Combinatorics of Feynman integrals

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This talk concerns a sequence of **Feynman integrals** that begins with two cases related to **enumeration of walks** on lattices. It contains cases related to **modular forms**. **L-series** associated with **Kloosterman sums** appear, with intricate **rational relations** to **determinants** of Feynman integrals. In general there are **quadratic** relations between integrals encoded by **rational matrices** associated with **Betti** and **de Rham** cohomology.

1. Walks on a **honeycomb** with **Gauss**
2. Walks in a **diamond** crystal with **Bessel**
3. **Modular forms** up to 6 loops
4. **Betti** and **de Rham** matrices for all loops
5. **L-series** up to 22 loops

Joint work with David P. Roberts, University of Minnesota Morris, USA

1 Walks on a honeycomb with Gauss

Let $W_3(2k)$ be the number of returning walks of length $2k$ on a **honeycomb**. In 1960, **Cyril Domb** (1920–2012) showed that

$$G_3(y) = \sum_{k=0}^{\infty} W_3(2k) y^{2k} = 1 + 3y^2 + 15y^4 + 93y^6 + 639y^8 + 4653y^{10} + O(y^{12})$$

is the reciprocal of an **arithmetic-geometric mean** (AGM).

For positive real $(a_0,\, b_0)$, Gauss evaluated the **elliptic** integral

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{(a_0\sin\theta)^2 + (b_0\cos\theta)^2}} = \frac{1}{\mathrm{agm}(a_0, b_0)}$$

by the rapidly converging process of his AGM:

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}, \quad \mathrm{agm}(a_0, b_0) \equiv a_\infty = b_\infty.$$

The honeycomb problem is solved by

$$G_3(y) = \frac{1}{\mathrm{agm}\left(\sqrt{(1-y)^3(1+3y)}, \sqrt{(1+y)^3(1-3y)}\right)}.$$

1.1 Two-loop sunrise diagram with Bessel and Gauss

The two-loop massive **sunrise** diagram in two spacetime dimensions, with external energy w and internal masses (a, b, c), gives the **Bessel** moment

$$\begin{aligned} I(w,a,b,c) &= 4\int_0^\infty I_0(wt)K_0(at)K_0(bt)K_0(ct)t\mathrm{d}t \\ &= \int_0^\infty\int_0^\infty \frac{\mathrm{d}x\,\mathrm{d}y}{P(x,y,1)} \end{aligned}$$

with $P(x,y,z) = (a^2x + b^2y + c^2z)(xy + yz + zx) - w^2xyz$ obtained from Schwinger parameters. Bailey, Borwein, Broadhurst and Glasser obtained

$$I(w,a,b,c) = 8\pi\int_{a+b+c}^\infty \frac{A(v)v\mathrm{d}v}{v^2 - w^2},$$

$$A(w) = 1/\mathrm{agm}\left(\sqrt{F(w)}, \sqrt{16abcw}\right),$$

$$F(w) \equiv (w + a + b + c)(w + a - b - c)(w - a + b - c)(w - a - b + c).$$

Since $F(w) - 16abcw = F(-w)$, the complementary elliptic integral is

$$B(w) = 1/\mathrm{agm}\left(\sqrt{F(w)}, \sqrt{F(-w)}\right).$$

1.2 The equal-mass case

With $a = b = c = 1$, we have $F(w) = (w-1)^3(w+3)$ and hence

$$w^2 B(w) = G_3(1/w), \quad w^2 A(w) = \widetilde{G}_3(1/w)$$

are given by complementary pair of **honeycomb** solutions

$$\begin{aligned} G_3(y) &= 1/\mathrm{agm}\left(\sqrt{(1-y)^3(1+3y)}, \sqrt{(1+y)^3(1-3y)}\right), \\ \widetilde{G}_3(y) &= 1/\mathrm{agm}\left(\sqrt{(1-y)^3(1+3y)}, 4y\sqrt{y}\right). \end{aligned}$$

