle v,x\rangle)^2 \\ &= \underbrace{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2}_{Q(v,w)} + 2\delta(\underbrace{\langle v,v\rangle\langle w,x\rangle - \langle v,w\rangle\langle v,x\rangle}_{=b}) + \delta^2(\underbrace{\langle v,v\rangle\langle x,x\rangle - \langle v,x\rangle^2}_{Q(v,x)}) \end{aligned}$$

□

Proposition 1.1.4 (K and R)**.** If $K = 0$ at p (i.e. $K(\Pi) = 0$ for every non-degenerate plane in T_pM), then $R = 0$ at p (i.e. $R_{xy}z = 0$ for all $x,y,z \in T_pM$).

Proof. We proceed in several steps.

1. Claim: $\langle R_{vw}v, w\rangle = 0$ for all $v,w \in T_pM$. If $\Pi = \text{span}(v,w)$ is non-degenerate, this follows from definition of K. Otherwise, by Lemma 1.1.3 there exist v_n, w_n such that span(v_n,w_n) is non-degenerate and $v_n \to v$, $w_n \to w$. Then
$$0 = \langle R_{v_n w_n} v_n, w_n\rangle \overset{n\to\infty}{\longrightarrow} \langle R_{vw}v, w\rangle\,.$$

2. Claim: $R_{vw}v = 0$ for all $v,w \in T_pM$. This claim follows by polarization. More precisely, for arbitrary x we have
$$\underbrace{\langle R_{v,w+x}v, w+x\rangle}_{=0, \text{ by 1.}} = \underbrace{\langle R_{vw}v,w\rangle}_{=0, \text{ by 1.}} + \langle R_{vw}v,x\rangle + \langle R_{vx}v,w\rangle + \underbrace{\langle R_{vx}v,x\rangle}_{=0, \text{ by 1.}}\,.$$
Therefore, $\langle R_{vw}v,x\rangle = 0$ for all x (since, by pair symmetry, $\langle R_{vw}v,x\rangle = \langle R_{vx}v,w\rangle$). Finally, non-degeneracy yields $R_{vw}v = 0$.

3. Claim: $R_{vw}x = R_{wx}v$ for all $v,w,x \in T_pM$. In order to prove this, we use polarization again:
$$\underbrace{R_{(v+x)w}(v+x)}_{=0, \text{ by 2.}} = \underbrace{R_{vw}v}_{=0, \text{ by 2.}} + \underbrace{R_{xw}v}_{=-R_{wx}v} + R_{vw}x + \underbrace{R_{xw}x}_{=0, \text{ by 2.}}\,.$$

Finally,

$$0 \overset{1^{\text{st}}\text{Bianchi i.}}{=} \underbrace{R_{vwx}}_{=R_{wx}v,\text{ by 3.}} + \underbrace{R_{xv}w}_{=R_{vw}x=R_{wx}v,\text{ by 3.}} + R_{wx}v = 3R_{wx}v$$

and therefore $R_{xy}z = 0$ for all $x, y, z \in T_pM$ i.e. $R = 0$ at p. □

Remark 1.1.5. Recall that a SRMF (M, g) is called *flat* if $R \equiv 0$. By Proposition 1.1.4, if $K \equiv 0$ then (M, g) is flat.

Example 1.1.6.

1. $\mathbb{R}^n_r$ is flat. By [3], 1.3.10, ii), since g is constant, $\Gamma^i_{jk} = 0$ $(1 \le i, j, k \le n)$ and so $R = 0$.
2. Every one-dimensional SRMF is flat. To see this let $X, Y \in \mathfrak{X}(M)$, where M is one-dimensional. Then $Y = f \cdot X$, where $f \in \mathcal{C}^\infty$. Therefore, $R_{XY} = fR_{XX} = -fR_{XX}$ and so $R_{XY} = 0$.

Definition 1.1.7. A multilinear map $F : (T_pM)^4 \to \mathbb{R}$ is called **curvature-like** if it has the same symmetries as $(x, y, v, w) \mapsto \langle R_{xy}v, w\rangle$.

Remark 1.1.8. By the proof of Proposition 1.1.4, we get that $F(v, w, v, w) = 0$ for all $v, w \in T_pM$ spanning a non-degenerate tangent plane at p, which is equivalent to $F = 0$.

Corollary 1.1.9 (Curvature-like Function)**.** Let F be curvature-like on T_pM such that

$$K(v, w) = \frac{F(v, w, v, w)}{Q(v, w)}$$

for all v, w with non-degenerate span. Then

$$F(v, w, x, y) = \langle R_{vw}x, y\rangle, \text{ for all } v, w, x, y \in T_pM.$$

Proof. The map $\Delta(v, w, x, y) := F(v, w, x, y) - \langle R_{xy}v, w\rangle$ is clearly curvature-like. By assumption,

$$\Delta(v, w, v, w) = 0$$

for all v, w spanning a non-degenerate plane. Remark 1.1.8 now gives the claim. □

Remark 1.1.10. In particular, for a given g, if R_1 and R_2 are Riemann-tensors with $K_1 = K_2$, then

$$\frac{R_1(v, w, v, w)}{Q(v, w)} = \frac{R_2(v, w, v, w)}{Q(v, w)} \implies R_1 = R_2.$$

Definition 1.1.11. We say that a SRMF (M, g) has **constant curvature** if K is constant on M.

Corollary 1.1.12 (R for Constant Curvature)**.** If M has constant curvature $K = C$, then

$$R_{XY}Z = C(\langle Z, X\rangle Y - \langle Z, Y\rangle X).$$

Proof. One easily checks that

$$F(x, y, v, w) := C(\langle v, x\rangle\langle y, w\rangle - \langle v, y\rangle\langle x, w\rangle)$$

is curvature-like. Moreover, $F(v, w, v, w) = C \cdot Q(v, w)$. If v and w span a non-degenerate tangent plane, then

$$K(v,w) = C = \frac{F(v,w,v,w)}{Q(v,w)}.$$

By Corollary 1.1.9, $F(x,y,z,w) = \langle R_{xy}z, w\rangle$ for all $v,w,x,z \in T_pM$, which proves the claim. □

1.2 Frame Fields

Definition 1.2.1. Let (M,g) be a SRMF.

- A **frame** at $p \in M$ is an ONB of T_pM.
- If $\dim(M) = n$, then a **frame field** is an n-tuple $E_1,\dots,E_n$ of pairwise orthogonal (smooth) vector fields. (Hence, $E_1(p),\dots,E_n(p)$ such that $\langle E_i(p), E_j(p)\rangle = \epsilon_j\delta_{ij}, \forall 1 \le i,j \le n$ provide a frame at any point p.)

Remark 1.2.2.

- Given a frame field any vector field (by [3], 1.1.13) $V \in \mathfrak{X}(M)$ can be expanded as

$$V = \sum_i \epsilon_i \langle V, E_i\rangle E_i, \text{ where } \epsilon_i = \langle E_i, E_i\rangle = \pm 1.$$

  Moreover, for $v,w \in \mathfrak{X}(M)$

$$\langle V, W\rangle = \sum_i \epsilon_i \langle V, E_i\rangle \langle W, E_i\rangle.$$

- We will prove later that frame fields always exist locally (but maybe not globally).
- Recall that the coordinate vector fields $(\partial_i|_p)_{i\in\{1,\dots,n\}}$ of Riemannian normal coordinate (RNC) system $x^1,\dots,x^n$ of p form an ONB at p (but not necessarily at any $q \ne p$) and that $\Gamma^i_{jk}(p) = 0$ for all i,j,k (see [3], 2.1.17). Therefore, as long as only pointwise operations are concerned, we may reduce formulas in frames to coordinate formulas.

Example 1.2.3.

1. Let $A \in \mathcal{T}^0_s(M)$, $E_1,\dots,E_n$ a frame field. Then,

$$(C_{ab}A)(X_1,\dots,X_{s-2}) = \sum_m \epsilon_m A(X_1,\dots,\underbrace{E_m}_{a^{\text{th}}\text{ sloth}},\dots,\underbrace{E_m}_{b^{\text{th}}\text{ sloth}},\dots,X_{s-2}). \tag{1.2.1}$$

   Indeed, since this is an equality of tensor fields, we only need to verify it pointwise. Therefore, let $p \in M$ and $x^1,\dots,x^n$ be RNC at p such that $\partial_i|_p = E_i|_p$ (choose $e_i := E_i|_p$ in [3], 2.1.17). By multilinearity, we only need to verify (1.2.1) for $X_i = \partial_i$. But, (1.2.1) reduces to the usual formula for $C_{ab}A$, as given in [3], (3.2.11), i.e. to

$$(C_{ab}A)_{j_1,\dots,j_{s-2}} = g^{mn}A_{j_1,\dots,\underbrace{m}_{b^{\text{th}}\text{ sloth}},\dots,\underbrace{n}_{b^{\text{th}}\text{ sloth}},\dots,j_{s-2}},$$

   which proves the claim since at p (by [3], 2.1.17) $g^{mn}(p) = \epsilon_m\delta^{mn}$.

2. Let $A : \mathfrak{X}(M)^s \to \mathfrak{X}(M)$ be a $\mathcal{C}^\infty(M)$-multilinear map. Then

$$(C^1_b A)(X_1,\dots,X_{s-1}) = \sum_m \epsilon_m \langle E_m, A(X_1,\dots,\underbrace{E_m}_{b^{\text{th}}\text{ sloth}},\dots,X_{s-1})\rangle.$$

Choose RNC as above and note that $dx^m(\partial_i) = \delta_i^m = \epsilon_m g_{im}(p) = \langle \epsilon_m \partial_m, \partial_i \rangle$. Therefore, $(\partial_m)^\flat = \epsilon_m dx^m$ (see [3], 1.3.3). As before, we only need to check the equality on coordinate vector fields:

$$\begin{aligned}
(C_b^1 A)(\partial_1, \dots, \partial_{s-1}) &\overset{[3],\ 1.3.15}{=} \sum_m \overline{A}(dx^m, \partial_1, \dots, \underbrace{\partial_m}_{b^{\text{th}}\text{ sloth}}, \dots, \partial_{s-1}) \\
&\overset{[3],\ (1.3.24)}{=} \sum_m dx^m(A(\partial_1, \dots, \partial_m, \dots, \partial_{s-1})) \\
&= \sum_m \epsilon_m (\partial_m)^\flat (A(\partial_1, \dots, \partial_m, \dots, \partial_{s-1})) \\
&= \sum_m \epsilon_m \langle \partial_m, A(\partial_1, \dots, \underbrace{\partial_m}_{E_m(p)}, \dots, \partial_{s-1}) \rangle.
\end{aligned}$$

Remark 1.2.4. Frame fields serve as an alternative tool for tensor calculations, offering an approach distinct from local coordinates. However, when proving an identity involving derivatives, a pointwise argument becomes unfeasible. Consequently, determining the most advantageous approach between utilizing frame advantages and other methods becomes a case-by-case decision.

$$\langle E_i, E_j \rangle = \delta_{ij} \epsilon_j \text{ (instead of } \langle \partial_i, \partial_j \rangle = g_{ij})$$

or use the advantages of a coordinate basis, for example

$$[\partial_i, \partial_j] = 0 \text{ (but } [E_i, E_j] \neq 0).$$

We initiate our discussion on the local existence of frames by initially focusing on frames along a curve.

Definition 1.2.5. Given a $\mathcal{C}^\infty$-curve $\alpha : I \to M$ we call a system $E_1, \dots, E_n \in \mathfrak{X}(\alpha)$ (where $E_i : I \to TM$, $\pi \circ E_i = \alpha$, see here [3], 1.3.26) a **frame field along** α if $\{E_i(t)\}_{i=1,\dots,n}$ is an ONB of $T_{\alpha(t)}M$, $\forall t \in I$.

Such frames always exist.

Proposition 1.2.6 (Parallel Frames). Let $\alpha : I \to M$ be a $\mathcal{C}^\infty$-curve and $e_1, ..., e_n$ an ONB at $\alpha(t_0)$ for some $t_0 \in I$. Then there exists a unique parallel frame field $E_1, \dots, E_n$ along α such that $E_i(t_0) = e_i$ for all $1 \le i \le n$.

Recall that *parallel* means that induced covariant derivatives of all E_i vanish along α (see [3], after (1.3.55)) i.e.

$$E_i'(t) = \frac{\nabla E_i}{dt} = 0$$

where

$$Z' = \left(\frac{dZ^k}{dt} + \Gamma_{ij}^k \dot{\alpha}^j Z^i \right) \partial_k \Big|_{c(t)}.$$

Proof. By [3], 1.3.28, there exist unique parallel vector fields $E_i \in \mathfrak{X}(\alpha)$ with $E_i(t_0) = e_i$ for all $1 \le i \le n$. Parallel transport is an isometry ([3], 1.3.30) and so $E_1(t), \dots, E_n(t)$ is an ONB for all $t \in I$. ☐

Corollary 1.2.7 (Local Frames). Every $p \in M$ possesses a neighborhood on which there is a frame field.

Proof. Let $\{e_1, \dots, e_n\}$ be an ONB of T_pM. Choose a small normal neighborhood U around p ([3], 2.1.14). For any radial geodesic ([3], 2.1.15) γ emanating from p we parallel transport $\{e_1, \dots, e_n\}$ and get a frame field $E_1, \dots, E_n$ on U (Proposition 1.2.6). Parallel transport is given as the solution to an IVP of a (linear) ODE ([3], 1.3.28) so the E_i depend smoothly on parameters and initial data. Therefore, $E_1, \dots, E_n \in \mathfrak{X}(U)$. □

In summary, within a local context, we constantly have access to both coordinates and frame fields, allowing us the flexibility to employ whichever best suits our calculations. Let's continue by exploring additional examples.

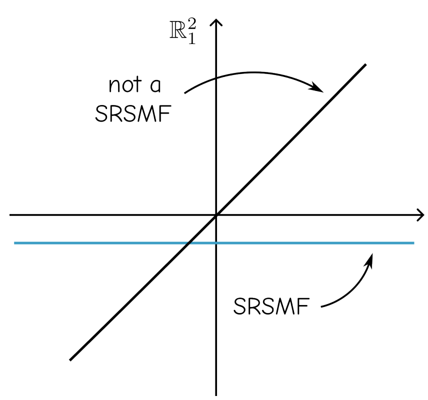
Example 1.2.8.

1. Divergence. For $X \in \mathfrak{X}(M)$
$$\text{div}X = C(\nabla X) \in \mathcal{C}^\infty(M),$$
(see, [3] 3.2.7). In a frame this equation takes the following form:
$$\text{div}X = \sum_i \epsilon_i \langle \nabla_{E_i} X, E_i \rangle .$$
Indeed, choosing RNC at a point p with $\partial_i|_p = E_i|_p$, we get
$$\begin{aligned} C(\nabla X)|_p &= \nabla X(dx^i, \partial_i)|_p = \nabla_{\partial_i} X(dx^i)|_p = dx^i(\nabla_{\partial_i} X)|_p \\ &\overset{1.2.3,2.}{=} \sum_i \epsilon_i \langle \nabla_{\partial_i} X, \partial_i \rangle = \sum_i \epsilon_i \langle \nabla_{E_i} X, E_i \rangle . \end{aligned}$$

2. Ricci-tensor. The Ricci-tensor is defined as Ric $= C_3^1(R)$ (see 3.3.1 in [3]). In a frame, it takes the form
$$\text{Ric}(X, Y) = \sum_m \epsilon_m \langle R_{XE_m} Y, E_m \rangle$$
since
$$\begin{aligned} \text{Ric}(X, Y) &= C_3^1(R)(X, Y) \overset{1.2.3}{=} \sum_m \epsilon_m \langle E_m, R(X, Y, E_m) \rangle \\ &= \sum_m \epsilon_m \langle R_{YE_m} X, E_m \rangle = \sum_m \epsilon_n \langle R_{XE_m} Y, E_m \rangle , \end{aligned}$$
where in the third equality we used the convention from 3.2.4 in [3] and in the fourth pair symmetry (i.e. property 4. from the beginning of this section).

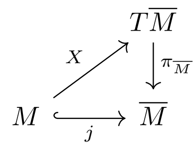
1.3 Semi-Riemannian Submanifolds

In this section, our focus lies on studying the (M, g) of a SRMF $(\overline{M}, \overline{g})$. Let $j : M \hookrightarrow \overline{M}$ be the inclusion. Then $g = j^*\overline{g}$.



We write $\langle\cdot,\cdot\rangle$ for both g and $\overline{g}$. Our goal is to relate curvature quantities in M and $\overline{M}$. We will compare $\overline{\nabla}$ with ∇, $\overline{R}$ with R, et cetera.

Definition 1.3.1. Let $M \subseteq \overline{M}$ be a smooth submanifold of a SRMF. Then we call X a vector field on $j : M \hookrightarrow \overline{M}$ (i.e. $X : M \to T\overline{M}$ is such that $\pi_{\overline{M}} \circ X = j$, compare with [3], 1.3.26)



an **$\overline{M}$-vector field** on M.
Furthermore, we write

$$\overline{\mathfrak{X}}(M) := \{X \in \mathcal{C}^\infty(M, T\overline{M}) : X \text{ is a vector field over } j\}.$$

Remark 1.3.2.

- $X \in \overline{\mathfrak{X}}(M)$ assigns to each $p \in M$ some $X_p \in T_p\overline{M}$ in a smooth way i.e. for all $f \in \mathcal{C}^\infty(M)$, $X(f) \in \mathcal{C}^\infty(M)$. Hence, the set $\overline{\mathfrak{X}}(M)$ is a $\mathcal{C}^\infty(M)$-module. $\mathfrak{X}(M)$ is a submodule of $\overline{\mathfrak{X}}(M)$. Also, if $X \in \mathfrak{X}(\overline{M})$, then $X|_M \in \overline{\mathfrak{X}}(M)$ (but, in general, not in $\mathfrak{X}(M)$).

- Since M is a SRSMF, by definition $\overline{g}|_{T_pM} = j^*\overline{g}(p)$ is non-degenerate and hence $T_pM \subseteq T_p\overline{M}$ is non-degenerate (see [3], p.5), so
$$T_p\overline{M} = T_pM \oplus T_pM^\perp. \tag{1.3.1}$$
Moreover, $T_pM^\perp$ is non-degenerate as well ([3], 1.1.10) and the dimension of $T_pM^\perp$ is the codimension of M in $\overline{M}$. The index of $\overline{g}|_{T_pM^\perp}$ is called the **coindex** of M in $\overline{M}$. We have
$$\operatorname{ind}(\overline{M}) = \operatorname{ind}(M) + \operatorname{coind}(M)$$
(see [3], 1.1.15).

- We call elements of T_pM **tangential** to M and those of $T_pM^\perp$ **normal** to M. By (1.3.1) every $x \in T_p\overline{M}$ has a unique decomposition
$$x = \underbrace{\tan(x)}_{\in T_pM} + \underbrace{\operatorname{nor}(x)}_{\in T_pM^\perp}$$
and we denote the respective orthonormal projections by $\tan : T_p\overline{M} \to T_pM$ and $\operatorname{nor} : T_p\overline{M} \to T_pM^\perp$. Those are clearly $\mathbb{R}$-linear.

Definition 1.3.3. A vector field Z is called **normal to M** if $Z_p \in T_pM^\perp$ for all $p \in M$.

Remark 1.3.4. The set of all such vector fields $\mathfrak{X}(M)^\perp$ is also a $\mathcal{C}^\infty(M)$-submodule of $\overline{\mathfrak{X}}(M)$.

Lemma 1.3.5 (Hicks, 1963.). Let $M^n \subseteq \overline{M}^{n+k}$ be a SRSMF and $p \in M$. Then there exists a neighborhood W of p and $E_1, \ldots, E_{n+k} \in \overline{\mathfrak{X}}(W)$ such that for all $q \in W$ $(E_i|_q)_{i=1,\ldots,n}$ is an ONB T_qM and $(E_i|_q)_{j=n+1,\ldots,k}$ is an ONB of $T_qM^\perp$.

Proof. For $\overline{M}$ a RMF consider an adapted chart $(x^1, \ldots, x^{n+k})$ around p and apply Gram-Schmidt to $\{\partial_1, \ldots, \partial_{n+k}\}$. For general SRMF $\overline{M}$ there might be null-vectors (during Gram-Schmidt process we normalize vectors, which could result in division by zero). Let $X_1, \ldots, X_{n+k}$ be an ONB for $T_p\overline{M}$ such that

$X_1, \dots, X_n$ is an ONB for T_pM and $X_{n+1}, \dots, X_{n+k}$ an ONB for $T_pM^\perp$ (see [3], 1.1.12).

1. There exists a chart $(\varphi = (x^1, \dots, x^{n+k}), \tilde{\mathcal{U}})$ of $\overline{M}$ at p such that $\frac{\partial}{\partial x^i}\big|_p = X_i$ where $1 \le i \le n+k$.
   To see this, let $(\tilde{\varphi}, \tilde{\mathcal{U}})$ be any chart of $\overline{M}$ at p and let $Y_i := \underbrace{T_p\tilde{\varphi}(X_i)}_{\in \mathbb{R}^{n+k}}$. Let $A \in GL(n+k)$ be such that

   $$A \cdot Y_i = e_i = \begin{pmatrix} 0 \\ \vdots \\ \underbrace{1}_{i^{\text{th}} \text{ place}} \\ \vdots \\ 0 \end{pmatrix}$$ for all $i \in \{1, \dots, n+k\}$ and $\varphi : \tilde{U} \to \mathbb{R}^{n+k}$ where $\varphi(\overline{q}) = A \cdot \tilde{\varphi}(\overline{q})$. Then $(\varphi, \tilde{U})$ is a chart at p and $T_p\varphi(X_i) = A \cdot \underbrace{T_p\tilde{\varphi}(X_i)}_{= Y_i} = e_i$, which implies that $X_i = (T_p\varphi)^{-1}(e_i) = \frac{\partial}{\partial x^i}|_p$.

2. There exists an adapted chart of $\overline{M}$ $\psi = (y^1, \dots, y^{n+k})$ at p such that $\frac{\partial}{\partial y_i}|_p = X_i$, for all $1 \le i \le n$. Indeed, let φ be as in 1. and write

   $$\varphi = (x^1, \dots, x^n, x^{n+1}, \dots, x^{n+k}) \equiv (\varphi', \varphi'') = (\underbrace{\mathrm{pr}_1}_{\mathbb{R}^{n+k} \to \mathbb{R}^n} \circ \varphi, \mathrm{pr}_2 \circ \varphi).$$

   $\frac{\partial}{\partial x^i}|_p = X_i$ are linearly independent and so $T_p(\varphi'|_M)$ is invertible. We have $T_p\varphi(X_i) = e_i$ for $1 \le i \le n$, and X_i is a basis of T_pM. Hence,

   $$T_p(\varphi'|_M)(X_i) = T_p(\mathrm{pr}_1 \circ \varphi|_M)(X_i) = \mathrm{pr}_1 \circ T_p(\varphi|_M)(X_i) = \mathrm{pr}_1 \circ T_p\varphi(X_i) = e_i$$

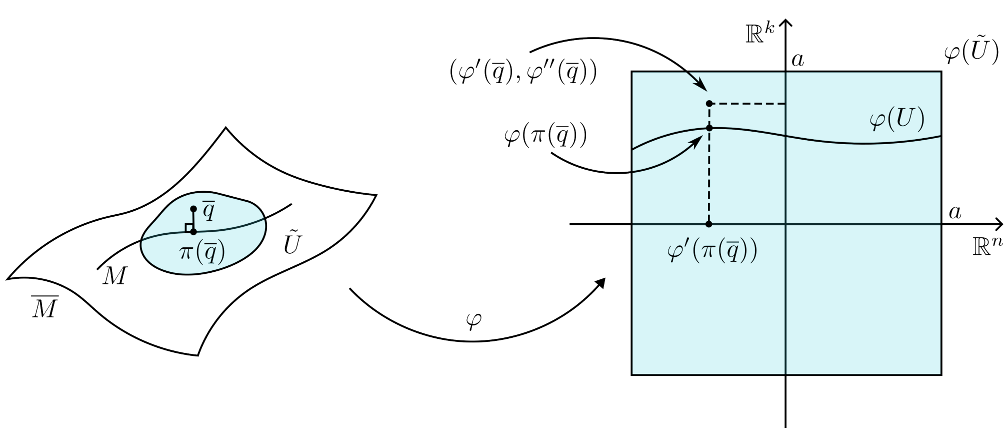
   for $1 \le i \le n$, implying that φ' a chart of M at p. Without loss of generality $\varphi(p) = 0 \in \mathbb{R}^{n+k}$. Furthermore, we can pick $a > 0$ so small that

   $$\varphi' : \mathcal{U} \to \{x \in \mathbb{R}^n : |x^i| < a \text{ for } 1 \le i \le n\}$$

   is bijective (Inverse Function Theorem). Making a even smaller we can also have that

   $$\varphi : \tilde{U} \to \{x \in \mathbb{R}^{n+k} : |x^i| < a \text{ for } 1 \le i \le n\}$$

   is bijective.



For $\overline{q} \in \tilde{\mathcal{U}}$ let $\pi(\overline{q})$ be the (unique) element of $\mathcal{U}$ such that

$$\varphi'(\pi(\overline{q})) = \varphi'(\overline{q}) \tag{1.3.2}$$

i.e. $\pi(\overline{q}) = (\varphi'|_{\mathcal{U}})^{-1}(\varphi'(\overline{q}), 0)$. Therefore, π is well-defined as well as smooth since

$$\underbrace{\varphi'|_{\mathcal{U}} \circ \pi}_{\mathrm{pr}_1 \circ \varphi} \circ \varphi^{-1} = \mathrm{pr}_1.$$

Let $\psi = (y^1, \dots, y^{n+k})$ be as follows:

$$\begin{cases} \psi' = \varphi' \\ \psi'' = \varphi'' - \varphi'' \cdot \pi. \end{cases}$$

Note that ψ is smooth on $\tilde{\mathcal{U}}$. We prove that

$$\mathcal{U} = \{\overline{q} \in \tilde{\mathcal{U}} : y^{n+1}(\overline{q}) = \dots = y^{n+k}(\overline{q}) = 0\} = \{\overline{q} \in \tilde{\mathcal{U}} : \psi''(\overline{q}) = 0\}.$$

If $\psi''(\overline{q}) = 0$ for some $\overline{q} \in \tilde{\mathcal{U}}$ then $\varphi''(\overline{q}) = \varphi''(\pi(\overline{q}))$ and $\varphi'(\overline{q}) = \varphi'(\pi(\overline{q}))$ (see (1.3.2)). Therefore,

$$\varphi(\overline{q}) = \varphi(\pi(\overline{q})) \text{ and so, since } \varphi \text{ is a chart, } \overline{q} = \pi(\overline{q}) \in \mathcal{U}.$$

Conversely, if $\overline{q} \in \mathcal{U}$ since $\pi|_{\mathcal{U}} = \mathrm{id}$, we are done.
$T_p\psi = (T_p\varphi', T_p\varphi'' - T_{\pi(p)}\varphi'' \cdot T_p\pi)$, where $\pi(p) = p$ since $p \in \mathcal{U}$. For $1 \le i \le n$

$$T_p\varphi(X_i) = \begin{pmatrix} T_p\varphi'(X_i) \\ T_p\varphi''(X_i) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \underbrace{1}_{i^{\text{th}} \text{ place}} \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix} \text{ for all } i \in \{1, \dots, n\}$$

and so $T_p\varphi''(X_i) = 0$ for $1 \le i \le n$. Since $\{X_i\}_{i=1,\dots,n}$ are a basis for T_pM, $T_p\varphi''|_{T_pM} = 0$ and $T_p\varphi'' \circ \underbrace{T_p\pi}_{T_p\overline{M} \to T_pM} = 0$. Therefore,

$$T_p\psi = T_p\varphi$$

(in particular, ψ is a chart around p), $T_p\psi(X_i) = T_p(X_i) = e_i$ for $1 \le i \le n+k$ and so $X_i = \frac{\partial}{\partial y_i}|_p$.

3. Existence of E_i.
   Shrink $\tilde{\mathcal{U}}$ so that $\left\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^i} \right\rangle$ is near ± 1 for all i and $\left\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right\rangle \approx 0$. Now apply Gram-Schmidt:

$$\begin{aligned} E_1 &= \frac{\partial}{\partial y^1} \\ E_2 &= \partial_{y_2} - \frac{\overbrace{\langle \partial y_2, E_1 \rangle}^{\approx 0}}{\underbrace{\langle E_1, E_1 \rangle}_{\approx 1}} E_1 \implies \langle E_2, E_2 \rangle \approx \langle \partial_{y_2}, \partial_{y_2} \rangle \approx \pm 1 \\ E_3 &= \partial_{y_2} - \frac{\langle \partial y_3, E_1 \rangle}{\langle E_1, E_1 \rangle} E_1 - \frac{\langle \partial y_3, E_2 \rangle}{\langle E_2, E_2 \rangle} E_2 \implies \langle E_3, E_3 \rangle \approx \langle \partial_{y_3}, \partial_{y_3} \rangle \approx \pm 1 \end{aligned}$$

□

Remark 1.3.6. As a consequence of the previous lemma, we have that for every $X \in \overline{\mathfrak{X}}(M)$

$$X = \underbrace{\sum_{i=1}^{n} \epsilon_i \langle X, E_i \rangle E_i}_{\text{tan}(X)} + \underbrace{\sum_{i=n+1}^{n+k} \epsilon_i \langle X, E_i \rangle E_i}_{\text{nor}(X)}$$

where $\text{tan}(X) \in \mathfrak{X}(M)$ and $\text{nor}(X) \in \mathfrak{X}(M)^{\perp}$. In other words, $\overline{\mathfrak{X}}(M) = \mathfrak{X}(M) \oplus \mathfrak{X}(M)^{\perp}$.

Moving forward, both V and W will consistently denote tangential vector fields to M, whereas Z will consistently represent a normal vector field.

Lemma 1.3.7 (Extensions of Functions and Vector Fields)**.** Let M^m, N^n be $\mathcal{C}^{\infty}$-manifolds, $j : M \to N$ an immersion and $p \in M$. Then

1. for all $f \in \mathcal{C}^{\infty}(M)$ there exists $\tilde{f} \in \mathcal{C}^{\infty}(N)$ such that near p

$$f = \tilde{f} \circ j.$$

2. for all $X \in \mathfrak{X}(j)$ there exists $\tilde{X} \in \mathfrak{X}(N)$ such that near p

$$X = \tilde{X} \circ j.$$

Proof.

1. From [2] we know that there exist charts φ of p and ψ of $j(p)$ such that $\text{pr} \circ \psi \circ j = \varphi$ where pr is a suitable projection from $\mathbb{R}^n$ to $\mathbb{R}^m$. Choose $\tilde{f} \in \mathcal{C}^{\infty}(N)$ such that near $j(p)$

$$\tilde{f} = f \circ \varphi^{-1} \circ \text{pr} \circ \psi$$

(use a partition of unity). Then near p

$$\tilde{f} \circ j = f \circ \varphi^{-1} \circ \underbrace{\text{pr} \circ \psi \circ j}_{= \varphi} = f.$$

2. Locally around p we have $X(q) = f^i(q) \partial_{y^i}|_{j(p)}$ where the y^i are coordinates with respect to ψ and f^i are $\mathcal{C}^{\infty}(M)$-functions. Extend f^i to $\tilde{f}^i$ as in 1. and set $\tilde{X} = \tilde{f}^i \partial y^i$. Then $\tilde{X} \circ j = X$ near p.

□

We now want to find a 'connection'

$$\nabla : \mathfrak{X}(M) \times \overline{\mathfrak{X}}(M) \to \overline{\mathfrak{X}}(M)$$

induced by the connection $\overline{\nabla}$ on $\overline{M}$. However, for $V \in \mathfrak{X}(M)$ and $X \in \overline{\mathfrak{X}}(M)$ $\overline{\nabla}_V X$ is a priori not defined since $\overline{\nabla}$ expects arguments from $\mathfrak{X}(\overline{M})$. The idea is to extend X and V as in Lemma 1.3.7 to vector fields $\overline{X}$ and $\overline{V}$ and set

$$\overline{\nabla}_V X := \overline{\nabla}_{\overline{V}} \overline{X}|_M, \text{ where } \overline{V}, \overline{X} \in \mathfrak{X}(\overline{M}). \tag{1.3.3}$$

Lemma 1.3.8 (Induced Connection)**.** The map defined in (1.3.2) is well-defined i.e. it does not depend on the choice of extensions $\overline{V}$, $\overline{X}$ of V and X. It is called the induced connection of $\overline{M}$ on M.

Proof. $\overline{\nabla}_V X$ is smooth since $\overline{V},\overline{X} \in \mathfrak{X}(\overline{M})$ (see [3], 1.3.1). To show independence of the choices of $\overline{V}$ and $\overline{X}$ fix $p \in M$ and pick an adapted chart $(\mathcal{U}, x^1, \dots, x^{n+k})$ of $\overline{M}$ around p ($\dim M = n$, $\dim \overline{M} = n+k$). Then

$$V = V^i \partial_i, X = X^j \partial_j \text{ where } V^i \in \mathcal{C}^\infty(M \cap \mathcal{U}) \text{ for } 1 \le i \le n \text{ and } X^i \in \mathcal{C}^\infty(M \cap \mathcal{U}) \text{ for } 1 \le i \le n+k$$

and

$$\overline{V} = \tilde{V}^i \partial_i \text{ where } \tilde{V}^i \in \mathcal{C}^\infty(\mathcal{U}), \tilde{V}^i|_{M\cap\mathcal{U}} = V^i$$
$$\overline{X} = \tilde{X}^j \partial_j \text{ where } \tilde{X}^j \in \mathcal{C}^\infty(\mathcal{U}), \tilde{X}^j|_{M\cap\mathcal{U}} = X^j.$$

For $q \in M \cap \mathcal{U}$

$$\overline{V}_q(\tilde{X}^j) = \underbrace{\tilde{V}^i(q)}_{= V^i(q)} \underbrace{\partial_i|_q(\tilde{X}^j)}_{\substack{= \partial_i|_q(X^j) \\ \text{for } 1\le i\le n}} = V_q(X^j) \tag{1.3.4}$$

and moreover

$$\overline{\nabla}_{\overline{V}}(\partial_j)|_q = \overline{\nabla}_{\overline{V}_q}(\partial_j) = \overline{\nabla}_{V_q}(\partial_j), \tag{1.3.5}$$

where in the first equality we used the fact that connection is always tensorial in the first slot. Finally,

$$(\overline{\nabla}_{\overline{V}} \underbrace{\overline{X}}_{=\tilde{X}^j\partial_j})|_q \overset{\text{RG, 1.3.1}}{=} \overline{V}(\tilde{X}^j)|_q \partial_j|_q + \tilde{X}^j(q)\overline{\nabla}_{\overline{V}}(\partial_j) \overset{(1.3.4),(1.3.5)}{=} V_q(X^j)\partial_j|_q + X^j(q)\overline{\nabla}_{V_q}(\partial_j)$$

and so we see that $\overline{\nabla}_{\overline{V}}\overline{X}$ only depends on V and X. □

Proposition 1.3.9 (Properties of the Induced Connection)**.** The induced connection $\overline{\nabla} : \mathfrak{X}(M) \times \overline{\mathfrak{X}}(M) \to \overline{\mathfrak{X}}(M)$ on $M \subseteq \overline{M}$ has the following properties for $V, W \in \mathfrak{X}(M)$, $X, Y \in \overline{\mathfrak{X}}(M)$ and $f \in \mathcal{C}^\infty(M)$:

($\nabla 1$) $\overline{\nabla}_V X$ is $\mathcal{C}^\infty(M)$-linear in V,

($\nabla 2$) $\overline{\nabla}_V X$ is $\mathbb{R}$-linear in X,

($\nabla 3$) $\overline{\nabla}_V(fX) = V(f)X + f\overline{\nabla}_V X$ (Leibniz rule),

($\nabla 4$) $[V, W] = \overline{\nabla}_V W - \overline{\nabla}_W V$ (torsion-free condition),

($\nabla 5$) $V\langle X, Y\rangle = \left\langle \overline{\nabla}_V X, Y \right\rangle + \left\langle X, \overline{\nabla}_V Y \right\rangle$ (metric property).

Proof. Locally extend V, W, X, Y to vector fields $\overline{V}, \overline{W}, \overline{X}, \overline{Y}$ on $\overline{M}$. Then for $\overline{V}, \overline{W}, \overline{X}, \overline{Y}$ properties ($\nabla 1$) – ($\nabla 5$) of $\overline{\nabla}$ are satisfied. The result follows from following observations:

1. $\overline{\nabla}_{\overline{V}}\overline{X}|_M = \overline{\nabla}_V X$ holds by definition (see (1.3.3)).
2. $\overline{V}(\tilde{f})|_M = V(f)$ (see (1.3.4)).
3. $\left\langle \overline{X}, \overline{Y} \right\rangle|_M = \langle X, Y\rangle$ (M is a SRSMF of $\overline{M}$).
4. $[\overline{V}, \overline{W}]|_M = [V, W]$ (since $V \sim_j \overline{V}$, $W \sim_j \overline{W}$, we get $[V, W] \sim_j [\overline{V}, \overline{W}]$; see [2], 2.3.16).

□

Note that for $V, W \in \mathfrak{X}(M)$, $\overline{\nabla}_V W \notin \mathfrak{X}(M)$ since the derivative might have non-tangential directions (see

[2], Section 3.2). Consequently, our interest lies in studying both tan$(\overline{\nabla}_V W)$ and nor$(\overline{\nabla}_V W)$. We'll discover that tan$(\overline{\nabla}_V W)$ corresponds to something already familiar, while nor$(\overline{\nabla}_V W)$ introduces a new concept.

Proposition 1.3.10 (tan$\overline{\nabla}$)**.** Let $V, W \in \mathfrak{X}(M)$ and M be a SRSMF of $\overline{M}$. Then we have

$$\text{tan}(\overline{\nabla}_V W) = \nabla_V W,$$

where ∇ is Levi-Civita connection on M.

Proof. Let $X \in \mathfrak{X}(M)$ and choose extensions $\overline{V}, \overline{W}$ and $\overline{X}$, as in Lemma 1.3.7. Then by the Koszul formula for $\overline{M}$ ([3], (1.3.10))

$$\begin{aligned} 2\left\langle \overline{\nabla}_{\overline{V}}\overline{W}, \overline{X}\right\rangle &= \overline{V}\left\langle \overline{W}, \overline{X}\right\rangle + \overline{W}\left\langle \overline{X}, \overline{V}\right\rangle - \overline{X}\left\langle \overline{V}, \overline{W}\right\rangle - \left\langle \overline{V}, [\overline{W}, \overline{X}]\right\rangle + \left\langle \overline{W}, [\overline{X}, \overline{V}]\right\rangle + \left\langle \overline{X}, [\overline{V}, \overline{W}]\right\rangle \\ &=: F(\overline{V}, \overline{W}, \overline{X}). \end{aligned} \tag{1.3.6}$$

As in the proof of Proposition 1.3.9, we find that

$$\begin{aligned} \left\langle \overline{\nabla}_{\overline{V}}\overline{W}, \overline{X}\right\rangle|_M &\overset{3.}{=} \left\langle \overline{\nabla}_{\overline{V}}\overline{W}|_M, \overline{X}|_M\right\rangle \overset{1.}{=} \left\langle \overline{\nabla}_V W, X\right\rangle, \\ (\overline{V}\left\langle \overline{W}, \overline{X}\right\rangle)|_M &\overset{2.}{=} V(\left\langle \overline{W}, \overline{X}\right\rangle|_M) \overset{3.}{=} V\left\langle W, X\right\rangle, \\ \left\langle \overline{V}, [\overline{X}, \overline{W}]\right\rangle|_M &\overset{3.}{=} (\left\langle V, [\overline{X}, \overline{W}]|_M\right\rangle) \overset{4.}{=} \left\langle V, [X, W]\right\rangle. \end{aligned}$$

So the restriction of equation (1.3.6) gives

$$2\left\langle \overline{\nabla}_V W, X\right\rangle = F(X, Y, Z) = 2\left\langle \nabla_V W, X\right\rangle \Longrightarrow \left\langle \overline{\nabla}_V W, X\right\rangle = \left\langle \nabla_V W, X\right\rangle,$$

where r.h.s. Koszul formula characterizes Levi-Civita connection on M (see [3], 1.3.4). Finally, since X is tangential ($X \in \mathfrak{X}(M)$!), we have

$$\left\langle \overline{\nabla}_V W, X\right\rangle = \left\langle \text{tan}\overline{\nabla}_V W + \text{nor}\overline{\nabla}_V W, X\right\rangle = \left\langle \text{tan}\overline{\nabla}_V W, X\right\rangle,$$

which, because of non-degeneracy, implies that tan$\overline{\nabla}_V W = \nabla_V W$. □

Lemma 1.3.11 (Second Fundamental Form)**.** The mapping

$$\mathbb{I} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)^\perp$$

$$\mathbb{I}(V, W) = \text{nor}(\overline{\nabla}_V W)$$

is $\mathcal{C}^\infty(M)$-bilinear and symmetric. We call it the **second fundamental form** or **shape tensor** of M in $\overline{M}$.

Proof. $\mathcal{C}^\infty(M)$-linearity in V and $\mathbb{R}$-linearity in W follow from $(\nabla 1)$ and $(\nabla 2)$ of $\overline{\nabla}_V W$ in Proposition 1.3.9. To prove $\mathcal{C}^\infty(M)$-linearity, we will need

$$\overline{\nabla}(fW) \overset{(\nabla 3)}{=} V(f)W + f\overline{\nabla}_V W$$

from Proposition 1.3.9 in order to obtain

$$\mathbb{I}(V, fW) = \text{nor}(\overline{\nabla}(fW)) = V(f)\underbrace{\text{nor}W}_{=0} + f\text{nor}(\overline{\nabla}_V W) = f\mathbb{I}(V, W),$$

where the second equality holds because nor is $\mathcal{C}^\infty(M)$-linear. Lastly, we prove symmetry:

$$\mathbb{I}(V,W) - \mathbb{I}(W,V) \overset{\text{bilinearity}}{=} \text{nor}(\overline{\nabla}_V W - \overline{\nabla}_W V) \overset{(\nabla 4)}{=} \text{nor}([V,W]) = 0,$$

since $[V,W] \in \mathfrak{X}(M)$. □

Remark 1.3.12. Even though $\mathbb{I}$ is not a tensor field on M (since it takes values in $\mathfrak{X}(M)^\perp$), $\mathcal{C}^\infty(M)$-bilinearity still implies, akin to the tensor case, that $\mathbb{I}$ behaves as a 'pointwise object'. This implies the ability to insert individual tangent vectors into it. Therefore, $\mathbb{I}(V,W)|_p$ depends only on $V(p)$ and $W(p)$ (proof as in the tensor case, see [2], 4.1.19). Consequently, at any point, $\mathbb{I}$ defines a bilinear map:

$$\mathbb{I}_p : T_pM \times T_pM \to T_pM^\perp$$
$$(v,w) \mapsto \mathbb{I}_p(v,w).$$

By Proposition 1.3.9 and 1.3.10, we have that

$$\overline{\nabla}_V W = \tan(\overline{\nabla}_V W) + \text{nor}(\overline{\nabla}_V W) = \underbrace{\nabla_V W}_{\mathfrak{X}(M)} + \underbrace{\mathbb{I}(V,W)}_{\mathfrak{X}(M)^\perp}, \ \forall V,W \in \mathfrak{X}(M) \tag{1.3.7}$$

We derive the fundamental result concerning the curvature of SRSMF, known as the *Gauss equation*, from this observation.

Theorem 1.3.13 (Gauss Equation)**.** Let M be a SRSMF of $\overline{M}$ with Riemann tensors R and $\overline{R}$, respectively. Then $\forall V,W,X,Y \in \mathfrak{X}(M)$

$$\langle R_{VW}X, Y\rangle = \langle \overline{R}_{VW}X, Y\rangle + \langle \mathbb{I}(V,X), \mathbb{I}(W,Y)\rangle - \langle \mathbb{I}(V,Y), \mathbb{I}(W,X)\rangle.$$

Proof. This is a tensor identity (in the slightly generalized sense of the previous remark) and so we can argue pointwise. We can assume that $[V,W] = 0$ (see the proof of [3], 3.1.2). Then

$$\langle \overline{R}_{VW}X, Y\rangle = -(VW) + (WV),$$

where

$$\begin{aligned}
(VW) &:= \langle \overline{\nabla}_V \overline{\nabla}_W X, Y\rangle \\
&\overset{(1.3.7)}{=} \langle \overline{\nabla}_V \nabla_W X, Y\rangle + \langle \overline{\nabla}_V \mathbb{I}(W,X), Y\rangle \\
&\overset{1.3.10,(\nabla 5)}{=} \langle \nabla_V \nabla_W X, Y\rangle + (V\langle \mathbb{I}(W,X), Y\rangle - \langle \mathbb{I}(W,X), \overline{\nabla}_V Y\rangle) \\
&= \langle \nabla_V \nabla_W X, Y\rangle (0 - \langle \mathbb{I}(W,X), \text{nor}(\overline{\nabla}_V Y)\rangle) \\
&= \langle \nabla_V \nabla_W X, Y\rangle - \langle \mathbb{I}(W,X), \mathbb{I}(V,Y)\rangle
\end{aligned}$$

where in the third equality the normal part vanishes because Y is tangential. Finally, we obtain

$$\begin{aligned}
\langle \overline{R}_{VW}X, Y\rangle &= (WV) - (VW) \\
&= \langle \underbrace{(\nabla_W \nabla_V - \nabla_V \nabla_W)X}_{R_{VW}X}, Y\rangle - \langle \mathbb{I}(V,X), \mathbb{I}(W,Y)\rangle + \langle \mathbb{I}(W,X), \mathbb{I}(V,Y)\rangle
\end{aligned}$$

□

As the Gauss equation represents a tensor identity, we have the flexibility to insert individual tangent vectors into the equation.

Corollary 1.3.14 (Gauss Equation for K). Let $v, w \in T_pM$ span a non-degenerate tangent plane of M. Then

$$K(v,w) = \overline{K}(v,w) + \frac{\langle \mathbb{I}(v,v), \mathbb{I}(w,w)\rangle - \langle \mathbb{I}(v,w), \mathbb{I}(v,w)\rangle}{Q(v,w)}.$$

Definition 1.3.15. Since $\mathbb{I}$ is a $(0,2)$-tensor with values in $\mathfrak{X}(M)^\perp$ it can be metrically contracted to give a normal tensor field on M. The result of this contraction is called the **mean curvature vector field**. At p in M it is given by

$$H_p := \frac{1}{n}\sum \epsilon_i \mathbb{I}(e_i, e_i),$$

where n is the dimension of M and $e_1, \dots, e_n$ any frame of T_pM (see here Example 1.2.3, 2.).

Example 1.3.16 (Sectional Curvature of Spheres). Sectional curvature of the n-sphere $S^n(r) := \{x \in \mathbb{R}^{n+1} : \|x\| = r\}$ is given by

$$K = \begin{cases} 0, & n = 1 \\ \frac{1}{r^2}, & n \geq 2. \end{cases}$$

Indeed, in the case of $n = 1$ the result is clear from Example 1.1.6 (2.). Therefore, let $n \geq 2$ and denote by

$$P(u) = \sum_{i=1}^{n+1} u^i \partial_i$$

the position vector field on $\mathbb{R}^{n+1}$. Then $P \perp S^n(r)$ at every point of $S^n(r)$. Let $\overline{\nabla}$ be the (flat) connection on $\mathbb{R}^{n+1}$. For all $\sum_{i=1}^{n+1} X(u^i) = X \in \mathfrak{X}(\mathbb{R}^{n+1})$ (like in [3], 1.3.10), we have

$$\overline{\nabla}_X P = \sum_{i=1}^{n+1} (X(u^i)\partial_i + u^i \underbrace{\overline{\nabla}_{\partial_i}(P)}_{=0}) = X.$$

Let $U := \frac{1}{r}P$ be the outer unit vector field on $S^n(r)$. Then

$$\langle \mathbb{I}(V,W), U\rangle = \langle \text{nor}\overline{\nabla}_V W, \underbrace{U}_{\in \mathfrak{X}(S^n)^\perp}\rangle = \left\langle \overline{\nabla}_V W, U\right\rangle = \frac{1}{r}\left\langle \overline{\nabla}_V W, P\right\rangle \overset{(\nabla 5)}{=} -\frac{1}{r}\left\langle W, \overline{\nabla}_V P\right\rangle = -\frac{1}{r}\langle V, W\rangle,$$

where the fourth equality follows from $(\nabla 5)$ since $0 = V\langle \underbrace{W}_{\in T_pM}, \underbrace{P}_{\in T_pM^\perp}\rangle$. Since $\mathbb{I}(V,W) \in \mathfrak{X}(S^n(r))^\perp$, it is proportional to U. Hence,

$$\mathbb{I}(V,W) = -\frac{1}{r}\langle V, W\rangle U.$$

Since $\mathbb{R}^{n+1}$ is flat ($\overline{K} = 0$), we obtain the following result from Corollary 1.3.14:

$$K(v,w) = 0 + \frac{\frac{1}{r^2}(\langle v,v\rangle \overbrace{\langle u,u\rangle}^{=1} \langle w,w\rangle - \langle v,w\rangle^2)}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2} = \frac{1}{r^2}.$$

As seen in the previous example, specific simplifications arise when M has a codimension of 1. We will explore this scenario in the upcoming discussion.

1.4 Semi-Riemannian Hypersurfaces (SRHSF)

Definition 1.4.1.

- A semi-Riemannian hypersurface (SRHSF) M of $\overline{M}$ is a SRSMF of codimension 1.
- The index of all the 1-dimensional spaces $T_pM^\perp$ is called co-index of M and it takes values in 0 or 1.

Definition 1.4.2. The signum ϵ of a SRHSF M of $\overline{M}$ is:

$+1$, if the coindex of M is 0 i.e. if for all $p \in M$ and for all $0 \neq z \in T_pM^\perp$

$$\langle z, z\rangle > 0.$$

-1, if the coindex of M is 1 i.e. if for all $p \in M$ and for all $0 \neq z \in T_pM^\perp$

$$\langle z, z\rangle < 0.$$

Remark 1.4.3. We have that

$$\text{ind}(M) = \text{ind}(\overline{M}) \iff \epsilon = 1 \text{ and } \text{ind}(M) = \text{ind}(\overline{M}) - 1 \iff \epsilon = -1.$$

If $\overline{M}$ is Riemannian, all SRHSFs are of signum 1 and hence M is also Riemannian. We call it Riemannian hypersurface (RHSF). If $\overline{M}$ is Lorentzian (i.e. if $\overline{M}$ has index equal to 1 according to [3], text under 1.2.1), we call M:

- spacelike if $\epsilon = -1$ (i.e. if M is Riemannian) and
- timelike if $\epsilon = 1$.

Hypersurfaces are often defined as zero sets of regular functions.

Proposition 1.4.4 (SRHSFs as Zero Sets). Let $f \in \mathcal{C}^\infty(\overline{M})$, $c \in f(\overline{M})$. Let $M := f^{-1}(c)$ and assume that $\langle \text{grad}(f), \text{grad}(f)\rangle > 0$ or $\langle \text{grad}(f), \text{grad}(f)\rangle < 0$ for all $p \in M$. Then M is a SRHSF of $\overline{M}$ with

$$\text{signum}(M) = \text{sgn}\,\langle \text{grad}(f), \text{grad}(f)\rangle$$

and $U := \frac{\text{grad}(f)}{\|\text{grad}(f)\|}$ is a unit normal vector for M.

Proof.

$$\text{grad}(f)|_p \neq 0 \implies df|_p \neq 0, \ \forall p \in M \implies f \text{ is regular}$$

and therefore M is a codimension 1 sub-manifold of $\overline{M}^{n+1}$ (see [2], 1.1.8). For every $v \in T_pM$, we have that

$$\langle \text{grad}(f), v\rangle = v(f) = v(\underbrace{f|_M}_{=c}) = 0 \tag{1.4.1}$$

i.e. $\text{grad}(f)|_p \in T_pM^\perp$. Since $T_pM^\perp$ is 1-dimensional, $\text{grad}(f)|_p$ spans it. To show that T_pM is non-degenerate it is (by [3], 1.1.10) enough to show that $T_pM^\perp$ is non-degenerate. Indeed, let $v, w \in T_pM^\perp$. Then there exist $\lambda, \mu \in \mathbb{R}$ such that $v = \lambda \cdot \text{grad}(f)|_p$ and $v = \mu \cdot \text{grad}(f)|_p$. If $\langle v, w\rangle = 0$ for all $w \in T_pM^\perp$, then

$$\lambda\mu \underbrace{\langle \text{grad}(f), \text{grad}(f)\rangle}_{\neq 0} = 0 \ \forall \mu,$$

$\lambda = 0$ and, consequently, $v = 0$. Finally, $\frac{1}{\|\text{grad}(f)\|}\text{grad}(f) = U \in T_pM^\perp$ by (1.4.1). □

Remark 1.4.5. If $\overline{M}$ is Lorentzian then, if grad is spacelike, M is timelike and, if grad is timelike, M is spacelike.

Example 1.4.6.

1. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ such that
$$f(u) = \sum (u^i)^2.$$
Then $f^{-1}(r^2) = S^n(r)$. $S^n(r)$ is Riemannian since $\overline{M} = \mathbb{R}^{n+1}$.
Also note that $\text{grad}(f) = 2P$, where P is position vector field from Example 1.3.16.

2. Observe that not every SRHSF is (globally) a level set of a $\mathcal{C}^\infty$-function f (for example, Möbius strip). However, locally, every SRHSF is of this form.

For SRHSFs the second fundamental form $\mathbb{I}$ can be described in simpler terms. Let U be a unit normal vector field, $p \in M$ and $V \in \mathfrak{X}(M)$. Then

$$T_pM \ni w \mapsto \langle \mathbb{I}(V_p, w), U_p \rangle \in \mathbb{R}$$

is linear. Hence, there exists a unique vector $S(V)_p \in T_pM$ such that

$$\langle S(V)_p, w \rangle = \langle \mathbb{I}(V_p, w), U_p \rangle, \quad \forall w \in T_pM.$$

Varying p we obtain $S(V) \in \mathfrak{X}(M)$ via a local frame E_i of M:

$$S(V) = \sum_i \epsilon_i \langle S(V), E_i \rangle E_i = \underbrace{\sum_i \epsilon_i \langle \mathbb{I}(V, E_i), U \rangle}_{\in \mathcal{C}^\infty} E_i \,.$$

Moreover, $V \mapsto S(V)$ is $\mathcal{C}^\infty(M)$-linear, hence $S(V) \in \mathcal{T}^1_1(M)$.

Definition 1.4.7. Let U be a unit normal vector field of the SRHSF M in $\overline{M}$. The $(1,1)$-tensor field S with

$$\langle S(V), W \rangle = \langle \mathbb{I}(V, W), U \rangle, \quad \forall V, W \in \mathfrak{X}(M)$$

is called the **shape operator** (derived from U) of M in $\overline{M}$.

Lemma 1.4.8 (Form of the Shape Operator)**.** Let S be the shape operator derived from U. Then

$$S(v) = -\overline{\nabla}_v U, \quad \forall v \in T_pM, \text{ where } p \in M$$

and $S_p : T_pM \to T_pM$ is self-adjoint.

Proof. Let $V, W \in \mathfrak{X}(M)$. Then

$$\langle S(V), W \rangle \overset{1.4.7}{=} \langle \mathbb{I}(V, W), U \rangle = \left\langle \overline{\nabla}_V W, U \right\rangle = -\left\langle W, \overline{\nabla}_V U \right\rangle \Longrightarrow S(V) = -\overline{\nabla}_V U,$$

where the second equality holds since U is normal and the third one follows from $(\nabla 5)$ since $V \underbrace{\langle W, U \rangle}_{=0} = 0$.

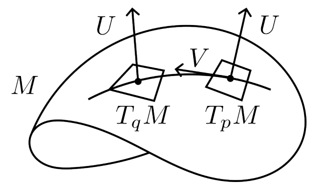
Finally, self-adjointness follows from symmetry of $\mathbb{I}$ (see Lemma 1.3.11)

$$\langle S(V), W \rangle = \langle \mathbb{I}(V, W), U \rangle = \langle \mathbb{I}(W, V), U \rangle = \langle S(W), V \rangle \,.$$

□

Remark 1.4.9.

1. S describes the shape of M in $\overline{M}$; $\overline{\nabla}_V U$ is the change of U in direction V. It describes the behavior of U and hence also of $T_p M$.



2. In classical terminology, sometimes the $(0,2)$-tensor

$$\mathcal{B} := \downarrow_1^1 S$$

is called 2^{nd} fundamental form.

3. If U is only locally defined and we replace U by $-U$, S changes sign as well. Therefore, even if there is no global unit normal vector field S is determined up to sign.

We close this section by deriving the form of the sectional curvature for SRHSF.

Corollary 1.4.10 (Gauss Equation). Let S be the shape operator of a SRHSF $M \subseteq \overline{M}$. If v, w span a non-degenerate tangent plane in $T_p M$ then

$$K(v,w) = \overline{K}(v,w) + \epsilon \frac{\langle Sv, v\rangle \langle Sw, w\rangle - \langle Sv, w\rangle^2}{Q(v,w)}.$$

Proof. U_p is an ONB of $T_p M^\perp$. Therefore,

$$T_p M^\perp \ni \mathbb{I}(v,w) = \epsilon \langle \mathbb{I}(v,w), U_p\rangle U_p = \epsilon \langle Sv, w\rangle U_p.$$

Insert this into Corollary 1.3.14 and note that $\langle U_p, U_p\rangle = \epsilon$. □

1.5 Geodesics in SRMFSs

We'll start by adapting $\overline{\nabla}_V W = \nabla_V W + \mathbb{I}(V,W)$ to vector fields on curves.

Proposition 1.5.1. Let $\alpha : I \to M^n \subseteq \overline{M}$ be a $\mathcal{C}^\infty$-curve and let $Y \in \mathfrak{X}(\alpha)$ with $Y(t) \in T_{\alpha(t)} M$ for all $t \in I$ (i.e. Y is tangential to M). Then

$$\dot{Y} = \underbrace{Y'}_{\text{tangent}} + \underbrace{\mathbb{I}(\alpha', Y)}_{\text{normal}}.$$

Here

$$\dot{Y}(s) = \frac{\overline{\nabla} Y}{ds} \text{ and } Y'(s) = \frac{\nabla Y}{ds}.$$

(Note that $\dot{\alpha} = \alpha' = \frac{d\alpha}{ds}$.)

Proof. Without loss of generality assume that α lies in a single adapted chart domain of $\overline{M}$. Then

$$Y = \sum_{i=1}^{n} Y^i \partial_i|_\alpha, \text{ where } Y^i : I \to \mathbb{R}.$$

[3] 1.3.27. implies that

$$\dot{Y} = \sum_{i=1}^{n} \frac{dY^i}{ds} \partial_i|_\alpha + \sum_{i=1}^{n} Y^i (\partial_i|_\alpha)^{\boldsymbol{\cdot}}.$$

Now

$$(\partial_i|_\alpha)^{\boldsymbol{\cdot}} = \overline{\nabla}_{\alpha'}(\partial_i) \overset{(1.3.7)}{=} \nabla_{\alpha'}(\partial_i) + \mathbb{I}(\alpha', \partial_i)$$

where the first equality holds by [3], 1.3.27, (iii). Therefore,

$$\dot{Y} = \sum_{i=1}^{n} \frac{dY^i}{ds} \partial_i|_\alpha + \sum_{i=1}^{n} Y^i \nabla_{\alpha'}(\partial_i|_\alpha) + \sum_{i=1}^{n} Y^i \mathbb{I}(\alpha', \partial_i|_\alpha)$$

proves the claim since, by [3], 1.3.27,

$$\sum_{i=1}^{n} \frac{dY^i}{ds} \partial_i|_\alpha + \sum_{i=1}^{n} Y^i \nabla_{\alpha'}(\partial_i|_\alpha) = Y'$$

and $\sum_{i=1}^{n} Y^i \mathbb{I}(\alpha', \partial_i|_\alpha) = \mathbb{I}(\alpha', Y)$. □

Corollary 1.5.2 (Acceleration). Let $\alpha : I \to M \subseteq \overline{M}$ be a $\mathcal{C}^\infty$-curve. Then

$$\ddot{\alpha} = \alpha'' + \mathbb{I}(\alpha', \alpha'),$$

where $\ddot{\alpha}$ denotes acceleration in $\overline{M}$ and α'' acceleration in M.

Proof. From Proposition 1.5.1 (recalling that $\dot{\alpha} = \alpha'$) we get that

$$\ddot{\alpha} = (\dot{\alpha})' + \mathbb{I}(\alpha', \dot{\alpha}) = \alpha'' + \mathbb{I}(\alpha', \alpha').$$

□

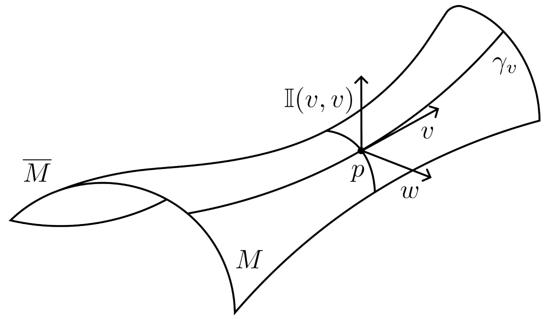
Remark 1.5.3. This gives us another nice way of seeing that $\mathbb{I}$ describes the shape of M in $\overline{M}$. Let $p \in M$, $v \in T_pM$ and γ_v an M-geodesic with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. Then γ_v is 'straight' in M i.e. $\gamma_v'' = 0$. Therefore, all curvature of γ_v in $\overline{M}$ comes from it being forced to stay in M i.e., by Corollary 1.5.2, $\ddot{\gamma}_v(0) = \mathbb{I}(v, v)$. According to Corollary 1.5.2 a curve γ is an M-geodesic if and only if $\ddot{\gamma}$ is normal to M. Indeed, assume γ is an M-geodesic. Then

$$\ddot{\gamma} = \gamma'' + \mathbb{I}(\gamma', \gamma'),$$

where $\gamma'' = 0$ since γ is an M-geodesic. Therefore, $\ddot{\gamma}$ is an element of $T_pM^\perp$. Conversely,

$$\gamma'' = \ddot{\gamma} - \mathbb{I}(\gamma', \gamma') \in T_pM^\perp,$$

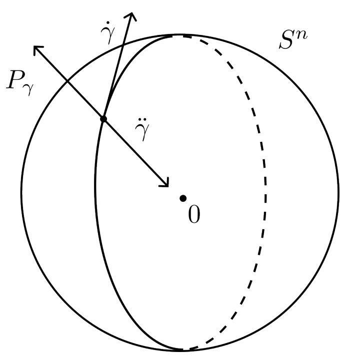
while $\gamma'' \in T_pM$. Therefore, $\gamma'' = 0$ and so γ is an M-geodesic.



Therefore, an M-geodesic moves freely within M but experiences a 'force' normal to M within $\overline{M}$ that confines it to remain on M.

Example 1.5.4 (Geodesics on the Sphere). The geodesics on $S^n(r)$ are precisely the great circles (parametrized with constant speed). A great circle on S^n is a circle $\Pi \cap S^n$ where Π is a 2-dimensional plane through $0 \in \mathbb{R}^{n+1}$. Let γ be such a curve (parametrized with unit speed). Then

$$c = \langle \dot\gamma, \dot\gamma \rangle \overset{\frac{\nabla}{dt}}{\Longrightarrow} \langle \dot\gamma, \ddot\gamma \rangle = 0 \Longrightarrow \ddot\gamma \perp \dot\gamma \text{ and } \dot\gamma, \ddot\gamma \in \Pi.$$

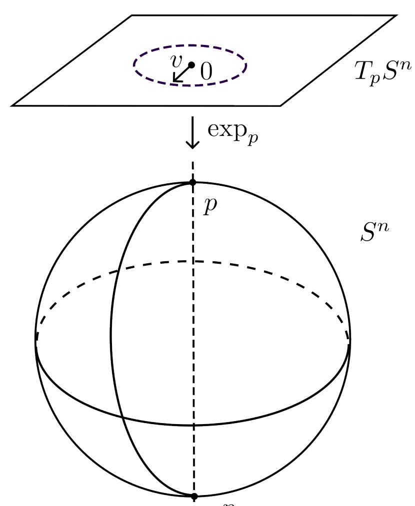


Let $P : x \mapsto x$ be the position vector field on $\mathbb{R}^{n+1}$. Then $P_\gamma \in \Pi$ but

$$P\gamma \perp \dot\gamma \Longrightarrow \ddot\gamma \propto P\gamma \Longrightarrow \ddot\gamma \perp S^n \Longrightarrow \gamma \text{ is a geodesic by Remark 1.5.3.}$$

Conversely, let γ be a non-constant geodesic on S^n and let Π be the plane through 0, $\gamma(0)$ and where $\dot\gamma(0)$ lies. Parametrize $\Pi \cap S^n$ as a suitable constant speed curve α such that $\alpha(0) = \gamma(0)$ and $\dot\alpha(0) = \dot\gamma(0)$. From the above, α is a geodesic. By unique solvability of the initial value problem for the geodesic equation, $\alpha = \gamma$.

Example 1.5.5 (Exponential Map of S^n at p).



Let $v \in T_pS^n$. Then $\exp_p$ maps (by Example 1.5.4) the straight line $tv \in T_pS^n$ to the great circle through p tangential to v. Hence,

$$\exp_p(tv) = \cos(t)p + \sin(t)v.$$

The $(n-1)$-spheres t = constant are mapped to 'spheres of latitude'; hence S^{n-1}-spheres in S^n which are given by the intersections of S^n with planes normal to the 'axis' $\{-p, p\}$. For $t = k\pi$ ($k \in \mathbb{Z}$) these spheres collapse to p and $-p$, respectively.
We explicitly see that $\exp_p$ is a diffeomorphism (see [3], 2.1.14) of

$$\mathcal{D}_\pi = \{v \in T_pS^n : \|v\| < \pi\} \text{ to } S^n \backslash \{-p\}.$$

Therefore, $S^n \backslash \{-p\}$ is a normal neighborhood of p.

Now, we turn our attention to the simplest SRMFSs—those that appear flat when observed from $\overline{M}$.

