DISPERSION ESTIMATES FOR ONE-DIMENSIONAL SCHRÖDINGER AND KLEIN–GORDON EQUATIONS REVISITED

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Dedicated to the memory of Boris Moiseevich Levitan

Abstract. We show that for a one-dimensional Schrödinger operator with a potential whose first moment is integrable the scattering matrix is in the unital Wiener algebra of functions with integrable Fourier transforms. Then we use this to derive dispersion estimates for solutions of the associated Schrödinger and Klein–Gordon equations. In particular, we remove the additional decay conditions in the case where a resonance is present at the edge of the continuous spectrum.

1. Introduction

We are concerned with the one-dimensional Schrödinger equation

$$\mathrm{i}\dot{\psi}(x,t) = H\psi(x,t), \quad H := -\frac{d^2}{dx^2} + V(x), \quad (x,t)\in\mathbb{R}^2, \tag{1.1}$$

and the Klein–Gordon equation

$$\ddot{\psi}(x,t) = -(H+m^2)\psi(x,t), \quad (x,t)\in\mathbb{R}^2, \quad m>0, \tag{1.2}$$

with real integrable potential V. In vector form equation (1.2) reads

$$\mathrm{i}\dot{\Psi}(t) = \mathbf{H}\Psi(t), \tag{1.3}$$

where

$$\Psi(t) = \begin{pmatrix}\psi(t)\\ \dot{\psi}(t)\end{pmatrix}, \quad \mathbf{H} = \mathrm{i}\begin{pmatrix}0 & 1\\ -H-m^2 & 0\end{pmatrix}. \tag{1.4}$$

More specifically, our goal is to provide dispersive decay estimates for these equations. This is a well-studied area and one of our main contribution is a strikingly simple proof which at the same time improves previous results. This approach depends on the fact that the scattering matrix minus the unit matrix is in the Wiener algebra (i.e. its Fourier transform is integrable). Since this result is of independent interest we prove it first in Section 2. Based on this we will then establish our main

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results. To formulate them we introduce the weighted spaces $L^p_\sigma = L^p_\sigma(\mathbb{R})$, $\sigma \in \mathbb{R}$, associated with the norm

$$\|\psi\|_{L^p_\sigma} = \begin{cases} \left(\int_{\mathbb{R}} (1+|x|)^{p\sigma}|\psi(x)|^p dx\right)^{1/p}, & 1 \le p < \infty, \\ \sup_{x\in\mathbb{R}}(1+|x|)^\sigma|\psi(x)|, & p = \infty. \end{cases}$$

Of course, the case $\sigma = 0$ corresponds to the usual L^p spaces without weight. We recall (e.g., [14] or [23, Sect. 9.7]) that for $V \in L^1_1$ the operator H has a purely absolutely continuous spectrum on $[0, \infty)$ plus a finite number of eigenvalues in $(-\infty, 0)$. At the edge of the continuous spectrum there could be a resonance if there is a corresponding bounded solution of $-\psi'' + V\psi = 0$ (equivalently, if the Wronskian of the two Jost solutions vanishes at this point).

The following two results hold for the Schrödinger equation:

Theorem 1.1. *Let $V \in L^1_1(\mathbb{R})$. Then the following decay holds*

$$\|\mathrm{e}^{-\mathrm{i}tH}P_c\|_{L^1\to L^\infty} = \mathcal{O}(t^{-1/2}), \quad t \to \infty. \tag{1.5}$$

Here $P_c = P_c(H)$ is the orthogonal projection in $L^2(\mathbb{R})$ onto the continuous spectrum of H.