Defining the elliptic **nome** $q = \exp(-\pi B(w)/A(w))$, we have

$$-\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^2 \left(\frac{I(w,1,1,1)}{24\sqrt{3}A(w)}\right) = \frac{w^2(w^2-1)(w^2-9)A(w)^3}{9\sqrt{3}}$$

as the differential equation, found by Broadhurst, Fleischer and Tarasov. Regarding w and $A(w)$ as functions of q, we obtain the **modular** functions

$$\frac{w}{3} = \left(\frac{\eta_3}{\eta_1}\right)^4 \left(\frac{\eta_2}{\eta_6}\right)^2, \quad 4\sqrt{3}A = \frac{\eta_1^6 \eta_6}{\eta_2^3 \eta_3^2},$$

$$\eta_n = q^{n/24} \prod_{k>0} (1 - q^{nk}) = \sum_{k \in \mathbf{Z}} (-1)^k q^{n(6k+1)^2/24}.$$

The two algebraic relations between $\{\eta_1, \eta_2, \eta_3, \eta_6\}$ give

$$\frac{w^2-1}{8} = \left(\frac{\eta_2}{\eta_1}\right)^9 \left(\frac{\eta_3}{\eta_6}\right)^3, \quad \frac{w^2-9}{72} = \left(\frac{\eta_6}{\eta_1}\right)^5 \frac{\eta_2}{\eta_3}.$$

Hence the BFT equation reduces to

$$-\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^2 \left(\frac{I}{24\sqrt{3}A}\right) = \frac{w}{3} f_{3,12} = \left(\frac{\eta_3^3}{\eta_1}\right)^3 + \left(\frac{\eta_6^3}{\eta_2}\right)^3$$

where $f_{3,12} \equiv (\eta_2\eta_6)^3$ is a weight-3 level-12 modular form. Let $\chi(n) = \pm 1$ for $n = \pm 1 \bmod 6$ and $\chi(n) = 0$ otherwise. Then

$$-\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^2 \left(\frac{I}{24\sqrt{3}A}\right) = \sum_{n>0} \frac{n^2(q^n - q^{5n})}{1-q^{6n}} = \sum_{n>0}\sum_{k>0} n^2 \chi(k) q^{nk}.$$

Integrating twice and using the known imaginary part on the cut, we get

$$\frac{I(w^2,1,1,1)}{4A(w)} = E_2(q) = -\pi \log(-q) - 3\sqrt{3}\sum_{k>0} \frac{\chi(k)}{k^2} \frac{1+q^k}{1-q^k} = -E_2(1/q).$$

This **elliptic dilogarithm** was obtained by Bloch and Vanhove.

2 Walks in a diamond crystal with Bessel

`boob redder testes teammate reappear intestines interessierten`

are **abelian squares**: words whose second halves are permutations of their first halves. There is a bijection between abelian squares of length $2k$ in an n-letter alphabet and **returning walks** of length $2k$ on the regular lattice in $n-1$ dimensions with n-valent vertices. Thus the number $W_n(2k)$ of walks of length $2k$ in $n-1$ dimensions is generated by

$$(I_0(2y))^n = \sum_{k=0}^{\infty} W_n(2k) \left(\frac{y^k}{k!}\right)^2$$

i.e. by the n-th power of the **Bessel** function $I_0(2y) = \sum_{k\geq 0}(y^k/k!)^2$.