Definition 1.5.6. A SRSMF M of $\overline{M}$ is called totally geodesic if $\mathbb{I} = 0$ on M.

Theorem 1.5.7 (Totally Geodesic SRSMF)**.** For a SRSMF M of $\overline{M}$ the following are equivalent:

1. M is totally geodesic i.e. $\mathbb{I} = 0$.
2. A curve γ in M is an M-geodesic if and only if it is an $\overline{M}$-geodesic.
3. Let $p \in M$, $v \in T_pM \subseteq T_p\overline{M}$. Then the $\overline{M}$-geodesic γ_v ($\gamma_v(0) = p$, $\dot{\gamma}_v(0) = v$) lies in M initially i.e. there exists an open interval $I \ni \{0\}$ such that $\gamma_v(t) \in M$ for all $t \in I$.
4. Let c be a $\mathcal{C}^\infty$-curve in M and $v \in T_{c(0)}M$. Then the M- and the $\overline{M}$-parallel transport of v along c coincide.

Proof.

$(1. \to 4.)$ Let V be the M-parallel vector field along c with $V(0) = v$. By Proposition 1.5.1

$$\dot{V} = V' + \underbrace{\mathbb{I}(c', V)}_{=0}$$

and V is also $\overline{M}$-parallel.

$(4. \to 2.)$ From [3] we know that γ is an M-geodesic if and only if γ' is M-parallel. By assumption, γ' is M-parallel if and only if γ' is $\overline{M}$-parallel, which is the case if and only if γ is an $\overline{M}$-geodesic.

$(2. \to 3.)$ Let $\alpha : I \to M$ be the M-geodesic with $\alpha(0) = \gamma_v(0) = p$ and $\alpha'(0) = v$. From 2. we have that α is also an $\overline{M}$-geodesic and from in [3] that $\alpha = \gamma_v|_I$. Observe here that γ_v can leave M. For example, consider an open disc in $\mathbb{R}^2$ and observe a radial geodesic $\gamma_v(t)$ starting at its centre. For some t, the geodesic leaves the disc.

$(3. \to 1.)$ Let $v \in T_pM$. From Corollary 1.5.2 we get that

$$\underbrace{\ddot{\gamma}_v}_{=0} = \underbrace{\gamma_v''}_{=0} + \mathbb{I}(v, v) \implies \mathbb{I}(v, v) = 0$$

and so $\mathbb{I} = 0$, by polarization. $0 = \mathbb{I}(v + w, v + w) = \mathbb{I}(v, v) + 2\mathbb{I}(v, w) + \mathbb{I}(w, w) = 2\mathbb{I}(v, w)$.

□

Example 1.5.8. For S^2 exactly the great circles are totally geodesic 1-dimensional submanifolds (see property 3. in the above theorem).

In the following result, we demonstrate that the number of totally geodesic SRSMFs is limited.

Proposition 1.5.9. Let M and N be SRMFSs of $\overline{M}$ that are complete, connected and totally geodesic. If there is $p \in M \cap N$ such that $T_pM = T_pN$, then $M = N$.

Proof. By symmetry, it suffices to prove that if M is connected, N complete and both are totally geodesic, then $M \subseteq N$. To this end, let γ be an M-geodesic connecting two points, p and q, in M. From Theorem 1.5.7 (2.) it follows that γ is also an $\overline{M}$-geodesic. $\dot{\gamma}(0) \in T_pM = T_pN$ by assumption and so, by Theorem 1.5.7 (3.) γ is an N-geodesic initially. Since N is complete, γ lies in N for all times. Hence, q is in N. By Theorem 1.5.7 (4.), $T_qM = T_qN$ (since parallel transport is an isometry by 1.3.30 from [3] and since all parallel transports agree). Since M is connected, every $q \in M$ can be reached by a broken geodesic (see

[3], 2.1.16). Therefore, any point in M lies in N as well, by iterating the above and so $M \subseteq N$. □

Example 1.5.10.

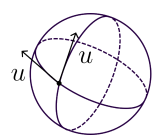
1. Let W be a k-dimensional subspace of $\mathbb{R}^n_r$. We call translates $W + x$, **k-planes**. The non-degenerate k-planes are totally geodesic SRSMFs of $\mathbb{R}^n_r$ by Theorem 1.5.7 (2.). Moreover, by Proposition 1.5.9, they are the only totally geodesic SRMFSs of $\mathbb{R}^n_r$ which are complete and connected.

2. The complete, connected, totally geodesic k-dimensional RSMFs of $S^n(r)$ are the **k-great spheres** i.e. the intersections $W \cap S^n(r)$ with W, where W is a $(k+1)$-plane through 0 in $\mathbb{R}^{n+1}$. Indeed, $W \cap S^n(r)$ is totally geodesic: $W \cap S^n(r)$ is a sphere in W. By Example 1.5.4 geodesics in $W \cap S^n(r)$ are (parts of) great circles in $W \cap S^n(r)$, hence in $S^n(r)$. The claim follows from Theorem 1.5.7 (2.). Conversely, if M is complete, connected, totally geodesic k-dimensional RSMF of $S^n(r)$, then choose a $(k+1)$-plane W through 0 such that $T_pM = T_p(W \cap S^n(r))$ (for example, let $W :=$ span(T_pM, p)). Proposition 1.5.9 implies that $M = W \cap S^n(r)$.

Definition 1.5.11. A point p in $M \subseteq \overline{M}$ SRSMF is called **umbilic** if there exists $z \in T_pM^\perp$, called **normal curvature vector**, such that

$$\mathbb{I}(v, w) = \langle v, w \rangle\, z, \ \ \forall v, w \in T_pM.$$

If $\overline{M}$ is Riemannian, for every unit vector u, we find that $\mathbb{I}(u, u) = z$. Consequently, at an umbilic point, M bends uniformly in all directions.



In the Lorentzian scenario, $\mathbb{I}(u, u) = \pm z$. Hence, within spacelike directions (i.e., when $\langle u, u \rangle = +1$), M bends toward z, while in timelike directions (i.e., when $\langle u, u \rangle = -1$), it bends away from z.

Definition 1.5.12. A SRSMF $M \subseteq \overline{M}$ is called **totally umbilic** if every $p \in M$ is umbilic.

In this case, we have

$$\mathbb{I}(V, W) = \langle V, W \rangle\, Z, \ \ \forall W, V \in \mathfrak{X}(M),$$

where $Z \in \mathfrak{X}(M)^\perp$ is called the **normal curvature vector field** of M. It is $\mathcal{C}^\infty$ since in a local frame of M

$$p \mapsto \epsilon_i Z|_p = \langle E_i, E_i \rangle\, |_p \cdot Z_p \overset{\text{def.}}{=} \mathbb{I}(E_i, E_i)|_p \in \mathcal{C}^\infty.$$

In particular, a totally geodesic SRSMF ($\mathbb{I} = 0$) is totally umbilic with vanishing Z.

Example 1.5.13. The spheres $S^n(r)$ are totally umbilic with $Z = -\frac{1}{r}U$, see Example 1.3.16.

In the case of hypersurfaces, things simplify significantly. In this scenario, the normal curvature vector reduces to a scalar quantity.

Proposition 1.5.14 (Characterizing Totally Umbilic SRHSFs)**.** Let $M \subseteq \overline{M}$ be a SRHSF. The following are equivalent:

1. M is totally umbilic.

2. The shape operator of M is scalar i.e. for any choice of unit normal vector U there is a scalar function k_U on the domain of U such that

$$S(V) = k_U V, \ \ \forall V \in \mathfrak{X}(M).$$

Proof.

$(1. \to 2.)$ Let Z be the normal curvature vector field of M in $\overline{M}$ (i.e. $\mathbb{I}(V,W) \overset{(\star)}{=} \langle V,W\rangle Z$). Then for all $V,W \in \mathfrak{X}(M)$, we have

$$\langle S(V),W\rangle \overset{1.4.7}{=} \langle \mathbb{I}(V,W),U\rangle \overset{(\star)}{=} \langle V,W\rangle \langle Z,U\rangle = \langle \langle Z,U\rangle V,W\rangle$$

hence, on the domain of U, $S(V) = \underbrace{\langle Z,U\rangle}_{=:\,k_U} V$.

$(2. \to 1.)$ As in the proof of Corollary 1.4.10,

$$\mathbb{I}(V,W) = \epsilon \langle \mathbb{I}(V,W),U\rangle U = \epsilon \langle S(V),W\rangle U \overset{2.}{=} \epsilon k_U \langle V,W\rangle U.$$

If we replace U by $-U$, $S(V)$ also changes sign (see Definition 1.4.7) and hence $k_{-U} = -k_U$. $Z := \epsilon k_U U$ is therefore globally well-defined and we have

$$\mathbb{I}(V,W) = \langle V,W\rangle Z.$$

□

Remark 1.5.15. The function k from the above proposition (defined up to sign) is often called the *normal curvature function* of M in $\overline{M}$.

1.6 The Normal Connection and the Codazzi-equation

We have studied the map $(V,W) \mapsto \overline{\nabla}_V W$ for $V,W \in \mathfrak{X}(M)$, where $\overline{\nabla}_V W = \nabla_V W + \mathbb{I}(V,W)$. We now want to study geometry normal to M.

Definition 1.6.1. The normal connection of a SRSMF $M \subseteq \overline{M}$ is the mapping $\nabla^\perp : \mathfrak{X}(M) \times \mathfrak{X}(M)^\perp \to \mathfrak{X}(M)^\perp$, where

$$\nabla^\perp_V Z := \mathrm{nor}(\overline{\nabla}_V Z), \ \forall V \in \mathfrak{X}(M), Z \in \mathfrak{X}(M)^\perp.$$

Remark 1.6.2. $\nabla^\perp_V Z$ is also called the cormal covariant derivative (of Z with respect to V); it measures the rate of change of Z in the normal direction when p moves tangentially in the direction of V.

Lemma 1.6.3 (Properties of $\nabla^\perp$). $\nabla^\perp$ satisfies the following:

($\nabla 1$) $\nabla^\perp_V Z$ is $\mathcal{C}^\infty(M)$-linear in V,

($\nabla 2$) $\nabla^\perp_V Z$ is $\mathbb{R}$-linear in Z,

($\nabla 3$) $\nabla^\perp_V (fZ) = V(f)Z + f\nabla^\perp_V Z$ (Leibnitz rule),

($\nabla 5$) $V\langle Y,Z\rangle = \langle \nabla^\perp_V Y, Z\rangle + \langle Y, \nabla^\perp_V Z\rangle$ (metric condition),

where $V \in \mathfrak{X}(M)$, $f \in \mathcal{C}^\infty(M)$ and $Z \in \mathfrak{X}(M)^\perp$.
(Note that there is no ($\nabla 4$) since $[V,Z]$ for $V \in \mathfrak{X}(M)$ and $Z \in \mathfrak{X}(M)^\perp$ is not defined!)

Proof. Properties ($\nabla 1$) – ($\nabla 3$) follow from Proposition 1.3.9, ($\nabla 1$) – ($\nabla 3$) since nor is $\mathcal{C}^\infty(M)$-linear. ($\nabla 5$)

also follows from Proposition 1.3.9 by noting that

$$\left\langle \overline{\nabla}_V Y, Z \right\rangle = \left\langle \text{nor}(\overline{\nabla}_V Y), Z \right\rangle \text{ for } Z \in \mathfrak{X}(M)^\perp.$$

□

We want to define $\nabla_V \mathbb{I}$.

Definition 1.6.4. Let $M \subseteq \overline{M}$ as before and $V, X, Y \in \mathfrak{X}(M)$. Then

$$(\nabla_V \mathbb{I})(X,Y) := \nabla_V^\perp(\underbrace{\mathbb{I}(X,Y)}_{\in \mathfrak{X}(M)^\perp}) - \mathbb{I}(\nabla_V X, Y) - \mathbb{I}(X, \nabla_V Y.)$$

Remark 1.6.5. It is easy to check that $\nabla_V \mathbb{I} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)^\perp$ is $\mathcal{C}^\infty$-bilinear and symmetric.

Recall that the Gauss equation from Theorem 1.3.13 describes $\tan(R_{VW}X)$ (since

$$\langle R_{VW}X, Y \rangle = \langle \tan(R_{VW}X), Y \rangle,$$

for $Y \in \mathfrak{X}(M)$) via $\mathbb{I}$.

Theorem 1.6.6 (Codazzi-equation)**.** Let $M \subseteq \overline{M}$ be a SRSMF and let $V, W, X \in \mathfrak{X}(M)$. Then we have

$$\text{nor}(R_{VW}X) = -(\nabla_V \mathbb{I})(W,X) + (\nabla_W \mathbb{I})(V,X).$$

Proof. Codazzi-equation is pointwise, so without loss of generality, we can assume that $[V,W] = 0$ (see the proof of Theorem 1.3.13). Then

$$\text{nor}(R_{VW}X) = -(VW) + (WV),$$

where

$$(VW) := \text{nor}(\overline{\nabla}_V \overline{\nabla}_W X) \overset{(1.3.7)}{=} \text{nor}(\overline{\nabla}_V \nabla_W X) + \text{nor}\overline{\nabla}_V(\mathbb{I}(W,X)).$$

By Lemma 1.3.11, $\text{nor}(\overline{\nabla}_V \nabla_W X) = \mathbb{I}(V, \nabla_W X)$ and, by Definition 1.6.1, $\text{nor}\overline{\nabla}_V(\mathbb{I}(W,X)) = \nabla_V^\perp(\mathbb{I}(W,X))$. Therefore, by Definition 1.6.4,

$$(VW) = \mathbb{I}(V, \nabla_W X) + (\nabla_V \mathbb{I})(W,X) + \mathbb{I}(\nabla_V W, X) + \mathbb{I}(W, \nabla_V X),$$

In $-(VW) + (WV)$ the first and last terms cancel in pairs and the third terms cancel due to $[V,W] = 0$. □

Definition 1.6.7. We call $Z \in \mathfrak{X}(M)^\perp$ normal parallel if $\nabla_V^\perp Z = 0$ for all $V \in \mathfrak{X}(M)$.

When dealing with constant curvature and hypersurfaces, matters simplify.

Corollary 1.6.8. Let $M \subseteq \overline{M}$ be a SRSMF with constant curvature. Then

1. the Codazzi-equation takes the simpler form

$$(\nabla_V \mathbb{I})(W,X) = (\nabla_W \mathbb{I})(V,X), \ \forall V, W, X \in \mathfrak{X}(M).$$

2. if M is a SRHSF with shape operator S, then

$$(\nabla_V S)(W) = (\nabla_W S)(V), \ \forall V, W \in \mathfrak{X}(M).$$

Proof. 1. By Corollary 1.1.12 $R_{VW}X$ is tangential. Now the result follows easily from Theorem 1.6.6.

2. Let S be derived from U (see Definition 1.4.7). For all $X \in \mathfrak{X}(M)$

$$0 = \nabla_X \langle U, U\rangle = 2\left\langle \overline{\nabla}_X U, U\right\rangle \implies \overline{\nabla}_X U \perp U,$$

where the first equality holds since $\langle U, U\rangle = \pm 1$. Since M is a SRHSF, $\overline{\nabla}_X U$ is tangential i.e. $\nabla^{\perp}_X U = 0$ for all X and so U is normal parallel. 2. is a pointwise equality so we can assume that all covariant derivatives of V, W, X with respect to each other vanish at p (see 2.1.17 in [3]). Then, at p,

$$\nabla_V(S(W)) = (\nabla_V S)(W) + S\underbrace{(\nabla_V W)}_{=0} \tag{1.6.1}$$

by the Leibniz rule for the tensor derivation ∇_V (recall that $S \in \mathcal{T}^1_1(M)$). Therefore,

$$\begin{aligned}
\langle(\nabla_V S)(W), X\rangle &\overset{(1.6.1)}{=} \langle \nabla_V(S(W)), X\rangle \\
&\overset{\nabla_V X|_p = 0,\ (\nabla 5)}{=} V(\langle S(W), X\rangle) \\
&\overset{1.4.7}{=} V\langle \mathbb{I}(W, X), U\rangle \\
&\overset{(\nabla 5)}{=} \left\langle \overline{\nabla}(\mathbb{I}(W, X)), U\right\rangle + \underbrace{\left\langle \mathbb{I}(W, X), \overline{\nabla}_V U\right\rangle}_{=0},
\end{aligned}$$

where the last equality holds since we showed earlier that $\overline{\nabla}_V U$ is tangential. Furthermore,

$$\left\langle \overline{\nabla}(\mathbb{I}(W, X)), U\right\rangle = \left\langle \mathrm{nor}\overline{\nabla}(\mathbb{I}(W, X)), U\right\rangle = \left\langle \nabla^{\perp}_V(\mathbb{I}(W, X)), U\right\rangle \overset{1.6.4}{=} \langle(\nabla_V \mathbb{I})(W, X), U\rangle,$$

where the last equality holds since $\nabla_V W = 0 = \nabla_V X$ at p. Hence,

$$\langle(\nabla_V S)(W), X\rangle = \langle(\nabla_V \mathbb{I})(W, X), U\rangle \overset{1.}{=} \langle(\nabla_W \mathbb{I})(W, X), U\rangle = \langle(\nabla_W S)(V), X\rangle$$

and the result follows since X was arbitrary.

□

Remark 1.6.9 (The Tensor $\tilde{\mathbb{I}}$)**.** Let $M \subseteq \overline{M}$ be a SRSMF.

1. We define the tensor $\tilde{\mathbb{I}}$ via

$$\begin{aligned}
\tilde{\mathbb{I}} : \mathfrak{X}(M) \times \mathfrak{X}(M)^{\perp} &\to \mathfrak{X}(M) \\
\tilde{\mathbb{I}}(V, Z) &:= \mathrm{tan}\overline{\nabla}_V Z
\end{aligned}$$

It is easy to check that $\tilde{\mathbb{I}}$ is $\mathcal{C}^{\infty}(M)$-bilinear;

$$\tilde{\mathbb{I}}(V, fZ) = \mathrm{tan}\overline{\nabla}_V(fZ) = \mathrm{tan}(f\overline{\nabla}_V Z) + \underbrace{V(f)Z}_{\substack{=0 \\ (Z \text{ is normal})}} = \mathrm{tan}(f\overline{\nabla}_V Z) = f\mathrm{tan}(\overline{\nabla}_V Z) = f\tilde{\mathbb{I}}(V, Z).$$

Hence, at every $p \in M$ we have a well-defined $\mathbb{R}$-bilinear map

$$\tilde{\mathbb{I}} : T_pM \times T_pM^{\perp} \to T_pM.$$

2. By definition, for all $V \in \mathfrak{X}(M)$ and all $Z \in \mathfrak{X}(M)^{\perp}$ we have the following analogue of (1.3.7):

$$\overline{\nabla}_V Z = \underbrace{\tilde{\mathbb{I}}(V, Z)}_{\text{tangential}} + \underbrace{\nabla^{\perp}_V Z}_{\text{normal}}. \tag{1.6.2}$$

3. $\tilde{\mathbb{I}}$ does not contain new information since for all $V, W \in \mathfrak{X}(M)$ and all $Z \in \mathfrak{X}(M)^\perp$, we have that

$$\left\langle \tilde{\mathbb{I}}(V,Z), W \right\rangle = -\left\langle \mathbb{I}(V,W), Z \right\rangle .$$

Indeed, by applying V to $\langle Z, W\rangle = 0$ (see $(\nabla 5)$, Proposition 1.3.9) we get that

$$\left\langle \overline{\nabla}_V Z, W \right\rangle = -\left\langle Z, \overline{\nabla}_V W \right\rangle .$$

Since

$$\left\langle \overline{\nabla}_V Z, W \right\rangle \overset{(1.6.2)}{=} \langle \tilde{\mathbb{I}}(V,Z) + \underbrace{\nabla^\perp_V Z}_{\text{normal}}, W \rangle = \left\langle \tilde{\mathbb{I}}(V,Z), W \right\rangle$$

and

$$\left\langle Z, \overline{\nabla}_V W \right\rangle \overset{(1.3.7)}{=} \langle Z, \mathbb{I}(V,W) + \underbrace{\nabla_V W}_{\text{tangential}} \rangle = \langle Z, \mathbb{I}(V,W)\rangle ,$$

the result follows.

4. If some $X \in \mathfrak{X}(M)^\perp$ is important (for what we are trying to calculate), we also use the notation

$$S_Z V = \tilde{\mathbb{I}}(V,Z).$$

From 3. above we get

$$\langle S_Z V, W\rangle = \langle \mathbb{I}(V,W), Z\rangle = \langle \mathbb{I}(W,V), Z\rangle = \langle S_Z W, V\rangle ,$$

because $\mathbb{I}$ is symmetric. Therefore, S_Z is a self-adjoint linear map. Furthermore, if M is a SRHSF and $Z \equiv U$ is a unit normal, then this notation is consistent with the one from Definition 1.4.7.

5. The normal connection induces a corresponding operation on vector fields over curves $\alpha : I \to M$, where vector fields are normal to M at every point. Let $Y \in \mathfrak{X}(\alpha)$ such that

$$Y(t) = T_{\alpha(t)}M^\perp, \text{ for all } t.$$

Then

$$Y' \equiv \frac{\nabla^\perp Y}{dt} := \text{nor}\frac{\overline{\nabla} Y}{dt}.$$

Analogs of the usual properties (cf. [3], 1.3.27) hold for $\frac{\nabla^\perp Y}{dt}$. Moreover, the analog of Proposition 1.5.1 here is

$$\frac{\overline{\nabla} Y}{ds} \equiv \dot{Y} = \underbrace{\tilde{\mathbb{I}}(\alpha', Y)}_{\text{tangent}} + \underbrace{Y'}_{\text{normal}}$$

Y is called *normal parallel* if $Y' = 0$, which induces *normal parallel transport* analogous to the one from [3], 1.3.28. For more details see [4], Chapter 4, 40.

2

IMPORTANT EXAMPLES OF LORENTZIAN MANIFOLDS

In this chapter, we'll delve into the most significant examples of Lorentzian manifolds. We'll start with the simplest—Minkowski space—and then proceed to explore de Sitter and anti-de Sitter spaces, comprising the trio of Lorentzian manifolds with constant curvature. Following this, we'll delve into Robertson-Walker spacetimes, crucial models in cosmology. Finally, we'll examine the Schwartzschield half-plane, an essential model in black hole physics.

2.1 Minkowski Space

Let's start by introducing some notation. For $x, y \in \mathbb{R}^n$ we write

$$\langle x, y \rangle := \sum_{i=1}^{n} x^i y^i$$

for the standard scalar product and for $x = (x^0, x^1, \dots, x^n), y = (y^0, y^1, \dots, y^n) \in \mathbb{R}^{n+1}$

$$\langle\langle x, y \rangle\rangle := -x^0 y^0 + \sum_{i=1}^{n} x^i y^i$$

for the Minkowski scalar product. Writing $x = (x^0, \hat{x}), y = (y^0, \hat{y}) \in \mathbb{R}^{n+1}$, we get

$$\langle\langle x, y \rangle\rangle = -x^0 y^0 + \langle \hat{x}, \hat{y} \rangle .$$

Definition 2.1.1 (Cf. [3], 1.1.5., (ii)). An $(n+1)$-dimensional Lorentzian manifold is called Minkowski space if it is isometric (cf. [3], 1.2.8) to $\mathbb{R}_1^{n+1}$ (cf. [3], 1.2.3, (ii)) with the metric

$$g = -dx^0 \otimes dx^0 + \sum_{i=1}^{n} dx^i \otimes dx^i = \epsilon_i dx^i \otimes dx^i,$$

where $\epsilon_i = \begin{cases} -1, & i = 0, \\ +1, & 1 \leq i \leq k. \end{cases}$

Remark 2.1.2.

1. If $n = 3$ Minkowski space is the spacetime of special relativity.

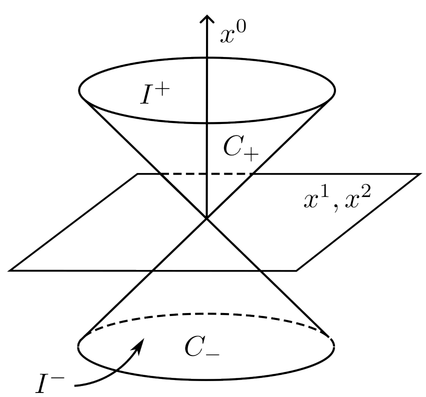
2. In 'natural coordinates' we have (for any n):

$$g_{ij} = \epsilon_i \delta_{ij} \text{ is constant}$$
$$\Longrightarrow \Gamma^i_{jk} = 0,\ \forall i,j,k \text{ (cf. [3], 1.3.9)}$$
$$\Longrightarrow R = 0 \text{ (cf. [3], 3.1.3)}$$
$$\Longrightarrow \text{Ric} = 0 \text{ (cf. [3], 3.3.1)}$$
$$\Longrightarrow S = 0, \text{ where } S \text{ is scalar curvature from [3], 3.3.2}$$
$$\Longrightarrow K = 0 \text{ (see Lemma 1.1.2).}$$

The geodesic equation hence is $\ddot{c} = 0$ (cf. [3], (2.1.2)) and the geodesics are affine-linear parametrized straight lines.

Definition 2.1.3 (Causality Relations in M).

- Light cone: $C := \{x \in \mathbb{R}^{n+1}_1 : \langle\langle x, x \rangle\rangle = 0\}$.
- Future/past light cone: $C_\pm := \{x \in C : \pm x^0 \geq 0\}$.
- Timelike vectors: $I := \{x \in \mathbb{R}^{n+1}_1 : \langle\langle x, x \rangle\rangle < 0\}$.
- Future/past timelike vectors: $I^\pm = I_\pm := \{x \in I : \pm x^0 > 0\}$.
- Vectors from $J := C \cup I = \{x : \langle\langle x, x \rangle\rangle \leq 0\}$ are called causal when they are different from 0.
- $J^\pm = J_\pm = C_\pm \cup I_\pm = \{x \in J : \pm x^0 \geq 0\}$.
- Null/lightlike vectors: $C \backslash \{0\}$.
- Spacelike vectors: $(\mathbb{R}^{n+1}_1 \backslash J) \cup \{0\}$. (Note here that null vector is spacelike by definition!)
- Future (past) pointing null vectors: $C_+ \backslash \{0\}$ $(C_- \backslash \{0\})$.
- Future (past) pointing causal vectors: $J_+ \backslash \{0\}$ $(J_- \backslash \{0\})$.



Lemma 2.1.4. Let $x, y \in I_+$ and $t > 0$. Then

1. $tx \in I_+$ and
2. $x + y \in I_+$.

The assertion 1. also holds for I_-, $C_\pm$, $J_\pm$ and $\mathbb{R}^{n+1}_1 \backslash J$. The assertion 2. also holds for I_- and $J_\pm$.

Proof. 1. Let $x \in I_+$. Then $\langle\langle x, x\rangle\rangle < 0$ and so

$$\langle\langle tx, tx\rangle\rangle = t^2 \langle\langle x, x\rangle\rangle < 0 \implies tx \in I.$$

Since $x^0 > 0$ and $t > 0$, we get that $tx \in I_+$. The proof is similar for all other cases.

2. Observe that $x \in I_+$ if and only if $x^0 > 0$ and $\|\hat{x}\|^2 < (x^0)^2$ (i.e. if and only if $\langle\langle x, x\rangle\rangle < 0$), where $\|\cdot\|$ is Euclidean norm. Let $x, y \in I_+$. Then $x^0, y^0 > 0$, $x^0 + y^0 > 0$ and

$$\begin{aligned}(x^0 + y^0)^2 &= (x_0)^2 + 2x^0y^0 + (y^0)^2 \\ &> \|\hat{x}\|^2 + 2\cdot \underbrace{\|\hat{x}\|\cdot\|\hat{y}\|}_{\substack{\geq 2\langle\hat{x},\hat{y}\rangle, \\ \text{by Cauchy-Schwarz}}} + \|\hat{y}\|^2 \\ &\geq \|\hat{x} + \hat{y}\|^2 = \|\widehat{(x+y)}\|^2.\end{aligned}$$

Therefore, $x + y \in I_+$.

□

Corollary 2.1.5. $I_\pm$ and $J_\pm$ are convex.

Proof. We only prove the claim for I_+ since for the other cases the proof is completely analogous. Let $x, y \in I_+$ and $t \in (0, 1)$. Then $t, 1 - t > 0$ and so, by Lemma 2.1.4 (1.), $tx, (1 - t)y \in I_+$. Finally, by Lemma 2.1.4 (2.), $tx + (1 - t)y \in I_+$. □

Next, we'll delve into the study of isometries within Minkowski space (refer to [3], 1.2.8).

Definition 2.1.6. A mapping $\phi : \mathbb{R}^{n+1}_1 \to \mathbb{R}^{n+1}_1$ is called a Lorentz transformation (L-transformation) if for all $x, y \in \mathbb{R}^{n+1}_1$ it holds that

$$\langle\langle \phi_x, \phi_y\rangle\rangle = \langle\langle x, y\rangle\rangle.$$

Remark 2.1.7. Recall that in a vector space V with scalar product g there always exists an ONB (cf. [3], 1.1.12) i.e. a dim(V)-tuple of pointwise orthogonal unit vectors $\{e_i\}_i$, where $v \in V$ is called a unit vector if

$$|v| := |g(v, v)|^{\frac{1}{2}} = 1$$

(cf. [3], 1.1.11). Then any vector $x \in V$ has a unique decomposition

$$x = \sum_{i=1}^{\dim(V)} \epsilon_i g(v, e_i)e_i, \text{ where } \epsilon_i := g(e_i, e_i),$$

(cf. [3], 1.1.13). An ONB $\{b_0, \ldots, b_n\}$ in $\mathbb{R}^{n+1}_1$ is often called *Lorentz-orthonormal*. Then,

$$\langle\langle b_0, b_0\rangle\rangle = -1,\ \langle\langle b_i, b_i\rangle\rangle = 1 \text{ for } 1 \leq i \leq n,\ \langle\langle b_i, b_j\rangle\rangle = 0 \text{ for } i \neq j.$$

A simple example is the standard basis $\{e_i\}^n_{i=0}$.

Proposition 2.1.8. A map $\phi : \mathbb{R}^{n+1}_1 \to \mathbb{R}^{n+1}_1$ is an L-transformation if and only if ϕ is linear and $\{\phi(e_i)\}^n_{i=0}$ is an orthonormal basis.

Proof. $(\to)$ Since $\langle\langle \phi(e_i), \phi(e_j)\rangle\rangle = \langle\langle e_i, e_j\rangle\rangle$ we conclude that $\{\phi(e_i)\}^n_{i=0}$ is an orthonormal basis. In order to show that ϕ is linear, we write

$$\phi(x) = \sum_{i=0}^{n} c_i \phi(e_i), \text{ for some } c_i \in \mathbb{R}.$$

Then

$$\begin{aligned} \langle\langle \phi(x), \phi(e_0) \rangle\rangle &= \sum c_i \langle\langle \phi(e_i), \phi(e_0) \rangle\rangle \\ &= \sum c_i \langle\langle e_i, e_0 \rangle\rangle = -c_0. \end{aligned}$$

On the other hand,

$$\langle\langle \phi(x), \phi(e_0) \rangle\rangle = \langle\langle x, e_0 \rangle\rangle = -x^0.$$

Therefore, $c_0 = x^0$ and, analogously, $c_i = x^i$ for all $i \in \{1, \dots, n\}$. Hence, $\phi(x) = \sum x^i \phi(e_i)$ i.e. ϕ is linear.
($\leftarrow$) Let ϕ be linear and $\{\phi(e_i)\}_{i=0}^n$ an ONB. Then

$$\langle\langle \phi(x), \phi(y) \rangle\rangle = \sum_{i,j=0}^{n} x^i y^j \langle\langle \phi(e_i), \phi(e_j) \rangle\rangle = \langle\langle x, y \rangle\rangle .$$

□

Let's now shift our focus to the matrix representations of Lorentz transformations. Let

$$J_n = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix},$$

where I_n is an $(n \times n)$-unit matrix. Then

$$\langle\langle x, y \rangle\rangle = \langle x, J_n y \rangle = -x^0 y^0 + \sum_{i=1}^{n} x^i y^i, \ \forall x, y \in \mathbb{R}_1^{n+1}$$

where $\langle \cdot, \cdot \rangle$ is the euclidean product in $\mathbb{R}^{n+1}$. Hence, $A \in GL(n+1, \mathbb{R})$ is the matrix of an L-transformation if and only if

$$\langle x, J_n y \rangle = \langle\langle x, y \rangle\rangle = \langle\langle Ax, Ay \rangle\rangle = \langle Ax, J_n Ay \rangle = \left\langle x, A^t J_n A y \right\rangle,$$

hence if and only if

$$A^t J_n A = J_n.$$

Proposition 2.1.9 (The Matrix of an L-transformation). For $A \in GL(n+1, \mathbb{R})$ the following are equivalent:

1. A is the matrix of a Lorentz transformation.
2. The columns of A are an ONB of $\mathbb{R}_1^{n+1}$.
3. $A^t J_n A = J_n$.

Proposition 2.1.10 (Lorentz Group).

1. The set of all Lorentz-transformations $\mathbb{R}_1^{n+1}$, denoted by $\mathcal{L}(n+1) = O(1,n)$, is a group (with respect to composition).
2. If A is the matrix of an L-transformation, then $\det(A) = \pm 1$.

Proof. 1. Let A, B be two matrices of Lorentz transformation. Then

$$(AB)^t J_n (AB) = B^t \underbrace{A^t J_n A}_{J_n} B = J_n,$$

i.e. AB is a Lorentz transformation as well. Furthermore,

$$A^{-1} \in O(1,n) \iff (A^{-1})^t J_n (A^{-1}) = J_n \iff (A^t)^{-1} J_n = J_n A \iff J_n = A^t J_n A.$$

2. $-1 = \det(J_n) = \det(A^t J_n A) = -\det(A)^2$.

□

Example 2.1.11 (L-transformations).

1. For $B \in \mathcal{O}(n,\mathbb{R})$ we have $A := \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \in O(1,n)$. Indeed,

$$A^t J_n A = \begin{pmatrix} 1 & 0 \\ 0 & B^t \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & B^t B \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} = J_n.$$

2. We call a matrix of the form

$$A := \begin{pmatrix} \cosh(\eta) & \sinh(\eta) & 0 \\ \sinh(\eta) & \cosh(\eta) & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix}$$

a Lorentz boost. It is a Lorentz transformation;

$$\begin{aligned} A^t J_n A &= \begin{pmatrix} \cosh(\eta) & \sinh(\eta) & 0 \\ \sinh(\eta) & \cosh(\eta) & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} \cosh(\eta) & \sinh(\eta) & 0 \\ \sinh(\eta) & \cosh(\eta) & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\eta) & \sinh(\eta) & 0 \\ \sinh(\eta) & \cosh(\eta) & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} -\cosh(\eta) & -\sinh(\eta) & 0 \\ \sinh(\eta) & \cosh(\eta) & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix} = J_n, \end{aligned}$$

where we have used the fact that $\cosh^2(\eta) - \sinh^2(\eta) = 1$.

3. $J_n, -I_{n+1} \in O(1,n)$;

$$(J_n)^t J_n J_n = J_n^3 \text{ and } (I_{n+1})^t J_n I_{n+1} = J_n.$$

4. For all $x \in I$ there exists $A \in O(1,n)$ such that $Ax = c \cdot e_0$, with $\pm c > 0$ if $x \in I_\pm$. Indeed, span(x) is a non-degenerate subspace of $\mathbb{R}^{n+1}_1$ (i.e. $\langle\!\langle \cdot, \cdot \rangle\!\rangle|_{\text{span}(x)}$ is non-degenerate). Using [3], 1.1.10, we get that

$$x^\perp := \{y \in \mathbb{R}^{n+1}_1 : \langle\!\langle x, y \rangle\!\rangle = 0\}$$

is non-degenerate and Euclidean (cf. [3], 1.1.15). Therefore, $x^\perp$ has an ONB $\{y_1, \dots, y_n\}$ and so $\{e_0, y_i\}_{i=1,\dots,n}$ is a basis of $\mathbb{R}^{n+1}_1$. Then

$$x = -\langle\!\langle x, e_0 \rangle\!\rangle e_0 + \sum_{i=1}^n \underbrace{\langle\!\langle x, y_n \rangle\!\rangle}_{=0} y_n = \underbrace{x_0}_{=c} e_0.$$

Choose A such that $\{e_0, e_1, \dots, e_n\} \mapsto \{e_0, y_1, \dots, y_n\}$.

Proposition 2.1.12. For $x, y \in \mathbb{R}^{n+1}_1 \backslash \{0\}$ the following are equivalent:

1. $\langle\langle x, x\rangle\rangle = \langle\langle y, y\rangle\rangle$.
2. There exists $A \in O(1, n)$ such that $y = Ax$.

Proof.

$(2. \to 1.)$ This is clear since $\langle\langle y, y\rangle\rangle = \langle\langle Ax, Ax\rangle\rangle = \langle\langle x, x\rangle\rangle$.

$(1. \to 2.)$ We only prove this for x and y timelike. The other cases are proven analogously Set

$$-c^2 := \underbrace{\langle\langle x, x\rangle\rangle}_{<0} = \langle\langle y, y\rangle\rangle .$$

By Example 2.1.11 (4.) it suffices to consider $y = c \cdot e_0$, for some $c > 0$. Without loss of generality, assume $x \in I_+$ (otherwise replace x by $J_n x$). Moreover, use $B \in \mathcal{O}(n)$ in order to obtain $B\hat{x} = d \cdot e_1$ (recall, $x = (x^0, \hat{x})$) and let, again without loss of generality, $x = (x^0, x^1, 0, \dots, 0)$. Then $-(x^0)^2 + (x^1)^2 = \langle\langle x, x\rangle\rangle = \langle\langle y, y\rangle\rangle = -c^2$ i.e.

$$-\left(\frac{x^0}{c}\right)^2 + \left(\frac{x^1}{c}\right)^2 = -1,$$

which is a hyperbola. We may parametrize it by $\eta \mapsto (c \cdot \cosh(\eta), c \cdot \sinh(\eta))$. Then

$$x = \begin{pmatrix} \cosh(\eta) & \sinh(\eta) & 0 \\ \sinh(\eta) & \cosh(\eta) & 0 \\ 0 & 0 & I_n \end{pmatrix} y = Ty.$$

From Example 2.1.11 (2.) we know that $T \in O(1, n)$. Then $A := T^{-1} \in O(1, n)$, by Proposition 2.1.10.

□

Lemma 2.1.13 (Cauchy-Schwarz Inequality). Let $z \in I$ and let $x, y \in \mathbb{R}^{n+1}_1$ with $\langle\langle x, z\rangle\rangle = \langle\langle y, z\rangle\rangle = 0$. Then

$$|\langle\langle x, y\rangle\rangle| \le \sqrt{|\langle\langle x, x\rangle\rangle|}\sqrt{|\langle\langle y, y\rangle\rangle|}, \tag{2.1.1}$$

with equality if and only if x and y are linearly dependent, and vice versa.

Proof. We can replace x, z, y by Ax, Ay, Az without changing (2.1.1), if $A \in O(1, n)$. By Example 2.1.11 (4.), without loss of generality, we may assume that $z = c \cdot e_0$, for $c \neq 0$. Then for $x = (x^0, \hat{x})$

$$\langle\langle x, z\rangle\rangle = 0 \iff x^0 = 0$$

and likewise for y. Hence,

$$\langle\langle x, y\rangle\rangle = \left\langle\!\!\left\langle \begin{pmatrix} 0 \\ \hat{x} \end{pmatrix}, \begin{pmatrix} 0 \\ \hat{y} \end{pmatrix} \right\rangle\!\!\right\rangle = \langle x, y\rangle$$

and the statement follows from from the usual Cauchy-Schwarz inequality. □

Remark 2.1.14. Note that without the assumption $\langle\langle x, z\rangle\rangle = \langle\langle y, z\rangle\rangle = 0$ the above lemma is in general false. For

$n = 1$ take $x = \left(\frac{1}{2}, 1\right)$ and $y = \left(-\frac{1}{2}, 1\right)$. Then $\langle\langle x, y\rangle\rangle = \frac{5}{4}$ and $\langle\langle x, x\rangle\rangle = \langle\langle y, y\rangle\rangle = \frac{3}{4}$ but

$$\frac{5}{4} \nleq \sqrt{\frac{3}{4}}\sqrt{\frac{3}{4}} = \frac{3}{4}.$$

Lemma 2.1.15 (Inverse Cauchy-Schwarz Inequality)**.** For $x, y \in I^+$ we have

$$|\langle\langle x, y\rangle\rangle| \geq \sqrt{|\langle\langle x, x\rangle\rangle|}\sqrt{|\langle\langle y, y\rangle\rangle|}$$

with equality if and only if x and y are linearly dependent.

Proof. Without loss of generality, we can assume $y = c \cdot e_0$ for $c > 0$. Then for $x = (x^0, \hat{x})$ and $y = (c, 0)$;

$$\begin{aligned}\langle\langle x, y\rangle\rangle^2 - |\langle\langle x, x\rangle\rangle| \cdot |\langle\langle y, y\rangle\rangle| &= (x^0)^2 c^2 - |\underbrace{-(x^0)^2 + \|\hat{x}\|^2}_{<0,\, x \in I^+}| \cdot c^2 \\ &= (x^0)^2 c^2 - \left((x^0)^2 - \|\hat{x}\|^2\right) \cdot c^2 = \|\hat{x}\|^2 c^2 \geq 0.\end{aligned}$$

Equality holds if and only if $\|\hat{x}\|^2 = 0$ i.e. if and only if $x = (x^0, 0)$, in which case x and y are linearly dependent. ☐

Proposition 2.1.16 (Time-preserving L-transformation)**.** Let $A = (a_{ij})_{i,j=0}^n \in O(1, n)$. Then this facts are equivalent:

1. $a_{00} > 0$.
2. $Ae_0 \in I^+$.
3. $A(I^+) \subseteq I^+$.

Proof.

$(1. \leftrightarrow 2.)$ $A \in O(1, n)$ preserves the causal character of vectors, hence $A(I) \subseteq I$ and so $Ae_0 \in I$. Moreover,

$$Ae_0 = \begin{pmatrix} a_{00} \\ a_{10} \\ \vdots \\ a_{n0} \end{pmatrix} \in I^+ \iff a_{00} > 0.$$

$(3. \to 2.)$ This is clear because $e_0 \in I^+$.

$(2. \to 3.)$ Let $x \in I^+$. By Corollary 2.1.5 I^+ is convex and so for all $t \in [0, 1]$

$$x(t) = tx + (1 - t)e_0 \in I^+ \implies Ax(t) \in I$$

since A preserves causal character. In particular, $(Ax(t))^0 \neq 0$, for all $t \in [0, 1]$. Since $(Ax(0))^0 = Ae_0 > 0$ by assumption, $(Ax(t))^0 > 0$ for all $t \in [0, 1]$. In particular, $A(x(1))^0 > 0$ and so $A(x(1)) = Ax \in I^+$.

☐

Definition 2.1.17. The elements of $\mathcal{L}^{\uparrow}(n+1) = O^{\uparrow}(1,n) = \{A \in O(1,n) : a_{00} > 0\}$ are called time-preserving or orthochronous Lorentz transformations.

Corollary 2.1.18. $\mathcal{L}^{\uparrow}(n+1)$ is a subgroup of $\mathcal{L}(n+1)$. Moreover, for $A \in \mathcal{L}^{\uparrow}(n+1)$ we have

$$A(I^+) = I^+ \text{ and } A(I^-) = I^-.$$

Proof. We first prove that $\mathcal{L}^{\uparrow}$ is a subgroup of $\mathcal{L}(n+1)$.

- Let $A, B \in \mathcal{L}^{\uparrow}$. Then
$$(A \circ B)(I^+) = A(B(I^+)) \overset{2.1.16}{\subseteq} A(I^+) \overset{2.1.16}{\subseteq} I^+$$
and so $A \circ B \in \mathcal{L}^{\uparrow}$, by Proposition 2.1.16 $(3. \to 1.)$.
- We know that $A^{-1}e_0 \in I$ since $A^{-1} \in O(1,n)$. Assume that $A^{-1}e_0 \in I^-$. Then $A^{-1}(-e_0) = -A^{-1}e_0 \in I^+$ and
$$\underbrace{-e_0}_{\in I^-} = \underbrace{A(A^{-1}(-e_0))}_{\in I^+} \in I^- \cap I^+ = \varnothing,$$
which is a contradiction. Therefore, $A^{-1}e_0 \in I^+$ and so $A^{-1}(I^+) \subseteq I^+$, by Proposition 2.1.16 $(2. \to 3.)$.

Since $I^+ = A(A^{-1}(I^+)) \subseteq A(I^+)$, from Proposition 2.1.16 it follows immediately that $A(I^+) = I^+$. Finally, since $I^- = -I^+$ and A is linear, we are done. ☐

Example 2.1.19 (Time-preserving Lorentz-transformations).

1. For $B \in \mathcal{O}(n)$ we have
$$A = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \in \mathcal{L}^{\uparrow}(n+1)$$
and
$$A = \begin{pmatrix} -1 & 0 \\ 0 & B \end{pmatrix} \in \mathcal{L}^{\downarrow}(n+1),$$
where $\mathcal{L}^{\downarrow}(n+1) := \mathcal{L}(n+1) \backslash \mathcal{L}^{\uparrow}(n+1)$ is called the time-reversing L-transformation.
2. All L-boosts
$$\begin{pmatrix} \cosh(\eta) & \sinh(\eta) & 0 \\ \sinh(\eta) & \cosh(\eta) & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix}$$
belong to $\mathcal{L}^{\uparrow}(n+1)$ since $\cosh(\eta) \geq 1$ for all $\eta \in \mathbb{R}$.

Remark 2.1.20. The following notations for the elements of $\mathcal{L}$ are in use:

1. $\mathcal{L}_{\pm}(n+1) := \{A \in \mathcal{L}(n+1) : \det(A) = \pm 1\}$.
2. $\mathcal{L}^{\uparrow}_{\pm}(n+1) := \mathcal{L}^{\uparrow}(n+1) \cap \mathcal{L}_{\pm}(n+1)$.
3. $\mathcal{L}^{\downarrow}_{\pm}(n+1) := \mathcal{L}^{\downarrow}(n+1) \cap \mathcal{L}_{\pm}(n+1)$.

Definition 2.1.21. A map $\phi : \mathbb{R}^{n+1}_1 \to \mathbb{R}^{n+1}_1$ is called Poincaré-transformation if it is of the form

$$\phi(x) = Ax + b,$$

where $A \in \mathcal{L}(n+1)$, $b \in \mathbb{R}^{n+1}_1$. We denote the set of such maps by $\mathcal{P}(n+1)$. Moreover, we write

$$\mathcal{P}^{\uparrow}(n+1) := \{\phi(x) = Ax + b : A \in \mathcal{L}^{\uparrow}(n+1)\}$$

and similarly for $\mathcal{P}^{\downarrow}(n+1)$.

Proposition 2.1.22 (Isometries of Minkowski Space). $\mathcal{P}(n+1)$ is the group of isometries of Minkowski space i.e. $\phi : \mathbb{R}_1^{n+1} \to \mathbb{R}_1^{n+1}$ is an isometry if and only if $\phi \in \mathcal{P}(n+1)$.

Note that $\phi \in \mathcal{P}(n+1)$ is not a linear isometry of $\mathbb{R}_1^{n+1}$ if $b \neq 0$! Also, distinguish isometries of vector spaces $\phi : (V, g) \to (W, \overline{g})$ (where $\phi : V \to W$ is linear and preserves the scalar product i.e. $\overline{g}(\phi(x), \phi(y)) = g(x, y)$) from isometries of SRMFs $\phi : (M, g) \to (N, h)$ (where $\phi : M \to N$ is a diffeomorphism and $\phi^* h = g$ i.e. $(\phi^* h)|_p(v, w) = h_{\phi(p)}(T_p\phi(v), T_p\phi(w)) = g_p(v, w)$).

Proof. Let $g_m = -dx^0 \otimes dx^0 + \sum_{i=1}^n dx^i \otimes dx^i$ be Minkowski metric. For all $\phi \in \mathcal{P}(n+1)$, where $\phi(v) = Av + b$, we have that $T_p\phi = A$ (for all p) and $A \in \mathcal{L}(n+1)$. Since A preserves scalar product, ϕ is an isometry (in the latter sense). Conversely, let ϕ be an isometry. Then, if c is a geodesic, so is $\phi \circ c$ and writing $c = t \mapsto \exp_p(tX)$, we obtain the following commutative diagram:

$$\begin{array}{ccc} T_p\mathbb{R}_1^{n+1} & \xrightarrow{T_p\phi} & T_{\phi(p)}\mathbb{R}_1^{n+1} \\ \downarrow{\scriptstyle \exp_p} & & \downarrow{\scriptstyle \exp_{\phi(p)}} \\ \mathbb{R}_1^{n+1} & \xrightarrow{\phi} & \mathbb{R}_1^{n+1}. \end{array}$$

Recall that (using $T_p\mathbb{R}_1^{n+1} \cong \mathbb{R}_1^{n+1}$) we have $\exp_p(X) = p + X$. Then we have for $p = 0$ from the diagram

$$\phi(X) = \exp_{\phi(0)}(T_0\phi(X)) = \underbrace{T_0\phi}_{=:A}(X) + \underbrace{\phi(0)}_{=:b}$$

where $\exp_0(X) = X \in T_0\mathbb{R}_1^{n+1} = \mathbb{R}_1^{n+1}$ and $A \in \mathcal{L}(n+1)$ since $T_0\phi$ is a linear isometry by definition. □

Remark 2.1.23 (Flat L-manifolds). Examples of L-manifolds with R_0:

1. Open subsets of $\mathbb{R}_1^{n+1}$.
2. Quotients of Minkowski space:
   - $\mathbb{R}_1^{n+1}/\mathbb{Z}^{n+1} \cong T^{n+1}$, $(n+1)$-dimensional torus.
   - $\mathbb{R}_1^{n+1}/\mathbb{Z}e_0 \cong S^1 \times \mathbb{R}^n$.
   - $\mathbb{R}_1^{n+1}/\mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_n \cong \mathbb{R} \times T^n$.

2.2 De Sitter Space

After discussing flat space, namely Minkowski space, in the previous section, let's now shift our focus to another fundamental spacetime or a family of spacetimes characterized by constant curvature. With this in mind, let's consider a function $f : \mathbb{R}_1^{n+1} \to \mathbb{R}$, where

$$\mapsto \langle\!\langle x, x \rangle\!\rangle = -(x^0)^2 + \sum_{i=1}^n (x^i)^2.$$

Then $f \in \mathcal{C}^\infty$ and

$$df|_x = -2x^0 dx^0 + 2\sum_{i=1}^n x^i dx^i \neq 0, \ \forall x \neq 0.$$

Therefore, every $c \in \mathbb{R}\backslash\{0\}$ is a regular value of f.

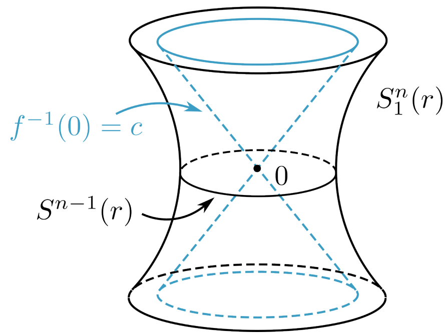
Definition 2.2.1. Let $r > 0$. Then we call the SRHSF (cf. Proposition 1.4.4)

$$S_1^n(r) := f^{-1}(r^2)$$

n-dimensional de Sitter space(time).

Remark 2.2.2 (Basic Properties of de Sitter)**.**

1. $S_1^n(r)$ is a *one-sheeted hyperboloid* in $\mathbb{R}_1^{n+1}$:



2. As a $\mathcal{C}^\infty$-manifold $S_1^n(r)$ is diffeomorphic to $\mathbb{R} \times S^{n-1}$ via

$$S_1^n(r) \to \mathbb{R} \times S^{n-1}$$
$$(x^0, \hat{x}) \mapsto \left(x^0, \frac{\hat{x}}{\sqrt{r^2 + (x_0)^2}}\right) = \left(x^0, \frac{\hat{x}}{\|\hat{x}\|_e}\right).$$

The inverse is given by

$$(y^0, \hat{y}) \mapsto (y^0, \sqrt{(y_0)^2 + r\hat{y}}).$$

3. A global unit normal can be derived from $\text{grad}(f)$. Indeed,

$$\text{grad}(f(x)) = -2x^0\left(-\frac{\partial}{\partial x^0}\right) + 2\sum_{i=1}^n x^i \frac{\partial}{\partial x^i} = 2\sum_{i=0}^n x^i \frac{\partial}{\partial x^i}$$

(cf. 3.2.6 in [3]), where $f(x) = \langle\langle x, x\rangle\rangle = -(x^0)^2 + \|\hat{x}\|^2$.
Therefore, $\text{grad}(f) = 2 \times$ the position vector field (cf. Example 1.3.16).

$$\langle\langle \text{grad}(f(x)), \text{grad}(f(x))\rangle\rangle = 4\langle\langle x, x\rangle\rangle = 4f(x) = 4r^2 > 0$$

and so $\epsilon = 1$, implying $\text{ind}(S_1^n(r)) = 1$. Hence, $\langle\langle \cdot, \cdot \rangle\rangle$ induces a Lorentzian metric on $S_1^n(r)$.

4. The outwards-pointing unit vector field is

$$\nu(x) \equiv U(x) = \frac{1}{2r}\text{grad}(f(x)) = \frac{1}{r}\sum_{i=0}^n x^i \frac{\partial}{\partial x^i} = \frac{1}{r}X,$$

where X is position vector field. Therefore, the shape operator of $S_1^n(n)$ is:

$$S(V) \overset{1.4.8}{=} -\overline{\nabla}_V U \overset{1.3.16}{=} -\frac{1}{r}V, \tag{2.2.1}$$

since $\overline{\nabla}_V(X) = \overline{\nabla}_V\left(\sum x^i \frac{\partial}{\partial x^i}\right) = \sum V(x^i)\frac{\partial}{\partial x^i} + \Gamma \ldots = V + 0$.
Now the 2^{nd} fundamental form takes the following form:

$$\mathbb{I}(V, W) = -\frac{1}{r}\langle\langle V, W\rangle\rangle U$$

since $\langle\!\langle \mathbb{I}(V,W),U\rangle\!\rangle \overset{1.4.7}{=} \langle\!\langle S(V),W\rangle\!\rangle = -\frac{1}{r}\langle\!\langle V,W\rangle\!\rangle$ and $\langle\!\langle U,U\rangle\!\rangle = +1$. In particular, we see that $S_1^n(r)$ is totally umbilic (cf. Definition 1.5.11) with the normal curvature vector field $Z = -\frac{1}{r}U$.

5. For the Riemannian curvature Gauss equation yields the following:

$$\begin{aligned}\left\langle\!\!\left\langle R^{S_1^n(r)}_{VW}X,Y\right\rangle\!\!\right\rangle &\overset{1.3.13}{=} \left\langle\!\!\left\langle R^{\mathbb{R}_1^{n+1}}_{VW}X,Y\right\rangle\!\!\right\rangle + \langle\!\langle \mathbb{I}(V,X),\mathbb{I}(W,Y)\rangle\!\rangle - \langle\!\langle \mathbb{I}(V,Y),\mathbb{I}(W,X)\rangle\!\rangle \\ &\overset{4.}{=} 0 + \frac{1}{r^2}(\langle\!\langle V,X\rangle\!\rangle\langle\!\langle W,Y\rangle\!\rangle - \langle\!\langle W,X\rangle\!\rangle\langle\!\langle V,Y\rangle\!\rangle)\end{aligned}$$

and so

$$R^{S_1^n(r)}_{VW}X = \frac{1}{r^2}(\langle\!\langle V,X\rangle\!\rangle W - \langle\!\langle W,X\rangle\!\rangle V).$$

6. For the sectional curvature we find for a non-degenerate tangent plane Π spanned by V and W that

$$\begin{aligned}K^{S_1^n(r)}(V,W) &\overset{1.3.14}{=} K^{\mathbb{R}_1^{n+1}}(V,W) + \epsilon\frac{\langle\!\langle S(V),V\rangle\!\rangle\langle\!\langle S(W),W\rangle\!\rangle - \langle\!\langle S(V),W\rangle\!\rangle^2}{Q(V,W)} \\ &\overset{(2.2.1),\ \epsilon=1}{=} 0 + \frac{1}{r^2}\frac{\langle\!\langle V,V\rangle\!\rangle\langle\!\langle W,W\rangle\!\rangle - \langle\!\langle V,W\rangle\!\rangle^2}{Q(V,W)} = \frac{1}{r^2}\frac{Q(V,W)}{Q(V,W)} = \frac{1}{r^2}.\end{aligned}$$

Therefore, de Sitter space has constant curvature $\frac{1}{r^2}$.

7. In order to derive the Ricci curvature we choose a frame $(E_i)_{i=1}^n$ of $T_pS_1^n(r)$ and calculate

$$\begin{aligned}\mathrm{Ric}(X,Y) &= \sum_{i=1}^n \epsilon_i\langle\!\langle R_{XE_i}Y,E_i\rangle\!\rangle \\ &\overset{5.}{=} \frac{1}{r^2}\sum_{i=1}^n \epsilon_i(\langle\!\langle X,Y\rangle\!\rangle \underbrace{\langle\!\langle E_i,E_i\rangle\!\rangle}_{=\epsilon_i} - \langle\!\langle E_i,Y\rangle\!\rangle\langle\!\langle X,E_i\rangle\!\rangle) \\ &= \frac{1}{r^2}(n\langle\!\langle X,Y\rangle\!\rangle - \langle\!\langle X,Y\rangle\!\rangle) \\ &= \frac{n-1}{r^2}\langle\!\langle X,Y\rangle\!\rangle.\end{aligned}$$

In other words, Ric $= \frac{n-1}{r^2}g$, implying

$$S = \frac{n(n-1)}{r^2},$$

where the scalar curvature $S = C(\mathrm{Ric})$ (cf. [3], 3.3.2).

8. The *Einstein tensor* (cf. [3], 3.3.5) is then equal to

$$G = \mathrm{Ric} - \frac{1}{2}Sg = \frac{1}{r^2}((n-1) - \frac{1}{2}n(n-1))g.$$

In particular, for $n = 4$,

$$G = -\frac{3}{r^2}g.$$

Setting $\Lambda := \frac{3}{r^2} > 0$, we obtain

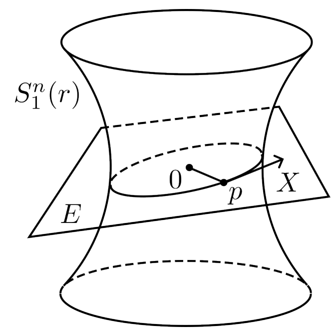
$$G + \Lambda g = 0 = 8\pi T,$$

where T is energy momentum tensor. $S_1^n(r)$ is a solution to the vacuum Einstein equations (cf. [3], (3.3.17)).

2.2.1 The Geodesics of de Sitter Spacetime

Remark 2.2.3.

1. In order to determine the geodesics geometrically, we consider $p \in S_1^n(r)$ and $X \in T_pS_1^n(r)\backslash\{0\}$. Those two vectors determine a plane E in $\mathbb{R}_1^{n+1}$ i.e. $E := \text{span}\{p, X\}$.



We first consider timelike or spacelike X. Then E is non-degenerate (since p is spacelike) and so $\mathbb{R}_1^{n+1} = E \oplus E^\perp$ (cf. [3], 1.1.9).

2. Define $A \in \mathcal{L}(n+1)$ as the reflection on E i.e. let

$$A|_E := \text{id}_E \text{ and } A|_{E^\perp} := \text{id}_{E^\perp}.$$

Then A is an L-transformation.

Indeed, let $x = x_E + x_{E^\perp}, y = y_E + y_{E^\perp} \in E \oplus E^\perp = \mathbb{R}_1^{n+1}$. Then

$$\langle\!\langle Ax, Ay \rangle\!\rangle = \langle\!\langle A(x_E + x_{E^\perp}), A(y_E + y_{E^\perp}) \rangle\!\rangle = \langle\!\langle x_E - x_{E^\perp}, y_E - y_{E^\perp} \rangle\!\rangle = \langle\!\langle x_E, y_E \rangle\!\rangle + \langle\!\langle x_{E^\perp}, y_{E^\perp} \rangle\!\rangle = \langle\!\langle x, y \rangle\!\rangle .$$

Moreover, A leaves $S_1^n(r)$ invariant.

Let $x \in S_1^n(r)$. Then

$$f(Ax) = \langle\!\langle Ax, Ax \rangle\!\rangle = \langle\!\langle x, x \rangle\!\rangle = f(x) = r^2$$

and so $A(S_1^n(r)) \subseteq S_1^n(r)$. Since $A^2 = \text{id}$, $A = A^{-1}$, yielding

$$A(S_1^n(r)) = S_1^n(r).$$

3. Therefore, $A|_{S_1^n(r)} \in \text{Isom}(S_1^n(r))$. We set

$$F(A|_{S_1^n(r)}) := \{x \in S_1^n(r) : Ax = x\} = S_1^n(r) \cap E.$$

F is the fixed point set of the isometry $A|_{S_1^n(r)}$. Lemma 2.2.4 will show that the connected components of $S_1^n(r) \cap E$ are (as point sets) geodesics of $S_1^n(r)$.

4. In case $X \in T_pS_1^n(r)$ is null we choose a sequence $(X_j)_j \in T_pS_1^n(r)$ such that X_j is not null for every j and $X_j \to X$ when $j \to \infty$. Then the planes E_j spanned by $\{p, X_j\}$ are non-degenerate (by 1.) and converge to E (as point sets). Moreover,

$$\exp_p(tX_j) \to \exp_p(tX), \forall t \ (\exp_p \in \mathcal{C}^\infty)$$

and so $\exp_p(tX)$ is a geodesic parametrizing $E \cap S_1^n(r)$. Hence, also in the null case are $S_1^n(r) \cap E$ are (null) geodesics (as point sets).

5. By uniqueness of geodesics, in this way we obtain all geodesics.

Lemma 2.2.4. Let ϕ be an isometry of a SRMF (M, g) and

$$F := \{x \in M : \phi(x) = x\}$$

its set of fixed points. Then if F is a $\mathcal{C}^\infty$-submanifold, it is totally geodesic.

Proof. In this proof we will use property 3. from Theorem 1.5.7. Let $p \in F$, $v \in T_pF \subseteq T_pM$ and let $c_v : I \to M$ be a geodesic of M with $c_v(0) = p$ and $\dot{c}_v(0) = v$. Then since ϕ is an isometry, also $\phi \circ c_v$ is a geodesic such that $\phi(c_v(0)) = \phi(p) = p$ and $(\phi \circ c_v)'(0) = T_p\phi(c_v'(0)) = T_p\phi(v) = v$ since $\phi|_F = \mathrm{id}_F$. Therefore, $\phi \circ c_v = c_v$ by uniqueness of geodesics (cf. [3], 2.1.5) and so $c_v(t) \in F$ for all $t \in I$ and so F is totally geodesic. □

Remark 2.2.5. In our case, Lemma 2.2.4 implies Remark 2.2.3 (3.). $F = S_1^n(r) \cap E$ is a 1-dimensional submanifold (as the zero set of $(f - r^2)|_E$); hence flat (cf. Example 1.1.6). Therefore, any parametrization with constant speed of the connected components of F is a geodesic in F, hence in $S_1^n(r)$.

Remark 2.2.6 (Geometric Interpretation of Geodesics)**.**

1. Assume that X is spacelike. Then $\langle\langle \cdot, \cdot \rangle\rangle|_E$ is positive definite. Hence,

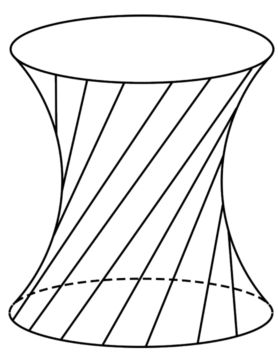
$$E \cap S_1^n(r) = \left\{ y \in E : \langle\langle y, y \rangle\rangle = r^2 \right\}$$

is an ellipse i.e. a closed curve (see the image in Remark 2.2.3).

2. If X is null then $\langle\langle \cdot, \cdot \rangle\rangle|_E$ is positive semi-definite, but degenerate. In this case $E \cap S_1^n(r)$ is a pair of parallel straight lines. Indeed,

$$\begin{aligned} E \cap S_1^n(r) &= \left\{ \alpha p + \beta X : \langle\langle \alpha p + \beta X, \alpha p + \beta X \rangle\rangle = r^2 \right\} \\ &= \{ \alpha p + \beta X : \alpha^2 \underbrace{\langle\langle p, p \rangle\rangle}_{= r^2} + 2\alpha\beta \underbrace{\langle\langle p, X \rangle\rangle}_{\substack{=0, \\ p \perp X}} + \beta^2 \underbrace{\langle\langle X, X \rangle\rangle}_{\substack{=0, \\ X \text{ is null}}} = r^2 \} \\ &= \left\{ \alpha p + \beta X : \alpha^2 = 1, \beta \in \mathbb{R} \right\} \\ &= \{ \pm p + \beta X : \beta \in \mathbb{R} \} \end{aligned}$$

These straight lines are precisely the generators of the hyperboloid as a ruled surface.



3. If X is timelike, then $\langle\langle \cdot, \cdot \rangle\rangle|_E$ is indefinite and non-degenerate. In this case $E \cap S_1^n(r)$ is a hyperbola with two connected components

$$E \cap S_1^n(r) = \left\{ \alpha p + \beta X : \alpha^2 r^2 + \beta^2 \langle\langle X, X \rangle\rangle = r^2 \right\} = \left\{ \alpha p + \beta X : \alpha^2 + \frac{\langle\langle X, X \rangle\rangle}{r^2} \beta^2 = 1 \right\},$$

which is the equation of hyperbola since $\langle\langle X, X \rangle\rangle < 0$.