Theorem 1.2. *Let $V \in L^1_2(\mathbb{R})$. Then, in the non-resonant case, the following decay holds*

$$\|\mathrm{e}^{-\mathrm{i}tH}P_c\|_{L^1_1\to L^\infty_{-1}} = \mathcal{O}(t^{-3/2}), \quad t \to \infty. \tag{1.6}$$

Note that for the free Schrödinger equation (1.1) with $V = 0$ the estimate (1.5) is immediate from the explicit formula for the time evolution (see e.g. [23, Sect. 7.3]). The dispersive decay (1.5) for the perturbed Schrödinger equation has been established by Goldberg and Schlag [8], improving earlier results from Weder [25], in the non-resonant case for $V \in L^1_1$ and in the resonant case under the more restrictive condition $V \in L^1_2$ (see also [1]). We emphasize that our approach does not require this additional decay in the resonant case. Moreover, our proof for Theorem 1.1 is a simple application of Fubini's theorem. To show that the extra decay in the resonant case is not needed we generalize an old (but obviously not so well known) result from Guseinov [9]. We also remark that in the half-line case the analogous result for the scattering data is well known (cf. Problem 3.2.1 in [14]) and was used by Weder [27] to prove a corresponding result in the half-line case.

Recall that (1.5) has some immediate consequences: Interpolating between unitarity of $\exp(-\mathrm{i}tH): L^2 \to L^2$ and (1.5) the Riesz–Thorin theorem gives

$$\|\mathrm{e}^{-\mathrm{i}tH}P_c\|_{L^{p'}\to L^p} = \mathcal{O}(t^{-1/2+1/p}) \tag{1.7}$$

for any $p \in [2, \infty]$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Using (1.7) we can also deduce the corresponding Strichartz estimates of [10, Theorem 1.2].

The dispersive decay (1.6) has been established by Schlag [21] in the case $V \in L^1_4$ and later refined by Goldberg [7] to the case $V \in L^1_3$. For $V \in L^1_2$ the estimate (1.6) as first obtained by Mizutani in [17]. Here we propose a proof of (1.6) based on a somewhat different approach.

Note that the decay (1.6) immediately implies the following long-time asymptotics in weighted norms:

$$\|\mathrm{e}^{-\mathrm{i}tH}P_c\|_{L^2_\sigma\to L^2_{-\sigma}} = \mathcal{O}(t^{-3/2}), \quad t \to \infty, \tag{1.8}$$

for any $\sigma > 3/2$. Asymptotics of type (1.8) in the non-resonant case were obtained by Murata [18] for more general (multi-dimensional) Schrödinger-type operators.

In particular, in the one-dimensional case these asymptotics were established for $\sigma > 5/2$ and $|V(x)| \le C(1+|x|)^{-\rho}$ with some $\rho > 4$ (see also [21] for an up-to-date review in this direction, in particular for higher dimensions).

In the second part of the present work we obtain some new dispersion estimates for the Klein–Gordon equation. To this end we introduce the Bessel potential

$$\mathcal{J}_\alpha = \mathcal{F}^{-1}(1+|\cdot|^2)^{\alpha/2}\mathcal{F},$$

where $\mathcal{F}$ is the Fourier transform. Then the generalized Sobolev space $H^{\alpha,1}_\sigma(\mathbb{R})$ (cf. [3, Definition 6.2.2]) is the space of all tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ for which the norm

$$\|f\|_{H^{\alpha,1}_\sigma} = \|\mathcal{J}_\alpha f\|_{L^1_\sigma}, \qquad \alpha,\sigma \in \mathbb{R}, \tag{1.9}$$

is finite. As before $H^{\alpha,1} = H^{\alpha,1}_0$.

Theorem 1.3. *i) Let $V \in L^1_1(\mathbb{R})$. Then the following decay holds*

$$\|[\mathrm{e}^{-\mathrm{i}t\mathbf{H}}\mathbf{P}_c]^{12}\|_{H^{\frac12,1}\to L^\infty} = \mathcal{O}(t^{-1/2}), \quad t\to\infty. \tag{1.10}$$

ii) Let $V \in L^1_2(\mathbb{R})$. Then, in the non-resonant case, the following decay holds

$$\|[\mathrm{e}^{-\mathrm{i}t\mathbf{H}}\mathbf{P}_c]^{12}\|_{H^{\frac12,1}_1\to L^\infty_{-1}} = \mathcal{O}(t^{-3/2}), \quad t\to\infty. \tag{1.11}$$

Here $\mathbf{P}_c$ is the orthogonal projection in $L^2(\mathbb{R})\oplus L^2(\mathbb{R})$ onto the continuous spectrum of $\mathbf{H}^2$ and $[\cdot]^{ij}$ denotes the ij entry of the corresponding matrix operator.