Identity for diamond: It was shown by **Geoffrey Joyce** in 1973 that

$$G_4(z) = \sum_{k=0}^{\infty} W_4(2k) z^{2k} = (1-y^2)(1-9y^2)G_3^2(y), \quad z^2 = \frac{-y^2}{(1-y^2)(1-9y^2)}.$$

2.1 Three-loop sunrise with Bessel and Gauss

Joyce's transformation of $G_4(z)$ to the **square** of $G_3(y)$ enables progress with the equal-mass three-loop sunrise diagram

$$J(w) = 8\int_0^\infty I_0(wt)K_0^4(t)t\mathrm{d}t$$

using the transformation

$$\frac{w^2}{64} = \frac{-y^2}{(1-y^2)(1-9y^2)}$$

which is solved by

$$y = \frac{2}{\sqrt{4-w^2}+\sqrt{16-w^2}}$$

with singularities at the pseudo-threshold $w = 2$ and the physical threshold $w = 4$. Then the solutions to the third-order homogeneous equation for $J(w)$ are $(yG_3(y))^2$, $(y\widetilde{G}_3(y))^2$ and $y^2G_3(y)\widetilde{G}_3(y)$. Thus a strategy for best presenting the inhomogeneous equation is to divide J by one of these 3 and to operate with $(q\mathrm{d}/\mathrm{d}q)^3$, where $\log(q)$ is proportional to the ratio of a pair of solutions. If the result is expressible as a simple q series, then the problem may be solved in the same manner as at two loops.

Let $y = 2/(\sqrt{4-w^2}+\sqrt{16-w^2})$ and $q = \exp(-\frac{2}{3}\pi\widetilde{G}_3(y)/G_3(y))$ with

$$G_3(y) = \frac{1}{\mathrm{agm}\left(\sqrt{(1-y)^3(1+3y)}, \sqrt{(1+y)^3(1-3y)}\right)},$$

$$\widetilde{G}_3(y) = \frac{1}{\mathrm{agm}\left(\sqrt{(1-y)^3(1+3y)}, 4y\sqrt{y}\right)}.$$

Then the differential equation is

$$\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^3\left(\frac{2J(w)}{y^2G_3^2(y)}\right) = -48 + 2\sum_{n>0}\sum_{k>0} n^3\psi(k)q^{nk}$$

with $\psi(k) = \psi(k+6) = \psi(6-k)$, and integers $\psi(1) = -48$, $\psi(2) = 720$, $\psi(3) = 384$, $\psi(6) = -5760$, that were found by Bloch, Kerr and Vanhove. We now integrate integrate 3 times. The constants of integration are determined by $J(0) = 7\zeta(3)$. The result is an **elliptic trilogarithm**:

$$\frac{2J(w)}{y^2G_3^2(y)} = E_3(q) = (-2\log(q))^3 + \sum_{k>0}\frac{\psi(k)}{k^3}\frac{1+q^k}{1-q^k} = -E_3(1/q).$$

3 Modular forms up to 6 loops

With $N = a + b$ **Bessel** functions and $c \geq 0$, I define **moments**

$$M(a,b,c) \equiv \int_0^\infty I_0^a(t) K_0^b(t) t^c \mathrm{d}t$$

that converge for $b > a \geq 0$. For $b = a = N/2$, we have convergence for $b > c + 1$. The L-loop on-shell **sunrise** diagram in $D = 2$ dimensions gives

$$2^L M(1, L+1, 1) = \int_0^\infty \ldots \int_0^\infty \frac{\Pi_{k=1}^L \mathrm{d}x_k / x_k}{(1 + \Sigma_{i=1}^L x_i)(1 + \Sigma_{j=1}^L 1/x_j) - 1}$$

as an integral over Schwinger parameters. $M(2, L, 1)$ is obtained by cutting an internal line. To obtain $M(1, L+1, 3)$ and $M(2, L, 3)$, we differentiate w.r.t. an external momentum, before taking the **on-shell** limit.