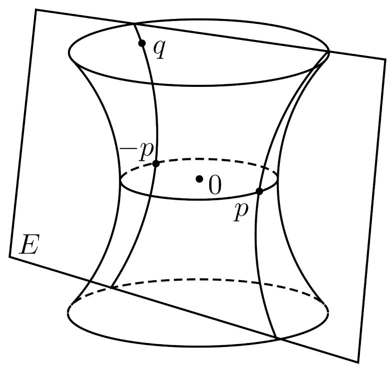
4. It follows that $S_1^n(r)$ is geodesically complete i.e. $\exp_p$ is defined on all of T_pM for every $p \in M$. Moreover, it is not totally geodesic since the geodesics of $\mathbb{R}_1^{n+1}$ are straight lines, which is the case only in (2.).

2.2.2 Geodesic Conectedness

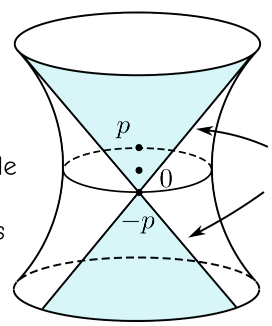
According to the Hopf-Rinow Theorem (cf. [3], 2.4.2), any connected and geodesically complete Riemannian manifold (RMF) ensures geodesic connectedness. However, De Sitter space contradicts this theorem within Lorentzian geometry. Precisely, $S_1^n(r)$ is geodesically complete but lacks geodesic connectedness.

Fix $p \in S_1^n(r)$ and consider which $q \in S_1^n(r)$ can be reached by a geodesic starting at p. If:

- $q = p$ or $q = -p$ the answer is trivial since there exist infinitely many planes containing p and $\pm p$.
- $q \neq \pm p$ then p and q are linearly independent and there exists a unique plane E containing them. If this plane is not spacelike and p and q happen to be in different connected components of $E \cap S_1^n(r)$, then there is no geodesic connecting them.



Moreover, all points q that can be reached from $-p$ by a causal geodesic cannot be reached from p.



Analytically, this is the set

$$\{q \in S_1^n(r) : \underbrace{\langle\langle q+p, p\rangle\rangle \leq 0}_{\langle\langle p,q\rangle\rangle \leq -r^2} \text{ and } q \neq p\}.$$

Indeed, pick an ONB $\{e_0, e_1\}$ in $E = \text{span}(p,q)$. Then $E \cap S_1^n(r)$ consists of two branches of hyperbola $-(x^0)^2 + (x^1)^2 = r^2$. If $p = p_0 e_0 + p_1 e_1$, then $-p_0^2 + p_1^2 = r^2$ and $-q_0^2 + q_1^2 = r^2$ and so we can parametrize p and q as

$$p = r(\sinh(t), \cosh(t))$$
$$q = r(\sinh(s), \pm\cosh(s)).$$

They are on different branches if and only if $q = r(\sinh(s), -\cosh(s))$. Then

$$\langle\langle p, q\rangle\rangle = -r^2 \sin(t)\sin(s) - r^2 \cosh(t)\cosh(s) = -r^2 \cosh(t+s) \le -r^2.$$

On the other hand, they are on the same branch if and only if $q = r(\sinh(s), +\cosh(s))$. Then

$$\langle\langle p, q\rangle\rangle = \cdots = r^2 \cosh(t-s) \ge r^2.$$

Therefore, p and q are on different branches if and only if $\langle\langle p, q\rangle\rangle \le -r^2$, which is the case if and only if $\langle\langle p+q, p\rangle\rangle < 0$.

To conclude this section, we'll delve into the study of isometries of $S_1^n(r)$.

Theorem 2.2.7 (Equality of Isometries). Let (M, g) be a connected SRMF and let ψ_1 and ψ_2 be isometries of M. If $\psi_1(p) = \psi_2(p)$ and $T_p\psi_1 = T_P\psi_2$ (for some M), then $\psi_1 = \psi_2$.

Proof. Set $\psi := \psi_2^{-1} \circ \psi_1$. Then $\psi \in \text{Isom}(M)$, $\psi(p) = p$ and $T_p\psi = \text{id}$. It remains to show that $\psi = \text{id}$. To that end, let

$$\mathcal{U} := \left\{q \in M : \psi(q) = q \text{ and } T_q\psi = \text{id}_{T_qM}\right\}.$$

Then,

- $\mathcal{U} \neq \varnothing$ since $p \in \mathcal{U}$.
- $\mathcal{U}$ is closed.
- $\mathcal{U}$ is open. To see this, let $q \in \mathcal{U}$. $\exp_q$ is a local diffeomorphism i.e. there exists a neighborhood $\mathcal{U}$ of $q \in M$ and a local neighborhood $\tilde{\mathcal{U}}$ of 0 in T_qM such that $\exp_q : \tilde{\mathcal{U}} \to \mathcal{U}$ is a diffeomorphism. We consider the following commutative diagram:

$$\begin{array}{ccc} \tilde{\mathcal{U}} & \xrightarrow{T_q\psi = \text{id}} & \tilde{\mathcal{U}} \\ {\scriptstyle \exp_q}\downarrow & & \downarrow{\scriptstyle \exp_q} \\ \mathcal{U} & \xrightarrow{\psi} & \mathcal{U}. \end{array}$$

(see Lemma 2.2.4). Therefore, on $\mathcal{U}$,

$$\psi = \exp_q \circ \text{id} \circ \exp_q^{-1} = \text{id}_{\mathcal{U}}.$$

Finally, since M is connected, $\mathcal{U} = M$ and so $\psi = \text{id}_M$. □

Proposition 2.2.8 (Isometries of De Sitter). The map

$$\begin{array}{ccc} \mathcal{L}(n+1) & \to & \text{Isom}(S_1^n(r)) \\ A & \mapsto & A|_{S_1^n(r)} \end{array} \tag{2.2.2}$$

is a group isomorphism. Moreover,

1. $\text{Isom}(S_1^n(r))$ acts transitively (i.e. for all $p, q \in S_1^n(r)$ there exists a $\psi \in \text{Isom}(S_1^n(r))$ such that $q = \psi(p)$). We say that $S_1^n(r)$ is homogeneous (or that no point is preferred).
2. For all $X, Y \in TS_1^n(r)$ with $\langle\langle X, X\rangle\rangle = \langle\langle Y, Y\rangle\rangle$ there exists $\psi \in \text{Isom}(S_1^n(r))$ such that $T\psi(X) = Y$. In this case we say that $S_1^n(r)$ is isotropic (or that no direction is preferred).

Proof. Let $A|_{S_1^n(r)} : S_1^n(r) \to S_1^n(r)$ be as before. Since $A|_{TS_1^n(r)}$ preserves $\langle\langle \cdot, \cdot \rangle\rangle$, $A|_{S_1^n(r)} \in \text{Isom}(S_1^n(r))$.

1. Let $p, q \in S_1^n(r)$. Then $\langle\langle p, p \rangle\rangle = r^2 = \langle\langle q, q \rangle\rangle$. By Proposition 2.1.12, there exists $A \in \mathcal{L}(n+1)$ such that $Ap = q$.

2. By 1. we may without loss of generality assume X and Y to have the same base point $p \in S_1^n(r)$. In $p^\perp = T_p S_1^n(r) \cong \mathbb{R}_1^n$ we find, according to Lemma 2.2.4, that there exists $B \in \mathcal{L}(n)$ such that $BX = Y$. As in Example 2.1.11,
$$\tilde{B} := \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{L}(n+1).$$
Now $\psi := \tilde{B}|_{S_1^n(r)} \in \text{Isom}(S_1^n(r))$ satisfies $\psi(p) = p$ and $T\psi(X) = BX = Y$. That (2.2.2) is injective is clear since $S_1^n(r)$ contains a basis of $\mathbb{R}^{n+1}$. Finally, we show surjectivity. To that end, let $\psi \in \text{Isom}(S_1^n(r))$ and fix $p \in S_1^n(r)$. For $q \in \psi(p) \in S_1^n(r)$ we find $A \in \mathcal{L}(n+1)$ such that $Ap = q$ (by 1.). Then p is a fixed point of
$$\psi_1 := A^{-1} \circ \psi \in \text{Isom}(S_1^n(r)).$$
Hence, $T_p\psi_1 : T_p S_1^n(r) \to T_p S_1^n(r)$ is a linear isometry. Choose $B \in \mathcal{L}(n+1)$ such that $B(p) = p$ and $T_p B|_{S_1^n(r)} = B|_{S_1^n(r)} = T_p\psi_1$ (cf. $\tilde{B}$ above). Define
$$\psi_2 := B^{-1} \circ A^{-1} \circ \psi.$$
Then $\psi_2 \in \text{Isom}(S_1^n(r))$, $\psi_2(p) = p$ and $T_p\psi_2 = \text{id}$. Theorem 2.2.7 implies that $\psi_2 = \text{id}$ and so $\psi = A \circ B$.

□

Remark 2.2.9. The isometry group of de Sitter space is considered 'maximal' because any $A \in \mathcal{L}(n+1)$ induces an isometry. It can be demonstrated that any Lorentzian manifold with a maximal isometry group exhibits constant curvature.

2.3 Anti-de Sitter Space

We now turn to a (family of) space(s) with Lorentzian metrics and constant negative curvature. We will introduce the respective space as a submanifold of a SRMF, which is no longer Lorentzian but has index 2. Consider $\mathbb{R}_2^{n+1}$ with the metric

$$ds^2 \equiv g = -dx^0 \otimes dx^0 - dx^1 \otimes dx^1 + \sum_{i=2}^n dx^i \otimes dx^i$$

and the map

$$\begin{aligned} f : \mathbb{R}^{n+1} &\to \mathbb{R} \\ x &\mapsto f(x) = g(x,x) = -(x^0)^2 - (x^1)^2 + \sum_{i=1}^n (x^i)^2. \end{aligned}$$

Here again $f \in \mathcal{C}^\infty$ and $T_x f = 0$ if and only if $x = 0$. Also, any $c \in \mathbb{R} \setminus \{0\}$ is a regular value of f.

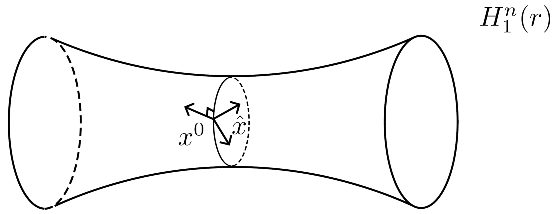
Definition 2.3.1. Let $r > 0$ then we call the SRHSF

$$H_1^n(r) = f^{-1}(-r^2)$$

n-dimensional **anti-de Sitter space**.

Remark 2.3.2. (Basic Properties of Anti-de Sitter)

1. $H_1^n(r)$ is a *one-sheeted hyperboloid* in $\mathbb{R}_2^{n+1}$.



2. The map

$$\begin{aligned} H_1^n(r) &\to S^1 \times \mathbb{R}^{n-1} \\ (x^0, x^1, \tilde{x}) &\mapsto \left(\frac{x^0}{\sqrt{\|\tilde{x}\| + r^2}}, \frac{x^1}{\sqrt{\|\tilde{x}\| + r^2}}, \tilde{x} \right) \end{aligned}$$

is a diffeomorphism with inverse

$$y \mapsto (\sqrt{\|\tilde{x}\| + r^2}y^0, \sqrt{\|\tilde{x}\| + r^2}y^1, \tilde{y}).$$

3. The normal bundle of $H_1^n(r)$ is generated by

$$\begin{aligned} \mathrm{grad}(f)(x) &= -2x^0 \left(-\frac{\partial}{\partial x^0} \right) - 2x^1 \left(-\frac{\partial}{\partial x^1} \right) + 2 \sum_{i=2}^n x^i \frac{\partial}{\partial x^i} \\ &= 2 \sum_{i=0}^n x^i \frac{\partial}{\partial x^i}, \end{aligned}$$

which again is proportional to the position vector field $X = \sum_{i=0}^n x^i \frac{\partial}{\partial x^i}$. Now

$$g(\mathrm{grad}(f)(x), \mathrm{grad}(f)(x)) = 2g(X, X) = 4f(x) = -4r^2 < 0$$

and so $\epsilon = -1$ and $\mathrm{ind}(H_1^n(r)) = 2 - 1 = 1$. Therefore, $H_1^n(r)$ is a Lorentzian manifold.

4. For the curvature quantities we obtain the following:

   - For the unit normal, we have

   $$\nu(x) \equiv U(x) = \frac{1}{2r}\mathrm{grad}(f)(x) = \frac{1}{r} \sum_{i=0}^n x^i \frac{\partial}{\partial x^i} = \frac{1}{r}X.$$

   Contrary to de-Sitter, now we have $\langle\!\langle U, U \rangle\!\rangle = \frac{1}{r^2} \langle\!\langle X, X \rangle\!\rangle = -1$.

   - Shape operator

   $$S(V) = -\overline{\nabla}_V U = -\frac{1}{r}\overline{\nabla}_V X = -\frac{1}{r}V.$$

   - $\mathbb{I}(V, W) = \frac{1}{r} \langle V, W \rangle U$ since

   $$\langle \mathbb{I}(V, W), U \rangle \overset{1.4.7}{=} \langle S(V), W \rangle = -\frac{1}{r} \langle V, W \rangle = \frac{1}{r} \langle V, W \rangle \langle U, U \rangle.$$

- Gauss equation yields for the Riemann curvature:

$$\begin{aligned}
\langle R_{VW}X,Y\rangle &\overset{1.3.13}{=} \underbrace{0}_{\mathbb{R}^{n+1}_2 \text{ is flat}} + \langle \mathbb{I}(V,X),\mathbb{I}(W,Y)\rangle - \langle \mathbb{I}(V,Y),\mathbb{I}(W,X)\rangle \\
&= \frac{1}{r^2}(\langle V,X\rangle\langle W,Y\rangle \underbrace{\langle U,U\rangle}_{-1} - \langle V,Y\rangle\langle W,X\rangle \underbrace{\langle U,U\rangle}_{-1}) \\
&= -\frac{1}{r^2}(\langle V,X\rangle\langle W,Y\rangle - \langle W,X\rangle\langle V,Y\rangle).
\end{aligned}$$

Therefore,

$$R_{VW}X = -\frac{1}{r^2}(\langle V,X\rangle W - \langle W,X\rangle V).$$

- Sectional curvature:

$$\begin{aligned}
K &\overset{1.3.14}{=} 0 + \epsilon\frac{\langle S(V),V\rangle\langle S(W),W\rangle - \langle S(V),W\rangle^2}{Q(V,W)} \\
&= -\frac{1}{r^2}\frac{\langle V,V\rangle\langle W,W\rangle - \langle V,W\rangle^2}{Q(V,W)} = -\frac{1}{r^2}.
\end{aligned}$$

This shows that $H_1^n(r)$ has constant negative curvature.

- Ricci curvature:

$$\begin{aligned}
\text{Ric}(X,Y) &= \sum_{i=1}^n \epsilon_i \langle R_{XE_i}Y,E_i\rangle \\
&= -\frac{1}{r^2}\sum_{i=1}^n \epsilon_i (\langle X,Y\rangle\langle E_i,E_i\rangle - \langle E_i,Y\rangle\langle E_i,X\rangle) \\
&= -\frac{n-1}{r^2}\langle X,Y\rangle,
\end{aligned}$$

where $\epsilon_i = \langle E_i,E_i\rangle$ and $\{E_i\}_{i=1}^n$ a local frame. Hence,

$$\text{Ric} = -\frac{n-1}{r^2}g.$$

- Scalar curvature:

$$S = C(\text{Ric}) = -\frac{n(n-1)}{r^2}.$$

- Einstein tensor:

$$G = \text{Ric} - \frac{n}{2}Sg = -\frac{1}{r^2}\left((n-1) - \frac{n}{2}(n-1)\right),$$

which is for $n = 4$ equal to $\frac{3}{r^2}g$ and so AdS (i.e. anti-de Sitter) is a solution of the vacuum Einstein equations with cosmological constant $\Lambda = -\frac{3}{r^2} < 0$ i.e. $G + \Lambda g = 0 = T$.

Remark 2.3.3. The geodesics of AdS are (again) given by the intersections of the form $E \cap H_1^n(r)$, where $E \subseteq \mathbb{R}^{n+1}$ is a 2-dimensional surface. Here again $H_1^n(r)$ is not geodesically connected (but it is geodesically complete!).

Remark 2.3.4. There exist closed timelike geodesics in $H_1^n(r)$, which often is unwanted. A way around it is to consider the universal cover $\tilde{H}_1^n(r) \cong \mathbb{R}^n$, where $\cong$ stands for 'diffeomorphic'. Sometimes AdS is understood to be $\tilde{H}_1^n(r)$.

2.4 Robertson-Walker Spacetimes

These are also referred to as Friedmann–Lemaître–Robertson–Walker spacetimes, commonly denoted as FLRW. These spacetimes represent pivotal cosmological models, portraying universes that are both homogeneous and isotropic. They encapsulate scenarios of universal expansion or contraction and are often regarded as a standard model in cosmology.

Definition 2.4.1. An $(n+1)$-dimensional Lorentzian manifold (M, g) is called a **FLRW-spacetime** if it is of the form

$$M = I \times S,$$

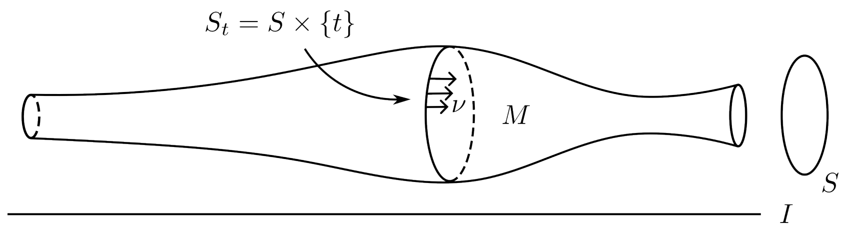
where I is an open interval of $\mathbb{R}$ and (S, g_s) a complete connected RMF with a constant curvature κ and of dimension n, with metric

$$g := -dt \otimes dt + f(t)^2 g_s, \tag{2.4.1}$$

where $f(t)^2$ is a scale factor and $f : I \to \mathbb{R}^+$ is a $\mathcal{C}^\infty$-function.

Remark 2.4.2.

- The above is an example of a *warped product*.
- A vivid picture of a FLRW-spacetime is:

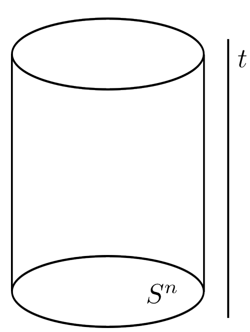


where $S_t = S \times \{t\}$ for some $t \in I$ interpreted as time, making S_t into a time-cut (time-slice). $\nu = \frac{\partial}{\partial t}$ is its normal vector in M.

- The most important model spaces are, in cases $\kappa = 1, 0, -1$, S^n, $\mathbb{R}^n$, and H^n. Other examples involve quotients of these spaces.

Example 2.4.3.

1. For $S = \mathbb{R}^n$, $\kappa = 0$, $I = \mathbb{R}$ and $f \equiv 1$ we have Minkowski space.
2. For $S = S^n$, $\kappa = 1$, $I = \mathbb{R}$ and $f \equiv 1$ we obtain *Einstein's static universe*. Einstein's static universe is diffeomorphic to a cylinder. In the 1920. the basic idea about the universe was that it is static (time-independent) and this is the prime model. To make it a solution to the field equations Einstein introduced the cosmological constant into his equations.



2.4.1 Geometry of the FLRW-spacetimes

1. Time slices. Set $S(t) \equiv S_t = \{t\} \times S$ (which is a level set of $h(t,x) = t$). Then
$$\nu = \frac{\partial}{\partial t} \tag{2.4.2}$$
is the unit normal to S_t. For the shape operator with respect to ν let X, Y, Z be tangential to S_t and let without loss of generality their Lie-brackets vanish. Then
$$\begin{aligned} \langle S(X), Y\rangle = \langle -\overline{\nabla}_X \nu, Y\rangle &= -\frac{1}{2}(X \underbrace{\langle \nu, Y\rangle}_{=0} + \nu \underbrace{\langle Y, X\rangle}_{f(t)^2 g_s(X,Y)} - Y \underbrace{\langle X, \nu\rangle}_{=0}) \\ &\overset{(2.4.2)}{=} -\frac{1}{2}\partial_t (f^2(t) g_s(X,Y)) \\ &= -f(t)\dot{f}(t) g_s(X,Y) = -\frac{\dot{f}}{f}\langle X, Y\rangle, \end{aligned}$$
where in the second equality we used the Koszul formula (cf. (1.3.10) in [3]). Finally,
$$S(X) = -\frac{\dot{f}}{f} X.$$

2. Cosmic observers. We will show that the curves $t \mapsto (t, x_0)$, for some fixed $x_0 \in S$, are geodesics. This models the worldline[1] of an observer (spaceship, galaxy...) often called a *cosmic observer*. Let $\gamma : t \mapsto (t, x_0)$. Since $\gamma'(t_0) = \frac{\partial}{\partial t}\big|_{t_0}$, it suffices to prove that
$$(\nabla^M_\nu \nu)(t_0, p_0) = 0, \forall (t_0, p_0) \in M$$
since $\nabla^M_\nu \nu = \nabla_{\gamma'}\gamma' = \gamma''$. Let $(x^1, \dots, x^n)$ be coordinates of S at p_0 such that $(t, x^1, \dots, x^n)$ are coordinates of M at (t_0, p_0). Using the Koszul-formula we obtain that
$$2\langle \nabla^M_\nu \nu, \partial_i\rangle = \nu\langle \nu, \partial_i\rangle + \nu\langle \partial_i, \nu\rangle - \partial_i \underbrace{\langle \nu, \nu\rangle}_{\equiv -1}$$
since $[\partial_i, \partial_j] = [\partial_i, \nu] = [\nu, \nu] = 0$. Because $\nu \in TS^\perp$ while $\partial_i \in TS$,
$$\langle \nabla^M_\nu \nu, \partial_i\rangle = 0$$
and so $\nabla^M_\nu \nu$ is perpendicular to S_{t_0}. Therefore, $\nabla^M_\nu \nu = \tan(\nabla^M_\nu \nu)$, where tan is the tangential projection of the SRSMF $I \times \{p_0\}$ of M. Proposition 1.3.10 implies that $\tan(\nabla^M_\nu \nu) = \nabla^I_\nu \nu$, which is equal to zero since I is flat. Therefore, $\nabla^M_\nu \nu = 0$, as claimed.

3. General geodesics. Let $c(s) = (t(s), \gamma(s))$, where $t(s) \in I$ and $\gamma(s) \in S$, be a curve in M such that $t'(s) \neq 0$ and $\gamma'(s) \neq 0$ for all s. Then $c'(s) = \underbrace{t'(s)\nu}_{\in (TS_t)^\perp} + \underbrace{\gamma'(s)}_{\in TS_t}$ and so
$$\begin{aligned} \nabla^M_{c'} c'(s) &= \nabla^M_{c'}(\underbrace{t'\nu}_{\in \mathfrak{X}(c)} + \gamma') \\ &\overset{(\nabla 3)}{=} t''\nu + t'\nabla^M_{t'\nu+\gamma'}\nu + \nabla^M_{t'\nu+\gamma'}\gamma' \\ &\overset{2.}{=} t''\nu + (t')^2 \underbrace{\nabla^M_\nu \nu}_{=0} + t'\nabla^M_{\gamma'}\nu + t'\nabla^M_\nu \gamma' + \nabla^M_{\gamma'}\gamma. \end{aligned}$$

[1] The world line (or worldline) of an object is the path that an object traces in 4-dimensional spacetime.

Using (1.3.7), we get

$$\nabla^M_{\gamma'}\gamma' = \nabla^S_{\gamma'}\gamma' + \mathbb{I}(\gamma',\gamma') = \nabla^S_{\gamma'}\gamma' - \langle \mathbb{I}(\gamma',\gamma'),\nu\rangle\,\nu = \nabla^S_{\gamma'}\gamma' - \langle S(\gamma'),\gamma'\rangle\,\nu.$$

We can extend γ' locally to a vector field X on S and lift this to M. Let T be a local extension of $t'\nu$ to a vector field on M. Then $[T,X]=0$ since in a basis ∂_t,∂_i, as above, T depends only on t and X only on X and $[\partial_t,\partial_i]=[\partial_i,\partial_j]=0$. Therefore, $\nabla_T X = \nabla_X T$ and so $\nabla^M_\nu\gamma' = \nabla^M_{\gamma'}\nu$. Now

$$\nabla^M_{c'}c'(s) = t''\nu - 2t'\underbrace{S(\gamma')}_{-\nabla^M_{\gamma'}\nu} + \nabla^S_{\gamma'}\gamma' - \langle S(\gamma'),\gamma'\rangle\,\nu.$$

Note that $t''\nu$ and $\langle S(\gamma'),\gamma'\rangle\,\nu$ are perpendicular to S_t while $S(\gamma')$ and $\nabla^S_{\gamma'}\gamma'$ tangential to it. Hence, c is a geodesic if and only if

$$\begin{aligned} 0 &= t'' - \langle S(\gamma'),\gamma'\rangle \overset{1.}{=} t'' + f'f\cdot g_s(\gamma',\gamma') \\ 0 &= \nabla^S_{\gamma'}\gamma' + 2t'\frac{f'}{f}\gamma' = \nabla^S_{\gamma'}\gamma' + 2\frac{(f\circ t)'}{f\circ t}\gamma'. \end{aligned}$$

Proposition 2.4.4 (Geodesics of FLRW). A smooth curve $c=(t,\gamma)$ in M is a geodesic if and only if

1. $t'' = -f'f\cdot g_s(\gamma',\gamma')$ and
2. $\nabla^S_{\gamma'}\gamma' = -2\frac{(f\circ t)'}{f\circ t}\gamma'$.

In particular, γ is a *pregeodesic* of S (cf. 2.1.10 in [3] or choose a parametrization such that $f\circ t$ is constant).

Corollary 2.4.5. For null geodesics (cf. 2.1.9 in [3]) we have that $t'(f\circ t)$ is constant.

Proof. Proof.

$$0 = f'\underbrace{g(c',c')}_{=0} \overset{(2.4.1)}{=} -f'(t')^2 + f'f^2 g_s(\gamma',\gamma') \overset{2.4.4}{=} -f'(t')^2 - ft'' = -(t'(f\circ t))'.$$

□

Remark 2.4.6 (Cosmological Redshift). Light emitted from a distant galaxy appears to undergo a shift towards lower frequencies when observed from Earth. The emitted wavelength remains assumed to be the same, suggesting that wavelengths have uniformly lengthened during transmission. The energy of a photon is given by

$$E = \hbar\omega = \frac{\hbar}{\lambda},$$

where $\hbar$ denotes Planck constant, ω frequency and λ wavelength. It is measured by an observer ν according to

$$E = -g(c',\nu) = t',$$

where $c=(t,\gamma)$ (and $c' = t'\nu + \gamma'$) is the null geodesic (i.e. light ray) emitted by the galaxy.

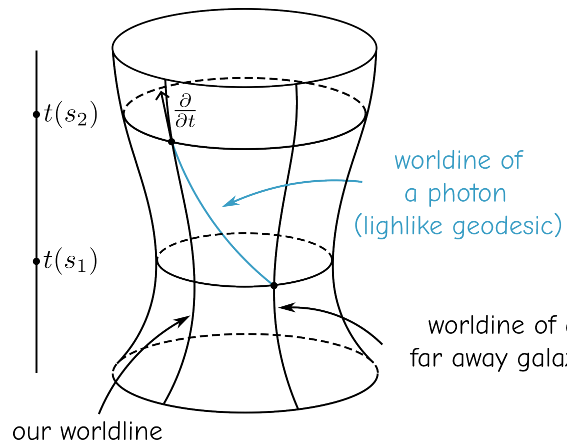
If the time of emission is s_1 and the time of absorption s_2, then $t'(s_1)f(t(s_1)) \overset{2.4.5}{=} t'(s_2)f(t(s_2))$ yields

$$\frac{t'(s_1)}{t'(s_2)} = \frac{f(t(s_2))}{f(t(s_1))}.$$

Since $\lambda = \frac{\hbar}{E} = \frac{\hbar}{t'}$,

$$\frac{E(s_1)}{E(s_2)} = \frac{\lambda(s_2)}{\lambda(s_1)}.$$

Therefore, if f is increasing (which corresponds to the universe expanding), then $\lambda(s_2) > \lambda(s_1)$. (For more details see 12.8 in [4].)



2.4.2 Curvature for FLRW Spacetimes

1. Let $S(t)$ be a time slice in M i.e. $S(t) = \{t\} \times S$, where S has a constant curvature κ. Then

$$\begin{aligned} R^S(X,Y)Z &\overset{1.1.12}{=} \kappa(g_S(Z,X)Y - g_S(Z,Y)X), \\ \text{Ric} &\overset{\text{cf. } 2.2.2}{=} \kappa(n-1)g_s \text{ and} \\ S &\overset{\text{cf. } 2.2.2}{=} \kappa(n-1)n, \end{aligned}$$

where $X,Y,Z \in \mathfrak{X}(S(t))$.

2. Claim: $\tan(R^M(X,Y)Z) = R^S_{XY}Z + \langle S(X),Z\rangle S(Y) - \langle S(Y),Z\rangle S(X)$. Let $X,Y,Z,W \in \mathfrak{X}(S(t))$. Then

$$\begin{aligned} \langle \tan(R^M(X,Y)Z), W\rangle &\overset{W\in\mathfrak{X}(S(t))}{=} \langle R^M(X,Y)Z, W\rangle \\ &\overset{1.3.13}{=} \langle R^S_{XY}Z, W\rangle \underbrace{- \langle \mathbb{I}(X,Z), \mathbb{I}(Y,W)\rangle + \langle \mathbb{I}(X,W), \mathbb{I}(Y,Z)\rangle}_{(\star)}. \end{aligned}$$

In general, $\mathbb{I}(X,Y) = \epsilon \langle S(X),Y\rangle \nu$ and so

$$\begin{aligned} (\star)|_{\text{free } W} &= -\langle \mathbb{I}(X,Z), \mathbb{I}(Y,\cdot)\rangle + \langle \mathbb{I}(X,\cdot), \mathbb{I}(Y,Z)\rangle \\ &= -\epsilon^2 \langle S(X),Z\rangle \underbrace{\langle \nu,\nu\rangle}_{=-1} S(Y) + \epsilon^2 \langle S(Y),Z\rangle \underbrace{\langle \nu,\nu\rangle}_{=-1} S(X) \\ &= \langle S(X),Z\rangle S(Y) - \langle S(Y),Z\rangle S(X). \end{aligned}$$

$$\begin{aligned} \tan(R^M(X,Y)Z) &= \kappa(g_s(Z,X)Y - g_s(Z,Y)X) + \left(-\frac{\dot f}{f}\right)^2 \underbrace{\langle X,Z\rangle}_{=f^2 g_s(X,Z)} Y - \left(\frac{\dot f}{f}\right)^2 \underbrace{\langle Y,Z\rangle}_{=f^2 g_s(Y,Z)} X, \\ &= \kappa(g_s(X,Z)Y - g_s(Y,Z)X) + \dot f^2 g_s(X,Z)Y - \dot f^2 g_s(Y,Z)X \\ &= (\kappa + \dot f^2)(g_s(X,Z)Y - g_s(Y,Z)X) \\ &= \left(\frac{\kappa}{f^2} + \left(\frac{\dot f}{f}\right)^2\right)(\langle X,Z\rangle Y - \langle Y,Z\rangle X) \end{aligned}$$

since $S(X) = -\frac{\dot f}{f}X$.

3. Claim: nor$(R^M_{XY}Z) = 0$. By the Codazzi-equation,

$$\text{nor}(R^M_{XY}Z) = \text{nor}(\overline{R}_{XY}Z) \overset{1.6.6}{=} -(\nabla_X \mathbb{I})(Z,Y) + (\nabla_Y \mathbb{I})(X,Z),$$

where

$$\nabla_X \mathbb{I}(Y,Z) = \epsilon \langle \nabla_X S(Y), Z\rangle \nu$$

as seen in the end of the proof of Corollary 1.6.8. Therefore,

$$\text{nor}(R^M_{XY}Z) = \langle(\nabla_X S)(Z), Y\rangle \nu - \langle(\nabla_Y S)(X), Z\rangle \nu = 0$$

since $S \in \mathcal{T}^1_1(S(t))$ and so $(\nabla_X S)(Z) \overset{(\nabla 3)}{=} \nabla_X(\underbrace{S(Z)}_{-\frac{\dot f}{f}Z}) - S(\nabla_X Z) \overset{(\nabla 3)}{=} \underbrace{X(-\frac{\dot f}{f})}_{=0} Z - \cancel{\frac{\dot f}{f}\nabla_X Z} + \cancel{\frac{\dot f}{f}\nabla_X Z}$.

Similarly, $(\nabla_Y S)(X) = 0$.

4. Claim: $R^M(X,\nu)\nu = \frac{f''}{f}X$. Indeed, 3.1.1 in [3] implies that

$$\begin{aligned}
R^M(X,\nu)\nu &= \nabla^M_{[X,\nu]}\nu - \nabla^M_X \underbrace{\nabla^M_\nu \nu}_{=0,\text{ by 2.}} + \nabla^M_\nu \nabla^M_X \nu \\
&= \nabla^M_{\nabla^M_X \nu - \nabla^M_\nu X} - \nabla^M_\nu(\underbrace{S(X)}_{\overset{1.4.8}{=} -\nabla^M_X \nu}) \\
&\overset{1.4.8}{=} S(S(X)) + S(\nabla^M_\nu X) - \nabla^M_\nu(S(X)) \\
&= \frac{\dot f^2}{f^2}X - \cancel{\frac{\dot f}{f}\nabla^M_\nu} X + \underbrace{\nabla^M_\nu\left(\frac{\dot f}{f}X\right)}_{=\partial_t\left(\frac{\dot f}{f}\right)X + \cancel{\left(\frac{\dot f}{f}\right)\nabla^M_\nu X}} \\
&= \frac{\dot f^2}{f^2}X + \frac{\ddot f f - (\dot f)^2}{f^2}X = \frac{\ddot f}{f}X.
\end{aligned}$$

5. Next we calculate Ric$^M(X,Y)$ for $X,Y \in \mathfrak{X}(S(t))$. To that end, let $(E_i)^n_{i=1}$ be an orthonormal frame for $S(t)$. Then $E_1,\dots,E_n$ is an orthonormal frame for M and so

$$\text{Ric}^M(X,Y) \overset{1.2.8,\ 2.}{=} \sum_{i=1}^n \langle R^M(X,E_i)Y, E_i\rangle - \underbrace{\langle R^M(X,\nu)Y,\nu\rangle}_{\substack{= -\langle R^M(X,\nu)\nu, Y\rangle \\ \overset{4.}{=} \frac{f''}{f}\langle X,Y\rangle}},$$

by skew-adjointness.

$$\begin{aligned}
\mathrm{Ric}^M(X,Y) &= \left\langle \tan(R^M(X,E_i)Y), E_i \right\rangle + \frac{f''}{f}\langle X,Y\rangle \\
&\overset{2.}{=} (\kappa + f^2)(\underbrace{g_s(X,Y)}_{=\frac{1}{f^2}\langle X,Y\rangle}\langle E_i,E_i\rangle - \underbrace{g_s(E_i,Y)}_{=\frac{1}{f^2}\langle E_i,Y\rangle}\langle X,E_i\rangle) + \frac{f''}{f}\langle X,Y\rangle \\
&= \left(\frac{\kappa}{f^2} + \left(\frac{\dot f}{f}\right)^2\right)\sum_{i=1}^n (\langle X,Y\rangle \underbrace{\langle E_i,E_i\rangle}_{=1} - \langle E_i,Y\rangle\langle X,E_i\rangle) + \frac{f''}{f}\langle X,Y\rangle \\
&= \left(\frac{\kappa}{f^2} + \left(\frac{\dot f}{f}\right)^2\right)(n-1)\langle X,Y\rangle + \frac{f''}{f}\langle X,Y\rangle = \left(\frac{n-1}{f^2}(\kappa + \dot f^2) + \frac{\ddot f}{f}\right)\langle X,Y\rangle .
\end{aligned}$$

6. $\mathrm{Ric}^M(X,\nu) = 0$. Indeed,

$$\mathrm{Ric}^M(X,\nu) = -\sum \underbrace{\left\langle R^M(X,E_i)E_i,\nu\right\rangle}_{=0 \text{ by 3.}} - \underbrace{\langle \overbrace{R^M(X,\nu)\nu}^{\propto X, \text{ by 4.}},\nu\rangle}_{=0},$$

because $X \perp \nu$.

7. Claim: $\mathrm{Ric}^M(\nu,\nu) = -n\frac{f''}{f}$ since

$$\mathrm{Ric}^M(\nu,\nu) = \sum_{i=1}^n \langle \underbrace{R^M(\nu,E_i)\nu}_{\overset{4.}{=} -\frac{f''}{f}E_i}, E_i\rangle - \underbrace{\left\langle R^M(\nu,\nu)\nu,\nu\right\rangle}_{\substack{=0,\\ \text{by the symm. of } R^M}} = -n\frac{f''}{f}.$$

8. Scalar curvature:

$$\begin{aligned}
S^M &= \sum_{i=1}^n \mathrm{Ric}^M(E_i,E_i) - \mathrm{Ric}^M(\nu,\nu) \\
&\overset{5.,7.}{=} n\left(\frac{n-1}{f^2}(\kappa + \dot f^2) + \frac{\ddot f}{f}\right) + n\frac{\ddot f}{f} \\
&= n(n-1)\left(\frac{\kappa}{f^2} + \left(\frac{\dot f}{f}\right)^2 + \frac{2}{n-1}\frac{\ddot f}{f}\right).
\end{aligned}$$

9. Finally, we can express the Einstein equations for $\Lambda = 0$. For X, Y tangential to $S(t)$,

$$G(X,Y) = \mathrm{Ric}^M(X,Y) - \frac{1}{2}S^M g(X,Y) \overset{5.,8.}{=} \underbrace{\left(\left(1-\frac{n}{2}\right)(n-1)\left(\frac{\kappa}{f^2} + \left(\frac{\dot f}{f}\right)^2\right) - (n-1)\frac{\ddot f}{f}\right)}_{p \text{ ... pressure}}\langle X,Y\rangle$$

and

$$G(X,\nu) = \mathrm{Ric}^M(\nu,\nu) - \frac{1}{2}S^M g(\nu,\nu) \overset{7.,8.}{=} \frac{n(n-1)}{2} \underbrace{\left(\frac{\kappa}{f^2} + \left(\frac{\dot f}{f}\right)^2\right)}_{\rho \text{ ... energy (= mass) density}} .$$

Since $\mathrm{Ric}^M(X,\nu) = 0$ (by 6.) and $g(X,\nu) = 0$, $G(X,\nu) = 0$. Therefore, for the Einstein equations, we get

$$G(X,Y) = T(X,Y).$$

The right hand side is a perfect fluid energy momentum tensor (cf. (3.3.17) in [3]) $T(\nu,\nu) = \rho$, $T(X,\nu) = 0$ and $T(X,Y) = p\langle X,Y\rangle$ i.e.

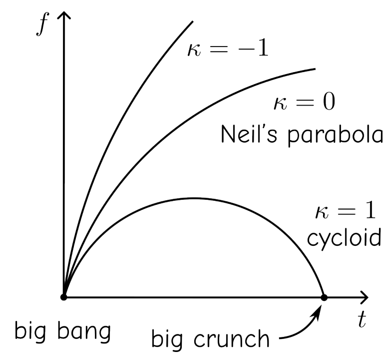
$$T = (\rho + p)(\nu^\flat \otimes \nu^\flat) + pg.$$

These equations describe the dynamics of our FLWR spacetime.

Definition 2.4.7. An FLRW spacetime is called a **Friedmann model** if $p = 0$.

Remark 2.4.8 (Friedmann Cosmology).
The condition $p = 0$ is, by 9. above, the second order ODE for f which can be solved explicitly. In case $n = 3$, we obtain:

1. $\kappa = 0$, $f(t) = C(t - t_0)^{\frac{3}{2}}$.
2. $\kappa = 0$, $t(\theta) = C(\theta - \sin(\theta))$, $f(t(\theta) - t_0) = C(1 - \cos(\theta))$.
3. $\kappa = -1$, $t(\eta) = C(\sinh(\eta) - \eta)$, $f(t(\eta) - t_0) = C(\cosh(\eta) - 1)$.



Remark 2.4.9 (Horizons). Consider a null geodesic $c(s) = (t(s), \gamma(s))$ in a FLRW spacetime. Then $c'(s) = t'(s)\nu + \gamma'$ and

$$0 = g(c', c') = -(t')^2 + f^2(t)\|\gamma'\|_s^2 \implies \|\gamma'\|_s = \frac{|t'|}{f(t)}.$$

In case $\gamma' > 0$ we have for the spatial 'component' γ:

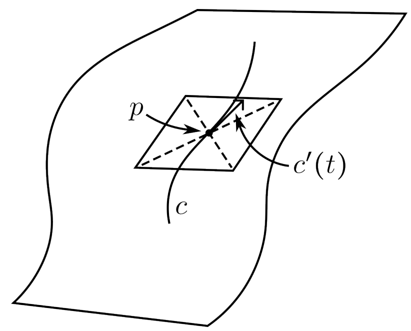
$$L^{g_s}(\gamma) = \int_{s_0}^{\infty} \|\gamma'\|_s ds = \int_{s_0}^{\infty} \frac{t'(s)}{f(t(s))} ds \overset{y=t(s)}{=} \int_{t(s_0)}^{\infty} \frac{dy}{f(y)}.$$

If f grows fast enough (e.g. $f(t) = t^2$, $f(t) = e^t$) then $L(\gamma) < \infty$ and hence γ is confined to some ball $B(\gamma(s_0), R)$ in S. This means that in this case some parts of the universe can never be seen i.e. are hidden behind a 'horizon'.

3

CAUSALITY

'Causality' refers to the broader inquiry concerning the connection between points within a Lorentzian manifold (LMF) and their ability to be linked by causal curves—those curves characterized by having a causal tangent vector. In General Relativity (GR), this inquiry delves into discerning which events can be influenced by a given event. While in some cases causality might not offer substantial insights, under appropriate conditions, it encapsulates the fundamental characteristics of an LMF. For instance, it provides sufficient conditions for points to be connected by causal geodesics or to be reached by normal causal geodesics originating from a spacelike hypersurface. Key aspects of causality theory have been developed within the context of singularity theorems by R. Penrose and S. Hawking.



3.1 Basic Notations

Definition 3.1.1. A time orientation on a LMF (M, g) is a map

$$\zeta : M \to \mathcal{P}(TM),$$

where $\mathcal{P}$ denotes the power set of TM, such that:

1. for every $p \in M$ $\zeta(p)$ is one of the connected components of the set of timelike vectors in T_pM.
2. for every $p \in M$ there exists a chart (U, x) of p such that $\frac{\partial}{\partial x^0}(q) \in \zeta(p)$ for every $q \in U$.

We call the pair (M, ζ) a time-oriented LMF and M time-orientable if it possesses a time orientation.

Proposition 3.1.2 (Time Orientability). For a LMF (M, g) these facts are equivalent:

1. M is time-orientable.
2. There exists a continuous timelike vector field on M.
3. There exists a $\mathcal{C}^\infty$ timelike vector field on M.

(Note that, in particular, there exists a nowhere vanishing $\mathcal{C}^0$, or $\mathcal{C}^\infty$, vector field on M.)

Proof.

$(3. \to 2.)$ Clear.

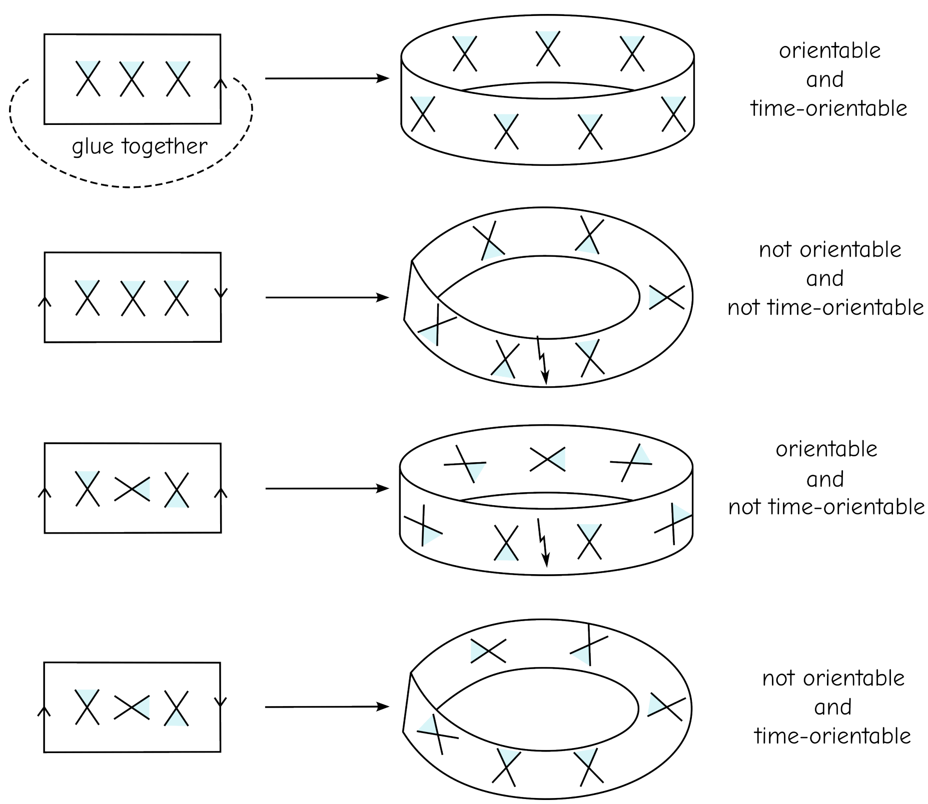
$(2. \to 1.)$ Let X be a continuous timelike vector field and define $\zeta(p)$ to be the connected component of $I(0)(\subseteq T_pM)$ containing $X(p)$. Pick a chart (x, U) at p such that $\frac{\partial}{\partial x^0}$ is timelike on U and $\frac{\partial}{\partial x^0}\big|_p \in \zeta(p)$. In a Lorentzian vector space let $x \in I^+(0)$, $y \in I$. Then $y \in I^+(0) \iff \langle x, y\rangle < 0$. Indeed, any Lorentz-transformation leaves I invariant so, without loss of generality, let $x = (a^2, 0)$ for some $a \in \mathbb{R}$. $y \in I^+(0)$ means that $y^0 > 0$ and so $\langle x, y\rangle = -y^0a^2 < 0$. Conversely, assume that $0 > \langle x, y\rangle = -a^2y^0$. Then $y^0 > 0$. By choice, $\left\langle X(p), \frac{\partial}{\partial x^0}\big|_p\right\rangle < 0$, hence also $\left\langle X(q), \frac{\partial}{\partial x^0}\big|_q\right\rangle < 0$, for all $q \in U$. By the remark in gray, $\frac{\partial}{\partial x^0}\big|_q \in \zeta(q)$ for all $q \in U$, which proves the claim.

$(1. \to 3.)$ Let ζ be a time orientation and (U_α, x_α) a covering of M by charts such that $\frac{\partial}{\partial x^0_\alpha}\big|_q \in \zeta(q)$ for all $q \in U_\alpha$, for every α. Let $(\rho_\alpha)_\alpha$ be a partition of unity subordinate to $(U_\alpha)_\alpha$ and set $X := \sum \rho_\alpha \frac{\partial}{\partial x^0_\alpha}$. Then $X \in \mathfrak{X}(M)$ and $X(p) = \sum \rho_\alpha \underbrace{\frac{\partial}{\partial x^0_\alpha}\Big|_p}_{\in\,\zeta(p)} \in \zeta(p)$, by the convexity of $\zeta(p)$ (see Corollary 2.1.5). In particular, X is timelike.

□

Remark 3.1.3.

- All previous examples of LMFs are time-orientable (Minkowski, de Sitter...).
- The concepts of orientability and time-orientability are independent, as illustrated in the following image.



Remark 3.1.4.

1. From now on we will only consider LMFs (M, g) that are T_2, 2^{nd} countable, connected and time-oriented. Such LMFs are called spacetimes.

2. A *curve* will always mean a piecewise $\mathcal{C}^\infty$-curve (i.e. a curve having finally many break points with two distinguished tangent vectors at break points).

3. A causal curve γ (i.e. a curve such that $\dot\gamma(t)$ is causal for all t) is called future directed causal curve if $\dot\gamma(t) \in \overline{\zeta(\gamma(t))}$ and past directed causal curve if $\dot\gamma(t) \in -\overline{\zeta(\gamma(t))}$.

Definition 3.1.5 (Causality Relations)**.** For $p, q \in (M, g)$ (spacetime) write

- $p \ll q$ if there exists future directed timelike curve from p to q.
- $p < q$ if there exists a future directed causal curve from p to q.
- $p \leq q$ if $p < q$ or $p = q$.

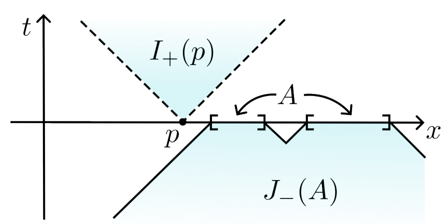
For $A \subseteq M$ define the set:

- $I^+(A) := \{q \in M : \exists p \in A \text{ such that } p \ll q\}$, called the chronological future.
- $J^+(A) := \{q \in M : \exists p \in A \text{ such that } p \leq q\}$, called the causal future of A.

$I^-(A)$ and $J^-(A)$ are defined analogously. Observe that $I^+(A) = \bigcup_{p\in A} I^+(p)$ and $J^+(A) = \bigcup_{p\in A} J^+(p)$.

Example 3.1.6.

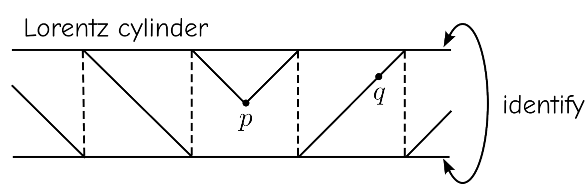
1. Let $M = \mathbb{R}^2_1$, then:



2. Let

$$(M, g) = \mathbb{R}^2_1/\mathbb{Z}e_0 \cong (S^1 \times \mathbb{R}, -d\theta^2 + df^2),$$

then:



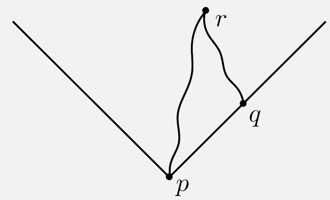
Remark 3.1.7 (Transitivity of $\ll$ and $\leq$)**.** Since one can concatenate pointwise $\mathcal{C}^\infty$-curves one easily sees that

$$\begin{aligned} p \leq q \wedge q \leq r &\implies p \leq r \text{ and} \\ p \ll q \wedge q \ll r &\implies p \ll r. \end{aligned}$$

However, we even have a stronger form of this transitivity often called the push-up principle.

Proposition 3.1.8 (Push-up Principle). In a spacetime (M, g) for all p, q, r, we have that

$$p \ll q \wedge q \leq r \implies p \ll r \text{ and}$$
$$p \leq q \wedge q \ll r \implies p \ll r.$$



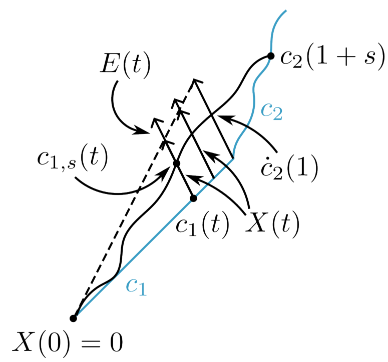
Proof. We only show the second assertion since the first one follows analogously. If $p = q$ then the statement is trivial. Therefore, let $p < q$. Then there exists a future directed causal curve

$$c_1 : [0, 1] \to M$$

such that $c_1(0) = p$ and $c_1(1) = q$. There also exists a future directed timelike curve

$$c_2 : [1, 2] \to M$$

such that $c_2(1) = q$ and $c_2(2) = r$. Let $E \in \mathfrak{X}(c_1)$ be the parallel transport of $\dot{c}_2(1)$ along c_1. Set $X(t) = t \cdot E(t)$. Then $X \in \mathfrak{X}(c_1)$.



We can find a 2-parameter map (cf. 2.1.19 in [3]), a variation of c_1, denoted $c_{1,s}$ with variational vector field X i.e.

$$\begin{aligned} c_{1,s} : [0,1] \times (-\epsilon, \epsilon) &\to M \\ (t, s) &\mapsto c_{1,s}(t) \end{aligned}$$

such that $c_{1,0}(t) = c_1(t)$, $c_{1,s}(0) = p$ for all s and

$$c_{1,s}(1) = c_2(1 + s) \text{ (for } s \geq 0). \tag{3.1.1}$$

Finally, we want that

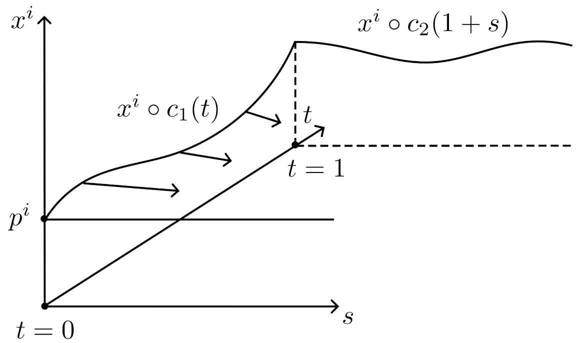
$$\partial_s c_{1,s}(t)\big|_{s=0} = X(t) = t \cdot E(t). \tag{3.1.2}$$

Is there such a variation? To convince ourselves that there is, we check for compatibility:

$$\partial_s\big|_0 c_{1,s}(1) \overset{(3.1.1)}{=} c_2'(1 + 0) \cdot 1 = c_2'(1) = E(1) = 1 \cdot E(1) = X(1). \checkmark$$

How to construct $c_{1,s}$?

Use Fermi coordinates along c_1; take a frame E_i along c_1 such that $\{\dot{c}_1(t), E_2(t), \dots, E_n(t)\}$ is an ONB of $T_{c_1(t)}M$ for all $t \in [0,1]$. In a tubular neighborhood of c_1 we can then define coordinates as follows: p has coordinates $(x^1, \dots, x^n) :\iff p = \exp_{c(x^1)}\left(\sum_{i=2}^n x^i E_i(x^1)\right)$. Now the above four conditions for $c_{1,s}$ prescribe for every $x^i = x^i(c_{1,s}(t)) \equiv x^i(t,s)$ $(i = 1, \dots, n)$ the following constraints:



This means that, for each $i = 1, \dots, n$, we have to find a smooth surface $x^i = x^i(t,s)$ as in the picture, where the s-derivative of x^i is prescribed, for $s = 0$, and the values at $t = 0$ and $t = 1$ are also imposed. This is obviously feasible for $t \in (0,1)$, and the above compatibility checks show that it is also possible at $t = 0$ and at $t = 1$ (since the s-derivatives match up in the 'corners' $(0,0)$ and $(1,0)$). Then we have

$$\left\langle \frac{\partial c_{1,s}}{\partial t}, \frac{\partial c_{1,s}}{\partial t} \right\rangle (t,0) = \langle \dot{c}(t), \dot{c}(t) \rangle \leq 0$$

since c_1 is a causal curve. Furthermore,

$$\begin{aligned}
\frac{\partial}{\partial s}\Big|_0 \left\langle \frac{\partial c_{1,s}}{\partial t}, \frac{\partial c_{1,s}}{\partial t} \right\rangle &\overset{(\nabla 5)}{=} 2\left\langle \frac{\nabla}{\partial s}\Big|_0 \frac{\partial}{\partial t} c_{1,s}, \underbrace{\frac{\partial}{\partial t} c_{1,0}}_{= \dot{c}_1(t)} \right\rangle \\
&\overset{2.1.20,[3]}{=} 2\left\langle \frac{\nabla}{\partial t} \frac{\partial}{\partial s}\Big|_0 c_{1,s}, \underbrace{\frac{\partial}{\partial t} c_{1,0}}_{= \dot{c}_1(t)} \right\rangle \\
&\overset{(3.1.2)}{=} 2\left\langle \frac{\nabla}{dt}(tE(t)), \dot{c}_1(t) \right\rangle \overset{\frac{\nabla}{dt}E \equiv 0}{=} 2\langle E(t), \dot{c}(t) \rangle,
\end{aligned}$$

since E is a parallel vector field (see 1.3.11 in [3]). $2\langle E(t), \dot{c}(t) \rangle < 0$ since $E(t)$ is a future directed timelike vector field (by definition) and $c_1(t)$ is a future directed causal curve. By Taylor's theorem, uniformly in $t \in [0,1]$ and for s small we get that $c_{1,s}$ is timelike. We define

$$c(t) := \begin{cases} c_{1,s}(t), & t \in [0,1] \\ c_2(t+s), & t \in [1, 2-s]. \end{cases}$$

$c(t)$ is piecewise $\mathcal{C}^\infty$ and timelike from p to r.

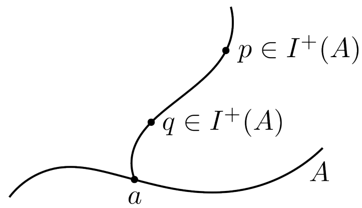
□

Corollary 3.1.9.

1. $I^+(A) = I^+(I^+(A)) = J^+(I^+(A)) = I^+(J^+(A))$.
2. $J^+(J^+(A)) = J^+(A), \forall A \subseteq M$.

Proof.

1. $I^+(A) \subseteq I^+(I^+(A)) \subseteq J^+(I^+(A)) \overset{3.1.8}{\subseteq} I^+(A)$.



Analogously, $I^+(A) \subseteq I^+(I^+(A)) \subseteq I^+(J^+(A)) \overset{3.1.8}{\subseteq} I^+(A)$.

2. $J^+ \subseteq J^+(J^+(A)) \subseteq J^+(A)$, where the first inclusion holds since $A \subseteq J^+(A)$ (by Definition 3.1.5) and the second one by transitivity (see Remark 3.1.7).

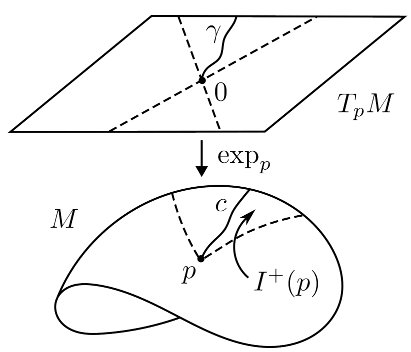
□

Proposition 3.1.10 (Gauss Lemma). Let M be a SRMF and let $p \in M$, $0 \neq x \in \mathcal{D}_p \subseteq T_pM$. Then for any $v_x, w_x \in T_x(T_pM)$ with v_x radial, we have

$$\left\langle (T_x \exp_p)(v_x), (T_x \exp_p)(v_x) \right\rangle = \langle v_x, w_x \rangle .$$

Proof. See the proof of Theorem 2.1.21 in [3]. □

Lemma 3.1.11. Let (M, g) be a spacetime and $p \in M$. Let $\gamma : [0, b] \to T_pM$ be a curve with $\gamma(0) = 0$ such that it is entirely contained in the domain of $\exp_p$. If $c := \exp_p \circ \gamma : [0, b] \to M$ is a future directed timelike curve, then $\gamma(t) \in I^+ \subseteq T_pM$ for all $t \in (0, b]$.



Proof.

- Define a map $q : T_pM \to \mathbb{R}$ where $q(x) = \langle x, x\rangle$. Choose RNC in p. Then

$$q(x) = -(x^0)^2 + \sum_{i=1}^{n}(x^i)^2$$

Let $\frac{\partial}{\partial x^0}$ be future directed. Now

$$d_q = -2x^0 dx^0 + 2\sum_{i=1}^{n} x^i dx^i$$

and so grad$(q) = 2\sum_{i=0}^{n} x^i \partial_i = 2x$, where 'gradient' is referring to the constant scalar product $\langle \cdot, \cdot \rangle = g_p$. Using Gauss Lemma 3.1.10 we obtain

$$\left\langle T_x \exp_p(\text{grad(q)}), T_x \exp_p(\text{grad(q)})\right\rangle = \langle \text{grad}(q(x)), \text{grad}(q(x))\rangle = 4q(x),$$

since grad(q) is radial (cf. [3], after 2.1.20). Set $P(x) := T_x \exp_p(\text{grad}(q))$. Then $x \in I^+ \subseteq T_pM$ implies that $P(x)$ is timelike and future directed. Indeed, $P(x)$ is future directed since

$$\langle P(x), \partial_{x^0}\rangle \overset{3.1.10}{=} \left\langle \text{grad}(q), e^0\right\rangle = 2\left\langle x, e^0\right\rangle < 0.$$

- Suppose that $\underline{\gamma \in \mathcal{C}^\infty}$ (without breaks). We have that $q(\gamma(0)) = q(0) = 0$ and $\dot{c}(0) = (\exp_p \circ \gamma)'(0) = \underbrace{T_0 \exp_p}_{=\text{id}}(\dot{\gamma}(0)) \overset{[3],(2.1.14)}{=} \dot{\gamma}(0)$. Therefore, $\dot{\gamma}(0)$ is future directed and timelike and so there exists $\epsilon > 0$ such that $\gamma(t) \in I^+$ for all $t \in (0, \epsilon)$. Indeed, suppose there exists $t_n \searrow 0$ such that $\gamma(t_n) \notin I^+$. Then $\frac{\gamma(t_n) - \overbrace{\gamma(0)}^{=0}}{t_n} \notin I^+$ and so

$$\dot{\gamma}(0) = \lim_{n\to\infty} \frac{\gamma(t_n) - \gamma(0)}{t_n} \notin I^+ : \lightning$$

To show that $\gamma(t) \in I^+ \subseteq T_pM$ for all $0 < t \leq b$ we first calculate

$$\begin{aligned} \frac{d}{dt} q(\gamma(t)) &= \left\langle \text{grad}(q)\big|_{\gamma(t)}, \dot{\gamma}(t)\right\rangle \\ &\overset{3.1.10}{=} \left\langle T_{\gamma(t)} \exp_p(\text{grad}(q)), T_{\gamma(t)} \exp_p(\dot{\gamma}(t))\right\rangle \\ &= \langle P(\gamma(t)), \dot{c}(t)\rangle \end{aligned}$$

Indirectly, assume that there exists some $t_1 \in (0, b]$ such that $q(\gamma(t_1)) = 0$. Without loss of generality assume t_1 to be minimal. Then $q(\gamma(0)) = q(\gamma(t_1))$ and so, by the mean value theorem, there exists $t_0 \in (0, t_1)$ such that

$$0 = \frac{d}{dt}\bigg|_{t=t_0} q(\gamma(t)) = \langle P(\gamma(t_0)), \dot{c}(t_0)\rangle < 0$$

since $\dot{c}(t)$ is future directed and timelike (by assumption) and $P(\gamma(t_0))$ is (by the previous point) also future directed and timelike (FDTL) for $\gamma(t_0) \in I^+$. But, this is clearly a contradiction.

- Finally, assume that $\underline{\gamma \text{ is piecewise } \mathcal{C}^\infty}$ and let $0 := b_0 < b_1 < \cdots < b_N := b$ be a partition with $\gamma|_{[b_i, b_{i+1}]} \in \mathcal{C}^\infty$. We proceed by induction.

  1. If $N = 1$, we are done (see the second point of this proof).

2. Step from $N-1 \to N$. By induction assumption $\gamma([0,b_{N-1}]) \subseteq I^+$. Then the previous point implies that
$$\frac{d}{dt}q(\gamma(b_{N-1}^+)) = \langle P(\gamma(b_{N-1})), \dot{c}(b_{N-1})\rangle < 0$$
since $\gamma(b_{N-1}) \in I^+$ by induction assumption and $\dot{c}(b_{N-1})$ is FDTL by assumption on c. Therefore, $\dot{\gamma}(b_{N-1}^+)$ is FDTL. As in the previous point, it follows that $\gamma([b_{N-1},b_N]) \subseteq I^+(\gamma(b_{N-1})) \subseteq I^+$.

□

Definition 3.1.12. Let $\Omega \subseteq M$ be open and $A \subseteq \Omega$. Then the relative future of A in Ω is

$$I^+_\Omega(A) := \{q \in \Omega : \exists p \in A \text{ such that } p \ll q\}$$

and the relative past

$$I^-_\Omega(A) := \{q \in \Omega : \exists p \in A \text{ such that } p \gg q\},$$

where $\ll$ and $\gg$ are referring to curves in Ω. We define $J^+_\Omega(A)$ and $J^-_\Omega(A)$ analogously.

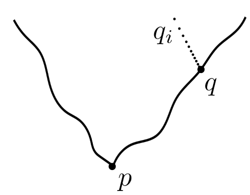
Corollary 3.1.13 (Local Causality). Let (M,g) be a spacetime, $p \in M$, Ω a normal neighborhood of p (with a starshaped set $\tilde{\Omega} \in T_pM$ such that $\exp_p : \tilde{\Omega} \to \Omega$ is a diffeomorphism, cf. 2.1.14 in [3]). Then

1. $I^\pm_\Omega(p) = \exp_p(I^\pm(0) \cap \tilde{\Omega})$.
2. $J^\pm_\Omega(p) = \exp_p(J^\pm(0) \cap \tilde{\Omega})$.

Proof.

1. ($\subseteq$) Let $q \in I^+_\Omega(p)$. Then by definition of $I^+_\Omega(p)$, there exists a FDTL curve c from p to q in Ω. Lemma 3.1.11 now implies that $\gamma := \exp_p^{-1} \circ c \in I^+(0) \cap \tilde{\Omega}$ and so $(\exp_p^{-1} \circ c)(1) = \exp_p^{-1}(q) \in I^+(0) \cap \tilde{\Omega}$. In other words, $q \in \exp_p(I^+(0) \cap \tilde{\Omega})$.
($\supseteq$) Let $x \in I^+(0) \cap \tilde{\Omega}$. Then $t \mapsto tx$ is, for $t \in (0,1]$ a segment in $I^+(0) \cap \tilde{\Omega}$. The Gauss Lemma 3.1.10 implies that $t \mapsto \exp_p(tx)$ is a FDTL geodesic from p to $\exp_p(x)$ in Ω and so $\exp_p(x) \in I^+_\Omega(p)$.