We remark that the corresponding decay for the other entries of the matrix operator $\mathrm{e}^{-\mathrm{i}t\mathbf{H}}\mathbf{P}_c$ can be obtained similarly.

We note that Theorem 1.3 (i) is frequently stated in terms of the Besov space $B^{\frac12}_{1,1}(\mathbb{R})$ defined in [3, Definitions 6.2.2]. Namely, recalling $B^{\frac12}_{1,1} \subset H^{\frac12,1}$ (see [3, Theorem 6.2.4]) shows that (1.10) holds with $B^{\frac12}_{1,1}$ in place of $H^{\frac12,1}$. Similarly, (1.11) holds with $B^{\frac12}_{1,1,1}$ in place of $H^{\frac12,1}$, where $B^{\frac12}_{1,1,1}$ is the corresponding weighted Besov space (to define $B^{\frac12}_{1,1,1}$ one need replace L^1 by L^1_1 in the definition of $B^{\frac12}_{1,1}$). As before this follows from $B^{\frac12}_{1,1,1} \subset H^{\frac12,1}_1$ (see for instance [16, Proposition 3.12]).

Moreover, as a consequence of (1.10) we obtain

$$\|[\mathrm{e}^{-\mathrm{i}t\mathbf{H}}\mathbf{P}_c]^{12}\|_{B^{\frac12-\frac3p}_{p',p'}\to L^p} = \mathcal{O}(t^{-\frac12+\frac1p}), \quad t\to\infty, \quad \frac{1}{p'}+\frac{1}{p}=1, \tag{1.12}$$

for any $p\in[2,\infty]$ under the same assumption $V \in L^1_1(\mathbb{R})$ (see Corollary 5.3).

In three space dimensions $W^{k,p}\to L^q$ estimates for the perturbed Klein–Gordon equation were established by Soffer and Weinstein [22] (see also [29] for general space dimensions $n\ge 3$). In the one-dimensional case $W^{k,p}\to W^{k,q}$ estimates were obtained by Weder [26] for $V\in L^1_\gamma$, where $\gamma>3/2$ in the non-resonant case and $\gamma>5/2$ in the resonant case. The dispersive estimate of type (1.12) (with $B^{\frac12-\frac3p}_{p',p}(V)$ instead of $B^{\frac12-\frac3p}_{p',p'}$) is shown in [1], but again requiring $V\in L^1_2$ in the resonant case.

For the one-dimensional Klein–Gordon equation the decay $t^{-3/2}$ in the weighted energy norms $H^1_\sigma\oplus L^2_\sigma \to H^1_{-\sigma}\oplus L^2_{-\sigma}$ with $\sigma>5/2$ has been obtained by Komech and Kopylova [11] (see also the survey [12]).

Note that dispersion estimates of type (1.5)–(1.12) play an important role in proving asymptotic stability of solitons in the associated one-dimensional nonlinear equations [2], [13]. For the discrete Schrödinger and wave equations we refer to [6].

2. Continuity properties of the scattering matrix

We first introduce the Banach algebra $\mathcal{A}$ of Fourier transforms of integrable functions

$$\mathcal{A} = \left\{ f(k) : \; f(k) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kp} \hat{f}(p) dp, \; \hat{f}(\cdot) \in L^1(\mathbb{R}) \right\}$$

with the norm $\|f\|_{\mathcal{A}} = \|\hat{f}\|_{L^1}$, plus the corresponding unital Banach algebra $\mathcal{A}_1$

$$\mathcal{A}_1 = \left\{ f(k) : \; f(k) = c + \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kp} \hat{g}(p) dp, \; \hat{g}(\cdot) \in L^1(\mathbb{R}), \; c \in \mathbb{C} \right\}$$

with the norm $\|f\|_{\mathcal{A}_1} = |c| + \|\hat{g}\|_{L^1}$. Evidently, $\mathcal{A}$ is a subalgebra of $\mathcal{A}_1$. The algebra $\mathcal{A}_1$ can be treated as an algebra of Fourier transforms of functions $c\delta(\cdot) + \hat{g}(\cdot)$, where δ is the Dirac delta distribution and $\hat{g} \in L^1(\mathbb{R})$. Note that if $f \in \mathcal{A}_1 \setminus \mathcal{A}$ and $f(k) \neq 0$ for all $k \in \mathbb{R}$ then $f^{-1}(k) \in \mathcal{A}_1$ by the Wiener theorem [28].