Recently, in [arXiv:1706.08308], **Yajun Zhou** gave a complete **proof** of my 10-year-old conjecture on the 5-Bessel matrix:

$$\mathcal{M}_5 \equiv \left[\begin{array}{cc} M(1,4,1) & M(1,4,3) \\ M(2,3,1) & M(2,3,3) \end{array} \right] = \left[\begin{array}{cc} \pi^2 C & \pi^2 \left(\frac{2}{15}\right)^2 \left(13C - \frac{1}{10C}\right) \\ \frac{\sqrt{15}\pi}{2} C & \frac{\sqrt{15}\pi}{2} \left(\frac{2}{15}\right)^2 \left(13C + \frac{1}{10C}\right) \end{array} \right].$$

The **determinant** $\det \mathcal{M}_5 = 2\pi^3/\sqrt{3^3 5^5}$ is **free** of the 3-loop constant

$$C \equiv \frac{\pi}{16}\left(1-\frac{1}{\sqrt{5}}\right)\left(\sum_{n=-\infty}^{\infty} e^{-n^2\pi\sqrt{15}}\right)^4 = \frac{1}{240\sqrt{5}\pi^2}\prod_{k=0}^{3}\Gamma\left(\frac{2^k}{15}\right)$$

with Γ values from the **square** of an elliptic integral [arXiv:0801.0891] at the 15th singular value. The **L-series** for $N=5$ Bessel functions comes from a **modular form** of weight 3 and level 15 [arXiv:1604.03057]:

$$\begin{aligned}
\eta_n &\equiv q^{n/24}\prod_{k>0}(1-q^{nk})\\
f_{3,15} &\equiv (\eta_3\eta_5)^3+(\eta_1\eta_{15})^3=\sum_{n>0}A_5(n)q^n\\
L_5(s) &\equiv \sum_{n>0}\frac{A_5(n)}{n^s}\quad \text{for } s>2\\
\Lambda_5(s) &\equiv \left(\frac{15}{\pi^2}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)L_5(s)=\Lambda_5(3-s)\\
L_5(1) &= \sum_{n>0}\frac{A_5(n)}{n}\left(2+\frac{\sqrt{15}}{2\pi n}\right)\exp\left(-\frac{2\pi n}{\sqrt{15}}\right)\\
&= 5C = \frac{5}{\pi^2}\int_0^\infty I_0(t)K_0^4(t)t\mathrm{d}t\,.
\end{aligned}$$

3.1 Magnetic moment of the electron at $N = 6$

Here the **modular form**, found with **Francis Brown** in 2010, is

$$f_{4,6} \equiv (\eta_1\eta_2\eta_3\eta_6)^2$$

with weight 4 and level 6. I discovered and **Zhou** proved that

$$2M(3,3,1) = 3L_6(2), \quad 2M(2,4,1) = 3L_6(3), \quad 2M(1,5,1) = \pi^2 L_6(2).$$

Stefano Laporta has evaluated 4-loop contributions to the **magnetic moment** of the **electron**. These engage the **first** row of the **determinant** [arXiv:1604.03057]

$$\det\begin{bmatrix} M(1,5,1) & M(1,5,3) \\ M(2,4,1) & M(2,4,3) \end{bmatrix} = \frac{5\zeta(4)}{32}.$$

It is notable that the hypergeometric series in

$$L_6(3) = \frac{\pi^2}{15}\, {}_4F_3\left(\begin{matrix} \frac{1}{3}, & \frac{1}{2}, & \frac{1}{2}, & \frac{2}{3} \\ \frac{5}{6}, & 1, & \frac{7}{6} & \end{matrix} \,\middle|\, 1\right)$$

does not appear in Laporta's final result, though it was present at intermediate stages.

3.2 Kloosterman sums over finite fields

For $a \in \mathbf{F}_q$, with $q = p^k$, we define Kloosterman sums

$$K(a) \equiv \sum_{x \in \mathbf{F}_q^*} \exp\left(\frac{2\pi \mathrm{i}}{p} \mathrm{Trace}\left(x + \frac{a}{x}\right)\right)$$

with a trace of Frobenius in $\mathbf{F}_q$ over $\mathbf{F}_p$. Then we obtain integers

$$c_N(q) \equiv -\frac{1 + S_N(q)}{q^2}, \quad S_N(q) \equiv \sum_{a \in \mathbf{F}_q^*} \sum_{k=0}^{N} [g(a)]^k [h(a)]^{N-k}$$