2. ($\supseteq$) Let $x \in J^+(0) \cap \tilde{\Omega}$. Then by applying the Gauss Lemma 3.1.10 we conclude that $t \mapsto \exp_p(tx)$ is a future directed causal geodesic from p to $\exp_p(x)$ in Ω, which implies that $\exp_p \in J^+_\Omega(p)$.
($\subseteq$) Let $q \in J^+_\Omega(p)$. Choose a sequence $(q_i)_{i\in\mathbb{N}}$ such that $q_i \gg q$ and $q_i \to q$ in Ω, for every i.



Now $p \le q \ll q_i$, where $\le$ and $\ll$ are relations in Ω. Applying Proposition 3.1.8, we conclude that $p \ll q_i$ and so $q_i \in I^+_\Omega(p)$. By 1., we get that $\exp_p^{-1}(q_i) \in I^+(0) \cap \tilde{\Omega}$ and so

$$\exp_p^{-1}(q) = \lim_{i\to\infty} \exp_p^{-1}(q_i) \subseteq \overline{I^+(0) \cap \tilde{\Omega}}^{\tilde{\Omega}} = J^+(0) \cap \tilde{\Omega},$$

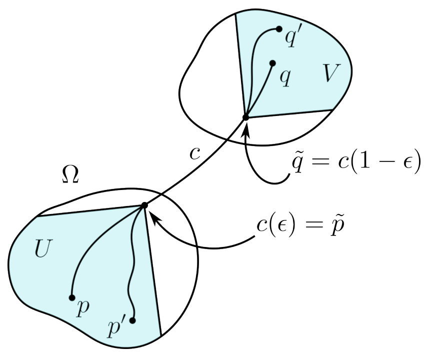
which clearly holds in a Minkowski case.

□

One essential fact in causality theory is that $I^{\pm}(A)$ are always open. In order to prove this, we first show the following proposition.

Proposition 3.1.14. $\ll$ is an open relation i.e. if $p \ll q$ then there exist neighborhoods U of p and V of q such that $p' \ll q'$ for all $p' \in U$ and $q' \in V$.

Proof. If $p \ll q$ we know that there exists a FDTL $c : [0,1] \to M$ such that $c(0) = p$ and $c(1) = q$. Let $\tilde{p} = c(\epsilon)$, where ϵ is so small that there exists a normal neighborhood Ω of $\tilde{p}$ with $p \in \Omega$ (for example, take Ω to be a convex neighborhood of p and ϵ so small that $c(\epsilon) \in \Omega$). Let $U := I^-_{\Omega}(\tilde{p}) \overset{3.1.13}{=} \exp_{\tilde{p}}(I^-(0) \cap \tilde{\Omega})$. Then U is open (since $\exp_p$ is a diffeomorphism) and $p \in U$. Therefore, U is an open neighborhood of p. Now choose V analogously around $\tilde{q} := c(1-\epsilon)$.



Finally, if $p' \in U$ and $q' \in V$ then $p' \ll \tilde{p} \ll \tilde{q} \ll q'$ and so $p' \ll q'$ and we are done. □

Corollary 3.1.15. For all $A \subseteq \Omega$, $I^{\pm}_{\Omega}(A) \subseteq \Omega$ is open.

Proof. From Proposition 3.1.14 we get that $I^{\pm}_{\Omega}(p) \subseteq \Omega$ are open.

$$I^{\pm}_{\Omega}(A) = \bigcup_p I^{\pm}_{\Omega}(p) \subseteq \Omega$$

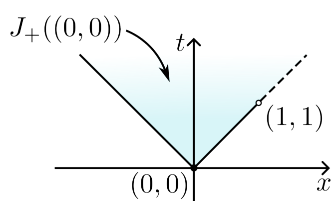
is open as it is a union of open sets. □

In what follows, we'll delve into the properties and interrelationships of sets I^+ and J^+.

Remark 3.1.16.

1. Even for $A \subseteq M$ closed $\not\Longrightarrow$ $I^{\pm}(A)$ is closed. For example, consider $I^{\pm}(p) \in \mathbb{R}^2_1$.

2. Also, $J^{\pm}(A)$ need not be closed. For example, consider $\mathbb{R}^2_1 \setminus \{1, 1\}$.



Proposition 3.1.17. Let $A \subseteq M$, then

1. $I^+(A) = (J^+)^\circ(A)$, where $^\circ$ denotes interior of a set.
2. $J^+(A) \subseteq \overline{I^+(A)}$, with equality if and only if $J^+(A)$ is closed.

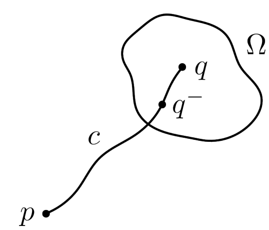
Proof. 1. Since $I^+(A) \subseteq J^+(A)$ is open and so $I^+(A) \subseteq J^+(A)^\circ$.
Conversely, let $p \in J^+(A)^\circ$. Choose $q \in (J^+)^\circ(A) \cap I^-(p) \neq \varnothing$ (because $p \in J^+(A)^\circ$ and $p \in I^-(p)$). Therefore, there exists an $r \in A$ such that $r \leq q \ll p$. Applying Proposition 3.1.8 we immediately get that $r \ll p$. In other words, $p \in I^+(A)$.

2. (a) If Ω is a normal neighborhood of p then $J^+_\Omega(p) = \overline{I^+_\Omega(p)}^\Omega$. This follows from Corollary 3.1.13 and the causality in Minkowski space.

   (b) It is sufficient to prove the claim for a single point i.e. to prove that $J^+(p) \subseteq \overline{I^+(p)}$ since then

$$J^+(A) = \bigcup_{p\in A} J^+(p) \subseteq \bigcup_{p\in A} \underbrace{\overline{I^+(p)}}_{\subseteq \overline{I^+(A)}} \subseteq \overline{I^+(A)}.$$

   In order to prove that $J^+(p) \subseteq \overline{I^+(p)}$ we first note that $p \in \overline{I^+(p)}$. Suppose that $p < q (\in J^+(p))$ i.e. suppose that there exists a future directed causal curve from p to q. Let Ω be a normal neighborhood of q. Choose q^- on c with $q^- \in J^-_\Omega(q)$. Then $q \in J^+_\Omega(q^-)$, where $J^+_\Omega(q^-) \overset{(a)}{=} \overline{I^+_\Omega(q^-)}^\Omega$. But,



$$I^+_\Omega(q^-) \subseteq I^+(J^+(p)) \overset{3.1.8}{=} I^+(p)$$

   and so $q \in \overline{I^+(p)}$.

   (c) The equality case: if $J^+(A) = \overline{I^+(A)}$ holds then $J^+(A)$ is clearly closed. Conversely, if $J^+(A)$ is closed, then

$$\overline{I^+(A)} \subseteq J^+(A) = J^+(A) \subseteq \overline{I^+(A)}.$$

□

In Riemannian geometry, compact manifolds are usually considered friendly objects. However, in Lorentzian geometry, they often serve as useful counterexamples due to the following proposition.

Proposition 3.1.18. A compact spacetime contains a closed timelike curve.

Proof. $\{I^+(p) : p \in M\}$ is an open cover of M. Hence, there exist $p_1, \ldots, p_N$ such that

$$M = \bigcup_{i=1}^N I^+(p_i),$$

where without loss of generality we may assume that

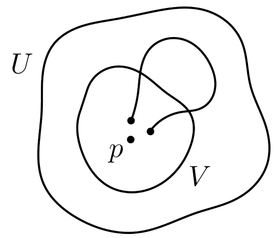
$$I^+(p_i) \notin I^+(p_j), \text{ for } i \neq j \tag{3.1.3}$$

(otherwise remove $I^+(p_i)$). If $p_1 \in I^+(p_i)$ for some $n \geq 2$ then $I^+(p_1) \subseteq I^+(p_1)$, which is a contradiction to (3.1.3). Therefore, $p_1 \in I^+(p_1)$ and so there exists a FDTL curve from p_1 to p_1. □

Now, we establish specific 'causality conditions' designed to prevent such instances.

Definition 3.1.19. A spacetime M is called:

1. **chronological**, if there do not exist closed timelike curves.
2. **causal**, if there do no exist closed causal curves.
3. **strongly causal**, if for every $p \in M$ and for every neighborhood U of p there exists a neighborhood V such that any causal curve starting and ending in V has to remain in U.



Remark 3.1.20.

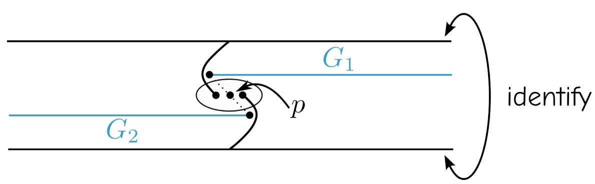
$$\text{strongly causal} \implies \text{causal} \implies \text{chronological}$$

however

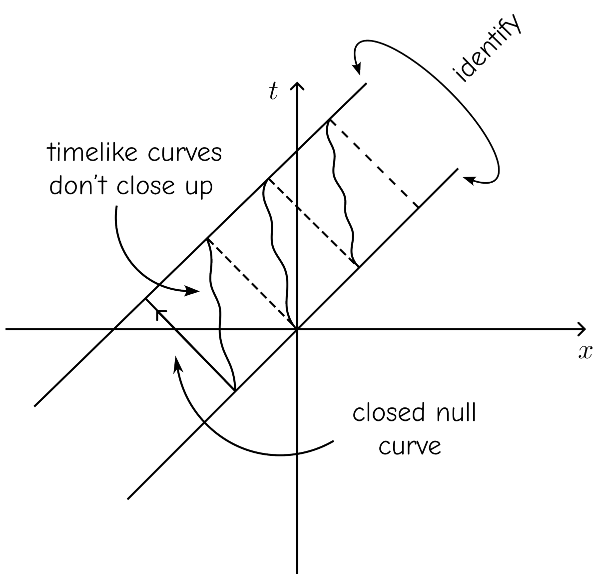
$$\text{strongly causal} \overset{1.}{\not\Longleftarrow} \text{causal} \overset{2.}{\not\Longleftarrow} \text{chronological}.$$

To see this let

1. $M = \left\{\mathbb{R}^2_1/\mathbb{Z}\cdot(1,0)\right\} \setminus (G_1 \cup G_2)$, where $G_1 = \left\{\left(\frac{1}{8}, s\right) : s \geq -\frac{1}{8}\right\}$ and $G_2 = \left\{\left(-\frac{1}{8}, s\right) : s \leq \frac{1}{8}\right\}$.



2. $M = \mathbb{R}^2_1/\mathbb{Z}\cdot(1,-1)$.



We now introduce the analog to the Riemannian distance function (see Section 2.3 in [3]).

Definition 3.1.21. The length of a curve $c : [a,b] \to M$ is

$$L(c) := \int_a^b \sqrt{|\langle \dot{c}(t), \dot{c}(t)\rangle|}dt.$$

Remark 3.1.22. Observe that null curves satisfy $L(c) = 0$.

Definition 3.1.23. For $p, q \in M$ define time separation (or Lorentzian distance) by

$$\tau(p,q) := \begin{cases} \sup\{L(c) : c \text{ FD causal from } p \text{ to } q\}, & \text{if } p < q \\ 0, & \text{if } p \not< q. \end{cases}$$

Remark 3.1.24.

- Observe that $\tau(p,q) = \infty$ is allowed here. For example, in Lorentz cylinder $\tau \equiv \infty$.
- Moreover, note that τ is not symmetric.

Example 3.1.25.

1. Minkowski space. For $p < q$ $\tau(p,q) = \sqrt{|\langle\langle p-q, p-q\rangle\rangle|}$. Indeed,

$$\gamma : t \mapsto qt + (1-t)p, \ (0 \le t \le 1)$$

is a causal future directed curve from p to q and we have

$$\tau(p,q) \ge L(\gamma) = \int_0^1 |\langle\langle q-p, q-p\rangle\rangle|^{\frac{1}{2}} dt = \sqrt{|\langle\langle q-p, q-p\rangle\rangle|}.$$

Conversely, if $p - q$ is timelike, then (using a Poincaré transformation), without loss of generality we can assume that $p = (0,\dots,0)$ and $q = (T, 0, \dots, 0)$. Let c be any future directed causal curve from p to q. Then $\dot{c}^0 > 0$. After reparametrization, $\dot{c}(t) = t$ and so $c(t) = (t, \hat{c}(t))$, with $\hat{c} : [0,T] \to \mathbb{R}^n$.

$$L(c) = \int_0^T \sqrt{|\langle\langle (1,\dot{\hat{c}}), (1,\dot{\hat{c}})\rangle\rangle|dt} = \int_0^T \sqrt{1 - \|\dot{\hat{c}}\|^2} dt \le \int_0^T 1 dt = T = \sqrt{|\langle\langle q-p, q-p\rangle\rangle|}.$$

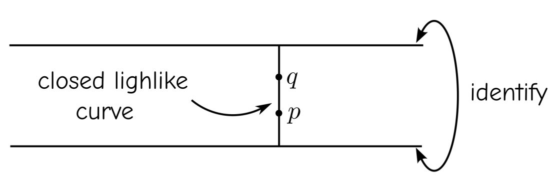
If case $p - q$ is null, without loss of generality let $p = 0$ and $q \in C^+(0)$. Then all causal curves from p to q are null and so

$$\tau(p,q) = 0 = \sqrt{|\langle\langle p-q, p-q\rangle\rangle|}.$$

2. In Lorentz cylinder $\mathbb{R}^2_1 \backslash \mathbb{Z}$, we have

$$\tau(p,q) = \infty$$

for all $p, q \in M$.



Proposition 3.1.26 (Properties of τ). In a spacetime (M, g) we have

1. $\tau(p,q) > 0$ if and only if $p \ll q$.
2. For $p \leq q$ and $q \leq r$ we have the reverse triangle inequality:

$$\tau(p,q) + \tau(q,r) \leq \tau(p,r).$$

3. $\tau : M \times M \to \mathbb{R}$ is lower semicontinuous i.e.

$$\forall p,q \in M \forall \epsilon > 0 \exists U(p), V(q) : \tau(p',q') > \tau(p,q) - \epsilon, \forall p' \in U \forall q' \in V.$$

Proof.

1. ($\leftarrow$) If $p \ll q$ then there exists a FDTL curve c from p to q. But, $L(c) > 0$ and so $\tau(p,q) > 0$.
($\rightarrow$) If $\tau(p,q) > 0$ Then, by definition, there exists a future directed causal curve c from p to q with $L(c) > 0$. c contains a timelike segment ($\int \|\dot{c}(t)\| dt > 0 \implies \exists t_0 \langle \dot{c}(t_0), \dot{c}(t_0) \rangle < 0 \implies \langle c(t), c(t) \rangle < 0$ on some interval). Choose p_1, q_1 on c with $p_1 \ll q_1$. Then $p \leq p_1 \ll q_1 \leq q$ and so $p \ll q$, by Proposition 3.1.8.

2.
   - Let $\tau(p,q) < \infty$ and $\tau(q,r) < \infty$. Let also $\epsilon > 0$. Then there exists a future directed causal curve c_1 from p to q with $L(c_1) \geq \tau(p,q) - \epsilon$ as well as a future directed causal curve c_2 from q to r with $L(c_2) \geq \tau(q,r) - \epsilon$.

$$\tau(p,r) \geq L(c_1 \cup c_2) = L(c_1) + L(c_2) \geq \tau(p,q) + \tau(q,r) - 2\epsilon,$$

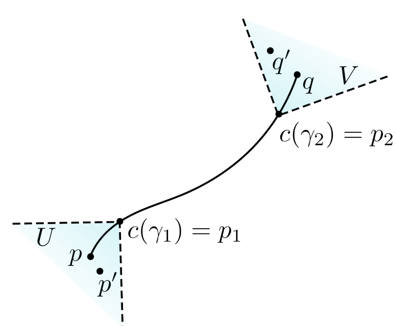
   which proves the result since ϵ may be chosen arbitrarily small.
   - Let $\tau(p,q) = \infty$ or $\tau(q,r) = \infty$. Without loss of generality assume that there exists a future directed causal curve from p to q of arbitrarily great length. Concatenation of this curve with any FD causal curve from q to r results in a curve from p to r with $\tau(p,r) = \infty$.

3.
   - If $\underline{\tau(p,q) = 0}$, there is nothing to show.
   - Let $\underline{0 < \tau(p,q) < \infty}$ and $0 < \epsilon < \frac{\tau(p,q)}{2}$. Choose $c : [0,1] \to M$ FD causal from p to q with $L(c) \geq \tau(p,q) - \frac{\epsilon}{2} \left(> \frac{3}{4}\tau(p,q)\right)$. Also choose $\delta_1 \in (0,1)$ such that

$$L\left(c|_{[0,\delta_1]}\right) < \frac{\epsilon}{4} \left(< \frac{\tau(p,q)}{8}\right) \text{ but greater than } 0$$

   and $\delta_2 \in (0,1)$ such that

$$L\left(c|_{[\delta_2,1]}\right) < \frac{\epsilon}{4} \left(< \frac{\tau(p,q)}{8}\right) \text{ but greater than } 0.$$



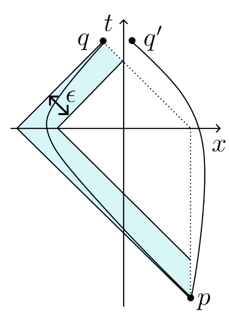
Let $p_1 = c(\delta_1)$ and $p_2 = c(1-\delta_2)$. Then we define $U := I^-(p_1)$ and $V := I^+(p_2)$. $L(c|_{[0,\delta_1]}) > 0$ and so $\tau(p, p_1) > 0$ i.e. $p \ll p_1$ (by 1.). Since $I^-(p_1)$ is open U is a neighborhood of p and, analogously, V is a neighborhood of q. Let $p' \in U$, $q' \in V$. Then we have

$$\begin{aligned}
\tau(p', q') &\overset{2.}{\geq} \tau(p', p_1) + \tau(p_1, p_2) + \tau(p_2, p') \\
&\geq 0 + L\left(c|_{[\delta_1, 1-\delta_2]}\right) + 0 \\
&= L(c) - L\left(c|_{[0,\delta_1]}\right) - L\left(c|_{[1-\delta_2, 1]}\right) \\
&\geq \tau(p,q) - \frac{\epsilon}{2} - \frac{\epsilon}{4} - \frac{\epsilon}{4} = \tau(p,q) - \epsilon.
\end{aligned}$$

- Consider $\tau(p,q) = \infty$. This condition implies the existence of future-directed causal curves from p to q of any length. Using a construction similar to the previous point, we establish neighborhoods around p and q. Within these neighborhoods, all points have arbitrarily large time separations from each other.

□

Remark 3.1.27. In general, τ is not (upper semi-)continuous. To see this let $M := \mathbb{R}^2_1 \setminus \{0 \times [-1, 1]\}$.



All causal curves from p to q have to pass through the 'tunnel' and so they are almost null and $\tau(p,q)$ small. But there are causal curves from p to q' which have greater length.

3.2 Variation of Curves

In this section, we'll explicitly delve into the relationship between geodesics and the largest curves. The primary tool employed is the *variation* of a given curve using a two-parameter map, as previously introduced.

Definition 3.2.1. A curve $c : [a, b] \to M$ is called a **pregodesic** if there exists a ($\mathcal{C}^\infty$-)function $\alpha : [a, b] \to \mathbb{R}$ such that for all $t \in [a, b]$

$$\frac{\nabla}{dt}\dot{c}(t) \equiv \ddot{c}(t) = \alpha(t)\dot{c}(t).$$

(Note that acceleration is colinear with velocity in the above equation.)

Remark 3.2.2 (On Pregeodesics).

1. Any geodesic is a pregeodesic with $\alpha \equiv 0$.

2. Any reparametrization of a geodesic is a pregeodesic. Indeed, assume c is a geodesic. Then for $\tilde{c} = c \circ \phi$ we have
$$\frac{\nabla}{dt}\dot{\tilde{c}}(t) = \frac{\nabla}{dt}((\dot{c} \circ \phi) \cdot \dot{\phi}) = \ddot{\phi} \cdot \dot{c} \circ \phi + \dot{\phi}^2 \underbrace{\frac{\nabla}{dt}\dot{c}}_{\ddot{c}=0} \circ \phi = \frac{\ddot{\phi}}{\dot{\phi}}\dot{\tilde{c}}.$$

3. Conversely, any pregeodesic can be reparametrized as a geodesic. To this end, let $\tilde{c}$ be a pregeodesic with
$$\frac{\nabla}{dt}\dot{\tilde{c}} = \alpha \dot{\tilde{c}}. \tag{3.2.1}$$
Set
$$\phi(t) := \int_a^t e^{\int_a^\tau \alpha(s)ds} d\tau.$$
Then $\dot{\phi} = e^{\int_a^t \alpha(s)ds}$, $\ddot{\phi} = \dot{\phi} \cdot \alpha(t)$ and so
$$\alpha = \frac{\ddot{\phi}}{\dot{\phi}}. \tag{3.2.2}$$
We show that $c = \tilde{c} \circ \phi^{-1}$ is a geodesic. First,
$$\dot{c}(s) = (\tilde{c} \circ \phi^{-1})^{\cdot}(s) = \dot{\tilde{c}} \circ \phi^{-1}(s) \frac{1}{\dot{\phi}(\phi^{-1}(s))}$$
and so
$$\begin{aligned}
\frac{\nabla}{ds}\dot{c} &= \frac{\nabla}{ds}\left(\dot{\tilde{c}} \circ \phi^{-1}(s) \frac{1}{\dot{\phi}(\phi^{-1}(s))}\right) \\
&= \left(\frac{\nabla}{ds}\dot{\tilde{c}}\right) \circ \phi^{-1}(s) \cdot \frac{1}{(\dot{\phi}(\phi^{-1}(s)))^2} + \dot{\tilde{c}} \circ \phi^{-1}(s) \cdot \frac{\overbrace{-\ddot{\phi}(\phi^{-1}(s)) \cdot \frac{1}{\dot{\phi}(\phi^{-1}(s))}}^{=\alpha, \text{ by } (3.2.2)}}{(\dot{\phi}(\phi^{-1}(s)))^2} \\
&\overset{(3.2.1)}{=} (\alpha \cdot \dot{\tilde{c}}) \circ \phi^{-1}(s) \cdot \frac{1}{(\dot{\phi}(\phi^{-1}(s)))^2} - (\alpha \cdot \dot{\tilde{c}}) \circ \phi^{-1}(s) \cdot \frac{1}{(\dot{\phi}(\phi^{-1}(s)))^2} = 0.
\end{aligned}$$

Remark 3.2.3. In the proof of Proposition 3.1.8 we saw that, if $c : [a, b] \to M$ is causal and $c_s : [a, b] \to M$ is a variation of c with $s \in (-\epsilon, \epsilon)$ and variation vector field $X = \frac{\partial c_s}{\partial s}\big|_{s=0}$ with $g\left(\frac{\nabla}{dt}X, \dot{c}\right) < 0$, then c_s is timelike for s small on $t \in [a, b]$. Indeed, recall that
$$\left\langle \frac{\partial c_s}{\partial t}, \frac{\partial c_s}{\partial t} \right\rangle \bigg|_{s=0} = \langle \dot{c}(t), \dot{c}(t) \rangle \le 0,$$
because c is causal. Furthermore,
$$\frac{\partial}{\partial s}\bigg|_0 \left\langle \frac{\partial c_s}{\partial t}, \frac{\partial c_s}{\partial t} \right\rangle = 2 \left\langle \frac{\nabla}{ds}\bigg|_0 \frac{\partial c_s}{\partial t}, \underbrace{\frac{\partial c_s}{\partial t}\bigg|_{s=0}}_{\dot{c}(t)} \right\rangle = 2 \left\langle \frac{\nabla}{dt} \frac{\partial c_s}{\partial s}\bigg|_0, \dot{c}(t) \right\rangle = 2 \left\langle \frac{\nabla}{dt}X, \dot{c} \right\rangle < 0,$$
which implies that $\left\langle \frac{\partial c_s}{\partial t}, \frac{\partial c_s}{\partial t} \right\rangle < 0$ for $s \in (-\epsilon_0, \epsilon_0)$ for all $t \in [a, b]$, where $\epsilon_0 > 0$ small enough.

Lemma 3.2.4 (Deforming Causal Curves Into Timelike Curves)**.** Let $c:[a,b]\to M$ be causal but not a null pregeodesic. Then arbitrarily close to c (in the compact-open topology) there is a timelike curve with the same endpoints.

Let K be a compact set in $[a,b]$ and let $U\subseteq M$ be an open set. Then the *compact-open topology* on $\{c:[a,b]\to M \mid c \text{ continuous}\}$ is generated by the subbase $V(K,U) := \{c:[a,b]\to M \mid c(K)\subseteq U\}$. Since M is metrizable this just amounts to locally uniform convergence.

Proof. Without loss of generality let $[a,b]=[0,1]$ (otherwise reparametrize the curve).

(a) If there exists $t_0\in[0,1]$ such that $\dot c(t_0)$ is timelike then c contains a timelike segment. By the proof of Proposition 3.1.8 there exists a deformation of c into a timelike curve with the same endpoints. Therefore, it suffices to consider the case where c is null everywhere.

(b) Let us first consider the case when c is null, $\mathcal{C}^\infty$, unbroken and is not a pregeodesic. Then

$$g(\dot c,\dot c)=0 \implies 0=\frac{d}{dt}g(\dot c,\dot c)=2g(\dot c,\ddot c)$$

and so $\ddot c(t)\perp\dot c$ for all $t\in[0,1]$. Since $\dot c(t)$ is null, $\dot c(t)^\perp=\mathbb{R}\dot c\oplus E(t)$ in $T_{c(t)}M$ for $E(t)$ spacelike. Therefore, for all t there exist $a(t),b(t)\in\mathbb{R}$ and $e(t)\in E(t)$such that

$$\ddot c(t)=a(t)\dot c(t)+b(t)e(t).$$

If $b(t)=0$ for all t then $\ddot c=a\cdot\dot c$ and so c is a pregeodesic, which contradicts our assumption. Therefore, there exists a t_0 such that $b(t_0)\neq 0$ and so

$$\langle\ddot c,\ddot c\rangle|_{t_0}=a(t_0)^2\underbrace{\langle\dot c,\dot c\rangle}_{=0}|_{t_0}+b(t_0)^2\underbrace{\langle e(t_0),e(t_0)\rangle}_{>0}>0.$$

Therefore,

$$\langle\ddot c,\ddot c\rangle\geq 0 \text{ but not } \equiv 0. \tag{3.2.3}$$

Choose $Y_0\in T_{c(0)}M$ timelike such that $\langle Y_0,\dot c(0)\rangle<0$ (this is possible since $\dot c(0)$ is null). Let Y be the parallel transport of Y_0 along c. Then Y is timelike and

$$\langle Y(t),\dot c(t)\rangle<0 \text{ for all } t. \tag{3.2.4}$$

Now set $X:=\alpha Y+\beta\ddot c$ where $\alpha,\beta\in\mathcal{C}^\infty$ are to be determined so that $\alpha(0)=\beta(0)=\alpha(1)=\beta(1)=0$ and $g(X',\dot c)<0$. Once we get that, we are done because then at a variation c_s of c (as in Remark 3.2.3); just set $c_s(t):=\exp_{c(t)}(sX(t))$. Then

$$\begin{aligned} c_0(t) &= c(t)\\ c_s(0) &= c(0)\\ c_s(1) &= c(1)\end{aligned}$$

and

$$\partial_s|_0 c_s(t)=X(t).$$

We have that $\langle\ddot c,\dot c\rangle=0$ and so

$$0=\frac{d}{dt}\langle\ddot c,\dot c\rangle=\langle\dddot c,\dot c\rangle+\langle\ddot c,\ddot c\rangle. \tag{3.2.5}$$

Now

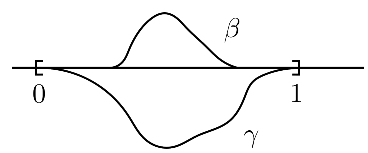
$$\langle X', \dot c\rangle \overset{\dot Y=0}{=} \dot\alpha \langle Y, \dot c\rangle + \dot\beta \underbrace{\langle \ddot c, \dot c\rangle}_{=0} + \beta\langle \ddot c, \dot c\rangle \overset{(3.2.5)}{=} \dot\alpha\langle Y, \dot c\rangle - \beta\langle \ddot c, \ddot c\rangle . \tag{3.2.6}$$

Let

$$\gamma := \frac{\langle \ddot c, \ddot c\rangle}{\langle Y, \dot c\rangle} \overset{(3.2.3),(3.2.4)}{\le} 0 \text{ but not } \equiv 0. \tag{3.2.7}$$

Then there exists a $\mathcal{C}^\infty$-function $\beta : [0,1] \to \mathbb{R}$ such that $\beta(0) = \beta(1) = 0$ and

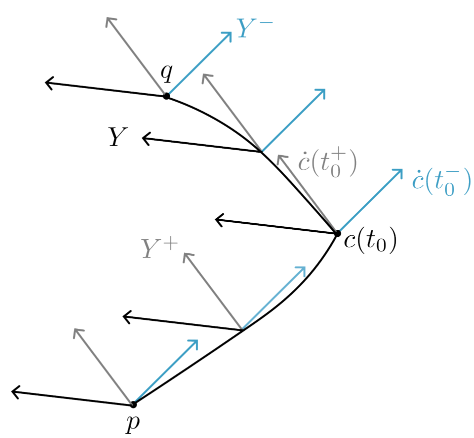
$$\int_0^1 \beta(t)\gamma(t)dt = -1.$$



Finally, set $\alpha(t) := \int_0^t (\beta\gamma + 1)(s)dt$. Then $\alpha(0) = 0 = \alpha(1)$ and we also have from (3.2.6) that

$$\langle X', \dot c\rangle = (\beta\gamma + 1) \underbrace{\langle Y, \dot c\rangle}_{<0, \text{ by } (3.2.4)} - \beta\langle \ddot c, \ddot c\rangle \overset{(3.2.7)}{<} \beta\langle \ddot c, \ddot c\rangle - \beta\langle \ddot c, \ddot c\rangle = 0.$$

(c) Consider c to be piecewise $\mathcal{C}^\infty$ and null. If one of the segments of c is not a pregeodesic then by (b), it can be deformed into a timelike segment and then by (a), c can be deformed into a timelike curve. We can assume, without loss of generality, that all segments are pregeodesics, except for c itself, which is not a null pregeodesic. According to (a), it suffices to consider the scenario where there is just one breakpoint, i.e., there exists $t \in (0,1)$ such that c is $\mathcal{C}^\infty$ on both $[0, t_0]$ and $[t_0, 1]$, where $\dot c(t_0^-)$ and $\dot c(t_0^+)$ are linearly independent (otherwise it wouldn't be a breakpoint). Denote by $Y^\pm$ the parallel transport of $\dot c(t_0^\pm)$ along c.



$c|_{[0,t_0]}$ is a pregeodesic $\implies$ $\dot c \parallel Y^-$ on $[0, t_0]$ and they have the same orientation

$c|_{[t_0,1]}$ is a pregeodesic $\implies$ $\dot c \parallel Y^+$ on $[t_0, 1]$ and they have the same orientation.

Set $Y := Y^+ - Y^-$. Then on $[0, t_0]$

$$\langle Y, \dot c\rangle = \underbrace{\langle Y^+, \dot c\rangle}_{<0} - \langle Y^-, \dot c\rangle < 0, \tag{3.2.8}$$

where $\langle Y^-, \dot c\rangle = 0$ since $Y^- \parallel \dot c$ and $\dot c$ is null. Analogously, on $[t_0, 1]$

$$\langle Y, \dot c\rangle = \langle Y^+, \dot c\rangle - \underbrace{\langle Y^-, \dot c\rangle}_{>0} > 0. \tag{3.2.9}$$

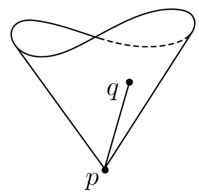
Let $\alpha : [0,1] \to \mathbb{R}$ be continuous and smooth on $[0, t_0]$ and $[t_0, 1]$ with $\alpha(0) = \alpha(1) = 0$, $\alpha' > 0$ on $[0, t_0]$, $\alpha' < 0$ on $[t_0, 1]$. Set $X = \alpha Y$. Then $X(0) = 0 = X(1)$ and

$$\langle X', \dot c\rangle \overset{Y'=0}{=} \alpha' \langle Y, \dot c\rangle \overset{(3.2.8),(3.2.9)}{<} 0$$

from this, as before, the claim follows.

□

Remark 3.2.5.
In general, a null pregeodesic cannot be deformed into a timelike curve with the same endpoints. For example, this holds true in Minkowski space $\mathbb{R}^3_1$.



Lemma 3.2.6. Let c be a null geodesic and let c_s be a variation of c with variation vector field $X \left(= \frac{\partial c_s}{\partial s}\big|_0\right)$ such that $X \perp \dot c$ in the endpoints. If there exists a sequence $s_i \to 0$ with c_{s_i} timelike then $X \perp \dot c$ everywhere.

Proof. Without loss of generality, we can assume that either $s_i > 0$ or $s_i < 0$ for all s_i. Then,

$$\lim_{i\to\infty} \frac{g(\dot c_s, \dot c_s)}{s_i} = \lim_{s_i \to 0} \frac{g(\dot c_{s_i}, \dot c_{s_i}) - \overbrace{g(\dot c, \dot c)}^{=0}}{s_i} = \frac{\partial}{\partial s}\bigg|_0 g(\dot c_s, \dot c_s)$$

and so

$$\frac{\partial}{\partial s} g(\dot c_s, \dot c_s)\bigg|_{s=0} \le 0 \text{ or } \frac{\partial}{\partial s} g(\dot c_s, \dot c_s)\bigg|_{s=0} \ge 0 \text{ for all } t \in [0,1].$$

But by Remark 3.2.3,

$$\frac{\partial}{\partial s} g(\dot c_s, \dot c_s)\bigg|_{s=0} = 2g\left(\frac{\nabla X}{dt}, \dot c\right) \implies g\left(\frac{\nabla X}{dt}, \dot c\right) \ge 0 \text{ or } g\left(\frac{\nabla X}{dt}, \dot c\right) \le 0 \text{ for all } t \in [0,1]. \tag{3.2.10}$$

Now

$$\int_0^1 g\left(\frac{\nabla X}{dt}, \dot c\right) dt \overset{\ddot c = 0}{=} \int_0^1 \left((\frac{d}{dt} g(X, \dot c)) - g(X, \ddot c) \right) dt = g(X, \dot c)\Big|_0^1 = 0,$$

by assumption. Using (3.2.10), for all $t \in [0,1]$

$$g\left(\frac{\nabla X}{dt}, \dot c\right) = \frac{d}{dt} g(X, \dot c),$$

which implies that $g(X, \dot c)$ is constant on $[0,1]$, hence equal to zero. □

Next, we introduce the notion of a *Jacobi field*. These fields adhere to a specific ordinary differential equation (ODE) and, simultaneously, they serve as variation vector fields for geodesic variations. More precisely, we define a variation c_s of c as a *geodesic variation* if all curves of the form $t \mapsto c_s(t)$ are geodesics. We denote by J the corresponding variational vector field, i.e., let

$$J(t) := \left.\frac{\partial c_s(t)}{\partial s}\right|_{s=0}.$$

Then

$$\frac{\nabla^2}{\nabla t^2} J = \frac{\nabla}{\partial t}\frac{\nabla}{\partial t}\left.\frac{\partial c_s}{\partial s}\right|_{s=0} \overset{[3],2.1.20}{=} \frac{\nabla}{\partial t}\frac{\nabla}{\partial s}\left.\frac{\nabla c_s}{\partial t}\right|_{s=0} \overset{[3],3.1.6}{=} \frac{\nabla}{\partial s}\frac{\nabla}{\partial t}\left.\frac{\partial c_s}{\partial t}\right|_{s=0} + R(J,\dot c)\dot c = 0,$$

where the term $\frac{\nabla}{\partial t}\frac{\partial c_s}{\partial t} = 0$ because c_s is a geodesic.

Definition 3.2.7. A vector field along a geodesic c is called a **Jacobi field** (JF) if it satisfies the **Jacobi equation** (JE)

$$\frac{\nabla^2}{dt^2} J + R(\dot c, J)\dot c = 0.$$

Remark 3.2.8 (On the Jacobi Equation)**.**

- The Jacobi equation is a linear ODE of 2^{nd} order. Hence, given $J(0)$ and $\frac{\nabla}{dt}J(0)$ there is a unique global solution to the Jacobi equation with these initial data J along all of c.
- The vector space of Jacobi fields on c has dimension $2n(= \dim(M))$.
- The Jacobi equation is sometimes also called *equation of geodesic deviation.*

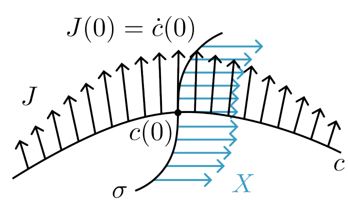
Lemma 3.2.9 (Jacobi Fields and Geodesic Varitions)**.** Let J be a $\mathcal{C}^\infty$-vector field along a geodesic c. These facts are equivalent:

1. J is a Jacobi field.
2. There is a geodesic variation of c with variation vector field J.

Proof.

$(2. \to 1.)$ See above Definition 3.2.7.

$(1. \to 2.)$ Choose any curve σ such that $c(0) = \sigma(0)$, $J(0) = \sigma'(0)$. Choose also $X \in \mathfrak{X}(\sigma)$ with $X(0) = \dot c(0)$ so that $\frac{\nabla X}{ds}(0) = \frac{\nabla}{dt}J(0)$.



Such an X exists. Indeed, let $A, B \in \mathfrak{X}(\sigma)$ be parallel with $A(0) = \dot c(0)$ and $B(0) = J'(0)$ and set $X(s) := A(s) + sB(s)$. Then $X(0) = A(0) = \dot c(0)$ and

$$\frac{\nabla X}{ds}(0) = \underbrace{A'(0)}_{=0} + B(0) + s \cdot 0 = B(0) = J'(0). \checkmark$$

Set $c_s(t) := \exp_{\sigma(s)}(tX(s))$. We show that c_s is a geodesic variation of J.

(a) We have $c_0(t) = \exp_{\sigma(0)}(tX(0)) = \exp_{c(0)}(t\dot c(0)) = c(t)$ and, clearly, all c_s are geodesics.

(b) Set $\tilde{J} := \frac{\partial c_s}{\partial s}\big|_0$ then by $(2. \to 1.)$ $\tilde{J}$ is a Jacobi field along c and we have to show that $\tilde{J} = J$. Hence, by Remark 3.2.8 it suffices to show that $J(0) = \tilde{J}(0)$ and $J'(0) = \tilde{J}(0)$. Indeed, we have

$$\tilde{J}(0) = \frac{d}{ds}\bigg|_{s=0} \exp_{\sigma(s)}(0) = \frac{d}{ds}\sigma(s)\big|_{s=0} = J(0)$$

and

$$\begin{aligned}\tilde{J}'(0) = \frac{\nabla}{dt}\frac{\partial}{\partial s}\exp_{\sigma(s)}(tX(s))\big|_{t=0=s} &\overset{2.1.20,[3]}{=} \frac{\nabla}{\partial s}\frac{\nabla}{\partial t}\exp_{\sigma(s)}(tX(s))\big|_{t=0=s} \\ &\overset{(2.1.14),[3]}{=} \frac{\nabla}{\partial s}T_0\exp_{\sigma(s)}(X(s))\big|_{s=0} \\ &= \frac{\nabla}{\partial s}X(s)\big|_{s=0} = J'(0).\end{aligned}$$

□

Example 3.2.10 (Jacobi Fields).

1. Trivial Jacobi fields. Let c be a geodesic, then

$$J(t) := (at+b)\dot{c}(t)$$

is a Jacobi field along c. Indeed,

$$\frac{\nabla^2}{dt^2}J(t) = \frac{\nabla}{dt}(a\dot{c} + (at+b)\ddot{c}) \overset{\ddot{c}=0}{=} a\ddot{c} = 0$$

and

$$R(\dot{c}, J)\dot{c} = (at+b)\underbrace{R(\dot{c},\dot{c})}_{=0}\dot{c} = 0.$$

Corresponding geodesic variations are:

$$\begin{aligned} c_s(t) := c(bs+t) &\implies \partial s|_0 c_s(t) = b\cdot\dot{c}(t) \\ c_s(t) := c((1+as)t) &\implies \partial_s|_0 c_s(t) = at\cdot\dot{c}(t)\end{aligned}$$

2. In $M = \mathbb{R}^n_n u$ (i.e. flat space with index ν) we have $R = 0$. Hence, the Jacobi equation is $\frac{\nabla^2}{dt^2}J = \frac{d^2}{dt^2}J = 0$ and so the general solution is

$$J(t) = tX(t) + Y(t)$$

with X, Y parallel i.e. X, Y constant. The corresponding geodesic variation is given by

$$c_s(t) = c(t) + s(tX(t) + Y(t)).$$

3. Let M have constant curvature i.e. let $R(X,Y) \overset{1.1.12}{=} \kappa(\langle Z,X\rangle Y - \langle Z,Y\rangle X)$. Let also c be a geodesic with $\eta := g(\dot{c},\dot{c})$ and $X, Y \in \mathfrak{X}(c)$ be parallel vector fields along c which are normal to $\dot{c}$. Let

$$J(t) := s_{\eta\kappa}(t)X(t) + c_{\eta\kappa}(t)Y(t)$$

with

$$s_\delta := \begin{cases}\sin(\sqrt{\delta}t), & \delta > 0 \\ t, & \delta = 0 \\ \sinh(\sqrt{|\delta|}t), & \delta < 0\end{cases}$$

and

$$c_\delta(t) := \begin{cases} \cos(\sqrt{\delta}t), & \delta > 0 \\ 1, & \delta = 0 \\ \cosh(\sqrt{|\delta|(t)}), & \delta < 0 \end{cases}$$

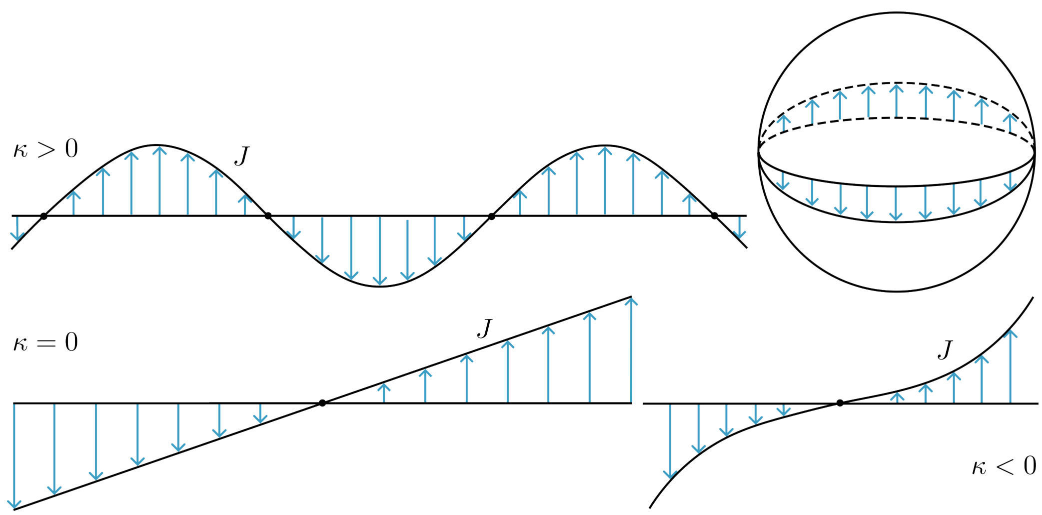
for $\delta \in \mathbb{R}$. Then $s_\delta'' = -\delta s_\delta$ and $c_\delta'' = -\delta c_\delta$. Therefore,

$$\frac{\nabla^2}{dt^2} J = -\eta\kappa \cdot J.$$

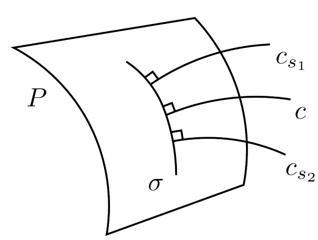
On the other hand,

$$R(\dot{c}, J)\dot{c} = \kappa(g(\dot{c}, \dot{c})J - g(J, \dot{c})\dot{c}) = \kappa\eta J,$$

since $g(\dot{c}, \dot{c}) = \eta$ and $g(J, \dot{c}) = 0$ since $J \perp c$. Therefore, the Jacobi equation holds. We can sketch the three cases:



Next, our focus will be on variations of geodesics that are perpendicular to a SRMF.



Remark 3.2.11 (Reminder on SRSMFs). Let P be a SRSMF of a SRMF M, then for $X, Y \in \mathfrak{X}(P)$, we have

$$\nabla_X^M Y(p) \overset{(1.3.7)}{=} \underbrace{\nabla_X^P Y(p)}_{\in T_pP} + \underbrace{\mathbb{I}(X(p), Y(p))}_{\in N_pP \equiv T_pP^\perp}$$

where the second fundamental form $\mathbb{I}_p : T_pP \times T_pP \to N_pP$ is bilinear, symmetric and given by

$$\mathbb{I}(X, Y) := \text{nor}(\nabla_X^M Y).$$

Similarly, we have $\tilde{\mathbb{I}}_p : T_pP \times N_pP \to T_pP$ (see (1.6.2)) defined via

$$\tilde{\mathbb{I}}(X,\nu) := \tan(\nabla^M_X \nu).$$

Then by Remark 1.6.9, 3.,

$$\left\langle \tilde{\mathbb{I}}_p(X,\nu), Y \right\rangle = -\langle \mathbb{I}_p(X,Y), \nu \rangle,$$

for $\nu \in \mathfrak{X}(P)^\perp$. Hence, for X fixed we have

$$\tilde{\mathbb{I}}_p(X,\cdot) = -(\mathbb{I}_p(X,\cdot))^t : N_pP \to T_pN.$$

Lemma 3.2.12. Let $P \subseteq M$ be a SRSMF. Let c be a geodesic in M with $c(0) = p \in P$ and $\dot{c}(0) \perp P$. Finally, let J be a Jacobi field along c. These facts are equivalent:

1. J is the variational vector field of a geodesic variation c_s of c with $c_s(0) \in P$ and $\dot{c}_s(0) \in N_{c_s(0)}P$ for all s.
2. We have $J(0) \in T_pP$ and $\tan\left(\frac{\nabla^M}{dt} J(0)\right) = \tilde{\mathbb{I}}(J(0), \dot{c}(0))$.

We call any such J a ***P*-Jacobi field**.

Proof.

$(1. \to 2.)$ We have $\sigma(s) := c_s(0) \in P$ and so $J(0) = \frac{\partial}{\partial s}\Big|_0 c_s = \sigma'(0) \in T_pP$. Let $X \in \mathfrak{X}(P)$. Then,

$$\langle \underbrace{X(\sigma(s))}_{\in T_{c(s)P}}, \underbrace{\dot{c}_s(0)}_{\perp P} \rangle = 0$$

for all s. Therefore,

$$\begin{aligned}
0 &= \frac{d}{ds}\Big|_{s=0} \langle X(\sigma(s)), \dot{c}_s(0) \rangle \\
&= \left\langle \frac{\nabla^M}{ds}\Big|_{s=0} X(\sigma(s)), \dot{c}(0) \right\rangle + \left\langle X(p), \frac{\nabla}{\partial s}\frac{\nabla}{\partial t} c_s \Big|_{s=0=t} \right\rangle \\
&\overset{1.6.9,4.}{=} \left\langle \underbrace{\frac{\nabla P}{ds}\Big|_0 X(\sigma(s))}_{\in T_pP} + \mathbb{I}_p(\dot{\sigma}(0), X(p)), \underbrace{\dot{c}}_{\perp P} \right\rangle + \left\langle X(p), \frac{\nabla}{\partial t}\frac{\nabla}{\partial s} c_s \Big|_{s=0=t} \right\rangle \\
&= \langle \mathbb{I}_p(\dot{\sigma}(0), X(p)), \dot{c}(0) \rangle + \left\langle X(p), \frac{\nabla}{\partial t} J(0) \right\rangle \\
&\overset{3.2.11,(3.2.11)}{=} -\left\langle \underbrace{\tilde{\mathbb{I}}_p(J(0), \dot{c}(0))}_{\text{tangential}} - \frac{\nabla}{\partial t} J(0), \underbrace{X(p)}_{\text{tangential}} \right\rangle \\
&= \left\langle \tan\left(\frac{\nabla}{dt} J(0)\right) - \tilde{\mathbb{I}}_p(J(0), \dot{c}(0)), X(p) \right\rangle
\end{aligned}$$

since X was arbitrary,

$$\tan\left(\frac{\nabla}{dt} J(0)\right) = \tilde{\mathbb{I}}_p(J(0), \dot{c}(0)).$$

As in Example 3.2.10 define c_s by

$$c_s(t) := \exp_{\sigma(s)}(tX(s))$$

with

$$\begin{aligned} \sigma(0) &= c(0) = p \in P \\ \dot\sigma(0) &= J(0) \\ X(0) &= \dot c(0) \\ \frac{\nabla}{ds}X(0) &= \frac{\nabla}{dt}J(0) \end{aligned}$$

where, in addition, we need $\sigma : [0,1] \to P$, $X(s) \in N_{\sigma(s)}P$ for all s. Once we have this we are done because then $c_s(0) = \sigma(s) \in P$, $\dot c_s(0) = X(s) \in N_{\sigma(s)}P$ and $\frac{\nabla}{ds}\big|_0 c_s(t) = J(t)$ exactly as in Example 3.2.10.

- To begin with we note that $\sigma(s) \in P$ can be achieved since, by assumption, $T_pP \ni J(0) = \dot\sigma(0)$.
- In order to construct X-transport $\dot c(0)$ normal-parallel along σ (i.e. nor$\frac{\nabla^M U}{ds} = 0$, see Remark 1.6.9, 5.) to obtain $U(0) = \dot c(0)$ and $U(s) \in N_{\sigma(s)}P$ for all s. Let $V(s) \in N_{\sigma(s)}P$ be the normal-parallel transport along σ of nor$\left(\frac{\nabla J}{dt}(0)\right)$, so that $V(s) \in N_{\sigma(s)P}$ for all s. Set
$$X(s) := U(s) + sV(s) \in N_{\sigma(s)}P.$$

Then

- $X(0) = U(0) = \dot c(0)$.
- we have to verify that
$$\frac{\nabla}{ds}X(0) = \frac{\nabla}{dt}J(0).$$
We start with normal components:
$$\text{nor}\left(\frac{\nabla X}{ds}(0)\right) = \text{nor}\left(\frac{\nabla U}{ds} + V(0)\right) = \text{nor}\left(\frac{\nabla J}{dt}(0)\right),$$
where nor$\left(\frac{\nabla U}{ds}\right) = 0$ because U is normal-parallel. Finally, in order to show that
$$\tan\left(\frac{\nabla X}{ds}(0)\right) = \tan\left(\frac{\nabla U}{ds}(0)\right)$$
we calculate
$$\tan\left(\frac{\nabla X}{ds}\right) = \tan\left(\frac{\nabla U}{ds} + V(0)\right) = \frac{\nabla U}{ds}(0), \tag{3.2.11}$$
since $V(0) \perp P$ and nor$\left(\frac{\nabla U}{ds}(0)\right) = 0$ because U is normal parallel. For $Y \in \mathfrak{X}(P)$, we have
$$\begin{aligned} \left\langle \tan\left(\frac{\nabla X}{ds}(0)\right), Y(0)\right\rangle &\overset{(3.2.11)}{=} \left\langle \frac{\nabla U}{ds}(0), Y(0)\right\rangle \\ &= \frac{d}{ds}\Big|_{s=0} \underbrace{\langle \overbrace{U(s)}^{\perp P} \overbrace{Y(s)}^{\parallel P}\rangle}_{=0} - \left\langle \overbrace{U(0)}^{\perp P}, \frac{\nabla Y}{ds}(0)\right\rangle \\ &\overset{1.5.1}{=} -\langle \underbrace{U(0)}_{\dot c(0)}, \mathbb{I}_p(\underbrace{\dot\sigma(0)}_{J(0)}, Y(0))\rangle \\ &= -\langle \underbrace{\dot c(0)}_{\text{normal}}, \mathbb{I}_p(J(0), Y(0))\rangle \\ &\overset{3.2.11,(3.2.11)}{=} \langle \tilde{\mathbb{I}}_p(J(0), \dot c(0)), Y(0)\rangle = \left\langle \tan\frac{\nabla^M}{dt}J(0), Y(0)\right\rangle, \end{aligned}$$
where the last equality holds by assumption. Since Y was arbitrary, we are done.

□

Definition 3.2.13.

1. Let c be a geodesic with $c(0) = p$. We say that $q = c(t)$ is **conjugate** to p of order μ if

$$\mu := \dim\{\text{nontrivial Jacobi fields } J \text{ along } c \text{ with } J(0) = 0 = J(t)\} > 0,$$

where by 'nontrivial' we mean 'not tangential to c'.

2. Let P be a SRSMF, c a geodesic of M with $c(0) = p \in P$ and $\dot{c}(0) \in N_pP$. We say that P has a **focal point** along c at t of order μ if

$$\mu := \dim\{\text{P-Jacobi fields along } c \text{ with } J(t) = 0\} > 0.$$

Recall here the definition of P-Jacobi field from Lemma 3.2.12.

Remark 3.2.14. If J is a JF along a geodesic c, $J(0)\perp\dot{c}(0)$ and $J(t_0)\perp\dot{c}(t_0)$ then $J(t)\perp\dot{c}(t)$ for all t. Indeed, $\langle J, \dot{c}\rangle' = \langle J', \dot{c}\rangle + 0$, where $J' = \frac{\nabla J}{dt}$. Therefore, $\langle J, \dot{c}\rangle'' = \langle J'', \dot{c}\rangle + 0 \overset{\text{JE}}{=} \langle R_{J\dot{c}}, \dot{c}\rangle = 0$ and so there exist $a, b \in \mathbb{R}$ such that $\langle J(t), \dot{c}(t)\rangle = a + tb$. If this function vanishes at two different t-values, it must be identically zero. In particular, if $\dot{c}(0)\perp P$ and J is a P-JF along c, then $J(t)\perp\dot{c}(t)$ for all t ($J(0) \in T_pP$ and so $J(0)\perp\dot{c}(0)$ also, if $J(t_0)$ where t_0 is a focal point, is equal to zero then $J(t_0)\perp\dot{c}(t_0)$).

Remark 3.2.15 (The Size of μ). Let $\dim(M) = n$ and $\dim(P) = m$. We know that $\dim\{J : \text{JF along } c\} = 2n$. If J is a P-JF then $J(0) \in T_pP$ and this reduces the dimension by $n - m$. Also, $\tan(J'(0)) = \tilde{\mathbb{I}}_p(J(0), \dot{c}(0))$ reduces further m dimensions. Therefore,

$$\dim\{\text{P-Jacobi fields along } c\} = 2n - (n - m) - m = n.$$

Since the trivial Jacobi field $J(t) = t\dot{c}(t)$ is a P-JF ($J(0) = 0 \in T_pP$, $J'(t) = \dot{c}(t) \implies J'(0) = \dot{c}(0)$ and $\tan(J'(0)) = 0 = \tilde{\mathbb{I}}_p(J(0), \dot{c}(0))$),

$$\mu \leq n - 1.$$

Example 3.2.16 (Focal Points).

1. Conjugate points can be viewed as focal points for $P = \{p\}$.

2. Let $M = S^n$ and $P = \{p\}$. Let c be a geodesic parametrized by unit speed. Then c has a conjugate point at $t = k\cdot\pi$ ($k \in \mathbb{N}$) of order $\mu = n - 1$. Indeed, let E be parallel along c. Then from Example 3.2.10 (3.), we know that $J(t) = \sin(t)\cdot E(t)$ is a JF along c and $J(0) = 0$, $J(k\pi) = 0$. Since there are $n - 1$ linearly independent E which are perpendicular to c, it follows that $\mu = n - 1$.

3. Let $P = S^n \subseteq \underbrace{\mathbb{R}^{n+1}}_{\text{RMF}} = M$. Let also $p \in S^n$ and $c(t) = (1 - t)p$. Then $\dot{c}(t) = -p$ and so c is a unit speed geodesic emanating orthogonally from P. Let $E \in T_pS^n$ and $E(t)$ parallel vector field along c with $E(0) = E$. By 3.2.10 (2.), we have that $J(t) = (1 - t)E(t)$ is a JF along c. In fact, J is a P-JF ($J(0) = E \in T_pP$) and

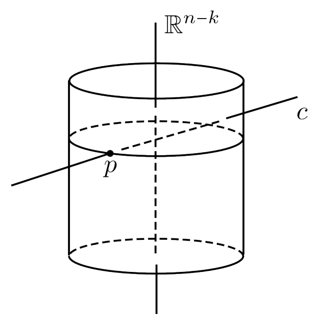
$$\tan\left(\frac{\nabla J}{dt}(0)\right) = \tan(-E) = -E.$$

By Example 1.3.16, $\mathbb{I}(X, Y) = \langle X, Y\rangle\,\dot{c}(0)$ (c points inward) and so

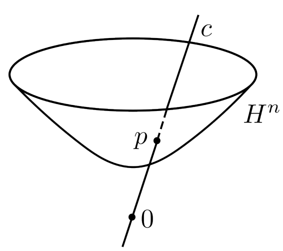
$$\left\langle \tilde{\mathbb{I}}(X, \dot{c}(0)), Y\right\rangle = -\langle \mathbb{I}(X, Y), \dot{c}(0)\rangle = -\langle X, Y\rangle\langle \dot{c}(0), \dot{c}(0)\rangle \overset{\|\dot{c}(t)\|=1}{=} -\langle X, Y\rangle.$$

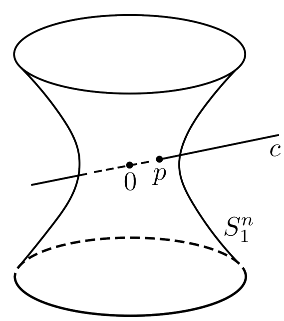
Hence, $\tilde{\mathbb{I}}(X, \dot{c}(0)) = -X$ and so $\tilde{\mathbb{I}}(\underbrace{J(0)}_{=E}, \dot{c}(0)) = -E$. We have that $J(1) = 0 = c(1)$ is a focal point of $P = S^n$. The order of $c(1)$ is n since there exist n linearly independent E's as above.

4. Cylinder. Let $P = S^k \times \mathbb{R}^{n-k} \subseteq \mathbb{R}^{k+1} \times \mathbb{R}^{n-k} = \mathbb{R}^{n+1} = M$. Similarly to 3. $c(t) = ((1-t)p_1, p_2)$ has a focal point at $t = 0$ of order k.



5. De Sitter or hyperbolic space. Similarly, one sees that zero is a focal point for geodesics $c(t) = (1-t)p$. Take $E(t)$ and $J(t)$ as above.





Proposition 3.2.17 (Null Geodesics Are Not Maximizing After Their First Focal Point)**.** Let P be a spacelike SMF of a spacetime M. Let $c : [0, b] \to M$ be a null geodesic with $c(0) = p \in P$ and $\dot{c}(0) \in N_p P$. Set $q := c(b)$. If P has a focal point along c before q i.e. if there exists some $t_0 \in (0, b)$ such that $c(t_0)$ is a focal point then there exists a TL curve from P to q arbitrarily close to c (in the compact-open topology).

Example 3.2.18. Let $M = \mathbb{R}^3_1$ and $P = \{1\} \times S^1 \subseteq \mathbb{R} \times \mathbb{R}^2 = \mathbb{R}^3 = M$. Consider $p := (1, 1, 0)$. Let $c(t) = (1-t)p = \begin{pmatrix} 1-t \\ 1-t \\ 0 \end{pmatrix}$. Then $c(t)$ is a null geodesic since $\langle\langle p, p \rangle\rangle = 0$ and $\dot{c}(0) \perp P$.

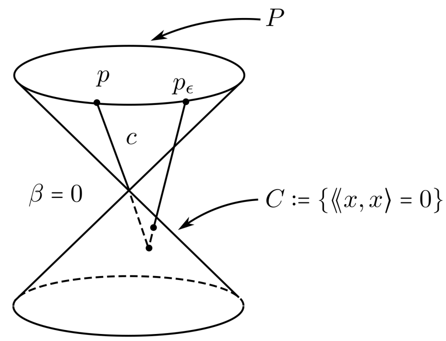
Indeed, write P as

$\{(1, \cos(t), \sin(t)) \; : \; t \in [0, 2\pi]\}$.

Then $T_pP = \langle (0,0,1) \rangle^\top$ and so $p \in N_pP$. Let $b > 1$ and set

$q := c(b) = (1-b, 1-b, 0) =: (\beta, \beta, 0)$.

Then $\beta < 0$. As before, we have that $(0,0,0)^\top$ is a focal point of P. Let also $p_\epsilon := (1, \cos(\epsilon), \sin(\epsilon))$. Then,



$$
\begin{aligned}
\langle\!\langle q - p_\epsilon, q - p_\epsilon \rangle\!\rangle &= \left\langle\!\!\left\langle (\beta - 1, \beta - \cos(\epsilon), -\sin(\epsilon))^\top, (\beta - 1, \beta - \cos(\epsilon), -\sin(\epsilon))^\top \right\rangle\!\!\right\rangle \\
&= -(\beta - 1)^2 + (\beta - \cos(\epsilon))^2 + \sin^2(\epsilon) = 2 \underbrace{\beta}_{<0} \underbrace{(1 - \cos(\epsilon))}_{>0} < 0
\end{aligned}
$$

and so the connecting line

$$
c_\epsilon(t) := p_\epsilon + \frac{t}{b}(q - p_\epsilon) = \begin{pmatrix} 1 - t \\ \cos(\epsilon) + \frac{t}{b}(\beta - \cos(\epsilon)) \\ \sin(\epsilon)\left(1 - \frac{t}{b}\right) \end{pmatrix}
$$

is a timelike geodesic from p_ϵ to q which for $\epsilon \to 0$ converges to c uniformly on $[0, b]$.

In preparation for proving Proposition 3.2.17, we require the following lemma.

Lemma 3.2.19. Let M be a SRMF, $c : [a, b] \to M$ smooth, $Z \in \mathfrak{X}(c)$ and let $P^c_{t,s} : T_{c(t)}M \to T_{c(s)}M$ be parallel transport along c. Then

$$
\frac{d}{dt} P^c_{t,s}(Z(t)) = P^c_{t,s}\left(\frac{\nabla Z}{dt}\right),
$$

with $t \mapsto P^c_{t,s}(Z(t))$ being a $\mathcal{C}^\infty$-curve in the finite-dimensional vector space $T_{c(s)}M$ where we take $\frac{d}{dt}$.

Proof. Let $E_i \in \mathfrak{X}(c)$, for $i = 1, \ldots, n$ be a parallel frame along c. Then we can write $Z(t) = Z^i(t)E_i(t)$, where $Z^i(t)$ are $\mathcal{C}^\infty$-functions and so

$$
P^c_{t,s}(Z(t)) = Z^i(t)E_i(s).
$$

Indeed, $\tau \mapsto Z^i(t)E_i(\tau)$ is parallel and has value $Z(t)$ at $\tau = t$ and so its value at $\tau = s$ is $P^c_{t,s}(Z(t))$. Therefore, $\frac{d}{dt} P^c_{t,s}(Z(t)) = (Z^i)'(t)E_i(s)$. Also, $\frac{\nabla Z}{dt}(t) \overset{(E_i)'=0}{=} (Z^i)'(t)E_i(t)$ and so $P^c_{t,s}\left(\frac{\nabla Z}{dt}\right) = (Z^i)'(t)E_i(t)$. □

Lemma 3.2.20. Let $P \subseteq M$ be a SRSMF of a spacetime, $c : [0, b] \to M$ a geodesic with $c(0) = p \in P$, $\dot{c}(0) \in N_pP$. Then

$$
\mathcal{T} := \{t \in (0, b] : P \text{ has focal point along } c \text{ in } t\}
$$

is compact.

Corollary 3.2.21. In the situation of Lemma 3.2.20 we have the following:

If $\mathcal{T} \neq \varnothing$ then there exists a minimum of $\mathcal{T}$ i.e. there is a first focal point.