Next we recall a few facts from scattering theory [5], [14] of the Schrödinger operator H, defined by formula (1.1). Under the assumption $V \in L^1_1$ there exist Jost solutions $f_\pm(x,k)$ of

$$H\psi = k^2 \psi, \quad k \in \overline{\mathbb{C}_+},$$

normalized according to

$$f_\pm(x,k) \sim \mathrm{e}^{\pm \mathrm{i}kx}, \quad x \to \pm\infty.$$

These solutions are given by

$$f_\pm(x,k) = \mathrm{e}^{\pm \mathrm{i}kx} h_\pm(x,k), \qquad h_\pm(x,k) = 1 \pm \int_0^{\pm\infty} B_\pm(x,y) \mathrm{e}^{\pm 2\mathrm{i}ky} dy, \tag{2.1}$$

where $B_\pm(x,y)$ are real-valued and satisfy (see [5, §2] or [14, §3.1])

$$|B_\pm(x,y)| \leq \mathrm{e}^{\gamma_\pm(x)} \eta_\pm(x+y), \tag{2.2}$$

$$|\frac{\partial}{\partial x} B_\pm(x,y) \pm V(x+y)| \leq 2\mathrm{e}^{\gamma_\pm(x)} \eta_\pm(x+y) \eta_\pm(x), \tag{2.3}$$

with

$$\gamma_\pm(x) = \int_x^{\pm\infty} (y-x)|V(y)| dy, \quad \eta_\pm(x) = \pm \int_x^{\pm\infty} |V(y)| dy. \tag{2.4}$$

Since $\eta_\pm(x + \cdot) \in L^1(\mathbb{R})$ we clearly have

$$h_\pm(x,\cdot) - 1, \; h'_\pm(x,\cdot) \in \mathcal{A}, \quad \forall x \in \mathbb{R}. \tag{2.5}$$

Let

$$W(\varphi(x,k), \psi(x,k)) = \varphi(x,k)\psi'(x,k) - \varphi'(x,k)\psi(x,k)$$

be the usual Wronskian, and set

$$W(k) = W(f_-(x,k), f_+(x,k)), \qquad W_\pm(k) = W(f_\mp(x,k), f_\pm(x,-k)).$$

The Jost solutions $f_\pm(x,k)$ and their derivatives do not belong to $\mathcal{A}_1$, as well as their Wronskian $W(k)$. However, the entries of the scattering matrix, that is, the transmission and reflection coefficients

$$T(k) = \frac{2\mathrm{i}k}{W(k)}, \quad R_\pm(k) = \mp \frac{W_\pm(k)}{W(k)},$$

turn out to be the elements of this algebra.

Theorem 2.1. *If* $V \in L^1_1$, *then* $T(k) - 1 \in \mathcal{A}$ *and* $R_\pm(k) \in \mathcal{A}$.

Proof. Since $|T(k)| \le 1$ for $k \in \mathbb{R}$ the Wronskian $W(k)$ can vanish only at the edge of continuous spectrum $k = 0$, which is known as the resonant case. Moreover, the zero is at most of the first order.
Step i) We first consider the non-resonant case $W(0) \ne 0$. Abbreviate $h_\pm(k) := h_\pm(0,k)$, $h'_\pm(k) := h'_\pm(0,k)$. Then (2.1) implies

$$W(k) = 2\mathrm{i}k h_+(k)h_-(k) + \tilde W(k), \quad \tilde W(k) := h_-(k)h'_+(k) - h'_-(k)h_+(k), \tag{2.6}$$

$$W_\pm(k) = h_\mp(k)h'_\pm(-k) - h_\pm(-k)h'_\mp(k). \tag{2.7}$$

Moreover, $\tilde W(k),\ W_\pm(k) \in \mathcal{A}$. Put

$$\nu(k) := \frac{1}{\mathrm{i}k - 1} = \int_0^\infty \mathrm{e}^{\mathrm{i}ky}\mathrm{e}^{-y}dy \tag{2.8}$$