with $K(a) = -g(a) - h(a)$ and $g(a)h(a) = q$. Then

$$Z_N(p, T) = \exp\left(-\sum_{k>0} \frac{c_N(p^k)}{k} T^k\right)$$

is a polynomial in T. For $N < 8$ and $s > (N-1)/2$, the L-series is

$$L_N(s) = \prod_p \frac{1}{Z_N(p, p^{-s})}.$$

With $N=7$ Bessel functions, the **local** factors at the **primes** in

$$L_7(s)=\prod_p \frac{1}{Z_7(p,p^{-s})} \quad \text{for } s>3$$

are given, for the **good** primes p coprime to 105, by the **cubic**

$$Z_7(p,T)=\left(1-\left(\frac{p}{105}\right)p^2T\right)\left(1+\left(\frac{p}{105}\right)(2p^2-|\lambda_p|^2)T+p^4T^2\right)$$

where $\left(\frac{p}{105}\right)=\pm 1$ is a **Kronecker** symbol and λ_p is a Hecke eigenvalue of a weight-3 newform with level 525. For the primes of **bad** reduction, I obtained **quadratics** from **Kloosterman** moments in **finite fields**:

$$Z_7(3,T)=1-10T+3^4T^2,\ Z_7(5,T)=1-5^4T^2,\ Z_7(7,T)=1+70T+7^4T^2.$$

Then **Anton Mellit** suggested a **functional equation**

$$\Lambda_7(s)\equiv\left(\frac{105}{\pi^3}\right)^{s/2}\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)L_7(s)=\Lambda_7(5-s)$$

that was validated at high precision and gave us the empirical result

$$24M(2,5,1)=5\pi^2L_7(2).$$

3.3 Subtleties at $N = 8$

With $N = 8$ Bessel functions, the L-series comes from the **modular form**

$$f_{6,6} \equiv \left(\frac{\eta_2^3\eta_3^3}{\eta_1\eta_6}\right)^3 + \left(\frac{\eta_1^3\eta_6^3}{\eta_2\eta_3}\right)^3$$

with weight 6 and level 6. I discovered and **Zhou** proved that

$$M(4,4,1) = L_8(3), \quad 4M(3,5,1) = 9L_8(4), \quad 4M(2,6,1) = 27L_8(5),$$

and $4M(1,7,1) = 9\pi^2 L_8(4)$ for the **6-loop sunrise** integral.

There are **two subtleties**. Kloosterman moments at $N = 8$ do **not** deliver the local factors directly: in $L_8(s) = \prod_p Z_4(p, p^{2-s})/Z_8(p, p^{-s})$ we remove factors from $N = 4$. Secondly, there is an infinite family of **sum rules**:

$$a(n) \equiv \left(\frac{2}{\pi}\right)^4 \int_0^\infty \left(\pi^2 I_0^2(t) - K_0^2(t)\right) I_0(t) K_0^5(t)\,(2t)^{2n-1}\mathrm{d}t$$

delivers the **integers** of `http://oeis.org/A262961` as was recently proved by **Zhou** in [arXiv:1706.01068].

3.4 Vacuum integrals and non-critical modular L-series

In the **modular** cases $N = 5, 6, 8$, L-series **outside** the critical strip are empirically related to **determinants** that contain **vacuum** integrals:

$$
\begin{aligned}
\det \int_0^\infty K_0^3(t) \left[\begin{array}{cc} K_0^2(t) & t^2 K_0^2(t) \\ I_0^2(t) & t^2 I_0^2(t) \end{array} \right] t\,\mathrm{d}t &= \frac{45}{8\pi^2} L_5(4) \\
\det \int_0^\infty K_0^4(t) \left[\begin{array}{cc} K_0^2(t) & t^2 K_0^2(t) \\ I_0^2(t) & t^2 I_0^2(t) \end{array} \right] t\,\mathrm{d}t &= \frac{27}{4\pi^2} L_6(5) \\
\det \int_0^\infty K_0^6(t) \left[\begin{array}{cc} K_0^2(t) & t^2(1-2t^2) K_0^2(t) \\ I_0^2(t) & t^2(1-2t^2) I_0^2(t) \end{array} \right] t\,\mathrm{d}t &= \frac{6075}{128\pi^2} L_8(7)\,.
\end{aligned}
$$