Proof. See 3.2.20.

a) Set $\mathcal{V} := \{$P-Jacobi fields along $c\}$, choose any Riemannian metric h on M and define for $J \in \mathcal{V}$

$$\|J\| := \sup_{t\in[0,b]}|J(t)|_h + \sup\left|\frac{\nabla J}{dt}(t)\right|_h.$$

Then $(\mathcal{V}, \|\cdot\|)$ is an n-dimensional (=dim(M)) normed vector space (see Remark 3.2.15) and $\|\cdot\|$ is a norm on $\mathcal{V}$ ($\|J\| = 0 \implies J = 0$).

b) Claim: $\mathcal{T}$ is closed in $(0, b]$. Indeed, let $t_i \in \mathcal{T}$, $t_i \to t \in (0, b]$. We want to show that $t \in \mathcal{T}$. There exists $J_i \in \mathcal{V}$, $J_i \neq 0$ with $J_i(t_i) = 0$ and without loss of generality $\|J_i\| = 1$ (otherwise normalize, if needed). Then the unit sphere in $\mathcal{V}$ is compact and so there exists a convergent subsequence, called J_i where $J_i \to J \in \mathcal{V}$ and $J_i(t_i) = 0$. Then $\|J\| = 1$ and so $J \neq 0$ is a P-JF. Consider parallel transport w.r.t. g, where g is the original metric, along c:

$$P^c_{s,t} : T_{c(s)}M \to T_{c(t)}M.$$

Then,

$$\begin{aligned}
|J(t)|_h &= |J(t) - P^c_{t_i,t}J(t_i) + P^c_{t_i,t}J(t_i) - P^c_{t_i,t}\overbrace{J_i(t_i)}^{=0}|_h \\
&\leq \underbrace{|J(t) - P^c_{t_i,t}\overbrace{J(t_i)}^{\to J(t_i)}|}_{\overset{i\to\infty}{\longrightarrow}} + C\|\underbrace{J(t_i) - J_i(t_i)}_{\to 0}\||_h \to 0,
\end{aligned}$$

where in the last inequality we used continuous dependence on initial conditions of parallel transport. Therefore, $J(t) = 0$ and so $t \in \mathcal{T}$.

c) Claim: There exists $\epsilon > 0$ such that $\mathcal{T} \subseteq [\epsilon, b]$. Assume that there exist $t_i \in \mathcal{T}$ such that $t_i \to 0$. Then as in b), there exist $J_i \in \mathcal{V}$ such that $J_i \to J$ in $\mathcal{V}$ with $J_i(t_i) = 0$, $\|J_i\| = 1 = \|J\|$, $J(0) = \lim_{i\to\infty} J_i(t_i) = 0$. Since J is a P-JF,

$$\tan\left(\frac{\nabla J}{dt}(0)\right) = \tilde{\mathbb{I}}(\underbrace{J(0)}_{=0}, \dot{c}(0)) = 0.$$

We show that also $\mathrm{nor}\left(\frac{\nabla J}{dt}(0)\right) = 0$ because then $\frac{\nabla J}{dt}(0) = 0$. Since $J(0) = 0$, we would get that $J \equiv 0$, which would be a contradiction to $\|J\| = 1$.

$$\left|\mathrm{nor}\left(\frac{P^c_{t_i,0}J(t_i) - J(0)}{t_i - 0}\right)\right|_h \to \left|\mathrm{nor}\left(\frac{\nabla J}{dt}(0)\right)\right|_h,$$

where $\frac{P^c_{t_i,0}J(t_i)-J(0)}{t_i-0}$ converges to $\frac{d}{dt}\Big|_0 P^c_{t,0}(J(t)) \overset{3.2.19}{=} P^c_{t,0}\left(\frac{\nabla J}{dt}(t)\right)\Big|_{t=0} = P^c_{0,0}(J'(0)) = J'(0)$. Furthermore,

$$\begin{aligned}\left|\text{nor}\left(\frac{P^c_{t_i,0}J(t_i)-J(0)}{t_i-0}\right)\right|_h &= \frac{1}{t_i}\left|\text{nor}(P^c_{t_i,0}(J(t_i)-J_i(t_i)))\right|_h\\ &= \frac{1}{t_i}\left|\text{nor}\left[\int_0^{t_i}P^c_{\tau,0}\left(\frac{\nabla J}{d\tau}(\tau)-\frac{\nabla J_i}{d\tau}(\tau)\right)d\tau+J(0)-J_i(0)\right]\right|_h\\ &\le \frac{1}{t_i}\int_0^{t_i}\left|P^c_{\tau,0}\left(\frac{\nabla(J-J_i)}{d\tau}(\tau)\right)\right|_h d\tau\\ &\le \frac{C}{t_i}\int_0^{t_i}\|J-J_i\|d\tau = C\|J-J_i\|\to 0\end{aligned}$$

(for some constant $C>0$), so $\text{nor}\left(\frac{\nabla J}{dt}(0)\right)=0$.

□

Proof. See 3.2.17. Let $t_0\in(0,b)$ be the first focal point of P along c. Let $J\neq 0$ be a P-JF along c with $J(t_0)=0$. Since t_0 is the first focal point, $J(t)\neq 0$ for all $t\in(0,t_0)$.

a) Claim: There exists a $\delta\in(0,b-t_0)$ such that J on $[0,t_0+\delta]$ can be written as $J=f\cdot U$, where U is a spacelike unit vector field along c and $f:[0,t_0+\delta]\to\mathbb{R}$ is $\mathcal{C}^\infty$ and $f>0$ on $(0,t_0)$, $f<0$ on $(t_0,t_0+\delta)$. Indeed, by Remark 3.2.14, since J is a P-JF, $J\perp\dot c$ on $[0,b]$. Since $\dot c$ is null, there could be points where J is proportional to $\dot c$. We show this is not the case. Suppose there exists $t_1\in(0,t_0)$ and that there exists some $\beta\in\mathbb{R}$ such that $J(t_1)=\beta\dot c(t_1)$. Set

$$\tilde J(t) := J(t)-\frac{\beta t}{t_1}\dot c(t).$$

Then $\tilde J$ is a trivial Jacobi field (see Example 3.2.10 (1.)). In fact, $\tilde J$ is even a P-JF:

- $\tilde J(0)=J(0)+0\in T_pP$. ✓
- $\tan\left(\frac{\nabla\tilde J}{dt}(0)\right)\overset{\ddot c=0,\dot c(0)\perp P}{=}\tan\left(\frac{\nabla J}{dt}(0)\right)=\tilde{\mathbb{I}}_p(J(0),\dot c(0))=\tilde{\mathbb{I}}_p(\tilde J(0),\dot c(0))$. ✓

But, $\tilde J(t_1)=0$ and so $c(t_1)$ is a focal point and $t_1<t_0$. ↯ Therefore, J is nowhere proportional to $\dot c$ on $(0,t_0)$ and so J is spacelike on $(0,t_0)$. Let $\{E_i\}_{i=1,\dots,n}$ be a parallel frame field along c and write $J(t)=J^i(t)E_i(t)$. $J(t_0)=0$ and so $J^i(t_0)=0$ for all i, implying that

$$J^i(t)=\underbrace{J^i(t_0)}_{=0}+\int_0^1\frac{d}{ds}J^i(t_0+s(t-t_0))ds=(t-t_0)\underbrace{\int_0^1\frac{dJ^i}{ds}(t_0+s(t-t_0))ds}_{=:Y^i(t)}.$$

Then setting $Y(t):=Y^i(t)E_i(t)\in\mathfrak{X}(c)$ yield

$$J(t)=(t_0-t)Y(t).$$

By the same argument, if $J(0)=0$ then $Y(0)=0$ and so $Y(t)=t\cdot\tilde Y(t)$ Altogether, we can write

$$J(t)=\gamma(t)Y(t),$$

where

$$\gamma(t)=\begin{cases}t(t_0-t), & \text{if } J(0)=0\\ (t_0-t), & \text{if } J(0)\neq 0.\end{cases}$$

This holds on $[0, t_0 + \delta]$ and $Y \in \mathfrak{X}(c)$ implies that Y is spacelike on $(0, t_0)$. If $J(0) \neq 0$ then $Y(0) \neq 0$. If $J(0) = 0$ then $\frac{\nabla J}{dt}(0) \neq 0$ (otherwise $J \equiv 0$ since J is a Jacobi field).

$$J'(t) = (\gamma(t)Y(t))' = (t_0 - 2t)Y + t(t_0 - t)Y' \tag{3.2.12}$$

and so $0 \neq J'(0) = t_0 Y(0)$ implies that $Y(0) \neq 0$. Similarly, $0 \neq J'(t_0) = -t_0 Y(t_0)$ implies $Y(t_0) \neq 0$. Claim: $Y(0)$ and $Y(t_0)$ are spacelike. If $J(0) \neq 0$ then $Y(0) = \frac{1}{\gamma(0)} J(0) \in T_pP$ is spacelike. If $J(0) = 0$ then $\langle J, \dot{c} \rangle = 0$ and so

$$0 = \frac{d}{dt} \langle J, \dot{c} \rangle \overset{\ddot{c}=0}{=} \langle J', \dot{c} \rangle \implies \frac{\nabla J}{dt} \perp \dot{c}.$$

We now show that $\frac{\nabla J}{dt}$ is not proportional to $\dot{c}$ at $t = 0$. Suppose $Y(0) = \beta \dot{c}(0)$. Then,

$$\frac{\nabla J}{dt}(0) \overset{(3.2.12)}{=} t_0 Y(0) = t_0 \beta \dot{c}(0)$$

and so $J(t) = t \cdot t_0 \beta \dot{c}(t)$. But then $J(t_0) = t_0^2 \beta \dot{c}(t_0) \neq 0$. ↯ ($t_0$ is a focal point.) Therefore, $Y(0)$ and hence also $J'(0)$ is spacelike. We now show that also $J'(t_0)$ is spacelike: $J' \perp \dot{c}$ and if we suppose that $Y(t_0)$ is proportional to $\dot{c}(t_0)$, $Y(t_0) = \beta \dot{c}(t_0)$. Then

$$J'(t) = \begin{Bmatrix} t_0 - 2t \\ -1 \end{Bmatrix} \cdot Y(t) + \begin{Bmatrix} t(t_0 - t) \\ t_0 - t \end{Bmatrix} \cdot Y'(t) \quad \begin{matrix} \longleftarrow \text{if } J(0) = 0 \\ \longleftarrow \text{if } J(0) \neq 0 \end{matrix}$$

and

$$J'(t_0) = \begin{Bmatrix} -t_0 \\ -1 \end{Bmatrix} \cdot Y(t_0) = \begin{Bmatrix} -t_0 \\ -1 \end{Bmatrix} \cdot \beta \dot{c}(t_0).$$

Just like before, we obtain

$$J(t) = (t_0 - t) \cdot \begin{Bmatrix} t_0 \\ -1 \end{Bmatrix} \cdot \beta \dot{c}(t_0)$$

but this contradicts, in the first case, $J(0) = 0$ and, in the second, $J(0)$ being spacelike if $J(0) \neq 0$ (since the vector we get is null). Therefore, $Y(t_0)$ is spacelike. Finally, Y is spacelike on $[0, t_0]$ and, by continuity, Y is spacelike even on $[0, t_0]$ for some small $\delta > 0$. Set $U := \underbrace{\frac{Y}{|Y|}}_{\langle U, U \rangle = 1}$ (since Y is a spacelike vector field, $|Y|$ is never zero) and

$$f := \gamma \cdot |Y|.$$

$f \cdot U = \gamma \cdot Y = J$ and so f is $\mathcal{C}^\infty$ and $f > 0$ on $(0, t_0)$ and $f < 0$ on $(t_0, t_0 + \delta)$.

b) Claim: There exists a $\delta \in (0, b - t_0)$ and $V \in \mathfrak{X}(c)$ such that $V(0) = J(0)$, $V(t_0 + \delta) = 0$, $V \perp \dot{c}$ on $[0, t_0 + \delta]$ and $\left\langle \frac{\nabla^2 V}{dt^2} + R(\dot{c}, V)\dot{c}, V \right\rangle > 0$ on $(0, t_0 + \delta)$. To this end, let $\delta > 0$ as in a) and take the following ansatz

$$V := (f + g) \cdot U = J + g \cdot U \tag{3.2.13}$$

with g to be determined. Then $V' = J' + g'U + gU'$ and $V'' = J'' + g''U + 2g'U' + gU''$. Therefore,

$$\begin{aligned} \frac{\nabla^2 V}{dt^2} + R(\dot{c}, V)\dot{c} &= \cancel{\frac{\nabla^2 J}{dt^2}} + g''U + 2g'U' + gU'' + \cancel{R(\dot{c}, J)\dot{c}} + gR(\dot{c}, U)\dot{c} \\ &\overset{J \text{ is a JF}}{=} g''U + 2g'U' + g\left(\frac{\nabla^2 U}{dt^2} + R(\dot{c}, U)\dot{c}\right) \end{aligned}$$

and so

$$\left\langle \frac{\nabla^2 V}{dt^2} + R(\dot c, U)\dot c, V \right\rangle \overset{(3.2.13)}{=} (f+g)(g'' \underbrace{\langle U,U\rangle}_{=1} + 2g' \underbrace{\langle U',U\rangle}_{=0} + g \underbrace{\langle U'' + R(\dot c, U)\dot c, U\rangle}_{:=l})$$
$$= (f+g)(g'' + g\cdot l).$$

Choose $a>0$ such that $l \ge -a^2$ on $[0, t_0+\delta]$ and set

$$g(t) := \tilde b(e^{at} - 1)$$

with $\tilde b > 0$ such that $g(t_0+\delta) = -f(t_0+\delta)$. This is possible since $f(t_0+\delta) < 0$. Then $V(t_0+\delta) \overset{(3.2.13)}{=} 0$ and $V(0) \overset{g(0)=0}{=} J(0)$. $(f+g) > 0$ on $(0, t_0]$ (since $g>0$ everywhere and $f>0$ on $(0,t_0)$) and $(f+g)(t_0+\gamma) = 0$. Without loss of generality, $t_0+\delta$ is the first zero of $f+g$. Then on $(0, t_0+\delta)$,

$$\left\langle \frac{\nabla^2 V}{dt^2} + R(\dot c, V)\dot c, V \right\rangle = (f+g)(g'' + g\cdot l),$$

since $V \perp \dot c$ ($J = f\cdot U$, $U\perp \dot c$ where $f \neq 0$ i.e. on $(0,t_0)\cup(t_0, t_0+\delta)$ and, by continuity, on $[0, t_0+\delta]$). Moreover, $(f+g)(g''+gl) > 0$ on $(0, t_0+\delta)$ because $g'' + gl = a^2 g + a^2\tilde b + gl \ge a^2 \tilde b > 0$.

c) Claim: There exists $A \in \mathfrak{X}(c)$ with $A(0) = \mathbb{I}(V(0), V(0))$, $A(t_0+\delta) = 0$ and

$$-\langle V'' - R_{V\dot c}\dot c, V\rangle + (\langle V, V'\rangle + \langle A, \dot c\rangle)' < 0$$

on $[0, t_0+\delta]$. We have

$$\langle \mathbb{I}(J(0), J(0)), \dot c(0)\rangle \overset{3.2.11}{=} -\left\langle \tilde{\mathbb{I}}(J(0), \dot c(0)), J(0)\right\rangle \overset{\text{J P-JF}}{=} -\langle \tan(J'(0)), \underbrace{J(0)}_{\text{tangential}} \rangle = \langle J'(0), J(0)\rangle \tag{3.2.14}$$

and $\langle V, V'\rangle = \left\langle J + \tilde b(e^{at}-1)U, J' + a\tilde b\cdot e^{at}U + \tilde b(e^{at}-1)U'\right\rangle$. Now, $\langle J, U'\rangle = f\underbrace{\langle U, U'\rangle}_{=0} = 0$ and so

$$\left\langle J + \tilde b(e^{at}-1)U, J' + a\tilde b\cdot e^{at}U + \tilde b(e^{at}-1)U'\right\rangle = \langle J, J'\rangle + \left\langle V, a\tilde b\cdot e^{at}U\right\rangle + \tilde b(e^{at}-1)\langle U, J'\rangle.$$

For $t=0$,

$$\langle V, V'\rangle(0) = \langle J, J'\rangle(0) + a\tilde b\langle \underbrace{V(0)}_{\overset{b)}{=} J(0)}, U(0)\rangle = \langle J, J'\rangle(0) + a\tilde b\cdot f(0) = \langle J, J'\rangle(0) + a\tilde b\cdot |J(0)|. \tag{3.2.15}$$

We distinguish the following three cases:

i) Case: $\langle \mathbb{I}(J(0), J(0)), \dot c(0)\rangle \neq 0$. Write

$$\mathbb{I}(J(0), J(0)) = \alpha X_0,$$

where $X_0 \in N_p P$ with $\langle X_0, \dot c(0)\rangle = -1$. Then

$$-\alpha = \langle \alpha X_0, \dot c(0)\rangle = \langle \mathbb{I}(J(0), J(0)), \dot c(0)\rangle \overset{(3.2.14)}{=} -\langle J'(0), J(0)\rangle$$

and so $\alpha = \langle J'(0), J(0)\rangle$. Let $X \in \mathfrak{X}(c)$ be parallel with $X(0) = X_0$. Then $\langle X(t), \dot c(t)\rangle = -1$ for all t (since $\langle X_0, \dot c(0)\rangle = -1$ and parallel transport is an isometry). Set

$$A(t) := \left(\langle V(t), V'(t)\rangle + \frac{a\tilde b \cdot |J(0)|}{t_0+\delta}(t - t_0 + \delta)\right) \cdot X(t).$$

Then

$$A(0) \overset{(3.2.15)}{=} (\underbrace{\langle J(0), J'(0)\rangle}_{=\alpha} + \cancel{a\tilde b \cdot |J(0)|} - \cancel{a\tilde b \cdot |J(0)|}) \cdot X(0) = \alpha X_0 = \mathbb{I}(\underbrace{J(0)}_{V(0)}, J(0)) \tag{3.2.16}$$

and

$$A(t_0+\delta) = (\langle V(t_0+\delta), V'(t_0+\delta)\rangle + 0)\, X(t_0+\delta) = 0,$$

since $V(t_0+\delta) = 0$. Also,

$$\begin{aligned}(VV' + \langle A, \dot c\rangle)' &\overset{\langle X,\dot c\rangle=-1}{=} (\cancel{\langle V, V'\rangle} - \cancel{\langle V, V'\rangle} - \frac{a\tilde b \cdot |J(0)|}{t_0+\delta}(t - t_0 - \delta))' \\ &= -\frac{a\tilde b |J(0)|}{t_0+\delta} < 0\end{aligned}$$

on $[0, t_0+\delta]$. This proves c) since, by b), $-\langle V'' - R_{V\dot c}\dot c, V\rangle \le 0$ on $[0, t_0+\delta]$.

ii) Case: $\langle \mathbb{I}(J(0), J(0)), \dot c(0)\rangle = 0$ and $J(0) \neq 0$. Choose $X \in \mathfrak{X}(c)$ parallel such that $\langle X, \dot c\rangle = -1$ and choose $X \in \mathfrak{X}(c)$ parallel such that $Z(0) = \mathbb{I}(J(0), J(0))$. Then by assumption, $\langle Z, \dot c\rangle \equiv 0$. Set

$$A(t) := \left(\langle V(t), V'(t)\rangle + \frac{a\tilde b \cdot |J(0)|}{t_0+\delta}(t - t_0 - \delta)\right) \cdot X(t) + \left(1 - \frac{t}{t_0+\delta}\right) \cdot Z(t).$$

Then,

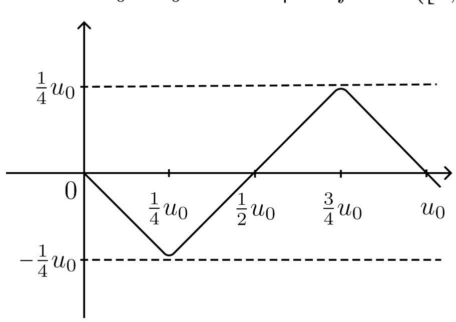
$$\begin{aligned} A(0) &\overset{(3.2.16)}{=} \langle J(0), J'(0)\rangle X(0) + Z(0) \\ &\overset{(3.2.14)}{=} -\underbrace{\langle \mathbb{I}(J(0), J'(0)), \dot c(0)\rangle}_{=0,\ \text{by assumption of ii)}} X(0) + \mathbb{I}(J(0), J(0)) \\ &= \mathbb{I}(J(0), J(0)), \\ A(t_0+\delta) &= 0 + 0 \cdot Z(t_0+\delta) = 0 \end{aligned}$$

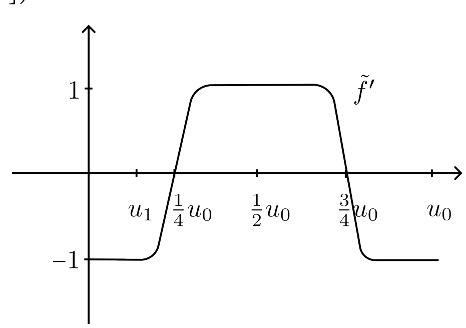
and

$$(\langle V, V'\rangle + \langle A, \dot c\rangle)' \overset{\text{i)}}{=} -\frac{a\tilde b \cdot |J(0)|}{t_0+\delta} < 0,$$

which proves the claim.

iii) $J(0) = 0$. Let $u_0 := t_0 + \delta$ and pick $f \in \mathcal{C}^\infty([0, t_0+\delta])$.





Pick u_1 and u_2 in $(0, t_0+\delta)$ such that $\tilde{f} \equiv -1$ on $[0, u_1]$ and on $[u_2, \underbrace{t_0+\delta}_{=u_0}]$. Let $\epsilon > 0$ such that

$$\epsilon < \min_{t\in[u_1,u_2]} \langle V'' - R_{V\dot{c}}\dot{c}, V\rangle (t),$$

which is possible by b). Now set

$$f := -\epsilon \cdot \tilde{f}$$

and define X as in ii) and set

$$A(t) := (\langle V, V'\rangle + f(t))X(t).$$

Then

$$A(0) = (\langle \underbrace{J(0)}_{=0} J'(0)\rangle + a\tilde{b}\cdot |\underbrace{J(0)}_{=0}| + \underbrace{f(0)}_{=0}) = 0 = \mathbb{I}(\underbrace{V(0)}_{=J(0)=0}, V(0)),$$

$$A(t_0+\delta) = (0 + \underbrace{f(t_0+\delta)}_{=0})X(t_0+\delta) = 0$$

and

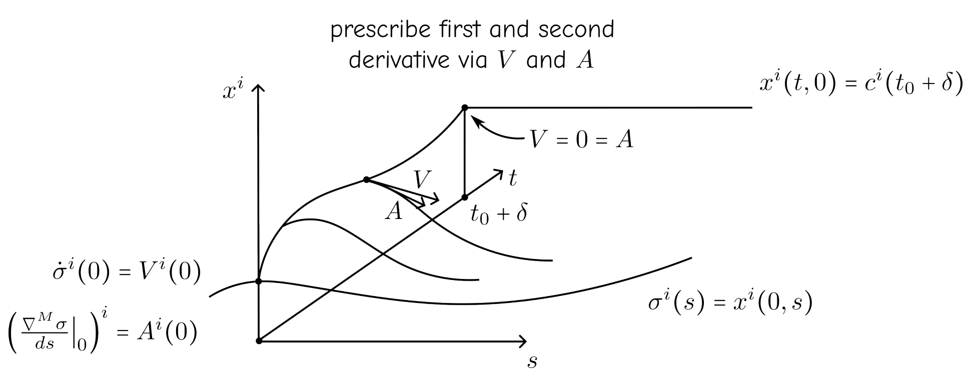
$$(\langle V, V'\rangle + \langle A, \dot{c}\rangle)' \overset{\langle X,\dot{c}\rangle=-1}{=} \epsilon \cdot \tilde{f}'.$$

Therefore,

$$\underbrace{-\langle V'' - R_{V\dot{c}}\dot{c}, V\rangle}_{\geq 0 \text{ on } [t_0, t_0+\delta]} + \underbrace{(\langle V, V'\rangle + \langle A, \dot{c}\rangle)'}_{=\epsilon\cdot\tilde{f}'} = \begin{cases} \leq \epsilon\cdot\tilde{f} = -\epsilon, & \text{on } [0, u_1] \\ < 0, & \text{on } [u_1, u_2], \text{ by definition of } \epsilon \\ \leq \epsilon\cdot\tilde{f} = -\epsilon, & \text{on } [u_2, u_0]. \end{cases}$$

Hence, $-\langle V'' - R_{V\dot{c}}\dot{c}, V\rangle + (\langle V, V'\rangle + \langle A, \dot{c}\rangle)' < 0$ on $[0, t_0+\delta]$.

d) We now construct a variation c_s of $c|_{[0,t_0+\delta]}$ such that $\dot{c}_s$ is timelike for $0 < |s|$ small. Set $\sigma(s) := \exp_p^P(sJ(0))$ where $\sigma : (-\epsilon, \epsilon) \to P$ is smooth, $\sigma(0) = 0$, $\dot{\sigma}(0) = J(0) \overset{b)}{=} V(0)$, $\frac{\nabla^P\dot{\sigma}}{ds}(0) = 0$ (in fact, $\frac{\nabla^P\sigma}{ds} \equiv 0$ since σ is a P-geodesic). Choose a variation c_s of c with $c_s(0) = \sigma(s)$, $c_s(t_0+\delta) = c(t_0+\delta) =: \tilde{q}$, $\frac{\partial c_s}{\partial s}\big|_0 = V$ (V is a transversal velocity) and $\frac{\nabla}{ds}\frac{\partial c_s}{\partial t}\big|_{s=0} = A$ (A is a transversal acceleration). Why is this possible? As in the proof of Proposition 3.1.8, pick Fermi-coordinates $x^1, \ldots, x^n$ along c and find for each $i = 1, \ldots, n$ a surface (depending on (t,s)) $x^i(t,s) = x^i \circ c_s(t)$.



By Proposition 1.5.1,

$$\left.\frac{\nabla^M \sigma}{ds}\right|_0 = \underbrace{\left.\frac{\nabla^P \dot\sigma}{ds}\right|_{s=0}}_{=0} + \mathbb{I}(\underbrace{\dot\sigma(0)}_{=J(0)}, \dot\sigma(0)) = \mathbb{I}(J(0), J(0)) = A(0).$$

Let $f : [0, t_0 + \delta] \times [-\epsilon_0, \epsilon_0] \to \mathbb{R}$ such that

$$f(t, s) := \langle \dot c_s(t), \dot c_s(t) \rangle .$$

By Taylor expansion in the s-variable we get

$$f(t, s) = f(t, 0) + s\frac{\partial f}{\partial s}(t, 0) + \frac{1}{2}s^2 \frac{\partial^2 f}{\partial s^2}(t, \theta \cdot s),$$

where $\theta = \theta(t, s) \in (0, 1)$. Here $f(t, 0) = \langle \dot c(0), \dot c(0)\rangle = 0$ (because c is null), $\left.\frac{\partial f}{\partial s}\right|_0 = 2\left\langle \left.\frac{\nabla}{\partial s}\right|_0 \frac{\partial c_s}{\partial t}, \dot c\right\rangle = 2\left\langle \frac{\nabla}{\partial t} \underbrace{\left.\frac{\partial c_s}{\partial s}\right|_{s=0}}_{=V} \dot c\right\rangle = 2\frac{d}{dt}\underbrace{\langle V, \dot c\rangle}_{\overset{b)}{=}0} - 2\langle V, \underbrace{\ddot c}_{=0}\rangle$ and

$$\begin{aligned}
\frac{1}{2}\left.\frac{\partial^2}{\partial s}\right|_0 f(t,s) = \left.\frac{\partial}{\partial s}\right|_0 \left\langle \frac{\nabla}{\partial s}\dot c_s, \dot c_s\right\rangle &= \left.\frac{\partial}{\partial s}\right|_0 \left\langle \frac{\nabla}{\partial t}\frac{\partial c_s}{\partial s}, \dot c_s\right\rangle \\
&= \left\langle \frac{\nabla V}{\partial t}, \frac{\nabla V}{\partial t}\right\rangle + \left\langle \left.\frac{\nabla}{\partial s}\frac{\nabla}{\partial t}\frac{\partial c_s}{\partial s}\right|_{s=0}, \dot c\right\rangle \\
&\overset{[3],3.1.6}{=} \left\langle \frac{\nabla V}{\partial t}, \frac{\nabla V}{\partial t}\right\rangle + \left\langle \frac{\nabla}{\partial t}\underbrace{\left.\frac{\nabla}{\partial s}\frac{\partial c_s}{\partial s}\right|_0}_{=A} + R(\dot c, V)V, \dot c\right\rangle \\
&= \left\langle \frac{\nabla V}{dt}, \frac{\nabla V}{dt}\right\rangle + \underbrace{\left\langle \frac{\nabla}{dt}A, \dot c\right\rangle}_{\overset{\ddot c = 0}{=}\langle A, \dot c\rangle'} - \langle R(\dot c, V)\dot c, V\rangle \\
&= (\langle V, V'\rangle + \langle A, \dot c\rangle)' - (\langle V'' - R(V, \dot c)\dot c, V\rangle) < 0,
\end{aligned}$$

on $[0, t_0 + \delta]$, by c). Since $[0, t_0 + \delta]$ is compact, there exists $\epsilon_0 > 0$ such that $\frac{\partial^2 f}{\partial s^2}(t, \theta \cdot s) < 0$ on $[0, t_0 + \delta] \times [-\epsilon_0, \epsilon_0]$ and so for $s \in [-\epsilon_0, \epsilon_0] \setminus \{0\}$, c_s is timelike. Finally,

$$\underbrace{c_s|_{[0,t_0+\delta]}}_{\text{timelike}} \cup \underbrace{c|_{[t_0+\delta,b]}}_{\text{null}}$$

can be 'pushed up' to a timelike curve from P to q arbitrarily near to $c_s \cup c$. Hence, altogether, there exists a timelike curve from P to q arbitrarily near c.

□

Lemma 3.2.22. Let $P \subseteq M$ be a spacelike SMF, $c : [0, b] \to M$ a null geodesic with $p := c(0) \in P$ but $\dot c(0) \notin N_p P$. Then arbitrarily close to c there exists a timelike curve from P to $q = c(b)$.

Proof. Since $\dot{c}(0) \notin N_pP$, there exists an $X \in T_pP$ such that $\langle X, \dot{c}(0)\rangle \neq 0$, without loss of generality let $\langle X, \dot{c}(0)\rangle > 0$. Let $X \in \mathfrak{X}(c)$ be the parallel transport of $X = X(0)$ along c and set

$$V(t) := \left(1 - \frac{t}{b}\right) X(t).$$

Then, $V(0) = X$ and $V(b) = 0$. Now construct a variation c_s of c with $\frac{\partial c_s}{\partial s}\big|_0 = V$, $c_s(0) \in P$ and $c_s(b) = q$ for all s. (to see that this is possible use Fermi coordinates). $\langle \dot{c}_s, \dot{c}_s\rangle|_{s=0} = \langle \dot{c}, \dot{c}\rangle = 0$ (c is null) and

$$\frac{\partial}{\partial s}\langle \dot{c}_s(t), \dot{c}_s(t)\rangle|_{s=0} = 2\left\langle \frac{\nabla V}{dt}, \dot{c}(t)\right\rangle = -\frac{2}{b}\langle X(t), \dot{c}(t)\rangle = -\frac{2}{b}\underbrace{\langle X, \dot{c}(0)\rangle}_{>0} \overset{\text{Taylor of } 1^{\text{st}} \text{ order}}{\Longrightarrow} \langle \dot{c}_s(t), \dot{c}_s(t)\rangle < 0$$

for $0 < s$ small and so c_s is timelike for such s. □

Combining Lemma 3.2.4, Lemma 3.2.20 and Lemma 3.2.22 yields the following theorem.

Theorem 3.2.23. Let M be a spacetime, $P \subseteq M$ spacelike SMF, $c : [0, b] \to M$ a causal curve with $p = c(0) \in P$. Then arbitrarily close to c there is a timelike curve from P to $q := c(b)$ unless c is (up to reparametrization) a null geodesic with $\dot{c}(0) \in N_pP$ without a focal point before b.

3.3 Convex Sets

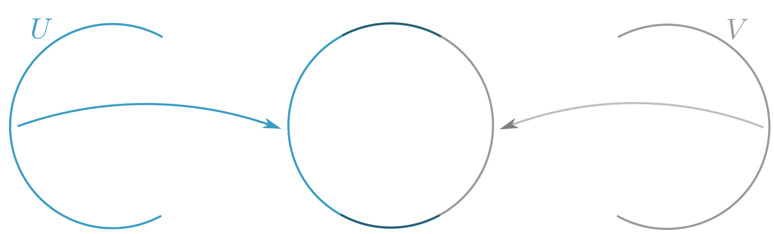
Definition 3.3.1 (Cf. [3], 2.2.4)**.** An open subset $U \subseteq M$ is called convex if it is a normal neighborhood of each of its points i.e. for every $p \in U$ there exists $\tilde{U} \subseteq T_pM$ starshaped and open such that $\exp_p : \tilde{U} \to U$ is a diffeomorphism.

Remark 3.3.2. If U is convex and $p, q \in U$ then there exists a unique geodesic in U from p to q (cf. [3], 2.1.15).

Proposition 3.3.3 (Existence of Convex Sets)**.** Every point p in a SRMF M possesses a basis of neighborhoods consisting of convex sets.

Proof. Cf. [3], 2.2.7. □

Remark 3.3.4. If $U, V \subseteq M$ are convex, then $U \cap V$ need not be convex.



Lemma 3.3.5 (Intersection of Convex Sets)**.** Let $C_1, C_2 \subseteq M$ be convex and suppose C_1 and C_2 are contained in a convex set $D \subseteq M$. Then the intersection $C_1 \cap C_2$ is convex.

Proof. Proof. Cf. [3], 2.2.11. □

Definition 3.3.6. An open covering $\mathcal{U} = \{U_\alpha\}_\alpha$ of a SRMF M is called a convex cover if

$$U_{\alpha_1} \cap \cdots \cap U_{\alpha_n} \text{ is convex } \forall \alpha_j, \forall n \in \mathbb{N}.$$

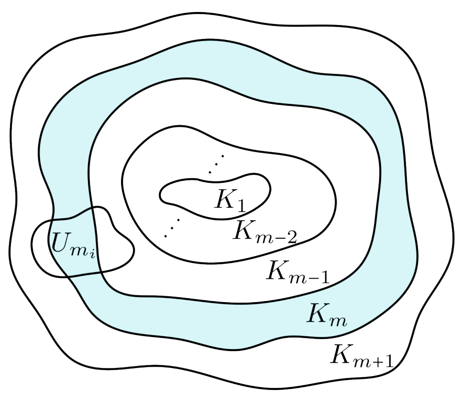
Remark 3.3.7. If (X, d) is a metric space and U is an open cover of X then there exists the Lebesgue number of $\mathcal{U}$ i.e. there exists $\delta > 0$ such that for any $A \subseteq X$ with $d(A) < \delta$, where d denotes the diameter of A, there exists some $U \in \mathcal{U}$ such that $A \subseteq U$.

Lemma 3.3.8. Let M be a $\mathcal{C}^\infty$-manifold (T2 and second-countable) and let $\mathcal{V}$ be an open cover of M. Then there exists an open cover $\mathcal{U}$ of M such that, if $U_1, U_2 \in \mathcal{U}$ with $U_1 \cap U_2 \neq \emptyset$, there exists a $V \in \mathcal{V}$ such that $U_1 \cup U_2 \subseteq V$. In particular, $\mathcal{U}$ is a refinement of $\mathcal{V}$.

Proof. Let d be a metric on M inducing the manifold topology (e.g. $d = d_g$ for some Riemannian metric g on M). Let (K_m) be a sequence of compact subsets of M such that $M = \bigcup_m K_m$ and $K_m \subseteq K_{m+1}^\circ$ for all m. For any m set

$$\mathcal{V}_m := \{V \cap K_m : V \in \mathcal{V}\}.$$

Then $\mathcal{V}_m$ is an open covering of the compact metric space K_m, hence there is a Lebesgue number δ_m and, without loss of generality, $\delta_{m+1} < \delta_m$ for all m. Set $K_{-1} = K_0 = \emptyset$.



For any $m \geq 1$ cover $K_m \backslash K_{m-1}^\circ$ by finitely many ($K_m \backslash K_{m-1}^\circ$ is compact) open sets U_{m_i} that lie in $K_{m+1}^\circ \backslash K_{m-2}$ and for which $d(U_{m_i}) < \delta_{\frac{m+3}{2}}$, $1 \leq i \leq i_m$. Then

$$\mathcal{U} := \{U_{m_i} : 1 \leq i \leq i_m, m \in \mathbb{N}\}$$

has the claimed property. Indeed, let $\mathcal{U}_{m_i} \cap U_{kj} \neq \emptyset$ and, without loss of generality, $m \leq k$. Since $\mathcal{U}_{m_i} \subseteq K_{m+1}^\circ$ and $U_{kj} \subseteq M \backslash K_{k-2}$, we must have $k - 2 < m + 1$. In particular, $k - 2 \leq m \leq k$. Now, $d(U_{m_i}) < \frac{\delta_{m+3}}{2} \leq \frac{\delta_{k+1}}{2}$ and $d(U_{kj}) < \frac{\delta_{k+3}}{2} < \frac{\delta_{k+1}}{2}$. Since $U_{m_i} \cap U_{k_j} \neq \emptyset$, $d(U_{m_i} \cup U_{kj}) \leq d(U_{m_i}) + d(U_{kj}) < \delta_{k+1}$ and, since $U_{m_i} \cup U_{kj} \subseteq K_{k+1}$, by the definition of Lebesgue number, there exists a $V \in \mathcal{V}$ such that $U_{m_i} \cup U_{k_j} \subseteq V \cap K_{k+1} \subseteq V$. □

Proposition 3.3.9. Let M be a SRMF, $\mathcal{V}$ an open cover. Then there exists a convex cover $\mathcal{U}$ of M that is a refinement of $\mathcal{V}$ (i.e. for all $U \in \mathcal{U}$ there is a $V \in \mathcal{V}$ such that $U \subseteq V$).

Proof. Let

$$\mathcal{U}_1 := \{U \subseteq M : U \text{ is convex and } \exists V \in \mathcal{V} \text{ such that } U \subseteq V\}.$$

Then, by Proposition 3.3.3, $\mathcal{U}_1$ is a cover of M and a refinement of $\mathcal{V}$. By Lemma 3.3.8, there exists an open cover $\mathcal{U}_2$ of M such that, if $U_1, U_2 \in \mathcal{U}_2$ and $U_1 \cap U_2 \neq \emptyset$, there exists $U \in \mathcal{U}_1$ with $U_1 \cup U_2 \subseteq U$. Now set

$$\mathcal{U} := \{U \subseteq M : U \text{ is convex and } \exists W \in \mathcal{U}_2 \text{ with } U \subseteq W\}.$$

Again, by Proposition 3.3.3, $\mathcal{U}$ is an open cover of M and $\mathcal{U}$ is a refinement of $\mathcal{U}_2$, hence of $\mathcal{U}_1$ and, hence, of $\mathcal{V}$. Finally, let $U_1, \dots, U_k \in \mathcal{U}$ and $U_1 \cap \dots \cap U_k \neq \emptyset$. $U_1 \in W_1$ and $U_2 \in W_2$, where $W_1, W_2 \in \mathcal{U}_2$, and $W_1 \cap W_2 \neq \emptyset$. By construction, there exists $U \in \mathcal{U}_1$ such that $U_1 \cup U_2 \subseteq W_1 \cup W_2 \subseteq \mathcal{U}$. By Lemma 3.3.5, $U_1 \cap U_2$ is convex. Moreover,

$$(U_1 \cap U_2) \cup U_3 \subseteq U_1 \cup U_3 \subseteq \tilde{U} \in \mathcal{U}_1$$

implies that $U_1 \cap U_2 \cap U_3$ is convex. Continuing in this way we get that $U_1 \cap \dots \cap U_k$ is convex. □

Definition 3.3.10. Let U be a convex subset of a spacetime M. For any $p, q \in U$ let σ_{pq} be the unique geodesic in U from p to q with $\sigma(0) = p$ and $\sigma(1) = q$. Then we call

$$\Delta(p,q) \equiv \overrightarrow{pq} := \dot{\sigma}_{pq}(0) = \exp_p^{-1}(q) \in T_pM$$

the **displacement vector** of p and q.

Remark 3.3.11.

- Recall from [3] 2.2.9 that $\exp_p^{-1}(q) = E^{-1}(p,q)$.
- Clearly, $\sigma(t) = \exp_p(t \cdot \exp_p^{-1}(q)) = \exp_p(t \cdot \overrightarrow{pq})$ for $t \in [0,1]$.

Lemma 3.3.12. Let M be a convex spacetime (e.g. a convex set in a given spacetime) and let $p, q \in M$ such that $p \neq q$. Then:

1. $q \in J^+(p) \iff \Delta(p,q) = \exp_p^{-1}(q) = \overrightarrow{pq} \in T_pM$ is FD causal.
2. $q \in I^+(p) \iff \Delta(p,q)$ is FD timelike.
3. $\overline{I^+}(p) = J^+(p)$.
4. The relation $\leq$ is closed (i.e. $p_n \to p$, $q_n \to q$ and $p_n \leq q_n$ for all n implies $p \leq q$).
5. Every causal curve $c : [0,b) \to M$ with its image contained in a compact set can be continuously extended to b.

Proof. By Corollary 3.1.13 we have for starshaped $\tilde{\Omega} \in T_pM$ a diffeomorphism $\exp_p : \tilde{\Omega} \to \Omega$ such that

$$\begin{aligned} J^+(p) &= \exp_p(J^+(0) \cap \tilde{\Omega}) \\ I^+(p) &= \exp_p(I^+(0) \cap \tilde{\Omega}). \end{aligned}$$

This immediately gives 1., 2. and 3. because those properties hold in Minkowski space.

4. $p = q$ is trivial so let $p \neq q$. Without loss of generality assume $p_n \neq q_n$. By [3] 2.2.9., $(p,q) \mapsto \overrightarrow{pq} = \Delta(p,q)$ is continuous. Since $\langle \overrightarrow{p_nq_n}, \overrightarrow{p_nq_n} \rangle \leq 0$ for all n, $\langle \overrightarrow{pq}, \overrightarrow{pq} \rangle \leq 0$ and so $\overrightarrow{pq}$ is causal. Moreover, if X is a timelike vector field on M then $\langle \overrightarrow{p_nq_n}, X \rangle < 0$ for all n. By continuity, $\langle \overrightarrow{pq}, X \rangle \leq 0$. If $\langle \overrightarrow{pq}, X \rangle$ were equal to zero then $\overrightarrow{pq}$ would be spacelike. ↯ Therefore, $\overrightarrow{pq}$ is FD causal if all $\overrightarrow{p_nq_n}$ are FD causal.
5. Let $t_j \to b$. Then, by compactness, $(c(t_j))_j$ has at least one cluster point. We need to show that there is only one such point (because then we have a continuous extension). Suppose p and q are cluster points. Then there exists $t_j \nearrow b$ such that $c(t_{2j}) \to p$ and $c(t_{2j+1}) \to q$ for $j \to \infty$. Without loss of generality assume c is future directed. Then

$$c(t_{2j}) \leq c(t_{2j+1}) \leq c(t_{2j+2}) \overset{4.}{\Longrightarrow} p \leq q \leq p.$$

1. now implies that $\Delta(p,q)$ is both past and future directed, that is $\Delta(p,q) = 0$, which yields $p = q$.

□

3.4 Quasi-Limits

The exploration of causality involves understanding the limits of causal curves. However, assuming these curves to be only pointwise $\mathcal{C}^\infty$ leads to challenges. The limit of a sequence of pointwise $\mathcal{C}^\infty$-curves may not itself be pointwise $\mathcal{C}^\infty$. To address this, we introduce the concept of a quasi-limit of a sequence of causal curves—these are imperfect approximations, akin to broken geodesics, where closeness is determined by a convex covering. Employing this concept allows us to simplify complex global causality problems into more manageable local ones.

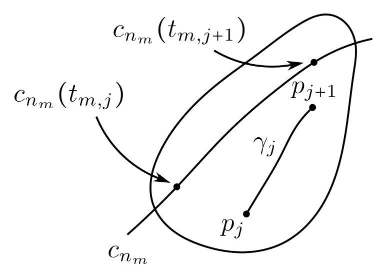
Definition 3.4.1. Let $\mathcal{K}$ be a convex covering of a spacetime M and let $(c_n)_n$ be a sequence of FD causal curves. A limit sequence of $(c_n)_n$ with respect to $\mathcal{K}$ is a finite or infinite sequence of points $p_0 < p_1 < p_2 < \ldots$ such that there exists a subsequence $(c_{n_m})_m$ and parameter values $t_{m,0} < t_{m,1} < t_{m,2} < \ldots$ such that

1. 1a) for every j

$$\lim_{m\to\infty} c_{n_m}(t_{m,j}) = p_j.$$

   2b) p_j, p_{j+1} and $c_{n_m}([t_{m,j}, t_{m,j+1}])$ lie in some $K \in \mathcal{K}$ for all $m \geq m(j)$.

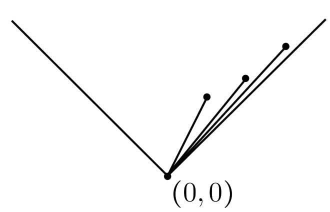
2. If $(p_i)_i$ is infinite, then $(p_i)_i$ does not converge. If $(p_i)_i$ is finite so that $p_0 < p_1 < \cdots < p_N$ then $N \geq 1$ and no strictly longer sequence satisfies 1.



Remark 3.4.2. For each j as above, since $K(j)$ is convex, there exists a unique causal radial geodesic γ_j in $K(j)$ from p_j to p_{j+1}. The causal geodesic polygon $\gamma := \gamma_0 \cup \gamma_1 \cup \gamma_2 \cup \ldots$ is called a quasi-limit of $(c_n)_n$.

Example 3.4.3.

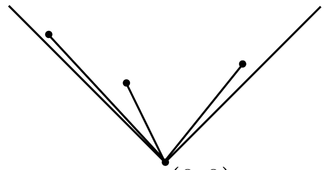
1. Consider $\mathbb{R}^2_1$ and C_n, where C_n is a straight line segment from $(0,0)$ to $\left(n + \frac{1}{n}, n\right)$. Every limit sequence lies on the null geodesic $\gamma(s) = (s,s)$. Up to reparametrization, γ is hence the unique quasi-limit of $(c_n)_n$.



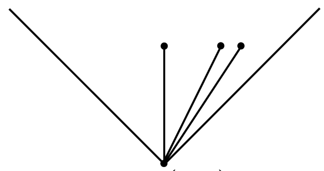
2. Consider $\mathbb{R}^2_1 \setminus \{(1,1)\}$ and C_n as above. Every accumulation point of $(c_n(t))$ for $t \in \mathbb{R}$ must be on $\{(s,s) : s \geq 0\}$. However, for any limit sequence $p_j = (s_j, s_j)$ we can never have $s_j < 1$, $s_{j+1} > 1$. Such points cannot both lie in a convex set (because there doesn't exist a geodesic connecting them). For example, $p_j = \left(1 - \frac{1}{j}, 1 - \frac{1}{j}\right)$ is a limit sequence and $s \mapsto (s,s)$ for $s \in [0,1]$ is a quasi-limit.

3. Consider now $\mathbb{R}^2_1$ and C_n a straight line segment from 0 to $\left(n + \frac{1}{n}, (-1)^n n\right)$. Both $s \mapsto (s,s)$ and

$s \mapsto (s, -s)$ for $s \geq 0$ are quasi-limits.



4. Lastly, consider $\mathbb{R}^2_1$ and let C_n be the straight line segment from 0 to $\left(1, 1 - \frac{1}{n}\right)$. Then $p_0 := (0,0) < p_1 := (1,1)$ is a limit sequence.



Proposition 3.4.4. Let $\mathcal{K}$ be a convex covering of M, let $c_n : [0, b_n) \to M$ (for $b_n \leq \infty$) or $c_n : [0, b_n] \to M$ (for $b_n \leq \infty$) be FD causal curves with $\lim_{n\to\infty} c_n(0) = p \in M$. Then the following facts are equivalent:

1. The sequence $(c_n)_n$ possesses a quasi-limit with respect to $\mathcal{K}$.
2. There exists a neighborhood U of p such that infinitely-many c_n are not entirely contained in U.

Proof.

$(1. \to 2.)$ Let $p := p_0 < p_1 < \dots$ be a limit sequence. Choose disjoint neighborhoods U_0, U_1 of p_0, p_1. Since $c_{n_m}(t_{m,1}) \to p_1$ almost all $c_{n_m}(t_{m,1}) \in U_1$ and, hence, not in U_0. Therefore, U_0 is the required neighborhood.

$(2. \to 1.)$

a) Let $\mathcal{U}$ be a locally finite refinement of $\mathcal{K}$ such that for all $U \in \mathcal{U}$ there exists a $K \in \mathcal{K}$ such that $\overline{U} \subseteq K$ and $\overline{U}$ is compact. (For example, let $(\chi_\alpha)_{\alpha\in A}$ be a partition of unity subordinate to $\mathcal{K}$ and let $\mathcal{U} := \{\text{supp}(\chi_\alpha)^\circ : \alpha \in A\}$.) By 2., we may assume that there exists $U_0 \in \mathcal{U}$, where U_0 is a neighborhood of p, and that infinitely many c_n leave U_0. Let $(c_n^{(1)})_n$ be the subsequence of curves which leave U_0 and set

$$t_{n,1} := \inf\left\{t > 0 : c_n^{(1)}(t) \notin U_0\right\}.$$

Then $c_n^{(1)}(t_{n,1}) \in \partial U_0$. Due to compactness, there exists a subsequence, denoted again by $c_n^{(1)}(t_{n,1})$, and we define p_1 to be the corresponding limit. Now since $c_n^{(1)}(0) < c_n^{(1)}(t_{n,1})$, by Lemma 3.3.12 (4.), $p_0 \leq p_1$. But, $p_1 \in \partial U_0$ while $p_0 \in U_0$ and so $p_0 \neq p_1$ (in particular, the limit sequence contains more than one point).) Now choose $U_1 \in \mathcal{U}$, a neighborhood of p_1. If infinitely many $c_n^{(1)}$ leave U_1, go on with the construction. When repeating obey the following selection criterion for U_i: If more than one $U \in \mathcal{U}$ contains p_i then select as U_i one which has been used the least times before p_i. $(\star)$
This construction yields 1. in Definition 3.4.1. Therefore, it remains to verify 2. from Definition 3.4.1.

b) If $p_0 < p_1 < \dots$ is infinite, it is not converging. Suppose to the contrary i.e. suppose $p_j \to q \in M$. Let $q \in V \in \mathcal{U}$. Now almost every p_j lies in V. $\overline{V}$ is compact by assumption and $\mathcal{U}$ is locally finite. Thus, only finitely many $U \in \mathcal{U}$ meet V while almost all U_j meet V (because $p_j \in U_j \cap V$ for almost every j) and so one of the U's has been selected infinitely often. However, V was always was a candidate for for these p_j but has only been selected finitely many times since only finitely many of the p_j lie in ∂V. ↯

c) Suppose the construction terminates after finitely many steps $p_0 < p_1 < \dots < p_k$ i.e. assume that only finitely many $(c_n^{(k)})_n$ leave U_k and so there exists a subsequence $(c_n^{(k+1)})_n \subseteq U_k$ (i.e. tail ends remain in U_k). $\overline{U}_k$ is compact and so, by Lemma 3.3.12 (5.), $c_n^{(k+1)}$ can be continuously extended to their endpoints b_n (if not anyways defined on $[0, \infty)$ - in this case, reparametrize so that c_n is defined on $[0, b_n)$ for $b_n < \infty$). Since $\overline{U}_k$ is compact, without loss of generality, $c_n^{(k+1)}(b_n) \to q \in \overline{U}_k$.

- Case 1: Assume $q = p_k$ and that the finite sequence is extendible by some $p_{k+1} > p_k$ such that $p_0 < p_1 \dots < p_k < p_{k+1}$ has property 1. Then on U_k we would have

$$\underbrace{c_n^{(k+1)}(t_{n,k+1})}_{\to p_{k+1}} < \underbrace{c_n^{(k)}(b_n)}_{\to q} \implies p_k < p_{k+1} \leq= p_k,$$

which, by Lemma 3.3.12 (4.), implies $p_k = p_{k+1}$ since $\overline{U_k} \subseteq K(k)$ for some $K(k) \in \mathcal{K}$. But this is a contradiction to Lemma 3.3.12 (1.). Therefore, p_{k+1} does not yield an extension and $p_0 < p_1 \dots < p_k$ is the limit sequence.

- Case 2: Let $q \neq p_k$. Set $p_{k+1} := q$. Then $p_{k+1} > p_k$ ($q \geq p_k$ by Lemma 3.3.12 and $q \neq p_k$) and $p_0 < p_1 < \dots < p_k < p_{k+1}$ satisfies 1. (from Definition 3.4.1) by construction and 2. (from the same definition) by Case 1, so it yields a limit sequence.

□

Remark 3.4.5. If $(p_j)_j$ is infinite, then the quasi-limit $\gamma = \gamma_1 \cup \gamma_2 \cup \dots$ is future-inextendible i.e. if γ is defined on $[0, b)$ then it can't be extended continuously to b. Indeed, let $p_i := \gamma(t_i)$, $p_i < p_{i+1}$. Then $t_i < t_{i+1}$. If γ could be extended to $[0, b]$ continuously then $t_i \nearrow \bar{t} \leq b$. But then $p_i = \gamma(t_i)$ converges, which is a contradiction to Definition 3.4.1 (2.).

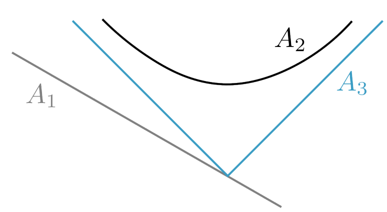
3.5 Cauchy Surfaces

Cauchy surfaces function as a sort of snapshot, offering a comprehensive view of how causality is organized within the spacetime framework.

Definition 3.5.1. A subset $A \subseteq M$ of a spacetime is called <u>**achronal**</u> if there are no $p, q \in A$ such that $p \ll q$. Hence, A is achronal if and only if any timelike curve meets A at most once i.e. if and only if $A \cap I^+(A) = \emptyset$.

Example 3.5.2. Let $M = \mathbb{R}^n_1$. Then the following three sets are all achronal:

1. Spacelike hyperplane A_1.
2. The $(n-1)$-dimensional hyperbolic space A_2.
3. Future cone A_3.

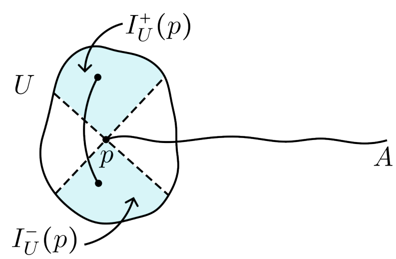


Remark 3.5.3. We note the following facts on achronal sets:

1. If $A \subseteq B$ and B is achronal, then A is achronal as well.
2. If A is achronal, then $\overline{A}$ is also achronal. To this end, suppose there exists $p \ll q \in \overline{A}$. Then there exist $p_n, q_n \in A$ such that $p_n \to p$ and $q_n \to q$. Since $\ll$ is open, by Proposition 3.1.14, $p_n \ll q_n$ for n large, which is a contradiction to A being achronal.
3. If A is a hypersurface, then A being spacelike does not imply that A is achronal. An example of this would be the Lorentz cylinder. On the contrary, if A is a hypersurface, A being achronal does not imply that it is spacelike. For example, consider the null cone in $\mathbb{R}^2_1$.

Definition 3.5.4. The edge of an anchoral subset A is defined as the following set:

$$\text{edge}(A) = \left\{ p \in \overline{A} \;\middle|\; \begin{array}{l} \text{for all open neighborhoods } U \text{ of } p \text{ there is a timelike} \\ \text{curve } \gamma \text{ in } U \text{ from } I^-_U(p) \text{ to } I^+_U(p) \text{ such that } \gamma \cap A = \varnothing \end{array} \right\}.$$

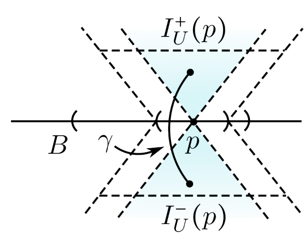


Example 3.5.5. Let M be an n-dimensional Minkowski space.

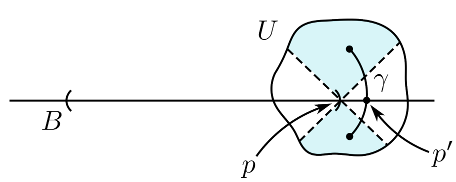
1. All $A_i's$ from Example 3.5.2 have $\text{edge}(A_i) = \varnothing$.
2. Let $A_4 := \{0\} \times B$ with $B \subseteq \mathbb{R}^{n-1}$. Then $\text{edge}(A_4) = \{0\} \times \partial B$. Indeed, let $A_4 \ni p = (0, b)$ for $b \in B^\circ$. Let $U_0 := B_\epsilon \subseteq B$, for $\epsilon > 0$ suitable and let

$$U = (I^+(U_0) \cup U_0 \cup I^-(U_0)) \cap \{(t, x) : |t| < \epsilon\}.$$

Every timelike curve $\gamma : I^-_U(p) \to I^+_U(p)$ intersects A_4, so $p \notin \text{edge}(A_4)$.



Let now $p = (0, b) \in \{0\} \times \partial B$. Then for any neighborhood V of $b \in \mathbb{R}^{n-1}$, $V \backslash B \neq \varnothing$ with respect to $\mathbb{R}^{n-1}$. Hence, for every open neighborhood U of p (with respect to $\mathbb{R}^n$) there exists $p' = (0, b') \in (\{0\} \times \mathbb{R}^{n-1} \backslash B) \cap U \;\hat{=}\; V \backslash B$ and a timelike curve γ connecting $I^-_U(p)$ and $I^+_U(p)$ which meets $\{0\} \times \mathbb{R}^{n-1}$ only in p' and, hence, it does not intersect A_4.



Remark 3.5.6. If A is achronal, then $\overline{A}\backslash A \subseteq \text{edge}(A)$. Indeed, let $p \in \overline{A}\backslash A$. Then for every open neighborhood U of p there exists a timelike curve γ from $I^-_U(p)$ to $I^+_U(p)$ passing through p. $\overline{A}$ is achronal (see Remark 3.5.3 (2.)) and so γ contains no further points on $\overline{A}$ and none on A. Therefore, $p \in \text{edge}(A)$.

Lemma 3.5.7. If $A \subseteq M$ is achronal, then $\text{edge}(A)$ is closed.

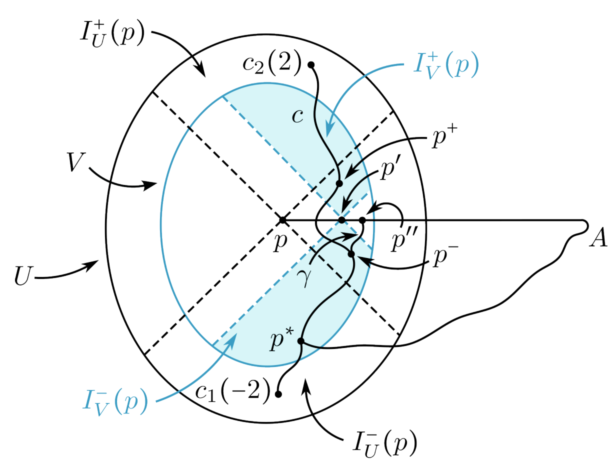
Proof. Let $p \in \overline{\text{edge}(A)}$. We show that $p \in \text{edge}(A)$. Let U be a neighborhood of p in M and let $V \subseteq U$ be an open neighborhood of p contained in $I^+_U(I^-_U(p)) \cap I^-_U(I^+_U(p))$. Since $p \in \overline{\text{edge}(A)}$, there is some $p' \in V \cap \text{edge}(A)$. It follows that there is a timelike curve $c : [-1,1] \to V$ with

$$p_\pm := c(\pm 1) \in I^\pm_V(p')$$

such that $c \cap A = \varnothing$. Note that $p_\pm \in V \subseteq I^+_U(I^-_U(p)) \cap I^-_U(I^+_U(p))$ implies that we can extend c to some timelike and future directed curve

$$c_1 : [-2,-1] \to U$$

such that $c_1(-2) \in I^-_U(p)$ and $c_1(-1) = p_-$. Similarly, we can also extend c to $c_2 : [1,2] \to U$ so that $c_2(2) \in I^+_U(p)$ and $c_2(1) = p_+$. Define $\tilde{c} := c_1 \cup c \cup c_2$. Now, if $\tilde{c} \cap A = \varnothing$, then $p \in \text{edge}(A)$. To see that this is the case, indirectly suppose that there exists some $p^* \in c_1 \cap A$. Since $p_- \in I^-_V(p')$, $p' \in I^+_V(p_-)$ and so (as $I^+_V(p_-)$ is open) $I^+_V(p_-)$ is a neighborhood of p'. $p' \in \text{edge}(A) \subseteq \overline{A}$ and so there exists a $p'' \in A \cap I^+_V(p)$ and so there exists a TLFD curve γ from p_- to p''. $c_1 \cup \gamma$ is FDTL and it intersects A in two points; in p^* and p'', which is a contradiction to A being achronal.

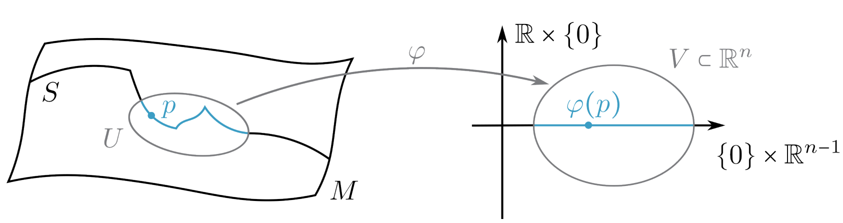


□

Our next aim is to see under which hypotheses (on the $\text{edge}(A)$) an achronal set is a $\mathcal{C}^0$-hypersurface.

Definition 3.5.8. A subset S of an n-dimensional differentiable manifold M is called topological hypersurface if for every $p \in S$ there is an open neighborhood U of p in M and a homeomorphism $\varphi : U \to V$, with some $V \subseteq \mathbb{R}^n$ open, such that

$$\varphi(U \cap S) = V \cap (\{0\} \times \mathbb{R}^{n-1}).$$



Example 3.5.9. The subset $S := C_+(0) \subseteq \mathbb{R}^n_1$ is a $\mathcal{C}^0$-hypersurface with, for example, $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, where

$$(x^0, \hat{x}) \mapsto (x^0 - \|\hat{x}\|, \hat{x}).$$

Theorem 3.5.10 (Brouwer)**.** Let $U \subseteq \mathbb{R}^n$ be open and $\varphi : U \to \mathbb{R}^n$ continuous and injective. Then $\varphi(U)$ is open in $\mathbb{R}^n$ and $\varphi : U \to \varphi(U)$ is a homeomorphism.

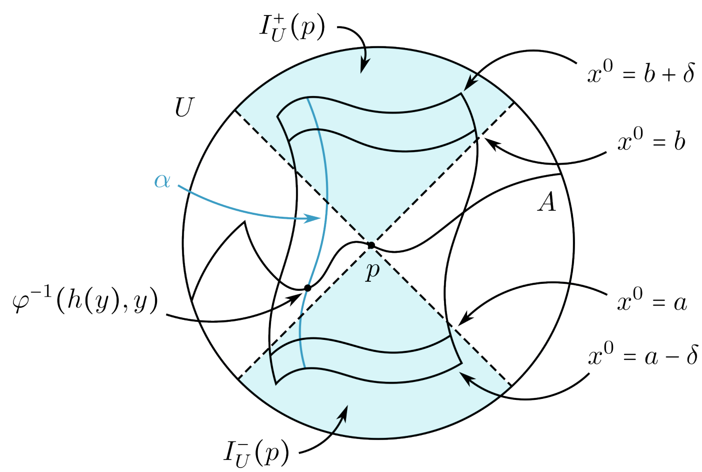
Proposition 3.5.11 (Achronal Hypersurfaces)**.** Let $A \subseteq M$ be achronal. Then the following facts are equivalent:

1. $A \cap \text{edge}(A) = \varnothing$.
2. A is a topological hypersurface (i.e. a $\mathcal{C}^0$-hypersurface).

Proof. $(1. \to 2.)$ Let A be a topological hypersurface with U, V, φ as in Definition 3.5.8 for some $p \in A$. Then $p \notin \text{edge}(A)$ (by assumption) and so there exists U, an open neighborhood of p such that any TLFD curve in U from $I_U^-(p)$ to $I_U^+(p)$ intersects A. Without loss of generality, U is a chart domain for a chart $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$, where $\varphi = (x^0, \dots, x^{n-1})$ and $\frac{\partial}{\partial x^0}$ is FD on U (for example, take $\varphi = \exp_p$). By shrinking U further, we obtain an open neighborhood of p, $V \subseteq U$, such that:

i) $\varphi(V) = (a - \delta, b + \delta) \times N \overset{\text{open}}{\subseteq} \mathbb{R} \times \mathbb{R}^{n-1}$ for some $a, b \in \mathbb{R}$, $\delta > 0$ and $N \subseteq \mathbb{R}^{n-1}$ open.

ii) $\{x \in V : x^0 = a\} \subseteq I_U^-(p)$ and $\{x \in V : x^0 = b\} \subseteq I_U^+(p)$.



Let $y \in N \subseteq \mathbb{R}^{n-1}$. Then the curve $\alpha : [a, b] \to V$ such that $s \mapsto \varphi^{n-1}(s, y)$ is timelike from $I_U^-(p)$ to $I_U^+(p)$ and, hence, it meets A. Since A is achronal, it does so precisely once. Let $h(y) \in [a, b]$ be such that $\varphi^{-1}(h(y), y) \in A$.

We claim that the map $h : N \to (a, b)$ is continuous. Indeed, let $(y_m)_m$ be a sequence in N such that $y_m \to y \in N$. Suppose $h(y_m) \not\to h(y)$. Since $h(N) \subseteq$ is contained in the compact interval $[a, b]$, without loss of generality, $h(y_m) \to r \neq h(y)$. Let $q := \varphi^{-1}(h(y), y) \in A$. The curve $s \mapsto \varphi^{-1}(s, y)$ is TL and both q and $\varphi^{-1}(r, y) \neq q$ are contained in it and so $\varphi^{-1}(r, y) \in \underbrace{I_V^-(q) \cup I_V^+(q)}_{\text{open}}$. Since $\varphi^{-1}(h(y_m), y_m) \to \varphi^{-1}(r, y)$, there exists m_0 with

$$\underbrace{\varphi^{-1}(h(y_{m_0}), y_{m_0})}_{\in A} \in I_V^-(\underbrace{q}_{\in A}) \cup I_V^+(q),$$

which is a contradiction to A being achronal. $V \cap A = \varphi^{-1}(\{(h(y),y) : y \in N\})$ i.e. in terms of φ A is the graph of h. Write $\varphi = (\varphi^0, \varphi')$ and let $\psi : V \to \mathbb{R}^n$, where

$$\psi(p) := (\varphi^0(p) - h(\varphi'(p)), \varphi'(p)). \tag{3.5.1}$$

Then ψ is continuous (since φ is) and bijective with the inverse

$$\psi^{-1}(x^0, x') = \varphi^{-1}(x^0 + h(x'), x'). \tag{3.5.2}$$

In fact,

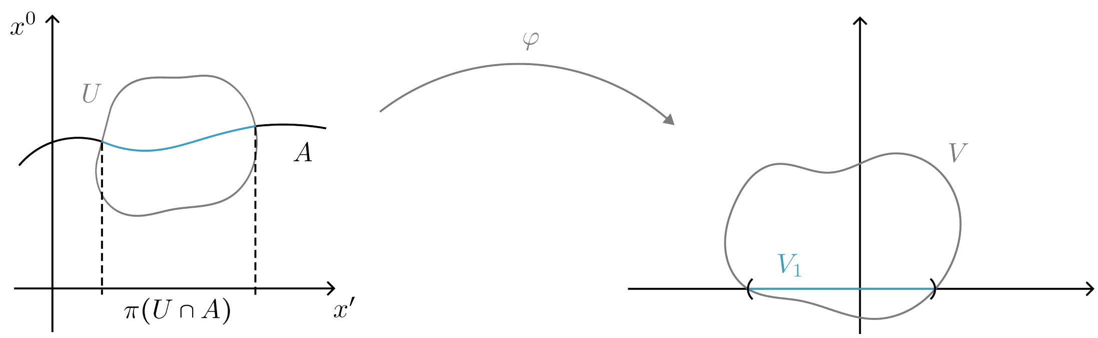
$$\psi \circ \varphi^{-1} : \underbrace{\varphi(V)}_{\text{open in } \mathbb{R}^n} \to \underbrace{\psi(V)}_{\subseteq \mathbb{R}^n}$$

is continuous and injective. Therefore, by Theorem 3.5.10, $\psi \circ \varphi^{-1}(\varphi(V)) = \psi(V)$ is a homeomorphism. Finally,

$$\begin{aligned}
\psi(V \cap A) &= \psi \circ \varphi^{-1}(\{(h(y),y) : y \in N\}) \overset{(3.5.1)}{=} \{(0,y) : y \in N\} \\
&= \{0\} \times N \overset{(3.5.2)}{=} \psi \circ \varphi^{-1}(\underbrace{(a-\delta, b+\delta) \times N}_{=\varphi(V)}) \cap (\{0\} \times N) \\
&= \psi(V) \cap (\{0\} \times \mathbb{R}^{n-1})
\end{aligned}$$

and so A is a $\mathcal{C}^0$-hypersurface.