and observe that $\nu(k) \in \mathcal{A}$, $k\nu(k) \in \mathcal{A}_1$ and, therefore, $\nu(k)W(k) \in \mathcal{A}_1$. Since $\nu(k)W(k) \to 2$ as $k \to \infty$ then $\nu(k)W(k) \in \mathcal{A}_1 \setminus \mathcal{A}$. Moreover, $\nu(k)W(k) \ne 0$ for all $k \in \mathbb{R}$, whence $(\nu(k)W(k))^{-1} \in \mathcal{A}_1$. Furthermore, $\nu(k)W_\pm(k) \in \mathcal{A}$ and we obtain

$$R_\pm(k) = \mp\frac{\nu(k)W_\pm(k)}{\nu(k)W(k)} \in \mathcal{A}, \quad T(k) = \frac{2\mathrm{i}k\nu(k)}{\nu(k)W(k)} \in \mathcal{A}_1.$$

Moreover, since $T(k) \to 1$ as $k \to \infty$ then $T(k) - 1 \in \mathcal{A}$.
Step ii) In the resonant case we need to work a bit harder. Introduce the functions

$$\Phi_\pm(k) := h_\pm(k)h'_\pm(0) - h'_\pm(k)h_\pm(0), \tag{2.9}$$

$$K_\pm(x) := \pm\int_x^{\pm\infty} B_\pm(0,y)dy, \quad D_\pm(x) := \pm\int_x^{\pm\infty} \frac{\partial}{\partial x}B_\pm(0,y)dy, \tag{2.10}$$

where $B_\pm(x,y)$ are the transformation operators from (2.1). Integrating (2.1) formally by parts we obtain

$$\begin{aligned} h'_\pm(k) &= \pm\int_0^{\pm\infty} \frac{\partial}{\partial x}B_\pm(0,y)\mathrm{e}^{\pm 2\mathrm{i}ky}dy = D_\pm(0) + 2\mathrm{i}k\int_0^{\pm\infty} D_\pm(y)\mathrm{e}^{\pm 2\mathrm{i}ky}dy \\ &= h'_\pm(0) + 2\mathrm{i}k\int_0^{\pm\infty} D_\pm(y)\mathrm{e}^{\pm 2\mathrm{i}ky}dy, \\ h_\pm(k) &= h_\pm(0) + 2\mathrm{i}k\int_0^{\pm\infty} K_\pm(y)\mathrm{e}^{\pm 2\mathrm{i}ky}dy. \end{aligned}$$

We emphasize that the above integrals have to be understood as improper integrals. Inserting them into (2.9) gives

$$\Phi_\pm(k) = 2\mathrm{i}k\Psi_\pm(k), \quad \Psi_\pm(k) := \int_0^{\pm\infty}(D_\pm(y)h_\pm(0) - K_\pm(y)h'_\pm(0))\mathrm{e}^{\pm 2\mathrm{i}ky}dy.$$

Lemma 2.2. *If* $V \in L^1_1$ *then* $\Psi_\pm(k) \in \mathcal{A}$.

Proof. Following [9] we will prove that the functions

$$H_\pm(y) := D_\pm(y)h_\pm(0) - K_\pm(y)h'_\pm(0)$$

satisfy $H_\pm \in L^1(\mathbb{R}_\pm)\cap L^\infty(\mathbb{R}_\pm)$. We simplify the original proof of [9] using the Gelfand–Levitan–Marchenko equation in the form proposed in [5]. Namely, as is known (§3.5 in [14]) the kernels $B_\pm(x,y)$ solve the equations

$$F_\pm(x+y)+B_\pm(x,y)\pm\int_0^{\pm\infty}B_\pm(x,t)F_\pm(x+y+z)dz=0, \tag{2.11}$$

where the functions $F_\pm(x)$ are absolutely continuous with $F_\pm'\in L^1(\mathbb{R}_\pm)$ and