3.5 Signpost

In work at $N > 8$ with **David Roberts** these features are notable:
local factors from **Kloosterman** moments, sometimes with adjustment;
guesses of Γ factors, signs and conductors in **functional equations**;
empirical fits of L-series to **determinants** of Feynman integrals;
quadratic relations between Bessel moments; **sum rules** when $4|N$.

4 Betti and de Rham matrices for all loops

Construction: Let v_k and w_k be the rational numbers **generated** by

$$\begin{aligned}
\frac{J_0^2(t)}{C(t)} &= \sum_{k\geq 0} \frac{v_k}{k!}\left(\frac{t}{2}\right)^{2k} = 1 - \frac{17t^2}{54} + \frac{3781t^4}{186624} + \ldots \\
\frac{2J_0(t)J_1(t)}{tC(t)} &= \sum_{k\geq 0} \frac{w_k}{k!}\left(\frac{t}{2}\right)^{2k} = 1 - \frac{41t^2}{216} + \frac{325t^4}{186624} + \ldots
\end{aligned}$$

where $J_0(t) = I_0(\mathrm{i}t)$, $J_1(t) = -J_0'(t)$ and

$$C(t) \equiv \frac{32(1 - J_0^2(t) - tJ_0(t)J_1(t))}{3t^4} = 1 - \frac{5t^2}{27} + \frac{35t^4}{2304} - \frac{7t^6}{9600} + \ldots$$

We construct rational bivariate polynomials by the **recursion**

$$\begin{aligned}
H_s(y,z) &= zH_{s-1}(y,z) - (s-1)yH_{s-2}(y,z) \\
&- \sum_{k=1}^{s-1} \binom{s-1}{k} \left(v_k H_{s-k}(y,z) - w_k z H_{s-k-1}(y,z)\right)
\end{aligned}$$

for $s > 0$, with $H_0(y,z) = 1$. We use these to define

$$d_s(N,c) \equiv \frac{H_s(3c/2, N+2-2c)}{4^s s!}.$$

Matrices: We construct rational **de Rham** matrices, with elements

$$D_N(a,b) \equiv \sum_{c=-b}^{a} d_{a-c}(N,-c)d_{b+c}(N,c)c^{N+1}$$

and a and b running from 1 to $k = \lceil N/2 - 1 \rceil$.
We act on those, on the left, with **period** matrices whose elements are

$$\begin{aligned}
P_{2k+1}(u,a) &\equiv \frac{(-1)^{a-1}}{\pi^u} M(k+1-u, k+u, 2a-1) \\
P_{2k+2}(u,a) &\equiv \frac{(-1)^{a-1}}{\pi^{u+1/2}} M(k+1-u, k+1+u, 2a-1)
\end{aligned}$$

and on the right with their `transposes`, to define **Betti** matrices

$$B_N \equiv P_N D_N P_N^{\mathtt{tr}}.$$

Conjecture: The Betti matrices have **rational** elements given by

$$\begin{aligned}
B_{2k+1}(u,v) &= (-1)^{u+k} 2^{-2k-2} (k+u)!(k+v)! Z(u+v) \\
B_{2k+2}(u,v) &= (-1)^{u+k} 2^{-2k-3} (k+u+1)!(k+v+1)! Z(u+v+1) \\
Z(m) &= \frac{1+(-1)^m}{(2\pi)^m} \zeta(m).
\end{aligned}$$