(2. $\to$ 1.) Let $p \in A$. Since 1. is local, we may suppose, by Corollary 3.1.13, that $M = \mathbb{R}^n_1$. Let (φ, U) be as in Example 3.5.9 with U connected and $\varphi : U \to V$ a homeomorphism such that $\varphi(U \cap A) = V \cap (\{0\} \times \mathbb{R}^{n-1}) =: V_1$. In particular, we have that $\varphi_1 := \varphi|_{U \cap A} : U \cap A \to V_1$ is a homeomorphism.



Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection $(x^0, x') \mapsto x'$. Then any vertical line $t \mapsto (t, x')$ is TL and, since A is achronal, it meets A at most once. Therefore, $\pi|_{U \cap A}$ is injective and $\pi \circ \varphi_1^{-1} : V_1 \to \pi(U \cap A)$ is continuous and bijective. Applying Theorem 3.5.10, we get that $\pi \circ \varphi_1^{-1}$ is a homeomorphism and $\pi(U \cap A)$ is open. Let now $f : \pi(U \cap A) \to \mathbb{R}$ where $f(x') := \underbrace{\mathrm{pr}_0}_{x \to x^0} \circ \pi^{-1}(x')$. f is continuous and $U \cap A = \mathrm{graph}(f) = \{(f(x'), x') : x' \in \pi(U \cap A)\}$. $U \backslash A$ decomposes into two connected components:

$$\begin{aligned}
U^+ &= \left\{(x^0, x') \in U : x^0 > f(x')\right\}, \\
U^- &= \left\{(x^0, x') \in U : x^0 < f(x')\right\}.
\end{aligned}$$

$I^-_U(p)$ and $I^+_U(p)$ are open and connected (cf. Corollary 3.1.13). Since A is achronal, they lie in $U \backslash A$. The vertical line through p meets $I^+_U(p)$ and $I^-_U(p)$ as well as U^- and U^+ (in the right order). Therefore, $I^-_U(p) \subseteq U^-$ and $I^+_U(p) \subseteq U^+$ and so any TLFD curve γ from $I^-_U(p)$ to $I^+_U(p)$ goes from U^- to U^+ and so, since the image of γ is connected, γ must meet $\partial U^- = \partial U^+ = U \cap A \subseteq A$ and so $p \notin \mathrm{edge}(A)$.

□

Corollary 3.5.12 (Closed Achronal Hypersurfaces)**.** Let $A \subseteq M$ be achronal. Then the following are equivalent:

1. edge$(A) = \varnothing$.
2. A is a closed topological hypersurface.

Proof.

$(1. \to 2.)$ $A \cap \text{edge}(A) = \varnothing$ and so A is a $\mathcal{C}^0$-hypersurface by Proposition 3.5.11. By Remark 3.5.6, $\overline{A} \backslash A \subseteq \text{edge}(A) = \varnothing$ and so $A = \overline{A}$, implying that A is closed.

$(2. \to 1.)$ By Proposition 3.5.11, $A \cap \text{edge}(A) = \varnothing$. Since $\text{edge}(A) \subseteq \overline{A} = A$, $\text{edge}(A) = \varnothing$.

□

Definition 3.5.13. $B \subseteq M$ is called a future set (or past set) if $I^+(B) \subseteq B$ (or $I^-(B) \subseteq B$).

Example 3.5.14. For $M = \mathbb{R}^n_1$ $B := \left\{x = (x^0, \hat{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} : x^0 - |\hat{x}| \geq 0\right\}$ is a future set in $\mathbb{R}^n_1$.

Remark 3.5.15. B is a future set if and only if $M \backslash B$ is a past set.

Corollary 3.5.16. Let $\varnothing \neq B \neq M$ be a future set. Then ∂B is an achronal closed $\mathcal{C}^0$-hypersurface.

Proof. We need to show that ∂B is achronal and edge$(\partial B) = \varnothing$.

- Let $p \in \partial B$ and $q \in I^+(p)$. Then $I^-(q)$ is an open neighborhood of $p \in \partial B$. Therefore, $I^-(q) \cap B \neq \varnothing$ and so $q \in I^+(B) \subseteq B$. $I^+(\partial B)$ is therefore contained in B°. Analogously, $I^-(\partial B) \subseteq (M \backslash B)^\circ$. $I^\pm(\partial B) \cap \partial B = \varnothing$ and so ∂B is achronal.
- According the first point, any TL γ from $I^-(p)$ to $I^+(p)$ has to meet ∂B and so edge$(\partial B) = \varnothing$.

□

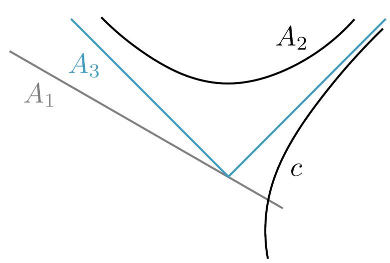
Definition 3.5.17. $B \subseteq M$ is acausal if for all $p, q \in B$ $p \nleq q$.

Remark 3.5.18.

- B is acausal if and only if every causal curve meets B at most once.
- B being acausal implies that B is achronal but the other direction is not true. For example, consider $C^+ \subseteq \mathbb{R}^n_1$.

Definition 3.5.19. A Cauchy hypersurface (or Cauchy surface) is a subset $S \subseteq M$ that is met by any inextendible timelike curve precisely once.

Example 3.5.20. Consider A_i from Example 3.5.2 for $i = 1, 2, 3$.

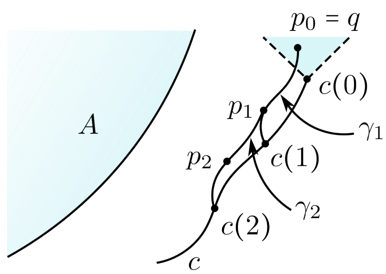


Lemma 3.5.21. Let $A \subseteq M$ be closed and $c : [0,b) \to M\backslash A$ be PD, causal and past inextendible with $c(0) := p$. Then,

1. for all $q \in I^+_{M\backslash A}(p)$ there exists a curve $\tilde{c} : [0,b) \to M\backslash A$ with $\tilde{c}(0) = q$ such that $\tilde{c}$ is PDTL and past inextendible.
2. unless c is a null pregeodesic without conjugate points, there will exist $\tilde{c} : [0,b) \to M\backslash A$ PDTL with $\tilde{c}(0) = p = c(0)$.

Proof. Without loss of generality assume $b = \infty$ and $(c(n))_{n\in\mathbb{N}}$ is non-convergent. Now choose a metric on M that induces the manifold topology (for example let $d = d_h$, where h is the Riemannian metric).

1. Set $p_0 := q \gg_{M\backslash A} p$. $c(1) \leq c(0) \ll p_0$ and so, by Proposition 3.1.8, $c(1) \ll p_0$ and there exists a TL curve γ_1 from p_0 to $c(1)$. Now pick p_1 on γ_1 such that $0 < d(p_1, c(1)) < 1$. $c(2) \ll p_1$ and so there exists a PD TL curve γ_2 from p_1 to $c(2)$. Pick a point p_2 on γ_2 so that $0 < d(p_2, c(2)) < \frac{1}{2}$.



Iterate this and get $c(k) \ll p_k \ll p_{k-1}$ and $d(c(k), p_k) < \frac{1}{k}$. Finally, obtain a PDTL curve $\tilde{c}$ starting in p_0 and containing all p_k. (All constructions are within $M\backslash A$.)
It only remains to show that $\tilde{c}$ is past inextendible. To this end, assume $\tilde{c}$ could be continuously extended to some endpoint $\tilde{p}$. Then $p_k \to \tilde{p}$ and

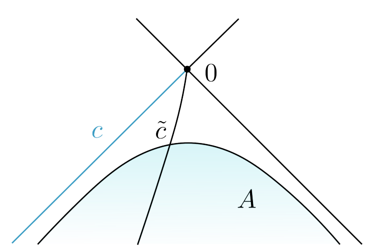
$$d(c(k), \tilde{p}) \leq \underbrace{d(c(k), p_k)}_{< \frac{1}{k}} + \underbrace{d(p_k, \tilde{p})}_{\to 0} \to 0,$$

which contradicts the assumption that $(c(k))_k$ does not converge.

2. Suppose that c is a null pregeodesic without conjugate points. Then there exists some $a > 0$ such that c is a null pregeodesic without conjugate points on $[0,a]$. By Theorem 3.2.23 (with $P = \{c(0)\}$ and $M\backslash A$ instead of M), there exists a PDTL curve from $c(0)$ to $c(a)$ in $M\backslash A$. Since $p = c(0) \gg c(a)$, $p \in I^+_{M\backslash A}(c(a))$. Now apply 1. with $q = c(0)$.

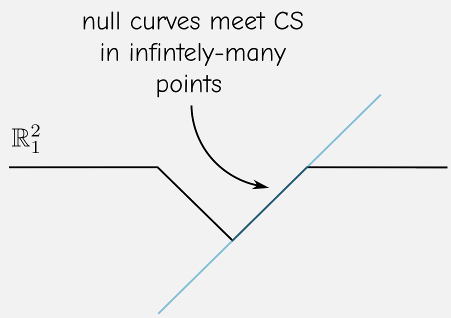
□

Remark 3.5.22.
The condition in 2. cannot be dropped. Let c be an inextendible null geodesic without conjugate points like in the picture below. Then no TLPD curve $\tilde{c}$ from $0 = c(0)$ avoids A.



Proposition 3.5.23 (Properties of Cauchy Surfaces)**.** Let S be a Cauchy surface (CS) for M. Then,

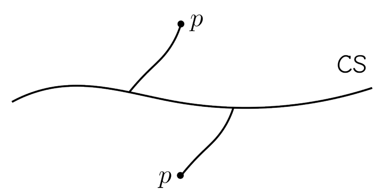
1. S is achronal.
2. S is a closed $\mathcal{C}^0$-hypersurface.
3. every inextendible causal curve meets S (non-uniquely, in general).



Proof.

1. If there were a TL curve c meeting S twice, then we could just extend it to the past and future i.e. obtain an inextendible TL curve $\tilde{\gamma}$ intersecting S twice. ↯

2. a) We show that $M = I^-(S)\dot\cup S\dot\cup I^+(S)$. In particular, $S = M\setminus(I^+(S)\cup I^-(S))$ is closed since $I^+(S)$ and $I^-(S)$ are open. $S\cap I^\pm(S) = \varnothing$ and also $I^+(S)\cap I^-(S) = \varnothing$ since through any $p\in M$ we find an inextendible timelike curve which has to intersect S.



   b) We now claim that $S = \partial I^+(S) = \partial I^-(S)$. To this end, note that $I^\pm(S)\cup S \overset{a)}{=} (I^\mp(S))^c$ is closed (since $I^\mp(S)$ is open) and so $\partial I^+(S) = \overline{I^+(S)}\cap\overline{M\setminus I^+(S)} \subseteq (I^+(S)\cup S)\cap(I^-(S)\cup S) = S$. Conversely, $S\subseteq\partial I^+(S)$ holds for any subset S of a Lorentzian manifold. Analogously, we obtain $\partial I^-(S) = S$.

   c) We now show that edge$(S) = \varnothing$ since then, by Corollary 3.5.12, S is a closed $\mathcal{C}^0$-hypersurface. By b), every TL curve from $I^-(S)$ to $I^+(S)$ must meet S (since a curve is always a connected set). Therefore, edge$(S) = \varnothing$.

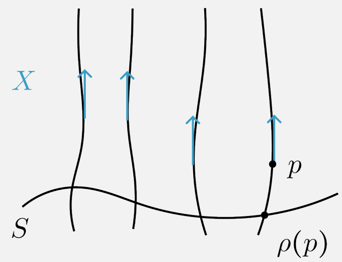
3. Suppose there exists a causal and inextendible curve c not meeting S. By a), without loss of generality let $c\subseteq I^+(S)$. Now choose $p\in c$ and $q\in I^+_{M\setminus S}(p)$. By Lemma 3.5.21, there exists a PDTL curve $\tilde{c}$ in $M\setminus S$ which is past inextendible. Maximally extend $\tilde{c}$ to the future in order to obtain a TL inextendible curve in $M\setminus S$. This lead us to a contradiction with Definition 3.5.19.

□

Theorem 3.5.24 (The Projection ρ)**.** Let X be a TL $\mathcal{C}^\infty$-vector field on a spacetime M and S a Cauchy surface in M. Define $\rho : M\to S$ where

$$p\mapsto\rho(p)$$

so that $\rho(p)$ is the unique point where the integral curve of X through p intersects S.



ρ is well-defined, continuous, $\mathcal{C}^0$, open and $\rho|_S = \mathrm{id}_S$. In particular, S is connected (since M is).

Proof.

a) Maximal integral curves are inextendible (and TL, since X is TL) by definition and so ρ is well-defined.

b) Let $\mathrm{Fl}^X : D \subseteq M \times \mathbb{R} \to M$ be the flow of X with maximal domain D (which is open). $S \subseteq M$ is a $\mathcal{C}^0$-hypersurface (by Proposition 3.5.23) and so $S \times \mathbb{R}$ is a $\mathcal{C}^0$-hypersurface in $M \times \mathbb{R}$. Then $D(S) := (S \times \mathbb{R}) \cap D$ is a $\mathcal{C}^0$-hypersurface. $\psi := \mathrm{Fl}^X|_{D(S)} : D(S) \to M$ is now continuous and bijective. If $p \in M$, then $t \mapsto \mathrm{Fl}^X_t(p)$ intersects S (by definition of CS) and so there exists a t_0 such that $\mathrm{Fl}^X_{t_0}(p) = q \in S$ i.e.

$$p = \mathrm{Fl}^X_{-t_0}(q) = \mathrm{Fl}^X \underbrace{(-t_0, q)}_{\in D(S)},$$

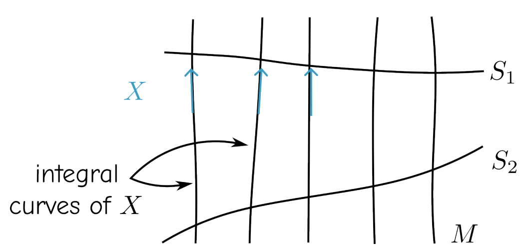
which shows that ψ is surjective. Suppose now that $\mathrm{Fl}^X(t_1, q_1) = \mathrm{Fl}^X(t_2, q_2)$, where $(t_i, q_i) \in D(S)$ for $i = 1, 2$. Then $q_2 = \mathrm{Fl}^X(t_1 - t_2, q_1)$ and so the flow line through q_1 meets S also in q_2. Since S is achronal, $q_1 = q_2$. But then $\mathrm{Fl}^X_{t_1}(q_1) = \mathrm{Fl}^X_{t_2}(q_1)$ (by uniqueness of integral curves) implies that $t_1 = t_2$ and so ψ is also injective.
$D(S)$ and M are both $\mathcal{C}^0$-manifolds and so, by Theorem 3.5.10, ψ is a homeomorphism and therefore open. Let $\pi : M \times \mathbb{R} \to \mathbb{R}$ so that $(p, t) \mapsto t$. Then π is open and continuous. $\rho = \pi \circ \psi^{-1}$ and so ρ is open and continuous.

c) If $p \in S$ then p is the unique intersection of $\mathrm{Fl}^X(p)$ with S and so $\rho(p) = p$.

□

Corollary 3.5.25. Any two Cauchy surfaces S_1, S_2 are homeomorphic.

Proof. Let $X \in \mathfrak{X}(M)$ be TL and define ρ_1, ρ_2 as in Theorem 3.5.24 with respect to S_1, S_2. Then $\rho_1|_{S_2} : S_2 \to S_1$ and $\rho_2|_{S_1} : S_1 \to S_2$ are inverses of one another and hence homeomorphisms.



3.6 Global Hyperbolicity

Global hyperbolicity stands as the most stringent among the causality criteria (refer to Definition 3.1.19), tightly linked to the presence of a Cauchy surface. In a way, globally hyperbolic spacetimes are counterparts of complete RMFs since a remnant of the Hopf-Rinow theorem holds. Within a globally hyperbolic set, for any $p < q$, there exists a causal geodesic that maximizes between p and q. Additionally, time separation is continuous on globally hyperbolic sets.

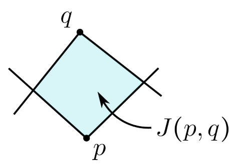
Definition 3.6.1. $X \subseteq M$ (where M is a spacetime) is called **globally hyperbolic** (GH) if

1. the strong causality condition holds on X i.e. for all $p \in X$ and all U neighborhoods of p there exists $V \subseteq U$ such that for all causal γ starting and ending in V, $\gamma \subseteq U$.

2. for every $p, q \in X$,

$$J(p,q) := J^+(p) \cap J^-(q)$$

is compact and a subset of X. $J(p,q)$ is sometimes called **causal diamond**.



Remark 3.6.2. In 1. it would suffice to suppose that X is causal (Bernal/Sanchez, 2007. [1]).

Lemma 3.6.3 (Non-imprisonment)**.** Let $K \subseteq M$ be compact and let strong causality hold on K. Furthermore let $c : [0,b) \to M$ $(b \leq \infty)$ be a future-inextendible causal curve starting in K i.e. with $c(0) \in K$. Then there exists $t_0 \in (0,b)$ such that for all $t \geq t_0$, $c(t) \notin K$.

Proof. Indirectly assume that there exist $s_i \in (0,b)$ such that $s_i < s_{i+1}$, $s_i \nearrow b$ and $c(s_i) \in K$ for all i. K is compact and so, without loss of generality, $c(s_i) \to p \in K$. c is future inextendible and therefore there exist $t_i \in (0,b)$, $t_i \nearrow b$ and $c(t_i) \not\to p$. Choosing subsequences we can get that there exists U, a neighborhood of p, such that $c(t_i) \notin U$ and $s_1 < t_1 < s_2 < t_2 < \dots$. For any $p \in V \subseteq U$, where U and V are neighborhoods of p and for i large enough $c(s_i), c(s_{i+1}) \in V$ while $c([s_i, s_{i+1}]) \notin U$, which is a contradiction to strong causality. □

Remark 3.6.4. Let $p \in M$, U a normal neighborhood of p, $\exp_p : \tilde{U} \to U$ diffeomorphism for $\tilde{U}$ star-shaped and $\tilde{P} \in \mathfrak{X}(\tilde{U})$ the position vector field $v \mapsto v_v$. Then $P := (\exp_p)_* \tilde{P} \in \mathfrak{X}(U)$ is called the position vector field on U. Let $\tilde{q} : T_pM \to \mathbb{R}$, where $\tilde{q}(v) = \langle v, v \rangle \equiv g_p(v,v)$ be the quadratic form corresponding to g_p and set $q := \tilde{q} \circ \exp_p^{-1} : U \to \mathbb{R}$. Both $\tilde{p}$ and p are radial and, by the proof of Lemma 3.1.11,

$$\text{grad}(\tilde{q}) = 2\tilde{P}. \tag{3.6.1}$$

Let $p_1 \in U$ and $w_{p_1} \in T_{p_1}U = T_{p_1}M$ and set $x_1 := \exp_p^{-1}(p_1) \in \tilde{U}$. $\exp_p$ is a diffeomorphism and so there exists a unique $w_{x_1} \in T_{x_1}(T_pM) \cong T_pM$ with $T_{x_1}\exp_p(w_{x_1}) = w_{v_1}$. Then

$$\begin{aligned} \langle \text{grad}(q)|_{p_1}, w_{p_1} \rangle &\overset{[3],(3.2.16)}{=} w_{p_1}(q) = T_{x_1}\exp_p(w_{x_1})(q) = w_{x_1}(q \circ \exp_p) \\ &= w_{x_1}(\tilde{q}) = \langle \text{grad}(\tilde{q}), w_{x_1} \rangle \overset{(3.6.1)}{=} 2\left\langle \tilde{P}(x_1), w_{x_1} \right\rangle. \end{aligned}$$

Since $\tilde{P}(x_1)$ is radial, by Proposition 3.1.10,

$$2\left\langle \tilde{P}(x_1), w_{x_1} \right\rangle = 2\left\langle T_{x_1}\exp_p(\tilde{P}(x_1)), T_{x_1}\exp_p(w_{x_1}) \right\rangle = \langle 2P(p_1), w_{p_1} \rangle$$

[1]Check https://arxiv.org/abs/gr-qc/0611138 for more information.

and so

$$\text{grad}(q) = 2P. \tag{3.6.2}$$

Moreover,

$$\langle P, P\rangle \circ \exp_p(x) = \left\langle T_x \exp_p(\tilde{P}(x)), T_x \exp_p(\tilde{P}(x))\right\rangle \overset{3.1.10}{=} \left\langle \tilde{P}(x), \tilde{P}(x)\right\rangle \text{ for } x \in \tilde{U}. \tag{3.6.3}$$

Proposition 3.6.5. Let M be a LMF, U a normal neighborhood of $p \in M$ and $\overline{p} \in U$. If there exists a TL curve from p to $\overline{p}$ then the radial geodesic σ from p to $\overline{p}$ is the unique (up to reparametrization) longest causal curve in U from p to $\overline{p}$.

Proof. We have $\sigma(t) = \exp_p(t \cdot \exp_p^{-1}(\overline{p}))$ for $t \in [0,1]$. Let $r(p') = |\exp_p^{-1}(p')|$ be the radius function on U. Then for $r \neq 0$, $U_1 := \frac{P}{r}$ is a unit vector field since

$$|\langle U_1, U_1\rangle| = |\langle P, P\rangle| \cdot \frac{1}{r^2} \overset{(3.6.3)}{=} (|\tilde{P}|^2 \circ \exp_p^{-1})\frac{1}{|\cdot|^2} \circ \exp_p^{-1} = 1.$$

Let $\alpha : [0,b] \to U$ be FDTL from p to $\overline{p}$. By Lemma 3.1.11, $\beta := \exp_p^{-1} \circ \alpha$ remains in $I^+(0) \subseteq T_pM$ (for $t > 0$). Therefore, $\alpha(t) \in \exp_p(I^+(0))$ for all $t > 0$. In particular, $\exp_p^{-1}(\overline{p}) \in I^+(0)$ and so σ is FDTL. Now write

$$\underbrace{\alpha'(t)}_{\text{TL}} = -\langle \underbrace{\alpha'(t)}_{\text{TL}}, \underbrace{U_1(\alpha(t))}_{\text{TL}}\rangle U_1(\alpha(t)) + N(t).$$

Hence $N(t)$ is spacelike (as it is orthogonal to U_1).

$$|\alpha'(t)| = -(\langle \alpha'(t), \alpha'(t)\rangle)^{\frac{1}{2}} = [\langle \alpha'(t), U_1(\alpha(t))\rangle^2 - \langle N(t), N(t)\rangle^2]^{\frac{1}{2}} \leq |\langle \alpha'(t), U_1(\alpha(t))\rangle|. \tag{3.6.4}$$

For any $x \in I^+(0)$ and $p' = \exp_p x$, $r(p') = |\exp_p^{-1}(p')| = |x| = \sqrt{-\tilde{q}(x)} = \sqrt{-q(p')}$. Therefore, $r = \sqrt{-q}$ and so

$$\text{grad}(r) = -\frac{1}{2\sqrt{-q}}\text{grad}(q) \overset{(3.6.1)}{=} -\frac{1}{2\sqrt{-q}} \cdot 2P = -\frac{1}{r}P = -U_1. \tag{3.6.5}$$

$\alpha'(t)$ and $U_1(\alpha(t))$ are TL and so

$$|\langle \alpha'(t), U_1(\alpha(t))\rangle| = -\langle \alpha'(t), U_1(\alpha(t))\rangle \overset{(3.6.5)}{=} \langle \text{grad}(r), \alpha'(t)\rangle = \frac{d(r\circ\alpha)}{dt}. \tag{3.6.6}$$

All of this holds wherever α is smooth, hence except for break points.

$$L(\alpha) = \int_0^b |\alpha'(t)|dt \overset{(3.6.4),(3.6.6)}{\leq} \int_0^b \frac{d(r\circ\alpha)}{dt}dt = r(\alpha(b)) - \underbrace{r(\overbrace{\alpha(0)}^{=p})}_{=0} = r(\overline{p}) = L(\sigma).$$

Equality holds if and only if for all t

$$|\alpha'(t)| \overset{(3.6.6)}{=} -\langle \alpha'(t), U_1(\alpha(t))\rangle \iff N(t) = 0. \tag{3.6.7}$$

Then $\alpha'(t) = -\langle \alpha'(t), U_1(\alpha(t))\rangle U_1(\alpha(t))$ and so

$$\beta'(t) = -\langle \alpha'(t), U_1(\alpha(t))\rangle \cdot \frac{\tilde{P}(\beta(t))}{|\beta(t)|}.$$

Indeed, $\alpha = \exp_p \circ \beta$ and so $\alpha' = T\exp_p \circ \beta'$. Finally,

$$\frac{\beta'}{-\langle \alpha, U_1 \circ \alpha \rangle} = (T\exp_p)^{-1} \circ U_1 \circ \exp_p \circ \beta = \frac{(\exp_p)^* P}{r \circ \exp_p} \circ \beta = \frac{\tilde{P}(\beta(t))}{|\beta(t)|}.$$

β' is proportional to $\beta(t)$ and so, from ODE theory we know that there exists a $\mathcal{C}^\infty$-function $h : (0,b) \to (0,\infty)$ and $\overline{y} \in T_pM$ where $|\overline{y}| = 1$, such that $\beta(t) = h(t) \cdot \overline{y}$. This holds for $t > 0$ but, since β is $\mathcal{C}^\infty$, it even holds on $[0,1]$. Let $\overline{x} := \Delta(p, \overline{p}) = \exp_p^{-1}(\overline{p})$. Since $\beta(p) = \overline{x}$ we get that $\overline{y} = \frac{\beta(b)}{h(b)} = \frac{\overline{x}}{h(b)}$ and so $\beta(t) = \frac{h(t)}{h(b)}\overline{x} \overset{|\overline{y}|=1}{=} \frac{h(t)}{|\overline{x}|}\overline{x}$. Therefore,

$$\alpha(b) = \exp_p(\beta(t)) = \exp_p\left(\frac{h(t)}{h(b)}\overline{x}\right) = \sigma\left(\frac{h(t)}{h(b)}\right) \Rightarrow \alpha'(t) = T_{\beta(t)}\exp_p\left(\frac{h'(t)}{h(b)}\overline{x}\right),$$

implying

$$|\alpha'(t)| = \frac{|h'(t)|}{h(b)}|\overline{x}| = |h'(t)|.$$

α' is TL and so $|\alpha'(t)| \neq 0$ for all t and so $h'(t) > 0$ for all t or $h'(t) < 0$ for all t. Now by (3.6.5), $h'(t) = (r \circ \alpha)'(t)$. Since $h(0) = 0$, $h(t) = (r \circ \alpha)(t)$ implies that $\alpha(t) = \sigma\left(\frac{r \circ \alpha(t)}{r(\overline{p})}\right)$ i.e. α is indeed a reparametrization of σ. σ is the unique longest TL curve from p to $\overline{p}$.
Finally, suppose there exists some causal curve c from p to $\overline{p}$ with $L(c) > L(r)$. $L(c) > 0$ and so c is not a null pregeodesic. According to Lemma 3.2.4, there exists a variation c_s of c with c_s TL for $0 < |s|$ small. Then $c_s \to c$ as $s \to 0$ in $\mathcal{C}^1$, implying that $L(c_s) \to L(c) > L(\sigma)$. Therefore, there exists s small such that $L(\underbrace{c_s}_{\text{TL}}) > L(\sigma)$. ↯ □

Lemma 3.6.6. Let $K \subseteq M$ compact such that strong causality holds on K. Let $c_n : [0,1] \to M$ be FD causal, $c_n(0) \to p$, $c_n(1) \to q \neq p$ and $c_n([0,1]) \subseteq K$. Then there exists a FD causal geodesic polygon γ from p to q and a subsequence $(c_{n_m})_m$ such that $\lim_{m\to\infty} L(c_n) \leq L(\gamma)$.

Proof. By Proposition 3.4.4, since all c_n have to leave a neighborhood of p there exists a limit sequence $p_0 = p < p_1 < \dots$.

a) We sow that the limit sequence is finite. Assume it was infinite. Then by Remark 3.4.5 there exists a FD and inextendible quasi-limit. By Definition 3.6.1, $c_n([0,1]) \subseteq J(p,q)$ for all $n \in \mathbb{N}$. Therefore, by Lemma 3.6.3 γ leaves K and never returns and so $p_i \notin K$ for i large. However, $c_{n_m}(t_{m,i}) \to p_i$ and so $c_{n_m}(t_{m,i}) \notin K$ for i, m large, which is a contraction to $c_n([0,1]) \subseteq K$.

b) By a), from Remark 3.4.5, there exists quasi-limit γ, a FD causal geodesic polygon from $p = p_0$ to $q = p_N$. By definition, $p_i, p_{i+1}, c_{n_m}([t_{m,i}, t_{m,i+1}]) \subseteq \tilde{K}$ for $\tilde{K}$ convex. Then by Proposition 3.6.5

$$L(c_{n_m}|_{[t_{m,i}, t_{m,i+1}]}) \leq |\underbrace{\Delta(p_{m,i}, p_{m,i+1})}_{\exp^{-1}_{p_{m,i}}(p_{m,i+1})}|$$

and so

$$L(c_{n_m}) \leq \sum_{i=0}^{N-1} |\Delta(p_{m,i}, p_{m,i+1})| \longrightarrow \sum_{i=0}^{N-1} |\Delta(p_i, p_{i+1})| = L(\gamma),$$

since Δ is continuous on $\tilde{K}$'s. Therefore, $0 \leq L(c_{n_m}) \leq 2L(\gamma)$ for m large and so there exists a subsequence, without loss of generality, c_{n_m} itself such that there exists $\lim L(c_{n_m})$ and $\lim L(c_{n_m}) \leq L(\gamma)$.

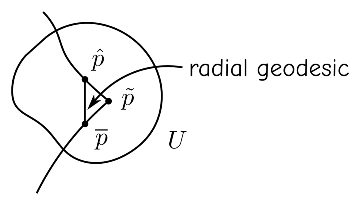
□

Theorem 3.6.7 (Avez-Seifert Theorem)**.** Let $p < q$ in (a spacetime) M with $J(p,q)$ compact and such that strong causality holds in each point of $J(p,q)$ (e.g. M globally hyperbolic). Then there exists a causal geodesic γ from p to q with

$$L(\gamma) = \tau(p,q)$$

i.e. of maximal length.

Proof. By definition of τ, there exists a FD causal $c_n : [0,1] \to M$ such that $c_n(0) = p$, $c_n(1) = q$ and such that $L(c_n) \to \tau(p,q)$. By Definition 3.6.1 $c_n([0,1]) \subseteq J(p,q)$ for all $n \in \mathbb{N}$. By Lemma 3.6.6, there exists FD causal geodesic polygon γ from p to q such that $\tau(p,q) = \lim_m L(c_{n_m}) \leq L(\gamma) \leq \tau(p,q)$. Therefore, $L(\gamma) = \tau(p,q)$. If γ had any break points then by Proposition 3.6.5 we could find a strictly longer curve.



Indeed, let $\tilde{p}$ be a break point, U a convex neighborhood around $\tilde{p}$, $\overline{p}$ a point before the break point and $\hat{p}$ a point after the break. Then we can replace γ between $\overline{p}$ and $\hat{p}$ by radial geodesic of greater length. □

Proposition 3.6.8. Let $X \subseteq M$ be open and globally hyperbolic. Then $\tau : X \times X \to \mathbb{R}$ (where τ is a time separation function in X) is finite and continuous.

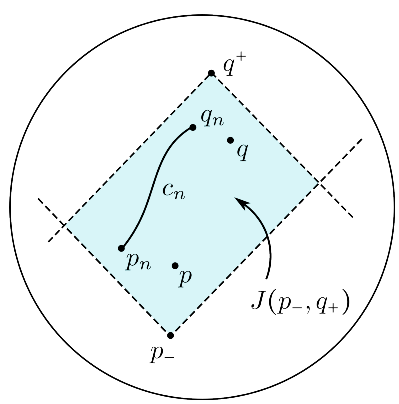
Proof. By Theorem 2, $\tau < \infty$ on $X \times X$ and, by Proposition 3.1.26, τ is lower semicontinuous. Suppose that τ is not upper semicontinuous. Then there exist $\delta > 0$, $p_n \to p$, $q_n \to q$ such that

$$\tau(p_n, q_n) \geq \tau(p,q) + \delta. \tag{3.6.8}$$

Choose $c_n : [0,1] \to X$, $c_n(0) = p_n$, $c_n(1) = q_n$ and

$$L(c_n) \geq \tau(p_n, q_n) - \frac{1}{n}. \tag{3.6.9}$$

X is open and therefore there exist $p_-, q_+ \in X$ such that $p_- \ll p, q_+ \gg q$. $p_n \in I^+(p_-)$, $q_n \in I^-(q_+)$ for n large. $c_n([0,1]) \subseteq I^+(p_-) \cap I^-(q_+) \subseteq J(p_-, q_+) =: K$, where K is compact in X by assumption.



By assumption, strong causality holds on K (see Definition 3.6.1). Therefore, by Lemma 3.6.6, there exists a FD causal geodesic polygon γ from p to q and a subsequence $(c_{n_m})_m$ such that

$$\lim_{m\to\infty} L(c_{n_m}) \leq L(\gamma) \leq \tau(p,q). \tag{3.6.10}$$

But, by (3.6.9), $L[c_{n_m}] \geq \tau(p_{n_m}, q_{n_m}) - \frac{1}{n_m} \overset{(3.6.8)}{\geq} \tau(p,q) + \delta - \frac{1}{n_m}$ and so (3.6.10) implies that $\tau(p,q) \geq \lim L[c_{n_m}] \geq \tau(p,q) + \delta$, which is a contradiction. □

Proposition 3.6.9. Let $X \subseteq M$ be open and globally hyperbolic. Then $\leq$ is closed on X.

Proof. Let $p_n \leq q_n$, $p_n \to p$, $q_n \to q$ in X. If $p = q$, we are done. Let $p \neq q$. Then there exists subsequences, without loss of generality the same ones, such that $p_n \neq q_n$ i.e. $p_n < q_n$. Choose FD causal $c_n : [0,1] \to X$ such that $c_n(0) = p_n$ and $c_n(1) = q_n$. As in Proposition 3.6.8, pick $p_-, p_+ \in X$ so that $c_n([0,1]) \subseteq J(p_-, q_+)$. Then by Lemma 3.6.6 there exists a FD causal geodesic polygon from p to q. In particular, $p \leq q$. □

3.7 Index Forms and Lengths of Curves

In this section we introduce tools on calculus of variations to study the length of curves in a variation $[a,b] \times (-\delta, \delta) \to M$

$$(t,s) \mapsto c_s(t) \equiv c(t,s)$$

of base curve $c(t) = c_0(t) = c(t,0)$. Then $V \in \mathfrak{X}(c)$, $V(t) = \partial_s|_0 c_s(t)$ is the variation vector field of c_s and for any $s \in (-\delta, \delta)$ let

$$L_c(s) := \int_a^b |\dot{c}_s(t)| dt$$

denote the length of c_s.

Definition 3.7.1. If $L = L_c$ is twice differentiable, we call

$$L'(0) := \left.\frac{dL}{ds}\right|_{s=0} \text{ and } L''(0) := \left.\frac{d^2L}{ds^2}\right|_{s=0}$$

first and second variation of arclength, respectively.

Remark 3.7.2. If $|c'| > 0$ everywhere, then either c is spacelike or timelike everywhere. We call

$$\epsilon := \operatorname{sgn} \langle c', c' \rangle = \pm 1$$

the signum of c.

Lemma 3.7.3. If c has signum ϵ then

$$L'(0) = \epsilon \int_a^b \left\langle \frac{\dot{c}(t)}{|\dot{c}(t)|}, V'(t) \right\rangle dt.$$

Proof. Since $|\langle \partial_t c(t,0), \partial_t c(t,0)\rangle| = |\langle \dot{c}(t), \dot{c}(t)\rangle| > 0$, there exists $\delta_0 > 0$ such that $|\langle \partial_t c(t,s), \partial_t c(t,s)\rangle| > 0$ for all $(t,s) \in [a,b] \times [-\delta_0, \delta_0]$ and so $|\partial_t c(t,s), \partial_t c(t,s)|^{\frac{1}{2}} = |\partial_t c(t,s)|$ is $\mathcal{C}^\infty$. Therefore,

$$L'(0) = \int_a^b \partial_s|_0 |\partial_t c_s(t)| dt.$$

For s small, sgn $\langle \partial_t c(t,s), \partial_t c(t,s)\rangle = \epsilon$ and so $|\partial_t c(t,0)| = (\epsilon \langle \partial_t c(t,s), \partial_t(t,s)\rangle)^{\frac{1}{2}}$.

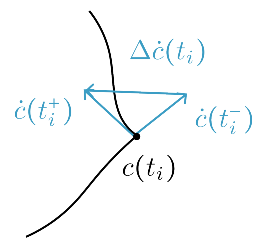
$$\begin{aligned} \partial_s |\partial_t c_s(t)| &= \frac{1}{\cancel{2}} (\epsilon \langle \partial_t c(t,s), \partial_t c(t,s)\rangle)^{-\frac{1}{2}} \cancel{2}\epsilon \left\langle \partial_t c_s, \underbrace{\frac{\nabla}{ds}\partial_t c_s}_{=\frac{\nabla}{dt}\partial_s c_s} \right\rangle \\ \partial_s|_0 |\partial_t c_s(t)| &= (\epsilon \langle \dot{c}(t), \dot{c}(t)\rangle)^{-\frac{1}{2}} \epsilon \langle \dot{c}(t), V'(t)\rangle = \epsilon \left\langle \frac{\dot{c}(t)}{|\dot{c}(t)|}, V'(t) \right\rangle. \end{aligned}$$

□

Definition 3.7.4.
Let $c : [a,b] \to M$ be a piecewise smooth curve. A variation c_s of c is called **piecewise smooth variation** if $c : (t,s) \mapsto c_s(t)$ is continuous and there exist break points

$$a = t_0 < t_1 < \cdots < t_{k+1} = b$$

such that $c|_{[t_{i-1},t_i]} \times (-\delta, \delta)$ is smooth for all $i \in \mathbb{N}$. Then $V = \partial_s|_0 c_s$ is piecewise smooth as well. $\dot{c}$ has breaks at t_i (for $1 \le i \le k$). Let



$$\Delta \dot{c}(t_i) \coloneqq \dot{c}(t_i^+) - \dot{c}(t_i^-) \in T_{c(t_i)M}.$$

Proposition 3.7.5. Let $c : [a,b] \to M$ be a piecewise smooth curve with constant velocity $v = |\dot{c}(t)|$ for all $t \in [1,b]$ and signum ϵ. Let $c_s(t)$ be a piecewise smooth variation of c with break points $t_1 < \cdots < t_k$. Then

$$L'(0) = -\frac{\epsilon}{v} \int_a^b \langle \ddot{c}(t), V(t)\rangle \, dt - \frac{\epsilon}{v} \sum_{i=1}^{k} \langle \Delta\dot{c}(t_i), V(t_i)\rangle + \frac{\epsilon}{v} \langle \dot{c}(t), V(t)\rangle \Big|_a^b.$$

Proof. From Lemma 3.7.3 we know that

$$L'(0) = \sum_{i=0}^{k} L'_{c|_{[t_i,t_{i+1}]}}(0) = \sum_{i=0}^{k} \frac{\epsilon}{v} \int_{t_i}^{t_i+1} \langle \dot{c}(t), V'(t)\rangle \, dt.$$

Here $\langle \dot{c}(t), V'(t)\rangle = \partial_t \langle \dot{c}, V\rangle - \langle \ddot{c}, V\rangle$ and so

$$\int_{t_i}^{t_i+1} \langle \dot{c}(t), V'(t)\rangle \, dt = \langle \dot{c}, V\rangle \big|_{t_i}^{t_{i+1}} - \int_{t_i}^{t_{i+1}} \langle \ddot{c}, V\rangle \, dt.$$

Summing over i we get the claim. □

In addition to the transversal velocity vector field V, we also consider the transversal acceleration vector field:

$$A = \frac{\nabla}{\partial_s}\Big|_0 \frac{\partial c_s}{\partial t} \in \mathfrak{X}(c).$$

If $|\dot{c}| > 0$, then any $Y \in \mathfrak{X}(c)$ can be uniquely decomposed as $Y = Y^\top + Y^\perp$, where $Y^\top = \frac{1}{\langle \dot{c}, \dot{c}\rangle} \langle Y, \dot{c}\rangle \dot{c}$ is the parallel part of c and $Y^\perp = Y - Y^\top$ the orthogonal one. If c is a geodesic i.e. if $\ddot{c} = 0$, then $\partial_t \left(\frac{1}{\langle \dot{c}, \dot{c}\rangle}\right) = 0$ yielding $(Y^\top)' = \frac{1}{\langle \dot{c}, \dot{c}\rangle} \langle Y', \dot{c}\rangle \dot{c} = (Y')^\top$. $Y' = \underbrace{(Y')^\top}_{=(Y^\top)'} + (Y')^\perp$ now so $(Y')^\perp = (Y - Y^\top) = (Y^\perp)'$. Hence, we express $\overset{\perp}{Y}'$ as a shorthand for $(Y')^\perp = (Y^\perp)'$.

Theorem 3.7.6 (Synge Formula). Let $\sigma : [a,b] \to M$ be a geodesic with velocity v and signum ϵ. If $(t,s) \mapsto \sigma(t,s)$ is a variation of σ (so $\sigma(t) \equiv \sigma(t,0)$) with velocity-vector field V and acceleration-vector field A, then

$$L''(0) = \frac{\epsilon}{v}\int_a^b [\langle \overset{\perp}{V}'(t), \overset{\perp}{V}'(t)\rangle - \langle R_{V\dot\sigma}V, \dot\sigma\rangle(t)]dt + \frac{\epsilon}{v}\langle \dot\sigma(t), A(t)\rangle\Big|_a^b.$$

Proof. Let $h(t,s) := |\partial_t\sigma(t,s)|$. Then $L(s) = \int_a^b h(t,s)dt$ and $L''(s) = \int_a^b \frac{\partial^2 h}{\partial s^2}dt$. The proof of Lemma 3.7.3 implies that $\frac{\partial h}{\partial s} = \frac{\epsilon}{h}\left\langle \partial_t\sigma, \frac{\nabla}{\partial s}\frac{\partial}{\partial t}\sigma\right\rangle$. Now

$$\begin{aligned}
\frac{\partial^2 h}{\partial s^2} &= \frac{\epsilon}{h^2}\left\{h\partial_s\left\langle \partial_t\sigma, \frac{\nabla}{\partial s}\frac{\partial\sigma}{\partial t}\right\rangle - \left\langle \partial_t\sigma, \frac{\nabla}{\partial s}\frac{\partial\sigma}{\partial t}\right\rangle\frac{\partial h}{\partial s}\right\} \\
&= \frac{\epsilon}{h}\left\{\left\langle \frac{\nabla}{\partial s}\frac{\partial\sigma}{\partial t}, \frac{\nabla}{\partial s}\frac{\partial\sigma}{\partial t}\right\rangle + \left\langle \frac{\partial\sigma}{\partial t}, \frac{\nabla}{\partial s}\frac{\nabla}{\partial s}\frac{\partial\sigma}{\partial t}\right\rangle - \frac{\epsilon}{h^2}\left\langle \partial_t\sigma, \frac{\nabla}{\partial s}\frac{\partial\sigma}{\partial t}\right\rangle^2\right\} \\
&= \frac{\epsilon}{h}\left\{\left\langle \frac{\nabla}{\partial t}\frac{\partial\sigma}{\partial s}, \frac{\nabla}{\partial t}\frac{\partial\sigma}{\partial s}\right\rangle + \left\langle \frac{\partial\sigma}{\partial t}, R\left(\frac{\partial\sigma}{\partial t}, \frac{\partial\sigma}{\partial s}\right)\frac{\partial\sigma}{\partial s}\right\rangle + \left\langle \frac{\partial\sigma}{\partial t}, \frac{\nabla}{\partial t}\frac{\partial}{\partial s}\frac{\partial\sigma}{\partial s}\right\rangle - \frac{\epsilon}{h^2}\left\langle \partial_t\sigma, \frac{\nabla}{\partial t}\frac{\partial\sigma}{\partial s}\right\rangle^2\right\},
\end{aligned}$$

where

$$\frac{\nabla}{\partial s}\frac{\nabla}{\partial s}\frac{\partial\sigma}{\partial t} = \frac{\nabla}{\partial s}\frac{\nabla}{\partial t}\frac{\partial\sigma}{\partial s} = \frac{\nabla}{\partial t}\frac{\partial}{\partial s}\frac{\partial\sigma}{\partial s} + R\left(\frac{\partial\sigma}{\partial t}\frac{\partial\sigma}{\partial s}\right)\frac{\partial\sigma}{\partial s}.$$

Now set $s = 0$. Then $h(t,0) = v$, $\frac{\partial\sigma}{\partial t}(t,0) = \dot\sigma(t)$, $\frac{\partial\sigma}{\partial s}(t,0) = V(t)$,

$$\frac{\nabla}{\partial t}\frac{\partial\sigma}{\partial s} = V'(t),\ \ \frac{\nabla}{\partial s}\frac{\partial\sigma}{\partial s}(t,0) = A(t),\ \ \frac{\nabla}{\partial t}\frac{\partial}{\partial s}\frac{\partial\sigma}{\partial s}(t,0) = A'(t)$$

and so

$$\frac{\partial^2 h}{\partial s^2}\Big|_{s=0} = \frac{\epsilon}{v}\left\{\langle V', V'\rangle - \langle R_{V\dot\sigma}V, \dot\sigma\rangle + \langle\dot\sigma, A'\rangle - \frac{\epsilon}{v^2}\langle\dot\sigma, V'\rangle^2\right\}.$$

We have $\langle\dot\sigma, A'\rangle \overset{\ddot\sigma=0}{=} \frac{d}{dt}\langle\dot\sigma, A\rangle$. $\frac{\dot\sigma}{v}$ is a unit vector field and so $(V')^\top = \epsilon\left\langle V', \frac{\dot\sigma}{v}\right\rangle\frac{\dot\sigma}{v}$, implying that

$$\begin{aligned}
V' = \frac{\epsilon}{v^2}\langle V', \dot\sigma\rangle + \overset{\perp}{V}' \quad &\Longrightarrow \quad \langle V', V'\rangle \overset{\langle\dot\sigma,\dot\sigma\rangle=\epsilon v^2}{=} \frac{\epsilon}{v}\langle V', \dot\sigma\rangle + \langle\overset{\perp}{V}', \overset{\perp}{V}'\rangle \\
&\Longrightarrow \quad \frac{\partial^2 h}{\partial s^2}\Big|_{s=0} = \frac{\epsilon}{v}\left\{\langle\overset{\perp}{V}', \overset{\perp}{V}'\rangle - \langle R_{V\dot\sigma}V, \dot\sigma\rangle + \frac{d}{dt}\langle\dot\sigma, A\rangle\right\} \overset{\int_a^b}{\Longrightarrow} \text{claim.}
\end{aligned}$$

□

Let P be a SRSMF of M, $q \notin P$ and set

$$\Omega(P,q) := \{\alpha : [0,b] \to M \mid \alpha \text{ pointwise } \mathcal{C}^\infty,\ \alpha(0) \in P,\ \alpha(b) = q\}.$$

A 'curve' in $\Omega(P,q)$ with initial value α is a piecewise $\mathcal{C}^\infty$-variation of α such that each longitudinal curve lies in $\Omega(P,q)$. Hence the first transversal curve lies in P and the last is equal to q. Such variations are called **(P,q)-variations**. The corresponding variation-vector fields can be viewed as the 'tangent space' of $\Omega(P,q)$ at α:

$$T_\alpha\Omega \equiv T_\alpha(\Omega(P,q)) := \left\{V : V \text{ pointwise } \mathcal{C}^\infty \text{ vector field along } \alpha,\ V(0) \in T_{\alpha(0)}P,\ V(b) = 0\right\}.$$

Set

$$T_\alpha\Omega^\perp := \{V \in T_\alpha\Omega \mid V \perp \dot\alpha\}.$$

By Lemma 3.2.12, any P-JF lies in $T_\alpha(\Omega(P,q))$ and, by Remark 3.2.14, if such a J has a zero focal point (i.e. $J(r) = 0$ at some $r > 0$), then $J \in T_\alpha(\Omega(P,q))^\perp$.

Lemma 3.7.7. Let $\sigma \in \Omega(P,q)$ be a geodesic and suppose that $L'(0) = 0$ for any (P,q)-variation of σ. Then $\dot\sigma$ is perpendicular to P i.e. $\dot\sigma(0) \in N_{\sigma(0)}P = T_{\sigma(0)}P^\perp$. In particular, this holds if $L(\sigma) = \max\{L(\alpha) : \alpha \in \Omega(P,q)\}$.

Proof. Let $y \in T_{\sigma(0)}P$ and pick $V \in T_\alpha\Omega$ with $V(0) = y$ and let $(t,s) \mapsto \sigma(t,s)$ be a (P,q)-variation of σ with variation vector field V (use Fermi-coordinates). Then since σ has no break points,

$$0 = L'(0) \overset{3.7.5}{=} \frac{\epsilon}{v}\langle\dot\sigma, V\rangle\Big|_0^b \overset{V(b)=0}{=} -\frac{\epsilon}{v}\langle\dot\sigma(0), y\rangle \implies \dot{\sigma(0)} \in N_{\sigma(0)}P.$$

Finally, if $L(\sigma) = \max\{L(\alpha) : \alpha \in \Omega(P,q)\}$ then $L(\sigma)$ is a local maximum of $s \mapsto L(s)$ for any (P,q)-variation of σ and so $L'(0) = 0$ for any such variation. □

Now let $\sigma \in \Omega(P,q)$ be a geodesic with $\sigma \perp P$ and $(t,s) \mapsto \sigma(t,s)$ a variation. By Theorem 3.7.6 we have

$$L''(0) = \frac{\epsilon}{v}\int_0^b [\langle \overset{\perp}{V}'(t), \overset{\perp}{V}'(t)\rangle - \langle R_{V\dot\sigma}V, \dot\sigma\rangle]dt + \frac{\epsilon}{v}\langle\dot\sigma(t), A(t)\rangle\Big|_0^b.$$

Now $\sigma(b,s) \equiv q$ implies that $A(t) = \frac{\nabla}{\partial s}\Big|_0 \frac{\partial\sigma(b,s)}{\partial s} = 0$. Let γ be the first transversal curve, $\gamma(s) = \sigma(0,s)$. Then $A(0) = \ddot\gamma(0)$ ($\ddot{}$ in M) and

$$\langle\dot\sigma(0), A(0)\rangle = \langle\sigma(0), \ddot\gamma(0)\rangle \overset{\dot\sigma\perp P}{=} \langle\dot\sigma(0), \mathrm{nor}(\ddot\gamma(0))\rangle \overset{1.5.2}{=} \langle\dot\sigma(0), \mathbb{I}(\dot\gamma(0), \dot\gamma(0))\rangle = \langle\dot\sigma(0), \mathbb{I}(V(0), V(0))\rangle.$$

Therefore,

$$L''(0) = \frac{\epsilon}{v}\int_0^b [\langle \overset{\perp}{V}'(t), \overset{\perp}{V}'(t)\rangle - \langle R_{V\dot\sigma}V, \dot\sigma\rangle]dt - \frac{\epsilon}{v}\langle\dot\sigma(0), \mathbb{I}(V(0), V(0))\rangle. \tag{3.7.1}$$

Definition 3.7.8. Let $\sigma \in \Omega(P,q)$ be a non-null geodesic with $\sigma \perp P$. The **index form** I_σ of σ is the unique bilinear form

$$I_\sigma : T_\sigma\Omega \times T_\sigma\Omega \to \mathbb{R}$$

such that $I_\sigma(V,V) = L''(0)$ for any P-variation of σ with variation vector field $V \in T_\sigma\Omega$.

Remark 3.7.9. Existence and uniqueness of I_σ follow from (3.7.1) by polarization. Explicitly,

$$I_\sigma(V,W) = \frac{\epsilon}{v}\int_0^b [\langle \overset{\perp}{V}'(t), \overset{\perp}{W}'(t)\rangle - \langle R_{V\dot\sigma}W, \dot\sigma\rangle]dt - \frac{\epsilon}{v}\langle\dot\sigma(0), \mathbb{I}(V(0), W(0))\rangle.$$

I_σ serves as a kind of 'Hessian' in this context.

Remark 3.7.10. Tangential parts can be disregarded,

$$I_\sigma(V,W) = I_\sigma(V^\perp, W^\perp).$$

Indeed, $((V^\perp)')^\perp = (V')^{\perp\perp} = (V')^\perp = \overset{\perp}{V}'$, $V(0) = V^\perp(0)$ since $V(0)$ is tangential to P and $\sigma \perp P$. Finally,

$$\langle R_{V\dot\sigma}W, \dot\sigma\rangle \overset{R_{V^\top\dot\sigma}=0}{=} \langle R_{V^\perp\dot\sigma}W, \dot\sigma\rangle \overset{\text{pair symm.}}{=} \left\langle R_{W\dot\sigma}V^\perp, \dot\sigma\right\rangle = \left\langle R_{W^\perp\dot\sigma}V^\perp, \dot\sigma\right\rangle = \left\langle R_{V^\perp\dot\sigma}W^\perp, \dot\sigma\right\rangle.$$

Proposition 3.7.11. Let $\sigma \in \Omega(P,q)$ be a non-null geodesic. If σ and $V \in T_\sigma\Omega$ have breaks at $t_1 < \cdots < t_k$ then for all $T_\sigma\Omega$

$$\begin{aligned} I_\sigma(V,W) &= -\frac{\epsilon}{v}\int_0^b \langle \overset{\perp}{V}'' - R(V^\perp,\dot\sigma)\dot\sigma, W^\perp\rangle dt - \frac{\epsilon}{v}\sum_{i=1}^{k}\langle \Delta \overset{\perp}{V}', W^\perp\rangle(t_i) \\ &= -\frac{\epsilon}{v}\langle V'(0), W(0)\rangle - \frac{\epsilon}{v}\langle \dot\sigma(0), \mathbb{I}(V(0),W(0))\rangle\,. \end{aligned}$$

Proof. Away from the break points we have

$$\langle \overset{\perp}{V}', \overset{\perp}{W}'\rangle = \frac{d}{dt}\langle \overset{\perp}{V}', W^\perp\rangle - \underbrace{\langle \overset{\perp}{V}'', W^\perp\rangle}_{(\star)}.$$

Moreover,

$$\langle R_{V\dot\sigma}W, \dot\sigma\rangle = -\langle R_{V\dot\sigma}\dot\sigma, W\rangle \overset{\text{cf. 3.7.10}}{=} \underbrace{-\left\langle R_{V^\perp\dot\sigma}\dot\sigma, W^\perp\right\rangle}_{(\star\star)}.$$

$(\star)$ and $(\star\star)$ give the integrand. We integrate the remaining term over $[t_i, t_{i+1}]$:

$$\int_{t_i}^{t_{i+1}} \frac{d}{dt}\langle \overset{\perp}{V}', W^\perp\rangle dt = \langle \overset{\perp}{V}', W^\perp\rangle\Big|_{t_i}^{t_{i+1}}.$$

$W(b) = 0$ and for $1 \le i \le k$, we get

$$\langle \overset{\perp}{V}'(t_i^-), \overset{\perp}{W}(t_i)\rangle - \langle \overset{\perp}{V}'(t_i^+), \overset{\perp}{W}(t_i)\rangle = -\langle \Delta\overset{\perp}{V}', W^\perp\rangle(t_i),$$

Summing over i yields the claim. ☐

Theorem 3.7.12. Let $\sigma \in \Omega(P,q)$ be a TL geodesic, $\sigma \perp P$ and suppose that there exists a focal point $\sigma(r)$ of P along σ with $0 < r < b$. Then there exists a TL curve γ in $\Omega(P,q)$ with $L(\gamma) > L(\sigma)$.

Proof.

a) There exists $z \in T_\sigma\Omega$ such that $I_\sigma(z,z) > 0$. According to Definition 3.2.13, there exists P-JF $0 \neq J \perp \sigma$ on $[0,b]$ with $J(r) = 0$. Let Y be the vector field along σ with

$$Y(t) := \begin{cases} J(t), & t \in [0,r] \\ 0, & t \in [r,b]. \end{cases}$$

Then $Y = Y^\perp$ (implying that $Y' = \overset{\perp}{Y}'$) and $Y'(r^-) = J'(r) \neq 0$ (since $J \not\equiv 0$). Since $Y'(r^+) = 0$, $(\Delta Y')(r) \neq 0$. Choose a $W \in T_\sigma\Omega$ with $W(r) = (\Delta Y')(r)$ and $W \perp \sigma$ (implying that $W = W^\perp$). We now show that, setting $Z := Y + \kappa W$ we have $I_\sigma(Z,Z) > 0$ for $\kappa > 0$ small. We have

$$I_\sigma(Y+\kappa W, Y+\kappa W) = I_\sigma(Y,Y) + 2\kappa I_\sigma(Y,W) + \kappa^2 I_\sigma(W,W).$$

By definition, Y is a JF (on $[0,r]$ and on $[r,b]$). Also, $Y(r) = 0$ i.e. Y vanishes at its only breakpoint (without loss of generality $v = 1$). Proposition now 3.7.11 reduces to:

$$\begin{aligned} I_\sigma(Y,Y) &= \langle Y'(0), Y(0)\rangle + \langle \dot\sigma(0), \mathbb{I}(Y(0),Y(0))\rangle \\ &= \langle J'(0), \underbrace{J(0)}_{\text{tangential to } P} \rangle + \underbrace{\langle \dot\sigma(0), \mathbb{I}(J(0),J(0))\rangle}_{\overset{1.6.9}{=} -\langle \tilde{\mathbb{I}}(J(0),\dot\sigma(0)),J(0)\rangle} \\ &= \left\langle \tan(J'(0)) - \tilde{\mathbb{I}}(J(0),\dot\sigma(0)), J(0)\right\rangle = 0, \end{aligned}$$

since J is a P-JF. Moreover, by definition of W we have (as in the equation above):

$$\begin{aligned} I_\sigma(Y,W) &= \langle \Delta Y'(r), \Delta Y'(r)\rangle + \langle \underbrace{\tan(J'(0)) - \tilde{\mathbb{I}}(J(0),\dot\sigma(0))}_{=0}, W(0)\rangle \\ &= \langle \Delta Y'(r), \Delta Y'(r)\rangle > 0, \end{aligned}$$

since $\Delta Y'(r) \perp \dot\sigma(r)$ ($\dot\sigma(r)$ is timelike and, therefore, $\Delta Y'(r)$ spacelike). Finally,

$$I_\sigma(Y+\kappa W, Y+\kappa W) = 2\kappa \langle \Delta Y'(r), \Delta Y'(r)\rangle + \kappa^2 I_\sigma(W,W) > 0$$

for $\kappa > 0$ small.

b) There exists a TL curve $\gamma \in \Omega(P,q)$ with $L(\gamma) > L(\sigma)$. Let $\sigma(t,s)$ be a variation of σ with variation-vector field Z, $\sigma(0,s) \in P$ and $\sigma(b,s) = q$ for all s (use Fermi-coordinates). By Proposition 3.7.5, $L'(0) = 0$ and, by Definition 3.7.8, $L''(0) = I_\sigma(Z,Z) > 0$. Hence, by Taylor:

$$L(\sigma_s) \equiv L(s) = L(0) + 0 + \frac{s^2}{2} I_\sigma(Z,Z) + \mathcal{O}(s^3) > L(\sigma)$$

for $0 < |s|$ small.

□

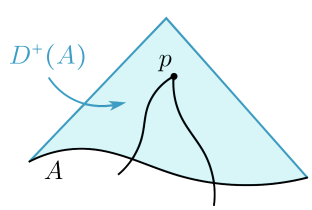
3.8 Cauchy Developments and Cauchy Horizons

The Cauchy development of an achronal set A refers to the portion of spacetime that is causally influenced by A—in other words, it comprises the events completely determined by A. An intriguing aspect lies in exploring the boundaries of this set, known as the *Cauchy horizon*, which marks the region where predictability stemming from A ceases. This section delves into examining the characteristics of these sets.

Definition 3.8.1. Let $A \subseteq M$ be achronal.

1. The **future Cauchy development** of A is defined as

$$D^+(A) := \{p \in M \mid \text{every past-inextendible causal curve through } p \text{ intersects } A\}.$$



2. The past Cauchy development is defined analogously;

$$D^-(A) := \{p \in M \mid \text{every future-inextendible causal curve through } p \text{ intersects } A\}.$$

3. The Cauchy development of A is defined as

$$D(A) := D^+(A) \cup D^-(A).$$

Remark 3.8.2.

1. $A \subseteq D^\pm(A) \subseteq A \cup I^\pm(A) \subseteq J^\pm(A)$. To see this, note that for $p \in D^+(A)$ and γ PDTL curve from p, $\gamma \cap A = \varnothing$. Therefore, $p \in I^+(A)$ unless $p \in A$ to begin with.
2. $D^\pm(A) \cap I^\mp(A) = \varnothing$. Suppose that $p \in D^\pm(A) \cap \in I^-(A)$. Then, since $p \in I^-(A)$, there exists a PDTL curve c from $a \in A$ to p. Extend c beyond p to the past via γ. Since $p \in D^+(A)$, $c \cup \gamma$ has to intersect A, which is in contradiction to the assumption that A is achronal.
3. $A = D^+(A) \cap D^-(A)$ since

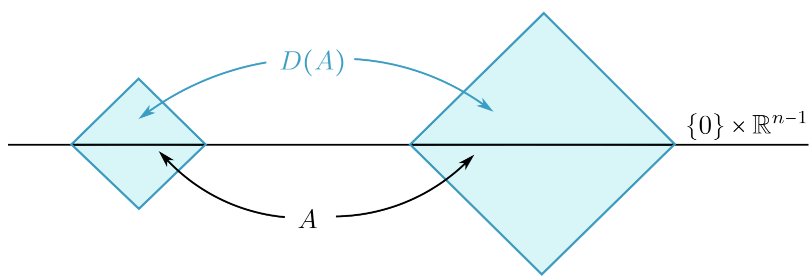
$$A \subseteq D^+(A) \cap D^-(A) \subseteq D^+(A) \cap (A \cup I^-(A)) \overset{2.}{=} D^+(A) \cap A = A.$$

4. $D(A) \cap I^\pm(A) = D^\pm(A) \backslash A$ since

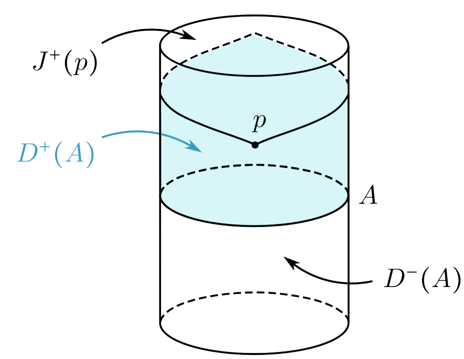
$$D(A) = D^+(A) \cup D^-(A) \implies D(A) \cap I^\pm(A) \overset{2.}{=} D^\pm(A) \cap I^\pm(A) = D^\pm(A) \backslash A.$$

Example 3.8.3.

1. Let $M = \mathbb{R}^n_1$ and $A := \{0\} \times \mathbb{R}^{n-1}$. Then $D^\pm(A) = J^\pm(A) = A \cup I^\pm(A)$.
2. Let M be any Lorentzian manifold with a Cauchy hypersurface S. By Proposition 3.5.23 (3.) every inextendible causal curve meets S. By the proof of Proposition 3.5.23 (1.), $M = I^-(S) \dot\cup S \dot\cup I^+(S)$ and so $D^\pm(S) = I^\pm(S) \cup S$ and $D(S) = M$ i.e. the Cauchy development of a Cauchy surface is all of spacetime. Conversely, if S is achronal and $D(S) = M$ then S is a Cauchy surface in M.
3. Let $M = \mathbb{R}^n_1$, $A = \{0\} \times B$ and $B \subseteq \mathbb{R}^{n-1}$. Then $D(A)$ is the double cone over B.



4. For $(M, g) = (\mathbb{R} \times S^1, -dt^2 + d\theta^2)$ and $A = \{0\} \times S^1$ we have that $D^\pm(A) = J^\pm(A)$. For $p \in I^+(A)$ and $\tilde{M} := M \backslash p$ we still have that $D^+(A) = J^-(A)$ but $D^+(A)$ is given by the union of A and the shaded region between A and future directed null geodesics emanating from p i.e. $D^+(A) = J^+(A) \backslash J^+(p)$.



Lemma 3.8.4. Let $A \subseteq M$ be achronal. Then:

1. Every PD causal curve starting in $D^+(A)$ and leaving it intersects A.
2. Every past (future) inextendible causal curve through $p \in D(A)^\circ$ intersects $I^-(A)$ $(I^+(A))$.

Proof.

1. Let $c : [0, b] \to M$ be a causal PD curve, $c(0) \in D^+(A)$ and $c(b) \notin D^+(A)$. Then there exists some past inextendible causal curve γ, which starts in $c(b)$ but does not hit A. Then the concatenation $c \cup \gamma$ yields a past inextendible causal curve through $c(0) \in D^+(0)$ which, by Definition 3.8.1, intersects A. Therefore, c intersects A.
2. By Remark 3.8.2 (1.) we have that $D(A) \subseteq A \cup I^+(A) \cup I^-(A)$. Let c be a PD causal inextendible curve starting in $p \in D(A)^\circ$. For $p \in I^-(A)$ the claim is trivial. Let, therefore, $p \in A \cup I^+(A)$ and choose $q \in I^+(p) \cap D(A)$. The proof of Lemma 3.5.21 (1.) with $A = \varnothing$ shows that there is a past inextendible timelike curve $\tilde{c}$ starting in q as constructed there. Then every point $\tilde{c}(s)$ has $I^-(\tilde{c}(s)) \cap c \neq \varnothing$. $q \in D^+(A)$ and so $\tilde{c}$ intersects A. Therefore, $I^-(A) \cap c \neq \varnothing$.

□

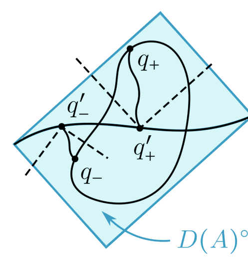
Theorem 3.8.5. Let $A \subseteq M$ be achronal. Then $D(A)^\circ$ is globally hyperbolic.

Proof.

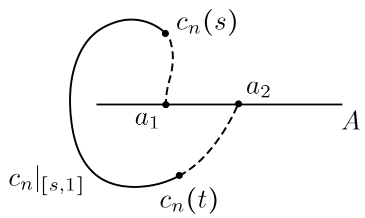
a) The causality condition holds at any $p \in D(A)^\circ$. Suppose there exists a causal loop c through $x \in D(A)^\circ$. Due to Lemma 3.8.4 (2.) we find points $q_\pm \in I^\pm(A)$ on c. Hence, there are $q'_\pm \in A$ such that $q_\pm \in I^\pm(q'_\pm)$, that is,

$$q'_+ \ll q_+ \leq q_- \ll q'_-.$$

By Proposition 3.1.8, $q'_+ \ll q'_-$, which is a contradiction to A being achronal. Therefore, there cannot exist such causal loops.



b) Strong causality holds at any $p \in D(A)^\circ$. Suppose strong causality fails at some $p \in D(A)^\circ$ i.e. suppose that there is a sequence of causal FD curves $c_n : [0, 1] \to M$, $n \in \mathbb{N}$ with $\lim_{n\to\infty} c(0) = p = \lim_{n\to\infty} c_n(1)$ and a neighborhood U of p such that for all n, c_n is not entirely contained in U. By Proposition 3.4.4 there exists a limit sequence $p =: p_0 < p_1 < \dots$ of $(c_n)_{n\in\mathbb{N}}$. If it is finite, then $p_N = p$ (because $c_n(1) \to p$), that is $p < p$ and hence we obtain a contradiction to a). Therefore, suppose the limit sequence is infinite and the corresponding quasi-limit γ future inextendible. According to Lemma 3.8.4 (2.), it meets $I^+(A)$ and does not leave it. That is, there exists some $p_{i_0} \in I^+(A)$. Possibly passing on to some subsequence and after a reparametrization, there exists $s \in (0, 1)$ such that $c_n(s) \to p_i$. In particular, we have $c_n(s) \in I^+(A)$ for n large enough. Let $\tilde{c}_n : [s, 1] \to M$ such that $\tilde{c}_n(t) := c_n(s+1-t)$ (i.e. $\tilde{c}_n = -c_n|_{[s,1]}$ in the homotopy sense). Then $\tilde{c}_n$ is causal and PD. Now apply Proposition 3.4.4 to $\tilde{c}_n$ and obtain a limit sequence $p := q_0 > q_1 > \dots$ (it starts at p because $\tilde{c}(s) = c_n(1) \to p$). That limit sequence is infinite since, otherwise, $p = q_0 > \dots > q_{N'} = p_{i_0} = \lim \tilde{c}_n(1) > p$, which contradicts a).



By Remark 3.4.5, there exists a past inextendible quasi-limit $\hat{\gamma}$ that starts at $p \in D(A)^\circ$. From Lemma 3.8.4 (2.) it follows that $\hat{\gamma}$ intersects $I^-(A)$ and so there exists $\tilde{t} \in (s,1]$ such that $\tilde{c}_n(\tilde{t}) \in I^-(A)$. Now $\tilde{c}_n(\tilde{t}) = c_n(s{+}1{-}\tilde{t})$. Let $(s,1] \ni t := s{+}1{-}\tilde{t}$. Then there exists $t \in (s,1]$ such that $c_n(t) \in I^-(A)$. But, $I^+(A) \ni c_n(s) \leq c_n(t) \in I^-(A)$ and so there exist $a_1, a_2 \in A$ such that $a_1 \ll c_n(s) \leq c_n(t) \ll a_2$, implying $a_1 \ll a_2$, which is a contraction to achronality of A.

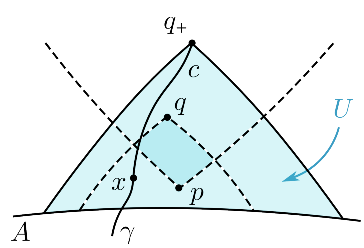
c) For all $p, q \in D(A)^\circ$ $J(p,q)$ is compact. If $p \nleq q$ then $J(p,q)$ is empty, which is compact and so we are done. If $p = q$ then $J(p,q) = \{p\}$ since if there were some $r \neq p$, $r \in J(p,q)$ then $p < r < p$ be a contradiction to a). Therefore, we consider the case when $p < q$ and show that every sequence $(x_n)_{n\in\mathbb{N}} \subseteq J(p,q)$ has a subsequence which converges in $J(p,q)$. Let $c_n : [0,1] \to M$ be causal FD curves from p to q through x_n. Furthermore, Let $\mathcal{K}$ be a cover of M by open convex subsets U such that the closures $\bar{U}$ are compact and contained in some open and convex set V. Then due to Proposition 3.4.4 we find a limit sequence $p =: p_0 < p_1 < \dots$ of $(c_n)_{n\in\mathbb{N}}$ relative to $\mathcal{K}$. We show that we can always find a finite one.

c1) There exists a finite limit sequence i.e. $p =: p_0 < \dots < p_N = q$. By the pigeonhole principle, there exists a subsequence (again denoted by) $(c_n)_{n\in\mathbb{N}}$ such that $x_n \in c_n([s_{n,i}, s_{n,i+1}])$ for all $n \in \mathbb{N}$ and $p_i = \lim c_n(s_i)$. In particular, $x_n \in \overline{U} \subseteq V$ and so $x_n \to x$, up to choosing a subsequence. $c_n(s_{n,i}) \leq x_n \leq c_n(s_{n,i+1})$, where $c_n(s_{n,i}) \to p_i$ and $c_n(s_{n,i+1}) \to p_{i+1}$. By Lemma 3.3.12 (4.), $p \leq p_i \leq x \leq p_{i+1} \leq q$ and so $p \leq x \leq q$, implying that $x \in J(p,q)$.

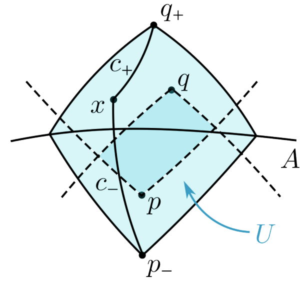
c2) Every limit sequence is infinite. As in b), there exists some s with $c_n(s) \to p_i \in I^+(A)$. q is the endpoint of each c_n and $p_0 < p_1 < \dots$ does not terminate. Therefore, $p_i \neq q$. Let $\tilde{c}_n := (c_n|_{[s,1]})$. Then $\tilde{c}_n$ is PD, $\tilde{c}_n(s) = c_n(1) \to q$ and $\tilde{c}_n(1) = c_n(s) \to p_{i_0} \neq q$. Therefore, by Proposition 3.4.4, there exists a limit sequence $q = q_0 > q_1 > \dots$. If this sequence were finite, then q_N would be equal to p_{i_0} and we would get a finite limit sequence $p = p_0 < p_1 < \dots < p_{i_0} = q_N < \dots < q_0 = q$, which would be a contradiction to the assumption that every limit sequence is infinite. In other words, $(q_i)_{i\in\mathbb{N}}$ does not terminate and so the corresponding quasi-limit is past inextendible. Therefore, by Lemma 3.8.4, it reaches $I^-(A)$. This leads us to a contradiction as in b).

d) For all $p, q \in D(A)^\circ$, $J(p,q) \subseteq D(A)^\circ$. As in c), without loss of generality, assume $p < q$. By Remark 3.8.2, the only possible cases are $p, q \in I^+(A)$ or $p, q \in I^-(A)$ or $p \in J^-(A)$, $q \in J^+(A)$.

d1) Let $p, q \in I^+(A)$. Choose $q_+ \in I^+(q) \cap D(A)$ and define $U := I^+(A) \cap I^-(q_+)$, an open neighborhood of $J(p,q)$. Indeed, $J(p,q) \subseteq J^+(I^+(A)) \cap J^-(I^-(A)) \overset{3.1.8}{=} I^+(A) \cap I^-(A) = U$. We show that $U \subseteq D(A)$.
To this end, for $x \in U$ let c be a timelike FD curve from x to q_+, which fails to intersect A due to achronality. Hence, for any past inextendible causal curve γ starting in x, the concatenation $c \cup \gamma$ yields a past inextendible causal curve, which starts in q_+ and, therefore, intersects A. It follows that γ intersects A and so $x \in D^+(A)$.



d2) Consider $p \in J^-(A)$ and $q \in J^+(A)$. Choose $q_+ \in I^+(q) \cap D(A)$ and $p_- \in I^-(p) \cap D(A)$ so that $U := I^-(p_-) \cap I^-(q_+)$ is again a neighborhood of $J(p,q)$. We show that $U \subseteq D(A)$. Let $x \in U$. Since for $x \in A$ the claim directly follows from $A \subseteq D(A)$, we assume $x \notin A$. Let c_- and c_+ be FDTL curves from p_- to x and from x to p_+, respectively. Due to achronality of A at least one of those curves does not intersect A.



□

Lemma 3.8.6. If M has a Cauchy surface, then M is globally hyperbolic.

Proof. Let S be a Cauchy surface in M. Then, by Proposition 3.5.23 (1.), S is achronal and, by Example 3.8.3, $D(S) = M$. Therefore, $D(S) = D(S)^\circ$. Finally, $D(S)^\circ$ is globally hyperbolic by Theorem 3.8.5. □

Lemma 3.8.7. Every spacelike achronal $\mathcal{C}^\infty$-hypersurface S is acausal.

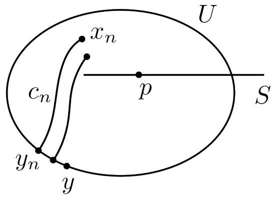
Proof. Suppose there exists some FD causal curve $c : [0,1] \to M$ such that $c(0), c(1) \in S$. By Theorem 3.2.23, there exists FDTL curve from S to S unless c is a null geodesic without focal point before $c(1)$ such that $\dot{c}(0) \perp S$. But S is spacelike and so $N_{c(0)}S$ is timelike, which contradicts the fact that $\dot{c}(0)$ is null ($\dim N_{c(0)}S = 1$). □

Proposition 3.8.8. Let $S \subseteq M$ be a closed acausal topological hypersurface. Then $D(S)$ is open and globally hyperbolic (by Theorem 3.8.5).

Proof. Recall that due to acausality of S, the union $I := I^-(S) \cup S \cup I^+(S)$ is disjoint, since if $I^-(S) \cap S$ or $I^+(S) \cap I^-(S)$ were not empty we would find timelike curves hitting S at least twice.

a) We show that $I(S) \subseteq M$ is open in M. Since $I^\pm(S)$ is open, it suffices to show that any $p \in S$ lies in the interior of $I(S)$. By Proposition 3.5.11, we know that $S \cap \text{edge}(S) = \emptyset$ and so $p \notin \text{edge}(S)$. Therefore, there exists a neighborhood U of p such that for all γ timelike in U from $I^-_U(p)$ to $I^+_U(p)$ γ must intersect S. Without loss of generality, $(U, x^0, \dots, x^{n-1})$ is a chart for RNCs $x^0, \dots, x^{n-1}$ around p with timelike x^0 and $|x^i| < \epsilon_i$ for some fixed $\epsilon_i > 0$. Choosing the ϵ_i's suitably small ensures $\left\{x^0 = \pm\epsilon_0\right\} \subseteq U$. Then the x^0-coordinate lines meet S and therefore run entirely in $I(S)$, so $\bigcap_{j=0}^{n-1} \left\{|x^i| < \epsilon_i\right\}$ yields an open neighborhood of p in $I(S)$.

b) We now show that $S \subseteq D(S)^\circ$. Suppose there exists some $p \in S \backslash D(S)^\circ$. Then by a) there exists U, an open neighborhood of p, such that $\overline{U} \Subset V \subseteq I(S)$ for V open and convex. Now $p \notin D(S)^\circ$ and so there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $M \backslash D(S)$ with $x_n \to p$. Without loss of generality, $x_n \in U$. $x_n \notin D(S)$ and so $x_n \notin D^+(S)$. Therefore, there exists a past inextendible causal curve c_n starting in x_n and not intersecting S.



By Lemma 3.3.12 (5.), every c_n intersects the boundary ∂U and we call the first intersection point y_n. Then we have $y_n \leq x_n$ and, due to compactness of ∂U, we find a subsequence converging to some $y \in \partial U$. $y_n \leq x_n$ so Lemma 3.3.12 (4.) implies that $y \leq p$ and even $y < p$ since $y \in \partial U$ while $p \in U^\circ$. In particular, $y \in \overline{U} \subseteq I(S)$.

b1) If $y \in I^+(S)$, then there exists some $q \in S$ such that $q \ll y \ll p$ and so $q \ll p$ (by Proposition 3.1.8). But, $p, q \in S$ so we get contradiction to acausality of S.

b2) If $y \in S$, then $y < p$, which is again a contradiction to acausality of S.

b3) If $y \in I^-(S)$, then (since $I^-(S)$ is open) there exists some n such that $c_n(t_n) = y_n \in I^-(S)$. But,

$$c_n([0,t_n]) \subseteq \overline{U} \subseteq I(S) = I^-(S) \mathring{\cup} S \mathring{\cup} I^+(S).$$

Recall that c_n does not meet S, so $c_n([0,t_n])$ has to be contained in $I^-(S)$ (c_n is connected), which contradicts the assumption $c_n(0) = x_n \in I^+(S)$.

c) We show that $D(S)$ is open if $S \subseteq M$ is closed. For S closed, it suffices to show that $D^+(S)\backslash S = I^+(S) \cap D(S)$ (see Remark 3.8.2 (4.)) is open because then also $D^-(S)\backslash S$ is open and

$$D(S) = (D^+(S)\backslash S) \cup S \cup (D^-(S)\backslash S) \overset{\text{b)}}{=} (D^+(S)\backslash S) \cup D(S)^\circ \cup (D^-(S)\backslash S) \subseteq D(S),$$

implying that $D(S)$ is open. Assume there exists some $p \in D^+(S)\backslash S$ that is not an interior point. Then there exists a sequence $(x_n)_{n\in\mathbb{N}} \notin D^+(S)\backslash S$ converging to p. For every n there exists a past inextendible causal curve $c_n : [0, b_n) \to M$ starting in x_n not meeting S (except maybe in $x_n \in S$). Due to b) and since S is closed $\{M\backslash S, D(S)^\circ\}$ yields an open cover of M. By Proposition 3.3.9, there exists a refinement $\mathcal{K}$ by open and convex sets such that for all $V \in \mathcal{K}$ there exists

$$\tilde{V} \text{ convex such that } \overline{V} \Subset \tilde{V} \text{ and } \tilde{V} \subseteq M\backslash S \text{ or } \tilde{V} \subseteq D(S)^\circ. \tag{3.8.1}$$

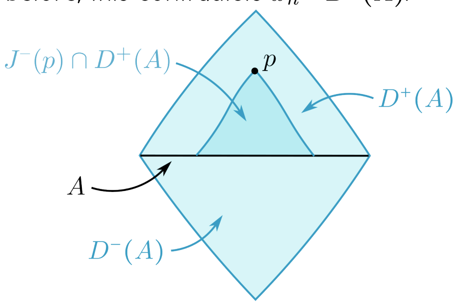
Choose $W \in \mathcal{K}$ such that $p \in W$ and $\overline{W} \Subset \tilde{W} \subseteq M\backslash S$. Lemma 3.3.12 (5.) now implies that all c_n must leave $\overline{W}$, otherwise they would be extendible. By Proposition 3.4.4 there exists γ, a quasi-limit of c_n with respect to $\mathcal{K}$, and every limit sequence is infinite (otherwise some subsequence would be extendible). This means that γ is past inextendible, PD causal and starting in p. $p \in D^+(S)$ so γ intersects S in precisely one point $\gamma(s)$. For the limit sequence $p = p_0 > p_1 > \dots$ let $p_i > \gamma(s) \geq p_{i+1}$. By our choice of W we have that $i \geq 1$. Hence, the element of $\mathcal{K}$ which contains the corresponding segment of γ meets S (in $\gamma(s)$) and is therefore contained in $D(S)^\circ$ by (3.8.1). Acausality of S implies $p_i \notin S$, that is, $p_i \in D^+(S)\backslash S = I^+(S) \cap D(S)$. It is even contained in the open set $I^+(S) \cap D(S)^\circ \subseteq D^+(S)$ so for n large enough, c_n has to meet $D^+(S)$. ↯

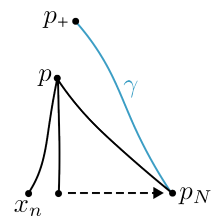
□

Lemma 3.8.9. For each achronal subset $A \subseteq M$ and $p \in D(A)^\circ \backslash I^-(A)$, the intersection $J^-(p) \cap D^+(A)$ is compact.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $J^-(p) \cap D^+(A)$. Furthermore, let c_n be a PD causal curve from p to x_n. If a subsequence of $(x_n)_{n\in\mathbb{N}}$ converges to p we are done. Otherwise, by Proposition 3.4.4 there exists a limit sequence $p := p_0 > p_1 > \dots$. Assume that the limit sequence is finite i.e. $p > p_1 > \dots > p_N$. Assume also that we can find a subsequence of $(x_n)_{n\in\mathbb{N}}$ converging to p_N, so it remains to show $p_N \in D^+(A)$. To that end, let $p_+ \in I^+(p) \cap D^+(A)$, that is $p_+ \gg p \geq p_N$ and thus $p_+ \gg p_N$ so there is a TLPD curve γ from p_+ to p_N. If γ does not meet A, we directly have $p_N \in D^+(A)$. If γ meets A, then $p_N \in A \cap I^-(A)$. But, $p_N \in I^-(A)$

would imply that also $x_n \in I^-(A)$ for n large enough, which contradicts $x_n \in D^+(A)$. If the limit sequence is infinite, the quasi-limit γ is a past-inextendible causal curve, which starts at p and meets $I^-(A)$ by Lemma 3.8.4 (2.). Therefore, $p_i \in I^-(A)$ for i large enough and, consequently, for n large enough, $x_n \in I^-(A)$. As before, this contradicts $x_n \in D^+(A)$.





□

Definition 3.8.10. For all $p \in M$ and $A \subseteq M$,

$$\tau(A,p) := \sup_{q \in A} \tau(p,q).$$

Theorem 3.8.11. Let $S \subseteq M$ be a closed, achronal, spacelike ($\mathcal{C}^\infty$-)hypersurface. For any $p \in D(S)$ there exists a geodesic c from S to p of length $\tau(S,p)$. c is normal to S, timelike and does not have a focal point before p unless $p \in S$.

Proof. Due to Lemma 3.8.7 S is acausal and, by Proposition 3.8.8, $D(S)$ is open and globally hyperbolic and so $\tau|_{D(S)\times D(S)}$ is finite and continuous (see Proposition 3.6.8). If $p \in S$ then $\tau(S,p) = 0$ since S is acausal (the only viable option is $c \equiv p$). Without loss of generality, we only consider $p \in D^+(S)$. From Lemma 3.8.9 we know that $J^-(p) \cap D^+(S)$ is compact and, hence

$$J^-(p) \cap S = J^-(p) \cap D(S) \cap S$$

is compact as well since S is closed. Proposition 3.6.8 ensures continuity of τ on $J^-(p) \cap S$, so the maximum of $\tau(\cdot,p)$ on $J^-(p) \cap S$ is attained at some q. By Theorem 2 there is a causal geodesic c of length $\tau(q,p)$ connecting q and p. If c was not orthogonal to S or if c had a focal point before p, this curve could be deformed into a longer timelike curve from S to p (see Theorem 3.7.12), which contradicts the maximality of the length of c. □

Definition 3.8.12. Let $A \subseteq M$ be achronal. Then we call

$$H^+(A) = \overline{D^+(A)} \setminus I^-(D^+(A)) = \left\{ p \in \overline{D^+(A)} : I^+(p) \cap D^+(A) = \varnothing \right\}$$

future Cauchy horizon of A. Analogously, one defines $H^-(A)$, the past Cauchy horizon of A. The Cauchy horizon of A is given by

$$H(A) := H^+(A) \cup H^-(A).$$

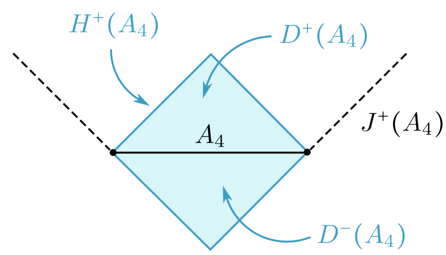
Example 3.8.13. Let $M := \mathbb{R}^n_1$.

1. For $A_1 := \{0\} \times \mathbb{R}^{n-1}$, we obtain $D^\pm(A_1) = J^\pm(A_1)$ and, consequently, $H(A) = \varnothing$.

2. For $A_2 := H^{n-1}$ the $(n-1)$-dimensional hyperbolic space (see Example 3.5.2), we have $D(A_2) = I^+(0)$ and so $H^+(A_2) = \varnothing$ and $H^-(A_2) = \overline{D^-(A_2)} \setminus \underbrace{I^+(D^-(A_2))}_{= I^+(0)} = C^+(0)$.

3. For $A_3 := C^+(0)$, we have $D(A_3) = D^+(A_3) = J^+(0)$ and $H^\pm(A_3) = H^\pm(A_2)$.

4. For $\dim(M) = 2$ we consider $A_4 := \{0\} \times (-1,1)$ and obtain $H^+(A_4)$ as in the picture. Note that $H^+(A_4) \notin J^+(A_4)$ since $(0, \pm 1) \in H^+(A_4) \backslash J^+(A_4)$.



Lemma 3.8.14 (Basic Properties of H)**.** For all achronal $A \subseteq M$, we have

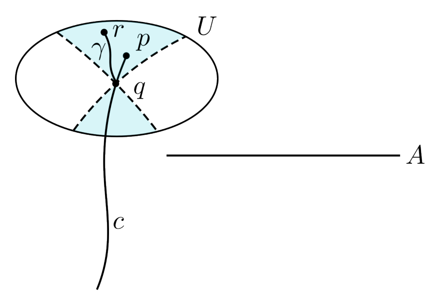
1. $H^\pm(A)$ is closed.
2. $H^\pm(A)$ is achronal.
3. If A is closed, then
$$\overline{D^+(A)} = \{p \in M : \text{every past-inextendible TL curve through } p \text{ meets } A\} := X.$$
4. If A is closed, then
$$\partial D^\pm(A) = A \cup H^\pm(A).$$

Proof.

1. $H^\pm(A) = \underbrace{\overline{D^\pm(A)}}_{\text{closed}} \backslash \underbrace{I^\mp(D^\pm(A))}_{\text{open}}$.

2. Since $I^+(H^+(A))$ is open and $I^+(H^+(A)) \cap D^+(A)$ empty by definition, we obtain $I^+(H^+(A)) \cap \overline{D^+(A)} = \varnothing$ and therefore $I^+(H^+(A)) \cap H^+(A) = \varnothing$ (since $H^+(A) \subseteq \overline{D^+(A)}$). This implies achronality of $H^+(A)$.

3. We first show that $\overline{D^+(A)} \subseteq X$. If there was a $p \in \overline{D^+(A)} \backslash X$, we would find a past-inextendible TL curve $c : [0,b) \to M$ which starts in p but does not intersect A. In particular, $p \notin A$ and, since A is closed (by assumption), there is an open and convex neighborhood U of p such that $U \cap A = \varnothing$. Choose $\epsilon > 0$ such that $q := c(\epsilon) \in U$ and thus $p \in I^+_U(q)$. Since $I^+_U(q)$ is an open neighborhood of p and $p \in \overline{D^+(A)}$, there is some $r \in I^+_U(q) \cap D^+(A)$ with γ the corresponding TL curve from r to q. Due to convexity, γ runs entirely in U, so it does not meet A. On the other hand, the concatenation $\gamma \cup c|_{[\epsilon,b)}$ yields a past-inextendible TL curve which starts in $r \in D^+(A)$, so it has to intersect A. ↯



We proceed with $X \subseteq \overline{D^+(A)}$. Let $p \notin \overline{D^+(A)}$ and choose $q \in I^-_{M \backslash \overline{D^+(A)}}(p)$ so in particular $q \notin \overline{D^+(A)}$. Therefore, we find a past-inextendible causal curve in M, which starts in p but does not intersect A. Lemma 3.5.21 now implies the existence of a past-inextendible TL curve $\tilde{\gamma}$ from p not meeting A. Then $p \notin X$.

4. a) We start with $A \subseteq \partial D^+(A)$. By Remark 3.8.2 (1.) we already know that $A \subseteq D^+(A) \subseteq \overline{D^+(A)}$. If there was some $p \in A \cap D^+(A)^\circ$ we could choose $q \in D^+(A)^\circ \cap I^-(p)$, which would imply the

existence of a past-inextendible timelike curve c starting in q. Since $q \in D^+(A)$, c must intersect A in some $r \in A$ i.e. $r \ll p$. On the other hand, $p, r \in A$, which yields a contradiction to A being achronal.

b) Without loss of generality, we now prove $H^+(A) \subseteq \partial D^+(A)$. By definition, we have $H^+(A) \subseteq \overline{D^+(A)}$. If there was any $p \in H^+(A) \cap D^+(A)^\circ$, the intersection $I^+(p) \cap D^+(A)$ would not be empty, which would contradict $p \in H^+(A)$.

c) It remains to show that $\partial D^+(A) \subseteq A \cup H^+(A)$. Suppose there was a $p \in \partial D^+(A) \backslash (A \cup H^+(A))$. Then, in particular, $p \in \overline{D^+(A)} \backslash A$ so 3. implies $p \in I^+(A)$. On the other hand, $p \in \overline{D^+(A)} \backslash H^+(A)$ and so there exists a $q \in I^+(p) \cap D^+(A)$ and $I^+(A) \cap I^-(q)$ is an open neighborhood of p. We now complete the proof by showing that this neighborhood is contained in $D^+(A)$. Then p would need to be an inner point, which contradicts $p \in \partial D^+(A)$. To this end let $I^+(A) \cap I^-(q)$ and c a past-inextendible causal curve starting in r. Furthermore, let γ be a TLPD curve from q to r, which necessarily stays in $I^+(A)$ due to $r \in I^+(A)$ and, therefore, fails to meet A since achronality of A demands that $A \cap I^+(A) = \emptyset$. On the other hand, $q \in D^+(A)$ implies that $\gamma \cup c$ intersects A, so c has to intersect A and hence $r \in D^+(A)$.

□

Proposition 3.8.15. Let $S \subseteq M$ be closed, acausal $\mathcal{C}^0$-hypersurface. Then

1. $H^+(S) = I^+(S) \cap \partial D^+(S) = \overline{D^+(S)} \backslash D^+(S)$ and $H^+(S) \cap S = \emptyset$.
2. $H^+(S)$ is an achronal closed $\mathcal{C}^0$-hypersurface.
3. In every point of $H^+(S)$ starts a past-inextendible null geodesic without conjugate points, which is entirely contained in $H^+(S)$.

Proof.

1. By definition,
$$H^+(S) \subseteq \overline{D^+(S)} \subseteq S \cup I^+(S)$$
(for the last inclusion see 3.8.14, 3.). If there was a $p \in H^+(S) \cap D^+(S)$, $I^+(p)$ would hit $D(S)$ since, according to Proposition 3.8.8 $D(S)$ is open, due to achronality of S, we have $I^+(p) \cap D^-(S) = \emptyset$. Therefore, $I^+(p)$ has to meet $D^+(S)$, which contradicts $p \in H^+(S)$ and thus $H^+(S) \cap D^+(S) = \emptyset$, that is $H^+(S) \subseteq \partial D^+(S)$. Moreover, from $S \subseteq D^+(S)$ follows that also $H^+(S) \cap S = \emptyset$, so the inclusion we started with implies $H^+(S) \subseteq I^+(S)$. Conversely, Lemma 3.8.14 (4.) provides
$$I^+(S) \cap \partial D^+(S) = I^+(S) \cap (S \cup H^+(S)) = H^+(S) \cap I^+(S) = H^+(S).$$
It remains to show $\overline{D^+(S)} \backslash D^+(S) \subseteq H^+(S)$. Hence, for all $p \in \overline{D^+(S)} \backslash D^+(S)$, we show $I^+(p) \cap D^+(S) = \emptyset$. Let $q \in I^+(p)$ and γ a PD curve from q to p. $p \notin S \cup I^-(S)$ since $S \subseteq D^+(S)$ and $D^+(S) \cap D^-(S) = \emptyset$ by Remark 3.8.2 (2.), implying $\overline{D^+(S)} \cap I^-(S) = \emptyset$ (since $I^-(S)$ is open). Therefore, γ does not meet S and there is a past-inextendible curve c starting in p which does not meet S either (since $p \notin D^+(S)$). $\gamma \cup c$ is a past-inextendible causal curve which starts in q but does not meet S, that is $q \notin D^+(S)$.

2. Let $B := D^+(S) \cup I^-(S)$.
   - B is a past set (cf. Definition 3.5.13). Indeed, let $q \in I^-(D^+(S))$ and let γ be PDTL from $p \in D^+(S)$ to q. If $q \in D^+(S)$, we are done. Otherwise, there exists some α PD causal from q not meeting S. $p \in D^+(S)$ so $\gamma \cup \alpha$ meets S but α does not so γ meets S in some r. If $r = q$ then $q \in S \subseteq D^+(S) \subseteq B$. Otherwise, $q \in I^-(r) \subseteq I^-(S) \subseteq B$.

- We can now apply Corollary 1 ($I^+(S) \notin B$ and $S \neq \varnothing$) and get that ∂B is an achronal $\mathcal{C}^0$-hypersurface.
- Also, $H^+(S) \stackrel{1.}{=} \partial D^+(S) \cap I^+(S) = \partial B \cap I^+(S)$. $I^-(S) \cap I^+(S) = \varnothing$ and so $\partial I^-(S) \cap I^+(S) = \varnothing$. Therefore, $\partial D^+(S) \cap I^+(S) = (\partial D^+(S) \cup \partial I^-(S)) \cap I^+(S) \supseteq \partial B \cap I^+(S)$ (where we have used $\partial(A_1 \cup A_2) \subseteq \partial A_1 \cup \partial A_2$). Conversely, $\partial D^+(S) \cap I^+(S) \subseteq I^+(S)$, where $\partial D^+(S) \cap I^+(S) = H^+(S)$ (by 1.) so it remains to show that $H^+(S) \subseteq \partial B$. $H^+(S) \subseteq \overline{D^+(S)} \subseteq \overline{D^+(S)} \cup \overline{I^-(S)} = \overline{B}$. On the other hand, $H^+(S) \subseteq B^c \subseteq (B^\circ)^c$. To see this let $p \in H^+(S)$. Then $p \notin D^+(S)$ by 1. and $p \notin I^-(S)$ since otherwise (again by 1.) we would have that $p \in I^-(S) \cap I^+(S) = \varnothing$. Therefore, $p \notin B$ and we are done since we showed that $H^+(S) \subseteq \overline{B} \backslash B^\circ = \partial B$.
- $H^+(S) = \partial B \cap I^+(S)$ is a relatively open subset of B (which is a $\mathcal{C}^0$-hypersurface), making it a $\mathcal{C}^0$-hypersurface. Moreover, $H^+(S)$ is achronal and closed by Lemma 3.8.14.

3. Let $p \in H^+(S) \subseteq \overline{D^+(S)} \backslash D^+(S)$. As $p \notin D^+(S)$ there exists a past-inextendible causal curve c from p not meeting S. By Lemma 3.8.14 (3.), since $p \in \overline{D^+(S)}$, such c cannot be TL so c cannot be deformed into a TL curve from p avoiding S. Therefore, by Lemma 3.5.21 (2.), c is a null geodesic without conjugate points. Finally, c remains in $H^+(S)$; if c were to intersect $D^+(S)$ it would also intersect S, which is a contradiction. If c were to leave $\overline{D^+(S)}$ i.e. if there existed some $c(s) \notin \overline{D^+(S)}$ then, by Lemma 3.8.14, there would exist a past-inextendible TL γ from $c(s)$ such that $\gamma \cap S = \varnothing$. Apply Lemma 3.5.21 (2.) to $c|_{[0,s]} \cup \gamma$ and obtain a PDTL curve from p not intersecting S. This contradicts $p \in \overline{D^+(S)}$.

□

Corollary 3.8.16. Let $S \neq \varnothing$ be a closed acausal $\mathcal{C}^0$-hypersurface. Then

1. S is a Cauchy hypersurface if and only if $H(S) = \varnothing$.
2. S is a Cauchy hypersurface if every inextendible null geodesic intersects S.

Proof.

1. By Proposition 3.8.8, we know that $D(S)$ is open and globally hyperbolic. We show that

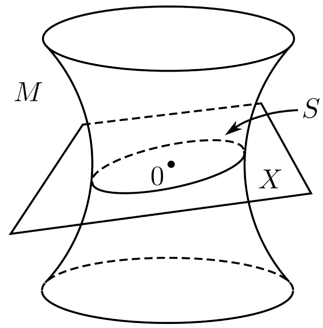
$$S = \overline{D^+(S)} \cap D^-(S). \tag{3.8.2}$$

(by symmetry, $S = \overline{D^-(S)} \cap D^+(S)$). $S \subseteq \overline{D^+(S)} \cap D^-(S)$ is clear. Conversely, suppose $p \in (\overline{D^+(S)} \cap D^-(S)) \backslash S$. Then every non-inextendible TL curve through p intersects S in the past and the future of p, which is a contradiction to S being acausal. Hence, $\partial D(S) = \overline{D(S)} \backslash D(S) = (\overline{D^+(S)} \cup \overline{D^-(S)}) \backslash D(S) = (\overline{D^+(S)} \backslash D(S)) \cup (\overline{D^-(S)} \backslash D(S)) \stackrel{(3.8.2)}{=} (\overline{D^+(S)} \backslash D^+(S)) \cup (\overline{D^-(S)} \backslash D^-(S)) \stackrel{3.8.15}{=} H^+(S) \cup H^-(S) = H(S)$. Therefore, $H(S) = \varnothing$ if and only if $\partial D(S) = \varnothing$. Since M is a spacetime, M is connected and so $\partial D(S) = \varnothing$. $\partial D(S) = \varnothing$ if and only if $D(S) = M$. The last statement is obviously equivalent to S being a Cauchy hypersurface.

2. Regarding 1., without loss of generality suppose that there exists some $p \in H^+(S)$. Then, according to Proposition 3.8.15 (3.) there is a past-inextendible null geodesic starting in p entirely contained in $H^+(S)$. By Proposition 3.8.15 (1.), $H^+(S) \cap S = \varnothing$ and so $c \cap S = \varnothing$. If the maximal extension of c to a future-inextendible geodesic met S in some $q \in S$, then $q \geq p$. On the other hand, $p \in H^+(S) \subseteq I^+(S)$ implies $q \in I^+(S) \cap S$, which contradicts achronality of S. Therefore, if $H(S) \neq \varnothing$, then there exists an inextendible null geodesic not intersecting S.

□

Example 3.8.17. Let $M = S_1^n(r)$ be de Sitter space and $S := M \cap X$, where $X \subseteq \mathbb{R}_1^{n-1}$ a spacelike hypersurface. Moreover, due to Section 2.2 every lightlike geodesic is of the form $M \cap E$ for some degenerate hyperplane E. The intersection is therefore a spacelike straight line hitting S. It now follows from Corollary 3.8.16 that S is a Cauchy hypersurface and, from Lemma 3.8.6, that it is globally hyperbolic.



3.9 Hawking's Singularity Theorem

Theorem 3.9.1 (Hawking, 1967.)**.** Let M be n-dimensional spacetime and assume that

1. Ric$(X, X) \geq 0$ for all $X \in TM$ TL.
2. There exists $S \subseteq M$ spacelike Cauchy surface and there exists $\beta > 0$ such that $\langle H, \nu \rangle \geq \beta$ where

$$H = \frac{1}{n-1} \sum_{i=1}^{n-1} \epsilon_j \mathbb{I}(e_j, e_j)$$

   is the mean curvature of S and ν is a future directed unit normal vector field on S.

Then every FDTL curve starting in S has length less or equal to $\frac{1}{\beta}$. In particular, M is future directed, timelike and geodesically incomplete.

Remark 3.9.2.

- The Hawking theorem models a 'cosmological' situation. Applied to the past direction it gives a strong evidence for a big bang. In the theorem, M corresponds to the spacetime (i.e. the universe) and S is a time-slice of our universe (i.e. of the current spatial universe).
- Condition of Theorem 3.9.1. The Einstein field equations are of the form

$$8\pi T = \text{Ric} - \frac{1}{2} g \cdot S,$$

  where T is energy-momentum tensor and S is the scalar curvature (!). For $n = 4$ this implies $8\pi \cdot \text{tr}(T) = S - \frac{1}{2} 4S = -S$, hence

$$8\pi T = \text{Ric} + g \cdot 4\pi \cdot \text{tr}(T).$$

  Now for all X TL,

$$\text{Ric}(X, X) \geq 0 \iff T(X, X) \geq \frac{1}{2} \text{tr}(T)(X, X).$$

This inequality is known as the *strong energy condition* (SEC). Here, $T(X,X)$ is interpreted as the energy density measured by an observer whose world line has the tangent vector X. The condition $\langle H,\nu\rangle \geq \beta$ stands for a spacelike universe which contracts at a rate at least β and so Hawking's theorem states that the time such universe exists is at most $\frac{1}{\beta}$, which then stands for the time a big crunch singularity would occur at the latest.

Proposition 3.9.3 (1^{st} and 2^{nd} Variation of Arc Length)**.** Let c_s be the variation of TL geodesic $c : [a,b] \to M$ with variational vector field $V := \frac{\partial c_s}{\partial_s}\Big|_{s=0}$ and acceleration vector field $A := \frac{\nabla}{\partial s}\frac{\partial c_s}{\partial s}\Big|_{s=0}$. Then we have

1. $\frac{d}{ds}L(c_s)\Big|_{s=0} = \left\langle V(a), \frac{\dot c(a)}{|\dot c(a)|}\right\rangle - \left\langle V(b), \frac{\dot c(b)}{|\dot c(b)|}\right\rangle$.
2. $\frac{d^2}{ds^2}L(c_s)\Big|_{s=0} = \left\langle A(a), \frac{\dot c(a)}{|\dot c(a)|}\right\rangle - \left\langle A(b), \frac{\dot c(b)}{|\dot c(b)|}\right\rangle - \int_a^b \frac{1}{|\dot c|}\left(\langle R(\dot c, V)V, \dot c\rangle + \left\langle \frac{\nabla V}{dt}, \frac{\nabla V}{dt}\right\rangle + \left\langle \frac{\nabla V}{dt}, \frac{\dot c}{|\dot c|}\right\rangle^2\right)dt.$

Proof. Let $c : [a,b] \to M$ be a timelike geodesic and c_s a smooth variation of c with variational vector field V and acceleration field A.

1. For $V_s = \frac{\partial c_s}{\partial s}$, we obtain

$$\begin{aligned}\frac{d}{ds}L(c_s) &= \frac{d}{ds}\int_a^b \sqrt{-\langle \dot c_s, \dot c_s\rangle}dt = \int_a^b \frac{-2\left\langle \frac{\nabla \dot c_s}{\partial s}, \dot c_s\right\rangle}{2\sqrt{-\langle \dot c_s, \dot c_s\rangle}}dt \\ &= -\int_a^b \left\langle \frac{\nabla}{\partial s}\frac{c_s}{\partial t}, \frac{\dot c_s}{|\dot c_s|}\right\rangle dt = -\int_a^b \left\langle \frac{\nabla}{\partial t}\frac{\partial c_s}{\partial s}, \frac{\dot c_s}{|\dot c_s|}\right\rangle dt = -\int_a^b \left\langle \frac{\nabla V_s}{\partial t}, \frac{\dot c_s}{|\dot c_s|}\right\rangle dt,\end{aligned}$$

which for $s = 0$ provides the claim:

$$\begin{aligned}\frac{d}{ds}L(c_s)\Big|_{s=0} &= -\int_a^b \left\langle \frac{\nabla V}{dt}, \frac{\dot c}{|\dot c|}\right\rangle dt \\ &= -\int_a^b \left(\frac{d}{dt}\left\langle V, \frac{\dot c}{|\dot c|}\right\rangle - \left\langle V, \underbrace{\frac{\nabla}{\partial t}\frac{\dot c}{|\dot c|}}_{=0}\right\rangle\right)dt = -\left\langle V, \frac{\dot c}{|\dot c|}\right\rangle\Big|_a^b.\end{aligned}$$

2. This claim follows from direct calculation of the second variation:

$$\begin{aligned}\frac{d^2}{ds^2}L(c_s)\Big|_{s=0} &= -\frac{d}{ds}\int_a^b \left\langle \frac{\nabla V_s}{\partial t}, \frac{\dot c_s}{|\dot c_s|}\right\rangle dt\Big|_{s=0} \\ &= -\int_a^b \left(\left\langle \frac{\nabla}{\partial s}\frac{\nabla V_s}{\partial t}\Big|_{s=0}, \frac{\dot c}{|\dot c|}\right\rangle + \left\langle \frac{\nabla V}{dt}, \frac{\nabla}{\partial s}\Big|_{s=0}\frac{\dot c_s}{|\dot c_s|}\right\rangle\right)dt \\ &= -\int_a^b \left(\left\langle R(V,\dot c)V + \frac{\nabla}{\partial t}\underbrace{\frac{\nabla V_s}{\partial s}\Big|_0}_{=A}, \frac{\dot c}{|\dot c|}\right\rangle + \left\langle \frac{\nabla V}{dt}, \frac{1}{|\dot c|^2}\left(|\dot c|\frac{\nabla \dot c_s}{\partial s}\Big|_0 - \frac{\partial |\dot c|_s}{\partial s}\Big|_0 \dot c\right)\right\rangle\right)dt \\ &= -\int_a^b \left(\left\langle R(V,\dot c)V, \frac{\dot c}{|\dot c|}\right\rangle + \frac{d}{dt}\left\langle A, \frac{\dot c}{|\dot c|}\right\rangle - \left\langle A, \underbrace{\frac{\nabla}{dt}\frac{\dot c}{|\dot c|}}_{=0}\right\rangle + \left\langle \frac{\nabla V}{dt}, \frac{1}{|\dot c|} + \frac{\left\langle \frac{\nabla V}{dt}, \dot c\right\rangle \dot c}{|\dot c|^3}\right\rangle\right)dt \\ &= -\left\langle A, \frac{\dot c}{|\dot c|}\right\rangle\Big|_a^b - \int_a^b \left(\left\langle R(V,\dot c)V, \frac{\dot c}{|\dot c|}\right\rangle + \frac{1}{|\dot c|}\left\langle \frac{\nabla V}{dt}, \frac{\nabla V}{dt}\right\rangle dt + \frac{1}{|\dot c|^3}\left\langle \frac{\nabla V}{dt}, \dot c\right\rangle^2 dt.\right.\end{aligned}$$

□

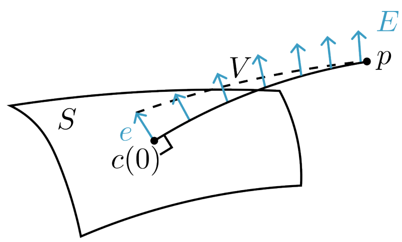
Proof. See 3.9.1. Let γ be a FDTL curve from S ending in $p \in M$. Then,

$$p \in I^+(S) = I^+(S) \cap \underbrace{D(S)}_{\overset{3.8.3}{=} M} \overset{3.8.2}{=} D^+(S)\backslash S.$$

From Theorem 3.8.11 we know that there exists a TL geodesic $c : [0,b] \to M$ with $c(0) \in S$, $\dot{c}(0) \perp S$, $c(b) = p$, $L(c) = \tau(S,p)$. Without loss of generality we may assume $|\dot{c}| = 1$ so that $L(c) = b$. Therefore, it remains to show $b \leq \frac{1}{\beta}$. Let $e \in T_{c(0)}S$ be a unit vector and let E be its parallel transport along c given by $E(0) = e$. Furthermore, let c_s be a variation of c with variation vector field

$$V(t) = \left(1 - \frac{t}{b}\right) E(t),$$

$c_s(0) \in S$ and $c_s(b) = p$ (Fermi coordinates allow such a choice).



Since c is the maximum of the length of TL curves from S to p, by Proposition 3.9.3, we have

$$\begin{aligned} 0 \geq \frac{d^2}{ds^2} L(c_s)\Big|_{s=0} &= \langle A(0), \dot{c}(0)\rangle - \underbrace{0}_{A(b)=0} - \int_0^b \Bigg[\left(1 - \frac{t}{b}\right)^2 \langle R(\dot{c}, E)E, \dot{c}\rangle \\ &\quad + \left\langle -\frac{1}{b}E, -\frac{1}{b}E \right\rangle \underbrace{\left\langle -\frac{1}{b}E, \dot{c}\right\rangle^2}_{\substack{=0, \\ e\perp\dot{c} \to E(t)\perp\dot{c}(t),\ \forall t \\ [3], 1.3.30}} \Bigg] dt \\ &= \left\langle \left(\frac{\nabla}{\partial s}\frac{\partial c}{\partial s}(0,0)\right), \dot{c}(0)\right\rangle - \int_0^b \left[\left(1 - \frac{t}{b}\right)^2 \langle R(\dot{c}, E)E, \dot{c}\rangle + \frac{1}{b^2}\right] dt = (\star). \end{aligned}$$

$s \mapsto c(0,s)$ is a curve in S, $Z(S) := \frac{\partial}{\partial s} c(0,s) \in \mathfrak{X}(c(0,\cdot))$ and so, by Proposition 1.5.1

$$\left(\frac{\nabla}{\partial s}\frac{\partial c}{\partial s}\right)(0,0) = \dot{Z}(0) = \underbrace{Z'(0)}_{\text{tan. to } S,\ \perp\dot{c}(0)} + \mathbb{I}\Bigg(\underbrace{\frac{\partial}{\partial s}\Big|_0 c(0,s)}_{=V(0)}, \underbrace{Z(0)}_{=V(0)} \Bigg).$$

Therefore,

$$\left\langle \left(\frac{\nabla}{\partial s}\frac{\partial c}{\partial s}\right)(0,0), \dot{c}(0)\right\rangle = \langle \mathbb{I}(V(0), V(0)), \dot{c}(0)\rangle,$$

where $\dot{c}(0) = \nu$ is a unit normal to S.

$$(\star) = \Big\langle \mathbb{I}(\underbrace{V(0)}_{=E(0)=e}, V(0)), \dot{c}(0) \Big\rangle + \int_0^b \left(1 - \frac{t}{b}\right)^2 \langle R(\dot{c}, E)\dot{c}, E\rangle\, dt - \frac{1}{b}.$$

Now for an ONB $\{e_1, \dots, e_{n-1}\}$ of $T_{c(0)}S$ we obtain corresponding $E_1, \dots, E_{n-1}$ as above. By summation,

$$0 \geq \sum_{j=1}^{n-1} \langle \mathbb{I}(e_j, e_j), \nu \rangle + \int_0^b \left(1 - \frac{t}{b}\right)^2 \cdot \underbrace{\sum_{j=1}^{n-1} \langle R(\dot{c}, E_j)\dot{c}, E_j \rangle}_{\overset{1.2.8}{=} \text{Ric}(\dot{c}, \dot{c}) \geq 0, \text{ by 1.}} dt - \frac{n-1}{b}.$$

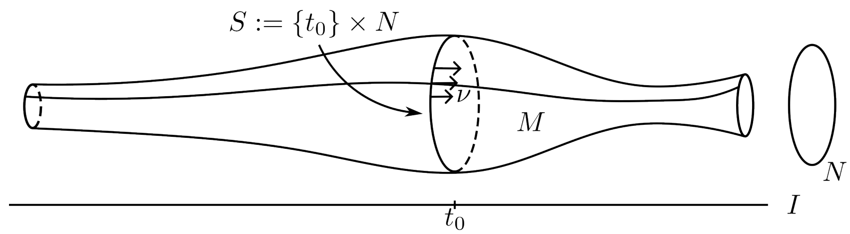
Therefore,

$$\begin{aligned} 0 &\geq (n-1)\langle H, \nu \rangle - \frac{n-1}{b} \\ &\overset{2.}{\geq} \cancel{(n-1)}\beta - \frac{\cancel{n-1}}{b} \implies b \leq \frac{1}{\beta}. \end{aligned}$$

□

Example 3.9.4.

1. • Let M be an $(n+1)$-dimensional Robertson-Walker spacetime.



By introduction to Section 2.4.2 (7.), Ric$(\nu, \nu) = -n\frac{f''}{f}$ for $\nu := \frac{\partial}{\partial t}$. For the proof of Theorem 3.9.1 we used Ric$(X, X) \geq 0$ for $X = \dot{c}(t)$, where c was a geodesic such that $\dot{c} \perp S$. Here, we actually have $\dot{c} = X = \nu = \frac{\partial}{\partial t}$, so the assumption Ric$(\nu, \nu) \geq$ is equivalent to $f'' \leq 0$ (since f is always positive by Definition 2.4.1) i.e. to f being concave.

• From Section 2.4.1 (1.) it follows that $S(X) = -\frac{\dot{f}}{f}X$ i.e. $S = -\frac{\dot{f}}{f}\text{id}$ and so

$$\langle \mathbb{I}(V, W), \nu \rangle \overset{1.4.7}{=} \langle S(V), W \rangle = \frac{\dot{f}}{f} \langle V, W \rangle \cdot \underbrace{\langle \nu, \nu \rangle}_{=-1} \implies \mathbb{I}(V, W) = \langle S(V), W \rangle = \frac{\dot{f}}{f} \langle V, W \rangle \nu.$$

Now,

$$H = \frac{1}{n-1} \sum_{j=1}^{n-1} \mathbb{I}(e_j, e_j) = \frac{1}{n-1} \sum_{j=1}^{n-1} \underbrace{\langle e_j, e_j \rangle}_{=1} \frac{\dot{f}}{f}\nu = \left(\frac{\dot{f}}{f}\right) \cdot \nu,$$

implying

$$\langle H, \nu \rangle = \frac{\dot{f}}{f} \underbrace{\langle \nu, \nu \rangle}_{=-1} = -\frac{\dot{f}}{f} \geq \beta \iff -\beta \geq \frac{\dot{f}}{f}(t_0) \;\; (S = \{t_0\} \times N).$$

2. Let M be n-dimensional Minkowski space.

• Ric $\equiv 0$.

• Consider $S := -H^{n-1}(r)$ where

$$H^{n-1}(r) := \{x \in M : \langle x, x \rangle = -r^2, \; x^0 > 0\}.$$

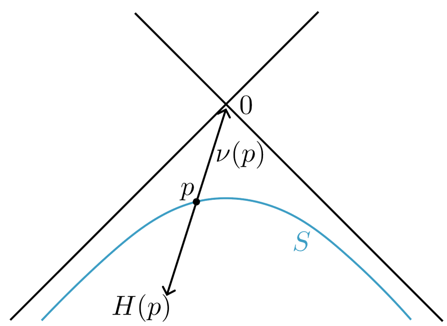
By Proposition 2.3.2 (4.)

$$\mathbb{I}(V,W) = -\frac{1}{r}\langle V,W\rangle \nu.$$

For any $p \in S$, $\nu = -\frac{1}{r}\cdot p$ so $H(p) = \frac{1}{n-1}\sum \mathbb{I}(e_i,e_i) \overset{\langle e_i,e_i\rangle=1}{=} \frac{1}{r^2}\cdot p$. Therefore,

$$\langle H(p),\nu\rangle = -\frac{1}{r^3}\underbrace{\langle p,p\rangle}_{=-r^2} = \frac{1}{r} =: \beta.$$

On the other hand, we know that the maximal timelike geodesics that start in S have infinite length. The reason for this is that S is not a Cauchy hypersurface in M, but it is in $D(S) = I_-(0)$, where the maximal timelike geodesics that start in S indeed have the maximal length $r = \frac{1}{\beta}$.



3.10 Penrose's Singularity Theorem

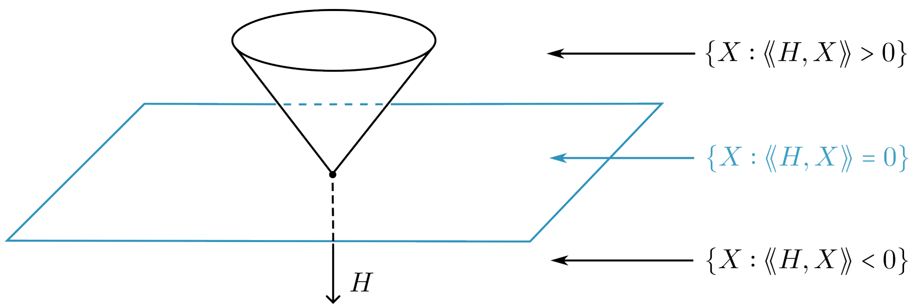
While Hawking's theorem models a cosmological scenario, providing evidence for the occurrence of a big bang or crunch, Penrose's theorem is tailored for a gravitational collapse situation. We begin by examining the 'initial condition,' specifically the concept of a *trapped surface*, a fundamental idea introduced in Penrose's 1965 paper, which initiated the study of singularity theorems. Unlike Hawking's case, we will now focus on spacelike, codimension 2 submanifolds. Our aim is to demonstrate that when both incoming and outgoing perpendicular null geodesics converge, gravity becomes very strong, to the extent that even light cannot escape.

Lemma 3.10.1. We call a spacelike submanifold $P \subseteq M$ a **future trapped surface** if one of the following equivalent conditions holds:

1. H is PDTL on P.
2. $k(X) := \langle H,X\rangle > 0$ for all $X \in TP$ FD null. We call $k(X)$ the **convergence** of X.
3. $k(X) = \langle H,X\rangle > 0$ for all $X \in TP$ FD causal.

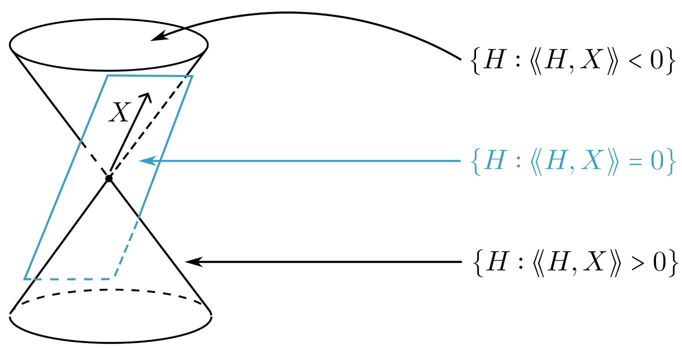
Proof. We work pointwise. Let $p \in P$, choose coordinates so that $g(p) = \langle\!\langle\cdot,\cdot\rangle\!\rangle$ and apply L-transformation such that $H = -ce_0$ (for $c > 0$).

$(1. \to 3.)$



$(3. \to 2.)$ Clear.

$(2. \to 1.)$



$$\implies \bigcap_{X \in C_+(0) \setminus \{0\}} \{H : \langle\!\langle H, X \rangle\!\rangle > 0\} = I_-(0).$$

□

Definition 3.10.2. A closed and achronal set $A \subseteq M$ is called future-trapped if its future horismos

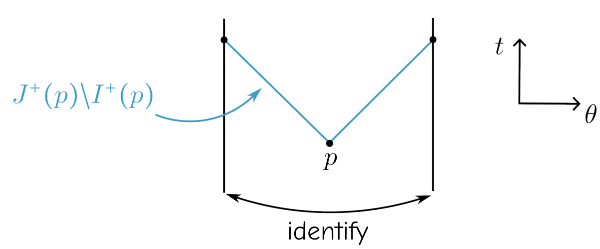
$$E^+(A) := J^+(A) \setminus I^+(A)$$

is compact. Analogously, A is called past-trapped $E^-(A) := J^-(A) \setminus I^-(A)$.

Example 3.10.3.
For

$$(M, g) := (\mathbb{R} \times S^1, -dt^2 + d\theta^2),$$

the subset $A := \{p\}$ is both future- and past-trapped.



Remark 3.10.4.

1. If A is future-trapped, then A is compact. Indeed, $A \subseteq J^+(A)$ and $A \cap I^+(A) = \varnothing$ (since A is achronal). Therefore, $A \subseteq J^+(A) \setminus I^+(A)$ is compact (since it is closed and $J^+(A)$ compact).

2. $E^+(A)$ is achronal. To see this, let $p \in I^+(E^+(A)) = I^+(J^+(A)\backslash I^+(A)) \subseteq I^+(J^+(A)) \overset{3.1.8}{=} I^+(A)$. Then $p \notin E^+(A)$ and $I^+(E^+(A)) \cap E^+(A) = \varnothing$ i.e. $E^+(A)$ is achronal.

Lemma 3.10.5. Let M be an n-dimensional Lorentzian manifold, $p \in M$, $l \in T_pM$ null, $e_1, \dots, e_{n-2} \in T_pM$ spacelike and orthonormal with $e_j \perp l$ for all j. Then

$$\mathrm{Ric}(l,l) = \sum_{j=1}^{n-2} \langle R(e_j,l)e_j, l\rangle\,.$$

Proof. Extend $e_1, \dots, e_{n-2}$ to an ONB $e_1, \dots, e_n$ so that e_{n-1} is spacelike, e_n TL and $e_{n-1} + e_n \propto l$. Then,

$$\mathrm{Ric}(l,l) \overset{1.2.8}{=} \sum_{j=1}^{n-1} \underbrace{\epsilon_j}_{=+1} \langle R(e_j,l)e_j, l\rangle + \underbrace{\epsilon_n}_{=-1} \langle R(e_n,l)e_n, l\rangle\,.$$

We need to show that

$$\langle R(e_{n_1},l)e_{n-1}, l\rangle - \langle R(e_n,l)e_n, l\rangle = 0.$$

Since $e_{n-1} + e_n$ is a multiple of l,

$$\langle R(e_{n-1}+e_n,l)e_{n-1}, l\rangle = 0 \text{ and } \langle R(e_{n-1}+e_n,l)e_n, l\rangle = 0.$$

Subtracting the second equation from the first one we get

$$\langle R(e_{n-1}+e_n,l)(e_{n-1}-e_n), l\rangle = 0,$$

which proves the claim since $\langle R(e_{n-1}+e_n,l)(e_{n-1}-e_n), l\rangle = \langle R(e_{n-1},l)e_{n-1}, l\rangle + \langle R(e_n,l)e_{n-1}, l\rangle - \langle R(e_{n-1},l)e_n, l\rangle - \langle R(e_n,l)e_n, l\rangle = 0$, where $\langle R(e_n,l)e_{n-1}, l\rangle - \langle R(e_{n-1},l)e_n, l\rangle = 0$ by pair symmetry (cf. introduction to Section 1.1). ☐

Proposition 3.10.6. Let M be a Lorentzian manifold, $P \subseteq M$ a spacelike submanifold of codimension 2 and $c: [0,b] \to M$ a null geodesic starting in $p \in P$ with $\dot c(0) \perp P$. If

- $\mathrm{Ric}(\dot c(t), \dot c(t)) \geq 0$ for all $t \in [0,b]$ and
- $\langle H_p, \dot c(0)\rangle \geq \frac{1}{b}$,

then c has a focal point in $(0,b]$.

Proof. Suppose that c has no focal point.

1. Let $e_1, \dots, e_{n-2}$ be an ONB of T_pP and consider Jacobi fields J_i $(1 \leq i \leq n-2)$ along c determined by the initial values $J_i(0) = e_i$ and $\frac{\nabla}{dt}J_i(0) = \tilde{\mathbb{I}}(e_i, \dot c(0))$. (according to Lemma 3.2.12, J_i is even a P-JF). In addition, let $J_0(t) := t \cdot \dot c(t)$ be the Jacobi field given by $J_0(0) = 0$ and $J_0'(0) = \dot c(0)$ (cf. Example 3.2.10).

2. We show that $J_0(t), \dots, J_{n-2}(t)$ are a basis of $\dot c(t)^\perp$ for all $t \in (0,b]$.
   - $\dot c(t)$ being null implies that $J_0(t) \perp \dot c(t)$ for all t.

- $J_i(t) \perp \dot{c}(t)$ for all t and for all $1 \le i \le n-2$:

$$\begin{aligned} \langle J_i, \dot{c} \rangle \big|_{t=0} &= \langle e_i, \dot{c}(0) \rangle = 0, \\ \frac{d}{dt}\Big|_0 \langle J_i, \dot{c} \rangle &= \langle J_i', \dot{c} \rangle \big|_{t=0} = \langle \overbrace{\tilde{\mathbb{I}}(e_i, \dot{c}(0))}^{\text{tangential to } P}, \underbrace{\dot{c}(0)}_{\perp P} \rangle = 0, \\ \frac{d^2}{dt^2} \langle J_i, \dot{c} \rangle &= \langle J_i'', \dot{c} \rangle \overset{J_i \text{ JF}}{=} \langle R(J_i, \dot{c})\dot{c}, \dot{c} \rangle = 0, \ \forall t. \end{aligned}$$

  Therefore,

$$\langle J_i, \dot{c} \rangle = \langle J_i, \dot{c} \rangle \big|_0 + t \langle J_i, \dot{c} \rangle' = 0 + 0 = 0.$$

- $J_0, J_1, \ldots, J_{n-2}$ are linearly independent for all $t \in (0, b]$. In order to see this, assume there exists $t_0 \in (0, b]$ where they are linearly dependent. Then there exist some $\alpha_0, \ldots, \alpha_{n-2} \in \mathbb{R}$ not all equal to zero, such that

$$\sum_{i=1}^{n-2} \alpha_i J_i(t_0) = 0.$$

  This provides a non-trivial Jacobi field $J := \sum_{i=0}^{n-2} \alpha_i J_i$ satisfying:

$$\begin{aligned} J(0) &= \sum_{i=1}^{n-2} \alpha_i e_i \in T_p P, \\ J(t) &= 0, \\ \tan(J'(0)) &= \tan\left(\sum_{i=1}^{n-2} \alpha_i \tilde{\mathbb{I}}(e_i, \dot{c}(0)) + \alpha_0 \dot{c}(0) \right) = \tan(\tilde{\mathbb{I}}(J(0), \dot{c}(0))). \end{aligned}$$

  Therefore, by Lemma 3.2.12, J is a P-JF and $J(t_0) = 0$. t_0 is thus a focal point, contradicting the initial assumption.