5 L-series up to 22 loops

Let $\Omega_{a,b}$ be the **determinant** of the $r \times r$ matrix with $M(a,b,1)$ at top left, size $r = \lceil (a+b)/4 - 1 \rceil$, powers of t^2 increasing to the right and powers of $I_0^2(t)$ increasing downwards. Thus $\Omega_{1,23}$ is a 5×5 determinant with the **22-loop sunrise** integral $M(1,23,1)$ at **top left** and $M(9,15,9)$ at bottom right. With $N = 4r+4$ Bessel functions, we discovered that

$$\begin{aligned}
L_8(4) &= \frac{2^2\,\Omega_{1,7}}{3^2\pi^2} \equiv \frac{4}{9\pi^2}\int_0^\infty I_0(t)K_0^7(t)t\mathrm{d}t \\
L_{12}(6) &= \frac{2^6\,\Omega_{1,11}}{3^4 \times 5\pi^6} \\
L_{16}(8) &= \frac{2^{14}\,\Omega_{1,15}}{3^7 \times 5^2 \times 7\pi^{12}} \\
L_{20}(10) &= \frac{2^{22} \times 11 \times \mathbf{131}\,\Omega_{1,19}}{3^{11} \times 5^6 \times 7^3\pi^{20}} \qquad \text{to 44 digits} \\
L_{24}(12) &= \frac{2^{29} \times \mathbf{12558877}\,\Omega_{1,23}}{3^{19} \times 5^9 \times 7^3 \times 11\pi^{30}} \qquad \text{to 19 digits,}
\end{aligned}$$

where boldface highlights **primes** greater than N. **30 GHz-years** of work gave 44-digit **precision** for $L_{20}(10)$. $L_{24}(12)$ agrees up to 19 digits.

With a **cut** of a line in the diagram at top left of the matrix, we found

$$\begin{aligned}
L_8(5) &= \frac{2^2\,\Omega_{2,6}}{3^3} \equiv \frac{4}{27}\int_0^\infty I_0^2(t)K_0^6(t)t\mathrm{d}t \\
L_{12}(7) &= \frac{2^5\times 11\,\Omega_{2,10}}{3^6\times 5^2\pi^2} \\
L_{16}(9) &= \frac{2^{14}\times 13\,\Omega_{2,14}}{3^9\times 5^3\times 7^2\pi^6} \\
L_{20}(11) &= \frac{2^{19}\times 17\times 19\times \mathbf{23}\,\Omega_{2,18}}{3^{13}\times 5^7\times 7^3\pi^{12}} \\
L_{24}(13) &= \frac{2^{27}\times 17\times 19^2\times 23^2\times \mathbf{46681}\,\Omega_{2,22}}{3^{23}\times 5^{12}\times 7^4\times 11^2\pi^{20}}\,.
\end{aligned}$$

At $N = 12, 16, 20$, with an **odd** sign in the functional equation, we found

$$\begin{aligned}
-L'_{12}(5) &= \frac{2^4\left(2^6\times \mathbf{29}\,\widehat{\Omega}_{2,10}+3\,\Omega_{2,10}\log 2\right)}{3^2\times 7\pi^6} \\
-L'_{16}(7) &= \frac{2^9\left(2^7\times \mathbf{83}\,\widehat{\Omega}_{2,14}+3\times 11\,\Omega_{2,14}\log 2\right)}{3^5\times 5\pi^{12}} \\
-L'_{20}(9) &= \frac{2^{17}\times 17\times 19\left(2^9\times 7\times \mathbf{101}\,\widehat{\Omega}_{2,18}+5\times 13\,\Omega_{2,18}\log 2\right)}{3^8\times 5^4\times 7^2\times 11\pi^{20}}
\end{aligned}$$

for **central derivatives**, using **enlarged** determinants $\widehat{\Omega}_{2,4r+2}$ of size $r+1$ with **regularization** of $M(2r+2, 2r+2, 2r+1)$ at bottom right.