3. We now claim $\langle J_i'(t), J_j(t) \rangle = \langle J_i(t), J_j'(t) \rangle$ for all i, j and for all t.

$$\frac{d}{dt}(\langle J_i', J_j \rangle - \langle J_i, J_j' \rangle) = \langle J_i'', J_j \rangle - \langle J_i, J_j'' \rangle \overset{3.2.7}{=} \langle R(J_i, \dot{c})\dot{c}, J_j \rangle - \langle J_i, R(J_j, \dot{c})\dot{c} \rangle \overset{\text{pair sy.}}{=} 0.$$

   For $t = 0$ and $i, j \ge 1$

$$\begin{aligned} \langle J_i'(0), J_j(0) \rangle - \langle J_i(0), J_j'(0) \rangle &\overset{1.}{=} \langle \tilde{\mathbb{I}}(e_i, \dot{c}(0)), e_j \rangle - \langle \tilde{\mathbb{I}}(e_j, \dot{c}(0)), e_i \rangle \\ &\overset{1.6.9, 3.}{=} -\langle \mathbb{I}(e_i, e_j), \dot{c}(0) \rangle + \langle \mathbb{I}(e_j, e_i), \dot{c}(0) \rangle \overset{\mathbb{I} \text{ is sym.}}{=} 0, \end{aligned}$$

   and moreover,

$$\langle J_0'(0), J_j(0) \rangle - \langle \overbrace{J_0(0)}^{=0}, J_j'(0) \rangle \overset{1.}{=} \langle \dot{c}(0), e_j \rangle - 0 = 0.$$

4. Let $V \in \mathfrak{X}(c)$, $V(t) \perp \dot{c}(t)$ for all t, $V(0) \in T_p P$, $V(b) = 0$. Then,

$$\int_0^b (\langle V', V' \rangle - \langle R(\dot{c}, V)\dot{c}, V \rangle) dt - \langle \dot{c}(0), \mathbb{I}(V(0), V(0)) \rangle \ge 0$$

   with equality if and only if V is parallel to c. By 2., there exist unique smooth functions $f_i : (0, b] \to \mathbb{R}$ with

$$V(t) = \sum_{i=0}^{n-2} f_i(t) J_i(t), \ \forall t \in (0, b].$$

Set $X := \sum_{i=0}^{n-2} f_i' J_i$ and $Y := \sum_{i=0}^{n-2} f_i J_i'$. Then $V' = X + Y$ and

$$\begin{aligned}
\frac{d}{dt}\langle V,Y\rangle &= \langle V',Y\rangle + \langle V,Y'\rangle \\
&= \langle X,Y\rangle + \langle Y,Y\rangle + \langle V,Y'\rangle \\
&= \langle X,Y\rangle + \langle Y,Y\rangle + \sum_{i,j=0}^{n-2}\langle f_jJ_j, f_i'J_i'\rangle + \left\langle V, \sum_{i=0}^{n-2} f_i R(J_i,\dot c)\dot c\right\rangle \\
&\overset{3.}{=} \langle X,Y\rangle + \langle Y,Y\rangle + \sum_{i,j=0}^{n-2} f_j f_i' \left\langle J_j', J_i\right\rangle + \langle V, R(V,\dot c)\dot c\rangle \\
&= \langle X,Y\rangle + \langle Y,Y\rangle + \underbrace{\sum_{i,j=0}^{n-2} f_j f_i' \left\langle J_j', J_i\right\rangle}_{=\langle X,Y\rangle} - \langle R(V,\dot c)V, \dot c\rangle \\
&= 2\langle X,Y\rangle + \langle Y,Y\rangle - \langle R(V,\dot c)V,\dot c\rangle \\
&= \langle X+Y, X+Y\rangle - \langle X,X\rangle - \langle R(V,\dot c)V,\dot c\rangle \\
&= \langle V',V'\rangle - \langle X,X\rangle - \langle R(V,\dot c)V,\dot c\rangle
\end{aligned}$$

for $\epsilon > 0$ implies

$$\begin{aligned}
\int_\epsilon^b \langle X,X\rangle\, dt &= \int_\epsilon^b (\langle V',V'\rangle - \langle R(V,\dot c)V,\dot c\rangle - \frac{d}{dt}\langle V,Y\rangle) dt \\
&\overset{V(b)=0}{=} \int_\epsilon^b (\langle V',V'\rangle - \langle R(V,\dot c)V,\dot c\rangle) dt + \langle V(\epsilon), Y(\epsilon)\rangle
\end{aligned}$$

Note that in addition to basis of $\dot c(t)^\perp$ for $t \in (0,b]$

$$J_0(t), J_1(t), \ldots, J_{n-2}(t),$$

we can find a further basis, namely

$$\frac{1}{t}J_0(t) = \dot c(t), J_1(t), \ldots, J_{n-2}(t).$$

This is a basis of $\dot c(t)^\perp$ also for $t = 0$ (at $t = 0$ this is equal to $\dot c(0), e_1, \ldots, e_{n-1}, \ldots$, which is ONB by assumption). Hence, $tf_0(t), f_1(t), \ldots, f_{n-2}(t)$ extends continuously to $t = 0$. Now

$$V(t) = \sum_{i=0}^{n-2} f_i(t)J_i(t) = t\cdot f_0(t)\dot c(t) + \sum_{i=1}^{n-2} f_i(t)J_i(t) \overset{t\to 0}{\longrightarrow} V(0),$$

since $V \in \mathfrak{X}(c)$. $V(0) \in T_pP$ so it has no $\dot c(0)$-component, implying $t\cdot f_0(t) \longrightarrow 0$ (as $t \to 0$), which yields

$$V(0) = \sum_{i=1}^{n-2} f_i(0)J_i(0) = \sum_{i=1}^{n-2} f_i(0)e_i.$$

$$\begin{aligned}
\langle V(\epsilon), Y(\epsilon)\rangle &= \left\langle V(\epsilon), f_0(\epsilon)\dot c(\epsilon) + \sum_{i=1}^{n-2} f_i(\epsilon)J_i'(\epsilon)\right\rangle \\
&\overset{V(t)\perp \dot c(t)}{=} \left\langle V(\epsilon), \sum_{i=1}^{n-2} f_i(\epsilon)J_i'(\epsilon)\right\rangle \overset{\epsilon\to 0}{\longrightarrow} \left\langle V(0), \sum_{i=1}^{n-2} f_i(0)J_i'(0)\right\rangle \\
\left\langle V(0), \sum_{i=1}^{n-2} f_i(0)J_i'(0)\right\rangle &\overset{1.}{=} \left\langle V(0), \sum_{i=1}^{n-2} f_i(0)\tilde{\mathbb{I}}(e_i, \dot c(0))\right\rangle = \left\langle V(0), \tilde{\mathbb{I}}(V(0), \dot c(0))\right\rangle \\
&= -\langle \mathbb{I}(V(0), V(0)), \dot c(0)\rangle,
\end{aligned}$$

hence,

$$\int_0^b (\langle V', V' \rangle - \langle R(V,\dot c)V, \dot c\rangle) dt - \langle \dot c(0), \mathbb{I}(V(0), V(0))\rangle = \lim_{\epsilon\to 0} \int_\epsilon^b \langle X, X\rangle \, dt.$$

X is a linear combination of the $J_i's$, which implies that $X \perp \dot c$ and so X cannot be TL. Therefore, $\langle X, X\rangle \geq 0$ for all $t > 0$, which proves the inequality in 4. Equality holds if and only if $\langle X, X\rangle(t) = 0$ for all t. In other words, if X is null for all t—that is, if X is proportional to $\dot c$—then equality holds if and only if $\dot f_1 = \cdots = \dot f_{n-2} = 0$. Moreover, $V(b) = 0$ implies $f_1(b) = \cdots = f_{n-2}(b) = 0$, where equality holds if and only if $f_1 = \cdots = f_{n-2} = 0$. This equivalence occurs if and only if $V(t) = t \cdot f_0(t)\dot c(t)$, in other words, if and only if V is parallel to $\dot c$.

5. Let $e \in T_pP$ be a unit vector and E its parallel transport along c. Set $V(t) := \left(1 - \frac{t}{b}\right) E(t)$. Then V satisfies the assumptions of 4. but is not tangential on c, hence

$$\begin{aligned} 0 &< \int_0^b (\langle V', V' \rangle - \langle R(V,\dot c)V, \dot c\rangle) dt - \langle \dot c(0), \mathbb{I}(V(0), V(0))\rangle \\ &= \int_0^b \left(\left\langle -\frac{1}{b}E, -\frac{1}{b}E \right\rangle - \left(1 - \frac{t}{b}\right)^2 \langle R(E,\dot c)E, \dot c\rangle \right) dt - \langle \dot c(0), \mathbb{I}(e,e)\rangle \\ &= \frac{1}{b} - \int_0^b \left(1 - \frac{t}{b}\right)^2 \langle R(E,\dot c)E, \dot c\rangle \, dt - \langle \dot c(0), \mathbb{I}(e,e)\rangle . \end{aligned}$$

Now set $e = e_i$ $(1 \leq i \leq n-2)$ and sum over i:

$$0 \overset{3.10.5}{<} \frac{n-2}{b} - \int_0^b \left(1 - \frac{t}{b}\right)^2 \underbrace{\mathrm{Ric}(\dot c, \dot c)}_{\geq 0} dt - (n-2) \underbrace{\langle \dot c(0), H(p)\rangle}_{\geq \frac{1}{b}} \leq 0,$$

which is a contradiction.

□

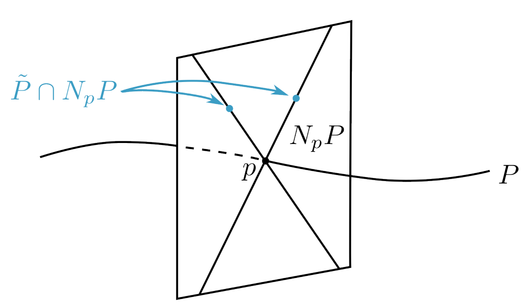
Proposition 3.10.7 (Trapped Sets from Trapped Surfaces)**.** Let $P \subseteq M$ be a compact achronal space-like sub-manifold of codimension 2 which is a future-trapped surface (H is past pointing TL). Let M be future null complete with $\mathrm{Ric}(X, X) \geq 0$ for all $X \in TM$ null. Then P is a future trapped set.

Proof.

a) Pick some Riemannian metric h on M and set

$$\tilde P := \{X \in NP \ : \ X \text{ is null FD and } h(X,X) = 1\}.$$

Then $\pi : NP \to P$ turns $\tilde P$ into a two-fold covering of P and so $\tilde P$ is compact.



More directly, both $h(X,X) = 1$ and X null are closed conditions. Let Z is a TL vector field inducing the time orientation of M. Then X is FD if and only if $\langle X, Z\rangle < 0$, which is the case if and only if $\langle X, Z\rangle \leq 0$. Thus, FD is a closed condition, implying $\tilde{P}$ is compact, as P is compact.

b) Let $X \in \tilde{P}$. Then, by Lemma 3.10.1, $\langle H(\pi(X)), X\rangle > 0$ and so there exists $b > 0$ such that $\langle H(\pi(X)), X\rangle \geq \frac{1}{b}$ for all $X \in \tilde{P}$. M is future null complete, so $t \mapsto c_X(t) := \exp(tX)$ is defined for all $t \geq 0$. In particular, c_X is defined on $[0,b]$ and, by Proposition 3.10.6, c_X has a focal point in $(0,b]$.

c) $E^+(P)$ is relatively compact. To see this, let $q \in E^+(P) = J^+(P)\backslash I^+(P)$. Since $q \in J^+(P)$ there exists a causal curve c from P to q and since $q \notin I^+(P)$, c is a null geodesic without a focal point before q and $\dot{c}(0)\perp P$. Therefore, $c = c_X$ for some $X \in \tilde{P}$ and so $q = c_X(t)$ for some $t \in (0,b]$, hence $E^+(P) \subseteq \exp(\{tX : 0 \leq t \leq b, X \in \tilde{P}\}) =: K$ is compact.

d) $E^+(P)$ is closed. Indeed, let $(q_n)_{n\in\mathbb{N}}$ be a sequence in $E^+(P)$, $q_n \to q$. Therefore, $q \in K \subseteq J^+(K)$. Suppose $q \in I^+(p)$, then $q_n \in I^+(P)$ for n large but $q_n \in E^+(P) = J^+(P)\backslash I^+(P)$, which is a contradiction. Therefore, $q \in J^+(P)\backslash I^+(P) = E^+(P)$ and so $E^+(P)$ is compact.

□

Lemma 3.10.8. Let $K \subseteq M$ be compact and M globally hyperbolic. Then $J^{\pm}(K)$ is closed.

Proof. Let $(p_n) \in J^+(K)$, $p_n \to p$ in M. We need to show that $p \in J^+(K)$. There exist $q_n \in K$ such that $q_n \leq p_n$ and so, since K is compact, without loss of generality $q_n \to q \in K$. $\leq$ is closed by Proposition 3.6.9, so $q \leq p$, implying $p \in J^+(q) \subseteq J^+(K)$. □

Theorem 3.10.9 (Penrose 1965.)**.** Let M be a spacetime with:

1. Ric$(X,X) \geq 0$ for all $X \in TM$ null.
2. There exists a non-compact Cauchy surface $S \subseteq M$.
3. There exists future trapped spacelike submanifold P of codimension 2 (by definition, P is compact, spacelike and achronal submanifold with H past pointing TL).

Then M is not future null complete.

Proof. Assume indirectly that M is future null complete.

1. $E^+(P)$ is an achronal compact $\mathcal{C}^0$-hypersurface. Indeed, S is a Cauchy surface and, therefore according to Lemma 3.8.6, M is globally hyperbolic. As P is compact, Lemma implies $J^+(P)$ is closed. Therefore,
$$E^+(P) = J^+(P)\backslash I^+(P) = \overline{J^+(P)}\backslash J^+(P)^\circ = \partial J^+(P).$$
$J^+(P)$ is a future set, $J^+(P) \neq \varnothing$ and $J^+(P) \neq M$. If $J^+(P) = m$ then let $q \in I^-(P)$. Now, $p_1 \gg q \geq p_2$ and so $p_1 \gg p_2$, which is a contradiction with P being achronal. Corollary now yields that $\partial J^+(P)$ is a closed achronal $\mathcal{C}^0$-hypersurface and Proposition 3.10.7 that $E^+(P) = \partial J^+(P)$ is compact.

2. $E^+(P) \neq \varnothing$. To this end, suppose $E^+(P) = \varnothing$. Then, $\varnothing = \partial J^+(P)$ and so $I^+(P) = J^+(P)^\circ = J^+(P) = \overline{J^+(P)}$ is open, closed nonempty (since $P \neq \varnothing$). Thus, $I^+(P) = M \supseteq P$, which is a contradiction to P being achronal.

3. Let $\rho : \partial J^+(P) \to S$ be the map defined by the flow of a TL vector field, as in Theorem 3.5.24. Then ρ is well-defined and $\partial J^+(P)$ is achronal so ρ is injective, implying that ρ is a continuous injective

map between $\mathcal{C}^0$-hypersurfaces. Since both $\partial J^+(P)$ and S are topological manifolds of dimension $n-1$, Theorem 3.5.10 implies that ρ is open and so $\rho(\partial J^+(P))$ is open in S. As $\partial J^+(P)$ is compact, $\rho(\partial J^+(P))$ is compact as well and, therefore, closed in S. M is connected and, by Theorem 3.5.24, S is connected as well so, $\rho(\partial J^+(P)) = S$. Therefore, S is compact, which is a contradiction to our assumption from the beginning.

□

3.11 Characterization of Global Hyperbolicity

Theorem 3.11.1 (Nomizu/Ozeki). On any connected $\mathcal{C}^\infty$-manifold M there exists a complete Riemannian metric.

Proof. Let h be any Riemannian metric on M. We construct a proper $\mathcal{C}^\infty$ -function $\rho : M \to \mathbb{R}$ (so, $\rho^{-1}(K)$ compact for all $K \subseteq \mathbb{R}$ compact). Let $(\chi_i)_{i\in\mathbb{N}}$ be a partition of unity such that supp$(\chi) \Subset M$ for all i and set

$$\rho := \sum_{i=1}^{\infty} i \cdot \chi_i.$$

Then $\rho \in \mathcal{C}^\infty(M)$ and, if $\rho(x) \in [-i, i]$, then $\chi_j(x) \neq 0$ at most for $j = 1, \ldots, i$, implying that

$$\rho^{-1}([-i,i]) \subseteq \bigcup_{j=1}^{i} \operatorname{supp}(\chi_j) \Subset M$$

i.e. that ρ is proper. Now set

$$g := h + d\rho \oplus d\rho.$$

g is then a Riemannian metric and, by Hopf-Rinow Theorem (cf. Theorem 2.4.2 in [3]), in order to show that g is complete, it suffices to show that any d_ρ-bounded and closed subset of M is compact. To this end, let $\gamma : [a,b] \to M$ be $\mathcal{C}^\infty$. Then,

$$L_g(\gamma) = \int_a^b \left(h(\dot\gamma,\dot\gamma) + \left(\frac{d}{dt}(\rho\circ\gamma)\right)^2 \right)^{\frac{1}{2}} dt \geq \int_a^b \left|\frac{d}{dt}(\rho\circ\gamma)\right| dt \geq \left| \int_a^b \frac{d}{dt}(\rho\circ\gamma) dt \right| = |\rho(\gamma(b)) - \rho(\gamma(a))|$$

and so

$$\forall p,q \in M, |\rho(p) - \rho(q)| \leq d_g(p,q). \tag{3.11.1}$$

Let $C \subseteq M$ be closed and bounded. Then $C \subseteq \rho^{-1}(\underbrace{\rho(C)}_{\subseteq K \Subset \mathbb{R},\text{ by } (3.11.1)})$ is compact in M (ρ proper!) and so C is compact. □

Proposition 3.11.2. Let (M,g) be a spacetime with a smooth spacelike Cauchy hypersurface S. Then M is diffeomorphic to $\mathbb{R} \times S$.

- If S' is another $\mathcal{C}^\infty$ spacelike Cauchy hypersurface, then S and S' are diffeomorphic.
- If S is merely a Cauchy hypersurface, then M is homeomorphic to $\mathbb{R} \times S$.

Proof.

- S is $\mathcal{C}^\infty$. Let $T_1 \in \mathfrak{X}(M)$ be TL, h a complete Riemannian metric on M (see Theorem 3.11.1). Set
$$T := T_1 / \|T_1\|_h .$$
Then T is complete. Indeed, let γ be an integral curve of T with maximal domain (t_-, t_+) and suppose, for example, $t_+ < \infty$. Then $\gamma|_{[0,t_+)}$ has length
$$\int_0^{t_+} |\underbrace{\dot\gamma(t)}_{T(\gamma(t))}|_h dt = t_+ < \infty$$
and so, since h is complete, γ remains in a compact set (see Theorem 2.4.2 in [3]). But, for $t \to t_+$, γ has to leave any compact set, which is a contradiction. Therefore, $t_+ = \infty$. $\mathrm{Fl}^T : \mathbb{R} \times M \to M$ and Fl^T is $\mathcal{C}^\infty$ and, similarly, $t_- = -\infty$. Thus, $\mathrm{Fl}^T : \mathbb{R} \times M \to M$ and Fl^T is $\mathcal{C}^\infty$. Let now $f := \mathrm{Fl}|_{\mathbb{R}\times S}$, $f : \mathbb{R} \times S \to M$. Then $f \in \mathcal{C}^\infty$ and f is bijective by Theorem 3.5.24. We show that f is a diffeomorphism. In order to do that, it suffices to show $T_{(t_0,x_0)} f$ is surjective for all $(t_0, x_0) \in \mathbb{R} \times S$ (which, at the same time, proves bijectivity). Let $\phi : M \to M$, where $\phi(p) := \mathrm{Fl}^T_{-t_0}(p)$. Then ϕ is a diffeomorphism and so is enough to show that $T_{(t_0,x_0)}(\phi \circ f)$ is surjective. Now,
$$(\phi \circ f)(t,x) = \mathrm{Fl}^T_{-t_0}(\mathrm{Fl}^T_t(x)) = \mathrm{Fl}^T_{t-t_0}(x). \tag{3.11.2}$$
Let $v \in T_{x_0} S$ and let $c : I \to S$ $\mathcal{C}^\infty$, $c(0) = x_0$, $c'(0) = v$. Then, $\gamma := t \mapsto (t_0, c(t))$ is $\mathcal{C}^\infty$, $\gamma(0) = (t_0, x_0)$, $\gamma'(0) = (0, v)$. Therefore,
$$(\phi \circ f \circ \gamma)(t) = (\phi \circ f)(t_0, c(t)) \overset{(3.11.2)}{=} \mathrm{Fl}^T_0(c(t)) = c(t)$$
and
$$v = c'(0) = \left.\frac{d}{dt}\right|_0 (\phi \circ f \circ \gamma)(t) = T_{(t_0,x_0)}(\phi \circ f)(\gamma'(0)) \in \mathrm{im}(T_{(t_0,x_0)}(\phi \circ f)),$$
implying $T_{x_0} S \subseteq \mathrm{im}(T_{(t_0,x_0)}(\phi \circ f))$. For
$$c(t) := (t_0 + t, x_0)$$
$c'(0) \in T_{(t_0,x_0)}(\mathbb{R} \times S)$ and
$$(\phi \circ f)(c(t)) = (\phi \circ f)(t_0 + t, x_0) \overset{(3.11.2)}{=} \mathrm{Fl}^x_t(x_0), \tag{3.11.3}$$
implying
$$T_{(t_0,x_0)}(\phi \circ f)(c'(0)) = \left.\frac{d}{dt}\right|_0 (\phi \circ f)(c(t)) \overset{(3.11.3)}{=} T(\underbrace{\mathrm{Fl}^x_0(x_0)}_{= x_0}) = T(x_0).$$
Therefore,
$$\mathrm{im}(T_{(t_0,x_0)}(\phi \circ f)) \supseteq \mathrm{span}(\{T(x_0)\} \cup T_{x_0} S) = T_{x_0} M$$
so $T_{(t_0,x_0)}(\phi \circ f)$ is indeed surjective and $f : \mathbb{R} \times S \to M$ a diffeomorphism. If S' is another $\mathcal{C}^\infty$ spacelike Cauchy hypersurface, then, as in Theorem 3.5.24, consider the mappings $\pi : \mathbb{R} \times S \to \mathbb{R}$, where $\pi(t,x) = x$, and $\rho : S' \to S$, where $\rho(x) = (\pi \circ f^{-1})(x)$. It follows that ρ is $\mathcal{C}^\infty$, and, by symmetry, ρ^{-1} is also $\mathcal{C}^\infty$. Thus, $\rho : S' \to S$ is a diffeomorphism.

- S is only a Cauchy surface. Again, let $f : \mathbb{R} \times S \to M$, where $f(t,x) := \mathrm{Fl}^T_t(x)$. Then, according to Theorem 3.5.24, f is a homeomorphism.

□

Lemma 3.11.3. Let M be globally hyperbolic and $K_1, K_2 \subseteq M$ compact. Then $J^+(K_1) \cap J^-(K_2)$ is compact.

Proof. Let $q_n \in K^+(K_1) \cap J^-(K_2)$. Then there exist $p_{n,i} \in K_i$ such that $p_{n,1} \le q_n \le p_{n,2}$. K_i are compact so, without loss of generality, we can assume $p_{n,i} \to p_i \in K_i$ $(i = 1, 2)$. Let $r_i \in M$ $(i = 1, 2)$ be such that $r_1 \ll p_1$ and $p_2 \ll r_2$. Then $r_1 \ll q_n \ll r_2$ for n large, implying $q_n \in J^+(r_1) \cap J^-(r_2)$, which is compact. Thus, there exists a convergent sequence, without loss of generality $q_n \to q$. '$\le$' is closed and so $p_1 \le q \le p_2$, implying $q \in J^+(K_1) \cap J^-(K_2)$. ☐

Remark 3.11.4 (Integration on M). Recall from analysis surface integral of a function f, where $S : \Omega \to \mathbb{R}^n$ for $\Omega \subseteq \mathbb{R}^n$:

$$\int_S f(y)dS(y) := \int_\Omega f(S(y))\sqrt{G(DS(y))}dy$$

(see Forster, Analysis, Vol. 3).
Here $G(DS) = G(\partial_{y_1} S, \dots, \partial_{y_m} S) = \det(\langle \partial_{y_i} S, \partial_{y_j} S\rangle)_{i,j=1}^m$ is Gramian determinant. Since $(y_1, \dots, y_m) \mapsto S(y_1, \dots, y_m)$ is a parametrization of S, $G(DS) = \det\langle \partial_{y_i}, \partial_{y_j}\rangle = \det g_{i,j}$.
If (M, g) is a SRMF and $(x^1, \dots, x^n, U)$ a chart with $g|_U = g(\partial_{x^i}, \partial_{x^j})dx^i \otimes dx^j$, we set

$$d \cdot \mathrm{vol}_g^U := \sqrt{|\det g_{i,j}|}.$$

Then, if $((y^1, \dots, y^n), V)$ is another chart,

$$\begin{aligned} d \cdot \mathrm{vol}_g^V &= \sqrt{|\mathrm{det}_g(\partial_{y^i}, \partial_{y^j})|} = \left|\det\left(\frac{\partial x^k}{\partial y^i}\partial x^k, \frac{\partial x^l}{\partial y^j}\partial_{x^l}\right)\right|^{\frac{1}{2}} \\ &= \left|\det \frac{\partial x^k}{\partial y^i}\right|\sqrt{|\det(\partial_{x^k}, \partial_{x^l}|)} = |\det D(\psi_U \circ \psi_V^{-1})| \cdot d \cdot \mathrm{vol}_g^U \end{aligned}$$

Therefore, if we set for $f \in \mathcal{C}_c^\infty(U)$

$$\int_M f \cdot d \cdot \mathrm{vol}_g := \int_{\psi_U(U)} (f \cdot d \cdot \mathrm{vol}_g^U) \circ \psi_U^{-1} d(x^1, \dots, x^n)$$

this gives a well-defined integral. We can extend it, via partitions of unity, to $f \in \mathcal{C}_c^\infty(M)$ as in [2]. If $\{(\psi_U, U)\}$ is oriented, $d \cdot \mathrm{vol}_g$ can be viewed as an n-form:

$$d \cdot \mathrm{vol}_g|_U = \sqrt{\mathrm{det}_g(\partial_{x^i}, \partial_{x^j})}dx^1 \wedge \dots \wedge dx^n.$$

Otherwise, $d \cdot \mathrm{vol}_g$ is still well-defined, a so-called 1-*density*.

Theorem 3.11.5. Let (M, g) be globally hyperbolic and oriented. Then there exists $\tau : M \to \mathbb{R}$ continuous and surjective such that τ is strictly increasing along all causal curves. If $\gamma : (t_-, t_+) \to M$ is causal and inextendible, then $\tau(\gamma(t)) \to \pm\infty$ as $t \to t_\pm \mp$. Analogous statements hold for γ only future/past inextendible. In particular, $\tau^{-1}(a)$ (for $a \in \mathbb{R}$) is an acausal Cauchy hypersurface.

Proof. Fix some Riemannian metric h on M. Let (χ_i) be a partition of unity on M with supp(χ) compactly contained in a chart domain. For a fixed i, let (U, x) be such a chart and set

$$m_i := \int \chi_i \cdot d_{\mu_h},$$

where $\mu_h = \sqrt{\det(h_{i,j})} dx^1 \wedge \cdots \wedge dx^n$. Now set

$$\omega := \sum_{i=1}^{\infty} \frac{1}{m_i 2^i} \chi_i \mu_h \in \Omega^n(M)$$

and let $\Lambda : C_c(M) \to \mathbb{R}$, where $\Lambda(f) := \int_M f \cdot \omega$. Then Λ is a positive linear functional (i.e. if $f \geq 0$, then $\Lambda(f) \geq 0$) and so, by Riesz Theorem (see in Elstrod, for example), there exists a positive Borel measure μ with $\Lambda(f) = \int_M f d\mu$ for all $f \in C_c(M)$. Let $h_i := \sum_{j=1}^{i} \chi_j$. Then, $h_i \nearrow 1$ and so, by monotone convergence from measure theory,

$$\underbrace{\int_M 1 d\mu}_{\mu(M)=1} = \lim_{i\to\infty} \int_M h_i d\mu = \lim_{i\to\infty} \int_M h_i \cdot \omega = \lim_{i\to\infty} \sum_{j=1}^{i} \frac{1}{2^j} = 1.$$

Thus, μ is probability measure on M.

1. If U is open and nonempty, then $\mu(U) > 0$. To see this, let $\phi \in \mathcal{C}_c^{\infty}(U)$ such that $\phi \equiv 1$ on some compact $K \subseteq U$ and $0 \leq \phi \leq 1$. Then, $0 < \int_M \phi \cdot \omega = \int_M \phi d\mu \leq \int_U d\mu = \mu(U)$.
2. Let $p \in M$, then for $C_p^+ := J^+(p) \backslash I^+(p)$ we have that $\mu(C_p^+) = 0$. Indeed, if $q \in C_p^+$ then by Avez-Seifert Theorem , there exists a causal geodesic γ from p to q. If γ were not null, then $q \in I^+(p)$, which is a contradiction. Therefore, there exists $v \in C^+ \subseteq T_pM$ such that $q = \exp_p(v)$ and so $C_p^+ \subseteq \exp_p(C^+)$. But, C^+ is a null set in T_pM and $\exp_p$ is $\mathcal{C}^{\infty}$ so Lipschitz now implies that $\mu(C_p^+) \leq \mu(\exp_p(C^+)) = 0$. Analogously, $\mu(C_p^-) = 0$. Set $f_-(p) := \mu(J^-(p))$, or $f_+(p) := \mu(J^+(p))$.
3. $f_\pm$ is continuous. First let $p_j \to p$ so that $p_j \ll p$. Then $J^-(p_j) \subseteq J^-(p)$ implies that $f_-(p_j) \leq f_-(p)$. By 2., $f_-(p) = \int_M \chi_{I^-(p)} d\mu$ and so

   $$\lim_{j\to\infty} \chi_{I^-(p_j)} = \chi_{I^-(p)}.$$

   If $q \in I^-(p)$, then $q \in I^-(p_j)$ for j large ($I^+(q)$ is a neighborhood of p, implying that for all $j \geq j_0$ $p_j \in I^+(q)$). By dominated convergence, $f_-(p_j) \to f_-(p)$. Next let $p_j \to p$ and $p_j \gg p$. Then, be any sequence with $p_j \to p$ and let $\epsilon > 0$. Then

   $$\lim_{j\to\infty} \chi_{J^-(p_j)} = \chi_{J^-(p)}.$$

   Indeed, in order to see this it suffices to to show that if $q \notin J^-(p)$, then $q \notin J^-(p_j)$ for j large. Assume there exists a subsequence p_{jk} with $q \in J^-(p_{jk})$, $q \leq p_{jk}$. Then, as '$\leq$' is closed, $q \leq p$, which is a contradiction. Therefore, by dominated convergence, $f_-(p_j) \to f_-(p)$. Finally, let p_j be any sequence with $p_j \to p$ and let $\epsilon > 0$. Then there exist exist q_1, q_2 such that $q_1 \ll p \ll q_2$ with

   $$f_-(p) \leq f_-(q_2) \leq f_-(p) + \epsilon$$

   and

   $$f_-(p) - \epsilon \leq f_-(q_1) \leq f_-(p).$$

   For large j, $q_1 \ll p_j \ll q_2$, so

   $$f_-(p) - \epsilon \leq f_-(q_1) \leq f_-(p_j) \leq f_-(q_2) \leq f_-(p) + \epsilon.$$

   Thus, f_- is continuous. Analogously, f_+ is continuous. Also, $f_\pm(p) > 0$ for all p by 1., since $J^\pm(p)$ contains open sets different from the empty set.

4. f_- is strictly increasing, while f_+ strictly decreasing along causal curves. Indeed, let $p < q$ then, since M is causal, $q \nleq p$ and so $q \notin J^-(p)$. Therefore, there exists a neighborhood U of q with $U \cap J^-(p) = \varnothing$. The set $J^-(q) \cap U$ contains a non-empty open set $I^-(q) \cap U$, implying that $f_-(p) < f_-(q)$. Analogously, $f_+(q) < f_+(p)$.

5. Let $\gamma : [0, a) \to M$ be causal and future inextendible. Then, $f_+(\gamma(t)) \nearrow 0$ as $t \nearrow a$. In order to see this, let $K \subseteq M$ be compact. We show that $K_t := J^+(\gamma(t)) \cap J^-(K) = \varnothing$ for t large. By Lemma 3.11.3, K_t is compact. Let now $C := K_0 \cup \{\gamma(0)\}$. Then K_0 compact and γ starts in C. By Lemma 3.6.3, there exists a t_0 such that $\gamma(t) \notin C$ for all $t > t_0$. Therefore, $\gamma(t) \notin J^-(K)$ for all $t > t_0$, as $\gamma(t) \in J^+(\gamma(0))$ for all $t \geq 0$. $K_t = \varnothing$ for $t > t_0$. Indeed, if $q \in K_t$, then there exists $k \in K$ such that $k \geq q \geq \gamma(t)$ and so $\gamma(t \in J^-(K))$, which is a contradiction. For any $i \geq 1$, set

$$C_i := \bigcup_{j=1}^{i} \operatorname{supp} \chi_i.$$

By construction, $\mu(M \backslash C_i) \leq 2^{-i}$. Since $J^+(\gamma(t)) \cap C_i \subseteq J^+(\gamma(t) \cap J^-(C_i)) = \varnothing$ for t large,

$$f_+(\gamma(t)) = \int_{J(\gamma(t))} d\mu = \mu(J^+(\gamma(t))) \leq \mu(M \backslash C_i) \leq 2^{-i}.$$

Therefore, $f_+ \circ \gamma(t) \to 0$ for $t \to b^-$. Analogously, if $\gamma : (a, 0] \to M$ is causal and past inextendible, $f_-(\gamma(t)) \to 0$ for $t \to a^+$. Finally, let $\tau : M \to \mathbb{R}$, $\tau(p) := \ln \frac{f_-(p)}{f_+(p)}$. f is continuous and strictly increasing along any FD causal curve. If $\gamma : (a, b) \to M$ causal and inextendible, then

$$(\tau \circ \gamma)(t) \to \begin{cases} -\infty, & t \to a^+ \\ +\infty, & t \to b^-. \end{cases}$$

Indeed, fix $t_0 \in (a, b)$. Then:

$$\begin{aligned} \frac{f_-(\gamma(t))}{f_+(\gamma(t))} &\overset{t \geq t_0}{\geq} \frac{f_- \gamma(t)}{f_+ \gamma(t)} \longrightarrow \infty \text{ (as } t \to b^-), \\ \frac{f_-(\gamma(t))}{f_+(\gamma(t))} &\overset{t \geq t_0}{\leq} \frac{f_- \gamma(t)}{f_+ \gamma(t)} \longrightarrow 0 \text{ (as } t \to a^+). \end{aligned}$$

6. Since any inextendible causal curve meets $\tau^{-1}(a)$ exactly once $\tau^{-1}(a)$ is an acausal Cauchy hypersurface.

□

Lemma 3.11.6. Let $V \subseteq U \subseteq \mathbb{R}^n$ be open. Let $f \in \mathcal{C}^\infty(U)$, $f(x) > 0$ for all $x \in V$, $f(x) = 0$ for all $x \in \partial V \cap U$. Set

$$h(x) := \begin{cases} e^{-\frac{1}{f(x)}}, & x \in V \\ 0, & x \notin V \end{cases}.$$

Then $h \in \mathcal{C}^\infty(U)$.

Proof. If $x \in V$ or $x \in U \backslash \overline{V}$, then h is $\mathcal{C}^\infty$ in a neighborhood of x. If $x \in \partial V \cap U$ and $x_j \to x$, then $f(x_j) \to 0$ and $h(x_j) \to 0 = h(x)$. Therefore, $h \in \mathcal{C}^0(U)$. For $x \in V$, any derivative of h can be expressed as a sum of terms, each taking the form of a smooth factor multiplied by

$$r(x) = \frac{1}{f(x)^k}\exp\left(-\frac{1}{f(x)}\right),$$

which goes to 0 as x tends to ∂V. We can thus extend r continuously to $U\backslash V$ by setting it 0 there. We now need to show that this extension is differentiable at any $x \in \partial V$ and has derivative 0 i.e. that

$$\forall x \in \partial V,\ \ \forall v \in \mathbb{R}^n \lim_{t\to 0, t\neq 0} \frac{r(x+tv)-r(x)}{t} = 0. \tag{3.11.4}$$

$r(x) = 0$, so (3.11.4) is equal to 0, unless $x + tv \in V$. $\sum \frac{1}{k!t^k} = e^{\frac{1}{t}}$ and so

$$\frac{1}{k!t^k} \le e^{\frac{1}{t}},\ \forall k \implies e^{-\frac{1}{t}} \le k!t^k \implies \frac{1}{t^{k+2}}e^{-\frac{1}{t}} \le (k+2)!.$$

Therefore,

$$\left|\frac{r(x+tv)}{t}\right| \le (k+2)!\frac{f^2(x+tv)}{|t|} \longrightarrow 0$$

since f is $\mathcal{C}^\infty$ and $f(x) = 0$. r is differentiable at every $x \in \partial V$ and, therefore, on all of U. Thus, $h \in \mathcal{C}^\infty(U)$. ☐

Remark 3.11.7. As in the Remark 3.6.4, let $\exp_p : \tilde{U} \to U$ be a diffeomorphism, U a normal neighborhood, $\tilde{q} : T_pM \to \mathbb{R}$, $\tilde{q}(v) = \langle v, v\rangle_{g_p}$ and $q = \tilde{q}\circ \exp_p^{-1} : U \to \mathbb{R}$. We have that $V := I_U^+(p) \overset{3.1.13}{=} \exp(I^+(0)\cap \tilde{U})$ is open in U and that $q(x) < 0$ for all $x \in V$. Now set

$$f(y) := \begin{cases} e^{\frac{1}{q(y)}}, & y \in V \\ 0, & y \in U\backslash V. \end{cases} \tag{3.11.5}$$

By Lemma 3.11.6, $f \in \mathcal{C}^\infty(U)$. Moreover,

$$\text{grad}(f) = -\frac{1}{q^2} f \cdot \underbrace{\text{grad}(q)}_{=2P}$$

when $f \neq 0$ (see Remark 3.6.4). Therefore, grad(f) is PDTL on V and 0 on $U\backslash V$.

Lemma 3.11.8. Let (M, g) be globally hyperbolic and let S be an acausal Cauchy hypersurface. Moreover, let $p \in S$ and let U be a convex neighborhood of $p \in M$. Then there exists a function $h_p : M \to [0,\infty)$, which is $\mathcal{C}^\infty$ and has the following properties:

1. $h_p(p) = 1$.
2. supp(h_p) is compact and contained in U.
3. If $r \in J^-(s)$ and $h_p(r) \neq 0$, then grad$(h_p)|_r$ is PDTL.

Proof. Let $\gamma : (-\epsilon, \epsilon) \to M$ be FDTL with $\gamma(0) = p$ and set $K_t : J^+(\gamma(t)) \cap J^-(S)$. By Lemma 3.8.9, since $D^-(S) = J^-(S)$, K_t is compact. Moreover, $K_t \subseteq K_s$ for $t \ge s$.

1. There exists a t_0 such that for all $t \in (t_0, 0)$, $K_t \subseteq U$. Indeed, otherwise there exist $s_j \nearrow 0$ and $p_j \in K_{s_j}\backslash U$. $p_j \in K_{s_1}$ so for all j, so without loss of generality $p_j \to r$ in $K_{s_1}\backslash U$. Now $\gamma(s_j) \le p_j$ and $\gamma(s_j) \to p$. Since '≤' is closed on a globally hyperbolic manifold, $p \le r$. But, $p \in S$ and $r \in J^-(S)$, which is a contradiction (to S being acausal).

2. Let $t \in (t_0, 0)$ and let $x := \gamma(t)$. Let $f \in \mathcal{C}^\infty(U)$ be the function from (3.11.5) with x instead of p. Then grad(f) is PDTL whenever $f \neq 0$. Choose K compact with $K_t \subseteq K^\circ \subseteq K \subseteq U$ and pick $\phi \in \mathcal{C}_c^\infty(U)$, $\phi : U \to [0,1]$ such that $\phi \equiv 1$ in a neighborhood of K. Let $H := \phi \cdot f$. Then $H \in \mathcal{C}_c^\infty(U) \subseteq \mathcal{C}^\infty(M)$. If $r \in J^-(s)$ and $H(r) \neq 0$, then $f(r) \neq 0$ and so, by 1., $r \in I^+\gamma(t) \subseteq K_t \subseteq K$. Hence, $r \in K$ and so $\phi \cdot f = f$ in a neighborhood of r. Also, $x \ll p$, so $H(p) = \underbrace{\phi(p)}_{=1} \cdot f(p) > 0$, by 1. from above. Now the function
$$h_p := \frac{H}{H(p)}$$
has the desired properties. It is evident that 1. and 2. hold. Regarding 3., if $r \in J^-(S)$ then $H = f$ near r, implying $(\text{grad}H)(r) = (\text{grad}f)(r)$.

□

Proposition 3.11.9. Let (M, g) be globally hyperbolic and S an acausal Cauchy hypersurface. If W is a neighborhood of S then there exists a function $h : M \to [0, \infty)$, $\mathcal{C}^\infty$ with the following properties:

1. supp$(h) \subseteq W$.
2. $h(p) > \frac{1}{2}$ for all $p \in S$.
3. grad(h) is PDTL on $h^{-1}((0,\infty)) \cap J^-(S)$.

Proof. Let d be the Riemannian distance induced by some complete Riemannian metric on M (cf. Theorem 3.11.1). By Hopf-Rinow Theorem (Theorem 2.4.2 in [3]), for any $\rho < \infty$, $\overline{B_\rho(p)}$ is compact. Now set $B_0(p) := \emptyset$. Fix any $p_0 \in M$ and for $l = 1, 2, \dots$ let $K_l := \overline{B_l(p_0)} \backslash B_{l-1}(p_0)$ and $R_l := K_l \cap S$. K_l and R_l are both compact (S is closed). For any $r \in S$, let U_r be a convex neighborhood of r with diam$(U_r) < 1$ and $U_r \subseteq W$. Let $h_r \in \mathcal{C}^\infty(U_r)$ be as in Lemma and set
$$V_r := h_r^{-1}\left(\left(\frac{1}{2}, \infty\right)\right).$$
V_r is then a neighborhood of r and $V_r \subseteq U_r$. For any l there exist finitely many $r_{l,1} \dots, r_{l,k} \in R_l$ such that the corresponding $V_{l,i} := V_{r_{l,i}}$ cover R_l (compact). Also, set $U_{l,i} := U_{r_{l,i}}$. If $|l - m| \geq 3$ then $U_{l,i} \cap U_{m,j} = \emptyset$ for all i, j as both have diameter less than 1 and by definition of R_l and R_m, $d(r_{l,i}, r_{m,j}) \geq 2$. Consequently,
$$h := \sum_{l=1}^{\infty} \sum_{i=1}^{k_l} h_{r_{l,i}} \in \mathcal{C}^\infty(M)$$
since $(\text{supp}h_{r_{l,i}})_{l,i}$ is locally finite ($h \equiv 0$ outside $U_{l,i} \subseteq W$). If $x \in S$, there exist l and i such that $x \in V_{l,i}$. Then $h(x) > \frac{1}{2}$. Moreover, since $(U_{l,i})_{l,i}$ is locally finite,
$$\text{supp}(h) \subseteq \bigcup_{l,i} \text{supp}(h_{r_{l,i}}) \subseteq W.$$
Finally, let $x \in h^{-1}((0,\infty)) \cap J^-(S)$. Then, grad$(h_{r_{l,i}})$ is PDTL if $h_{r_{l,i}}(x) \neq 0$. If $h_{r_{l,i}}(x) = 0$, then x is a minimum of $h_{r_{l,i}}$ and grad$(h_{r_{l,i}}(x)) = 0$. Since there exist some l, i with $h_{r_{l,i}}(x) > 0$, grad$(h(x))$ is PDTL. □

Definition 3.11.10. A time function on a Lorentzian manifold M is a function $\tau : M \to \mathbb{R}$ that is strictly increasing along any FD causal curve. A temporal function is a function $\tau : M \to \mathbb{R}$ such that grad(τ) is everywhere PDTL.

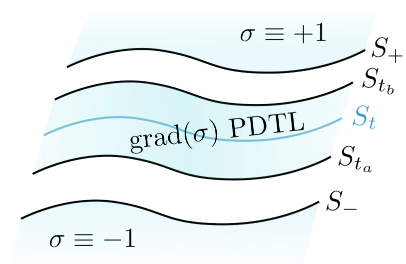
Remark 3.11.11.

- Any temporal function is a time function.
- Let γ be FD causal. Then,
$$\frac{d}{dt}(\tau \circ \gamma)(t) = \langle \underbrace{\text{grad}(\tau)|_{\gamma(t)}}_{\text{PDTL}}, \gamma'(t)\rangle > 0$$
and so $\tau \circ \gamma$ is strictly increasing.
- We will show that on any globally hyperbolic spacetime (M, g) there exists a temporal function all of whose level sets are Cauchy hypersurfaces.

Moving forward, we will operate under the general assumption that (M, g) is globally hyperbolic and oriented, and τ denotes the function from Theorem 3.11.5. Furtehrmore, define $S_t = \tau^{-1}(t)$.

Definition 3.11.12. Fix $t_- < t_a < t < t_b < t_+$ and set $S_\pm := S_{t\pm}$. Then $\sigma : M \to \mathbb{R}$ is called a temporal step function around t compatible with $t\pm$ and t_a, t_b if:

1. grad(σ) is PDTL on
$$V := \{p \in M : \text{grad}(\sigma)(p) \neq 0\}.$$
2. $\sigma(p) \in [-1, 1]$ for all $p \in M$.
3. $\sigma(p) = -1$ for $p \in J^-(S_-)$, $\sigma(p) = 1$ for $p \in J^+(S_+)$.
4. $S_{t'} \subseteq V$ for all $t' \in (t_a, t_b)$.



Lemma 3.11.13. Let $t_- < t < t_+$. Then there exists an open set U such that $J^-(S_t) \subseteq U \subseteq I^-(S_{t+})$ and a function $h^+ : M \to \mathbb{R}$ with $h^+ \geq 0$ and supp$(h^+) \subseteq I^+(S_{t-})$ such that:

1. if $p \in U$ and $h^+(p) > 0$, then grad$(h_+)(p)$ is PDTL.
2. $h^+(p) > \frac{1}{2}$ for all $p \in J^+(S_t) \cap U$.

Proof. Let h^+ be the function from Proposition 3.11.9 with $S := S_t$ and $W = I^-(S_{t+}) \cap I^+(S_{t-})$. If $x \in S_t$, then $h^+(x) > \frac{1}{2}$ and grad$(h^+)(x)$ is PDTL. Therefore, there exists an open neighborhood V_x of x where both conditions hold. Now set $U := I^-(S_t) \cup \bigcup_{x \in S_t} V_x$. □

Lemma 3.11.14. Fix $t, t_+ \in \mathbb{R}$ with $t < t_1$ and let $U \subseteq I^-(S_{t_+})$ be an open neighborhood of $J^-(S_t)$. Then there exists a function $h^- \in \mathcal{C}^\infty(M, \mathbb{R})$, $-1 \leq h^- \leq 0$ such that:

1. supp$(h^-) \subseteq U$.
2. if grad$(h^-)(p) \neq 0$ at $p \in U$, then it is PDTL.
3. $h^-(p) = -1$ for all $p \in J^-(S_t)$.

Proof. Reverse time orientation and construct h as in Proposition 3.11.9 with $S = S_t$ and $W = U$. Then $h : M \to [0, \infty)$ is $\mathcal{C}^\infty$ and supp$(h) \subseteq U$. If $h(p) > 0$ and $p \in J^+(S_t)$ then grad$(h)(p)$ is FDTL and $h(p) > \frac{1}{2}$

for all $p \in S_t$. Next set $h_1 := -h$ and let $\phi : \mathbb{R} \to \mathbb{R}$ be $\mathcal{C}^\infty$ such that:

$$\begin{cases} \phi(t) = -1, & t \leq -\frac{1}{2} \\ \phi'(t) > 0, & -\frac{1}{2} < t < 0 \\ \phi(t) = 0, & t \geq 0 \\ \phi(t) \in [-1,1], & \forall t \end{cases}$$

Let

$$f(t) = \begin{cases} e^{-\frac{1}{t}}, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

Set $\psi(t) = f\left(t + \frac{1}{2}\right) f(-t)$. Then $\psi \in \mathcal{C}^\infty$, $\psi \geq 0$, $\psi(t) > 0$ for $t \in \left(-\frac{1}{2}, 0\right)$ and $\psi(t) = 0$ for $t \notin \left(-\frac{1}{2}, 0\right)$. Also set:

$$\phi(t) := \int_{-\frac{1}{2}}^{t} \psi(s) ds \left[\int_{-\frac{1}{2}}^{0} \psi(s) ds \right]^{-1} - 1$$

and

$$h^{-1}(p) = \begin{cases} \phi \circ h_1(p), & p \in J^+(S_t), \\ -1, & p \in J^-(S_t). \end{cases}$$

h^- now has the required properties. Indeed, h^- is $\mathcal{C}^\infty$ since $J^+(S_t) \cap J^-(S_t) = S_t$ and $h_1|_{S_t} < -\frac{1}{2}$. Moreover, grad$(h^-)(p) = \phi'(h_1(p)) \cdot$ grad$(h_1(p))$, with grad$(h_1(p))$ PDTL where it is not equal to zero. ☐

Proposition 3.11.15. Let $t_- < t < t_+$. Then there exists a function $\sigma \in \mathcal{C}^\infty(M, \mathbb{R})$ with properties 1.,2. and 3. from Definition 3.11.12, such that $S_t \subseteq \{p \in M : \text{grad}(\sigma)(p) \neq 0\}$.

Proof. Let h^+ and U be as in Lemma 3.11.13 and h^- for this U as in Lemma 3.11.14. Then $h^+ > \frac{1}{2}$ on $U \cap J^+(S_t)$ and $h^- = -1$ on $J^-(S_t)$ and so $h^+ - h^- > \frac{1}{2}$ on U. We can set:

$$\sigma = \begin{cases} 2 \cdot \frac{h^+}{h^+ - h^-} - 1, & \text{on } U \\ 1, & \text{on } M \backslash \text{supp}(h^-) \end{cases}$$

to get $\sigma \in \mathcal{C}^\infty(M, \mathbb{R})$. $\sigma(p) \in [-1, 1]$ for all $p \in M$. For $p \in J^-(S_{t_-})$, we have that $h^+(p) = 0$, since supp$h^+ \subseteq I^+(S_{t_-})$. Therefore, $\sigma(p) = -1$. For $p \in J^+(S_{t_+})$, $p \notin U$ implying $h^-(p) = 0$. Thus, $\sigma(p) = 1$, proving 2. and 3. from Definition 3.11.12. Finally, note that for any $F : \mathbb{R}^2 \to \mathbb{R}$ smooth,

$$\text{grad}(F)(h^+, h^-) = \partial_1 F \cdot \text{grad}(h^+) + \partial_2 F \cdot \text{grad}(h^-).$$

Applying this to $F(x, y) := \frac{x}{x-y}$, we obtain that

$$\text{grad}(\sigma) = 2 \cdot \frac{h^+ \cdot \text{grad}(h^-) - h^- \cdot \text{grad}(h^+)}{(h^+ - h^-)^2}$$

is PDTL whenever it is different from 0, implying that 1. from Definition 3.11.12 also holds. ☐

Corollary 3.11.16. Let $t \in \mathbb{R}$, $t_\pm = t \pm 1$. Also, let $t_- < t_a < t_b < t_+$ and let K be a compact subset of $\tau^-([t_a, t_b])$. Then there exists a function $\sigma : M \to \mathbb{R}$ with properties 1.-3. from Definition 3.11.12 and $K \subseteq \{p \in M : \text{grad}(\sigma)(p) \neq 0\}$.

Proof. For each $s \in [t_a, t_b]$ let σ_s be the function from Proposition 3.11 with $t_- < t < t_+$ replaced by $t_- < s < t_+$ and let

$$V_s := \{p : \text{grad}(\sigma_s)(p) \neq 0\}.$$

Then $\{V_s : s \in [t_a, t_b]\}$ is an open covering of K (if $p \in K$ then there exists $s \in [t_a, t_b] : p \in S_s$ and Proposition implies $\text{grad}(\sigma_s)(p) \neq 0$). Thus, there exist $s_1, \dots, s_k$ such that $(V_{s_i})_{i=1}^k$ covers K. Define $\sigma := \frac{1}{k}\sum_{i=1}^k \sigma_{s_i}$; this choice successfully accomplishes the task. □

Lemma 3.11.17. Let (v_i) be a sequence of TL vectors; all FD or all PD. If $v = \sum_{i=1}^\infty v_i$ exists then it is also TL.

Proof. Set $w := \sum_{i=2}^\infty v_i$. Then, by continuity, w is causal and FD (or PD) and so

$$v = \underbrace{v_1}_{\text{TL}} + w$$

is timelike. □

Theorem 3.11.18. Let $t \in \mathbb{R}$, $t_\pm = t \pm 1$. Then for any t_a, t_b such that $t_- < t_a < t_b < t_+$ there exist a compatible temporal step function.

Proof. Let G_j (for $j = 1, 2, \dots$) be open sets such that $\overline{G_j}$ is compact, $\overline{G_j} \subseteq G_{j+1}$ for all j and $M = \bigcup_{j\geq 1} G_j$. Set $K_j := \overline{G}_j \cap J^+(S_{t_a}) \cap J^-(S_{t_b})$. K_j is compact and $K_j \subseteq \tau^{-1}([t_a, t_b])$, hence Corollary 3.11.16 implies that there exists σ_j for K_j. Let $(v_i)_{i\geq 1}$ be a locally finite covering of M such that each v_i is contained in a chart domain and let U_i be open, $\overline{U}_i \Subset V_i$, U_i open covering of M. Let $x_i^1, \dots, x_i^n$ be coordinates on V_i and let $A_j > 1$ be constants such that for all $1 \leq i \leq j$ and for all $0 \leq m < j$,

$$\left| \frac{\partial^m \sigma_j}{\partial x_i^{l_1} \dots \partial x_i^{l_m}} \right| < A_j$$

on $\overline{U_i}$ for all $l_1, \dots, l_m \in \{1, \dots, n\}$. Now define

$$\sigma := \sum_{j=1}^\infty \frac{1}{2^j A_j} \sigma_j. \tag{3.11.6}$$

That series converges absolutely, implying that σ is continuous, in fact, it is even $\mathcal{C}^\infty$. Indeed, let $p \in M$. Then there exists i such that $p \in U_i$. To show that $\sigma \in C^l$ ($l \geq 1$, arbitrary) choose a $j > i, l$. Then $\frac{\sigma_j}{2^j A_j}$ and all its derivatives of order less or equal to l (with respect to x_i) are bounded by 2^{-j}. $\sigma \in C^l$ for all l, implies that $\sigma \in \mathcal{C}^\infty$. By construction, σ is constant and negative. Say $\sigma \equiv \sigma_- < 0$ on $J^-(S_{t-})$ and $\sigma \equiv \sigma_+$ on $J^+(S_+)$. Let $\psi \in \mathcal{C}^\infty(\mathbb{R})$, $\psi(t) = -1$ for $t \leq \sigma_-$, $\psi(t) = +1$ for all $t \geq \sigma_+$, $\psi'(t) > 0$ for $t \in (\sigma_-, \sigma_+)$ and $\psi(t) \in [-1, 1]$ for all t. Then $\psi \circ \sigma$ has all the claimed properties:

1. $\text{grad}(\psi \circ \sigma) = (\psi' \circ \sigma) \cdot \text{grad}(\sigma)$ and $\text{grad}(\sigma) = \sum_{j=1}^\infty \frac{1}{2^j A_j} \text{grad}(\sigma_j)$ is PDTL where it is different from zero, by Lemma 3.11.17.

2., 3. Clear.

4. Let $t' \in (t_a, t_b)$ and $p \in S_{t'}$. Then there exists j such that $p \in K_j$, implying $\text{grad}(\sigma_j)(p) \neq 0$ and so $\text{grad}(\sigma)|_p \neq 0$.

□

Theorem 3.11.19. Let (M, g) be (connected and oriented) globally hyperbolic spacetime. Then there exists $\mathcal{T} \in \mathcal{C}^\infty(M, \mathbb{R})$ such that grad$(\mathcal{T})(p)$ is PDTL for all $p \in M$ and for any FD causal inextendible $\gamma : (t_-, t_+) \to M$, $\mathcal{T}(\gamma(t)) \to \pm\infty$ (as $t \to t_\pm\mp$). In particular, each level set $\mathcal{T}^{-1}(t)$ is a smooth spacelike Cauchy surface.

Proof. Let $t_k := \frac{k}{2}$, $t_{k,\pm} = t_k \pm 1$, $t_{k,a} = t_k - \frac{1}{2}$ and $t_{k,b} = t_k + \frac{1}{2}$. Let σ_k be the function from Theorem 3.11.18 with $t \leftrightarrow t_k$, $t_\pm \leftrightarrow t_{k,\pm}$ and $t_{m,b} \leftrightarrow t_{k,a}, t_{k,b}$. Then for all $p \in M$ there exists a k such that $\tau(p) \in (t_{k,a}, t_{k,b})$. Now set

$$\mathcal{T} := \sigma_0 + \sum_{k=1}^{\infty} (\sigma_{-k} + \sigma_k). \tag{3.11.7}$$

For $k \geq 3$ we have that $(\sigma_{-k} + \sigma_k)(p) = 0$ if $-\frac{k}{2} + 1 \leq \tau(p) \leq \frac{k}{2} - 1$ (since then $\sigma_k(p) = -1$ and $\sigma_{-k}(p) = +1$ by Definition 3.11.12, 3.). Now any $p \in M$ has a neighborhood where the sum in 3.11.7 is finite and so $\mathcal{T} \in \mathcal{C}^\infty(M, \mathbb{R})$. $(\star)$
σ_k has PDTL gradient for $\tau(p) \in (t_{k,a}, t_{k,b})$ by Definition 3.11.12 (4.), so grad$(\mathcal{T})$ is PDTL for all $p \in M$. $(\star\star)$
Let $\gamma : (t_-, t_+) \to M$ be FD causal and inextendible. By $(\star\star)$, we know that $\mathcal{T} \circ \gamma$ is strictly monotonically increasing. Since every S_t is a Cauchy hypersurface according to Theorem 3.11.5, for any $m \geq 1$ there exists some $s_m \in (t_-, t_+)$ such that $\tau(\gamma(s_m)) = m$. Let $l := 2(m+1)$, then by $(\star\star)$ with $p = \gamma(s_m)$ we get that $(\sigma_{-k} + \sigma_k)(\gamma(s_m)) = 0$ for $k \geq l$. Indeed,

$$-\frac{2(m+1)}{2} \leq \tau(\gamma(s_m)) = m \leq \frac{2(m+1)}{2} - 1.$$

Hence, $\mathcal{T}(\gamma(s_m)) = \left(\sigma_0 + \sum_{k=1}^{l} (\sigma_k + \sigma_{-k})\right)(\gamma(s_m))$. Since $m \geq 1$, $\sigma_0(\gamma(s_m)) = 1$ (cf. Definition 3.11.12, 3.). Also, $(\sigma_k + \sigma_{-k})(\gamma(s_m)) \geq 0$ for all $k \geq 1$ because $\sigma_{-k}(\gamma(s_m)) = 1$ and $\sigma_k(\gamma(s_m)) \geq -1$ (cf. Definition 3.11.12, 2.) for $k \geq 1$. Moreover, for $1 \leq k \leq 2(m-1)$, $\frac{1}{2} \leq \frac{k}{2} \leq m - 1$ i.e. $m \geq \frac{k}{2} + 1$, implying $\sigma_k(\gamma(s_m)) = 1$ and $\sigma_{-k}(\gamma(s_m)) = 1$ $(m \geq -\frac{k}{2} + 1)$. Therefore, $(\sigma_k + \sigma_{-k})(\gamma(s_m)) = 2$ (cf. Definition 3.11.12, 3.). Altogether, $\mathcal{T}(\gamma(s_m)) \geq 1 + 2 \cdot 2(m-1) = 4(m-1) + 1$. Now, $\mathcal{T}(\gamma(s)) \to +\infty (s \to t_+) \geq \mathcal{T}(\gamma(s_m)) \geq 4(m-1) + 1$ for $s \in [s_m, t_+)$ and so $\mathcal{T}(\gamma(s)) \to +\infty$ as $(s \to t_+)$ and, analogously, $\mathcal{T}(\gamma(s)) \to -\infty$ as $(s \to t_-)$. □

Theorem 3.11.20 (Bernal/Sanchez, 2004./2005.). Let (M, g) be a connected, oriented spacetime. TFAE:

1. M is globally hyperbolic.
2. M has a Cauchy surface.
3. M has a smooth spacelike Cauchy hypersurface (CH).
4. M is isometric to $(\mathbb{R} \times S, -\beta d\tau^2 + g_\tau)$, where $\beta : \mathbb{R} \times S \to (0, \infty)$ is $\mathcal{C}^\infty$ and g_τ is a smooth family of Riemannian metrics on S. Then $\{\tau_0\} \times S$ is a $\mathcal{C}^\infty$ spacelike CH for all $\tau_0 \in \mathbb{R}$.

Proof.

$(4. \to 3. \to 2. \to 1.)$ Clear. The last implication follows from Lemma 3.8.6.

$(1. \to 4.)$ Let τ be what we called $\mathcal{T}$ in Theorem 3.11.19. Set $S := S_0 = \tau^{-1}(0)$ and $X :=$ grad$(\tau) \equiv \nabla\tau$. Let $\Phi : M \to \mathbb{R} \times S_0$, where $q \mapsto (\tau(q), \Pi(q))$ for $\Pi(q)$ the unique intersection of $\mathrm{Fl}.^X(q)$ with S_0. From the proof of Proposition 3.11.2 we know that

$$\Psi := (t, q) \mapsto \mathrm{Fl}_t^X(q) : \mathbb{R} \times S_0 \to M$$

is a diffeomorphism, implying that $\Pi = \mathrm{pr}_2 \circ \Psi^{-1}$ is $\mathcal{C}^\infty$ and so Φ is $\mathcal{C}^\infty$. Moreover, Ψ^{-1} is given by

$$\Psi^{-1} : \mathbb{R} \times S_0 \to M, \text{ where } (t,p) \mapsto \mathsf{Fl}^X_{s(t,p)},$$

where $s(t,p)$ is the unique number such that $t = \tau(\mathsf{Fl}^X_{s(t,p)}(p))$ i.e. $t = \tau(\Phi^{-1}(t,p))$. $(\star)$
Indeed,

$$\Phi(\Phi^{-1}(t,p)) = \Phi(\mathsf{Fl}^X_{s(t,p)}(p)) = (\tau(\mathsf{Fl}^X_{s(t,p)}(p)), \Pi(\mathsf{Fl}^X_{s(t,p)}(p))) = (t,p)$$

and

$$\Phi^{-1}(\Phi(q)) = \Phi^{-1}(\tau(q), \Pi(q)) = \mathsf{Fl}^X_{s(\tau(q),\Pi(q))}(\Pi(q)) = q. \tag{3.11.8}$$

Note that $\mathsf{Fl}.^x : (\Pi(q))$ is the flow line through $\Pi(q)$, hence through q. This curve meets $S_{\tau(q)}$ precisely once, namely where $\tau(\mathsf{Fl}^X_s(\Pi(q))) = \tau(q)$ To verify (3.11.8), it suffices to see that

$$\tau(\mathsf{Fl}^X_{s(\tau(q),\Pi(q))})(\Pi(q)) = \tau(q),$$

which is clear by $(\star)$. To see that Φ^{-1} is $\mathcal{C}^\infty$, it suffices to show that s is $\mathcal{C}^\infty$ in (t,p). We have

$$\partial_s(\tau(\mathsf{Fl}^X_s(p))) = d\tau(X(\mathsf{Fl}^X_s(p))) \overset{X=\nabla\tau}{=} \langle X, X\rangle|_{\mathsf{Fl}^X_s(p)} \neq 0 \tag{3.11.9}$$

and so the claim follows from the implicit function theorem. Using Φ as a 'chart' (Φ goes to $\mathbb{R} \times S_0$ and not to $\mathbb{R} \times \mathbb{R}^{n-1}$), we have that

$$\left.\frac{\partial}{\partial\tau}\right|_q = \partial_r|_0 \Phi^{-1}(\Phi(q) + (r,0)) = \partial r|_0 \Phi^{-1}(\tau(q) + r, \Pi(q)). \tag{3.11.10}$$

Here, $\Phi^{-1}(\tau(q)+r, \Pi(q)) = \mathsf{Fl}^X_{s(\tau(q)+r,\Pi(q))}(\Pi(q))$ and so

$$\partial_r|_0 \Phi^{-1}(\tau(q)+r, \Pi(q)) \overset{\text{chain r.}}{=} X(\underbrace{\mathsf{Fl}_{s(\tau(q),\Pi(q))}(\Pi(q))}_{\Phi^{-1}(\Phi(q)) = q}) \cdot \left.\frac{\partial}{\partial r}\right|_0 s(\tau(q)+r, \Pi(q)) \tag{3.11.11}$$

Thus, $\left.\frac{\partial}{\partial\tau}\right|_q \propto X(q)$. Moreover,

$$\begin{aligned}
\left\langle \frac{\partial}{\partial\tau}, X\right\rangle &= \left\langle \frac{\partial}{\partial\tau}, \nabla\tau\right\rangle = d\tau\left(\frac{\partial}{\partial\tau}\right) \\
&= d\tau(\partial_r|_0 \Phi^{-1}(\tau(q)+r, \pi(q))) \\
&= \partial_r|_0 \underbrace{\tau(\Phi^{-1}(\tau(q)+r,\pi(q)))}_{\overset{(\star)}{=} \tau(q)+r} = 1.
\end{aligned}$$

Thus,

$$\frac{\partial}{\partial\tau} = -\left\langle \frac{\partial}{\partial\tau}, \frac{X}{|X|}\right\rangle \frac{X}{|X|} = -\underbrace{\left\langle \frac{\partial}{\partial\tau}, X\right\rangle}_{=1} \frac{X}{|X|^2} = -\frac{\nabla\tau}{|\nabla\tau|^2}.$$

Now pick local coordinates in the spacelike (and, hence, Riemannian) hypersurface $s_{\tau(q)}$ containing q. Then $\frac{\partial}{\partial\tau} \propto \nabla\tau \perp \partial_i$ (grad is always $\perp$ to level sets) and

$$\left\langle \frac{\partial}{\partial\tau}, \frac{\partial}{\partial\tau}\right\rangle = \frac{1}{|\nabla\tau|^2}\langle\nabla\tau, \nabla\tau\rangle = -\frac{1}{|\nabla\tau|^2} =: -\beta.$$

In those coordinates $g = -\beta\tau^2 + g|_{s_{\tau(q)}}$. Moreover,

$$\begin{aligned}
\Phi_*\left(\frac{\partial}{\partial\tau}\bigg|_q\right) &\overset{(3.11.10)}{=} T\Phi(\partial_r|_0\Phi^{-1}(\tau(q)+r,\Pi(q))) \\
&= \partial_r|_0\cancel{\Phi}(\cancel{\Phi^{-1}}(\tau(q)+r,\pi(q))) \\
&= (1,0) \mathrel{\hat{=}} \frac{\partial}{\partial\tau} \text{ on } \mathbb{R}\times S_0.
\end{aligned}$$

Also, $\Phi_*(\partial_i)$ are coordinate vector fields on S_0 because $\Phi|_{s(\tau(q))} : s_{\tau(q)} \to s_0$ is a diffeomorphism. Consequently, $\Phi_* g = -\beta\circ\Phi^{-1}dt^2 + g_\tau$ since $\Phi_*(d\tau) = d(\Phi_*\tau) = d(\tau\circ\Phi^{-1}) = dt$ (note that g_τ is a Riemannian metric on S_0, depending smoothly on τ). Finally, $\Phi^{-1}(\{\tau_0\}\times s_0) = \tau^{-1}(\tau_0)$ is a Cauchy hypersurface in M by Theorem 3.11.19 and so $\{t_0\}\times s_0$ is a Cauchy hypersurface in $(\mathbb{R}\times S, -\beta d\tau^2 + g_\tau)$ since Φ is an isometry.

□

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