In the cases with $N = 4r+2$, we obtained

$$\begin{aligned}
L_6(2) &= \frac{2\,\Omega_{1,5}}{\pi^2} \equiv \frac{2}{\pi^2}\int_0^\infty I_0(t)K_0^5(t)t\mathrm{d}t \\
L_6(3) &= \frac{2\,\Omega_{2,4}}{3} \equiv \frac{2}{3}\int_0^\infty I_0^2(t)K_0^4(t)t\mathrm{d}t \\
L_{10}(4) &= \frac{2^7\,\Omega_{1,9}}{3^2\pi^6} \\
L_{10}(5) &= \frac{2^4\,\Omega_{2,8}}{3\times 5\pi^2} \\
L_{14}(6) &= 0 \\
L_{14}(7) &= \frac{2^{10}\times 11\times 13\,\Omega_{2,12}}{3^6\times 5^2\times 7\pi^6} \\
L_{18}(8) &= \frac{2^{21}\times 17\times \mathbf{19}\,\Omega_{1,17}}{3^5\times 5^4\times 7\pi^{20}} \\
L_{18}(9) &= \frac{2^{12}\times 13\times 17\times \mathbf{41}\,\Omega_{2,16}}{3^8\times 5^3\times 7^2\pi^{12}} \\
L_{22}(10) &= 0 \\
L_{22}(11) &= \frac{2^{23}\times 17\times 19\times \mathbf{11621}\,\Omega_{2,20}}{3^{14}\times 5^7\times 7^3\pi^{20}}
\end{aligned}$$

with central vanishing from an odd sign at $N = 14$ and $N = 22$.

For cases with odd N, we obtained

$$
\begin{aligned}
L_5(2) &= \frac{2^2\,\Omega_{2,3}}{3} \equiv \frac{4}{3}\int_0^\infty I_0^2(t)K_0^3(t)t\mathrm{d}t \\
L_7(2) &= \frac{2^3\times 3\,\Omega_{2,5}}{5\pi^2} \equiv \frac{24}{5\pi^2}\int_0^\infty I_0^2(t)K_0^5(t)t\mathrm{d}t \\
L_9(4) &= \frac{2^6\,\Omega_{2,7}}{3\times 5\pi^2} \\
L_{11}(4) &= \frac{2^8\times 5\,\Omega_{2,9}}{3\times 7\pi^6} \\
L_{13}(6) &= \frac{2^7\times \mathbf{149}\,\Omega_{2,11}}{3^3\times 5\times 7\pi^6} \\
L_{15}(6) &= \frac{2^8\times 7\times \mathbf{53}\,\Omega_{2,13}}{3^2\times 5\pi^{12}} && \text{to 43 digits} \\
L_{17}(8) &= \frac{2^{15}\times \mathbf{29}\,\Omega_{2,15}}{3^5\times 5^2\times 7\pi^{12}} && \text{to 23 digits} \\
L_{19}(8) &= \frac{2^{14}\times \mathbf{1093}\times \mathbf{13171}\,\Omega_{2,17}}{3^4\times 5^4\times 7\times 11\pi^{20}} && \text{to 14 digits.}
\end{aligned}
$$

Comment: We also have results relating Bessel moments $M(a,b,c)$ with **even** c to L-series from Kloosterman moments with a quadratic **twist**.

6 Summary

1. Moments of 4 Bessel functions relate to walks on a **honeycomb**.
2. Moments of 5 Bessel functions relate to walks in a **diamond** crystal.
3. The L-series for 5, 6 and 8 Bessel functions are **modular**.
4. There are **quadratic** relations of the form $P_N D_N P_N^{\mathrm{tr}} = B_N$ with **period**, **de Rham** and **Betti** matrices that we have specified.
5. Relations between **determinants** of Feynman integrals and **L-series** have been discovered up to 22 loops and presumably go on for **ever**.

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