$$|F_\pm(x)|\le C\eta_\pm(x),\quad \pm x\ge 0, \tag{2.12}$$

with $\eta_\pm$ from (2.4). Now differentiate (2.11) with respect to x and set $x=0$. Also set $x=0$ in (2.11) and then integrate both equations with respect to y from x to $\pm\infty$. Then (2.10) implies

$$\pm\int_x^{\pm\infty}F_\pm(y)dy+K_\pm(x)+\int_0^{\pm\infty}B_\pm(0,z)\int_x^{\pm\infty}F_\pm(y+z)dy\,dz=0$$

and

$$\begin{aligned}\mp F_\pm(x)+D_\pm(x)+\int_0^{\pm\infty}\frac{\partial}{\partial x}B_\pm(0,z)\int_x^{\pm\infty}F_\pm(y+z)dy\,dz\\ -\int_0^{\pm\infty}B_\pm(0,z)F_\pm(x+z)dz=0.\end{aligned}$$

To get rid of double integration here, we apply (2.10) and the equalities

$$\frac{\partial}{\partial z}\int_x^{\pm\infty}F_\pm(y+z)dy=-F_\pm(x+z).$$

The integration by parts yields

$$\begin{aligned}&\pm(1+K_\pm(0))\int_x^{\pm\infty}F_\pm(y)dy+K_\pm(x)\mp\int_0^{\pm\infty}K_\pm(z)F_\pm(x+z)dz\\ &=K_\pm(x)\pm h_\pm(0)\int_x^{\pm\infty}F_\pm(y)dy\mp\int_0^{\pm\infty}K_\pm(z)F_\pm(x+z)dz=0\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}&\mp F_\pm(x)+D_\pm(x)\pm h_\pm'(0)\int_x^{\pm\infty}F_\pm(y)dy\mp\int_0^{\pm\infty}D_\pm(z)F_\pm(x+z)dz\\ &-\int_0^{\pm\infty}B_\pm(0,z)F_\pm(x+z)dz=0.\end{aligned} \tag{2.14}$$

Multiplying (2.13) by $h_\pm'(0)$ and (2.14) by $h_\pm(0)$ and subtracting, we get integral equations

$$H_\pm(x)\mp\int_0^{\pm\infty}H_\pm(y)F_\pm(x+y)dy=G_\pm(x), \tag{2.15}$$

where

$$G_\pm(x)=h_\pm(0)\Big(\int_0^{\pm\infty}B_\pm(0,y)F_\pm(x+y)dy\pm F_\pm(x)\Big).$$

The bounds (2.2) and (2.12) imply

$$|G_\pm(x)|\le C\eta_\pm(x),\quad \pm x\ge 0. \tag{2.16}$$

Furthermore, for sufficiently large $N>0$ represent (2.15) in the form

$$H_\pm(x)\mp\int_{\pm N}^{\pm\infty}H_\pm(y)F_\pm(x+y)dy=G_\pm(x,N), \tag{2.17}$$

where

$$G_\pm(x,N)=G_\pm(x)\pm\int_0^{\pm N}H_\pm(y)F_\pm(x+y)dy.$$

Formulas (2.10) and estimates (2.2)–(2.4) give $H_\pm\in L^\infty(\mathbb{R}_\pm)\cap C(\mathbb{R}_\pm)$. Then $|G_\pm(x,N)|\le C(N)\eta_\pm(x)$ by (2.16) and monotonicity of $\eta_\pm(x)$. Applying the method of successive approximations (cf. [14, Chapter 3, Section 2]) to (2.17) we obtain $H_\pm\in L^1(\mathbb{R}_\pm)$. □

Now we can continue the proof of Theorem 2.1 in the resonant case. Since the Jost solutions are linear dependent at $k=0$, i.e. $h_+(x,0)=c\,h_-(x,0)$, we distinguish two cases: $h_+(0)h_-(0)\ne0$ and $h_+(0)=h_-(0)=0$. In the first case $h_+(0)h_-(0)\ne0$ we have

$$\begin{aligned}\tilde W(k)=\tilde W(k)-\tilde W(0)&=\frac{h_+(k)}{h_-(0)}\Phi_-(k)-\frac{h_-(k)}{h_+(0)}\Phi_+(k)\\&=2\mathrm{i}k\Big(\frac{h_+(k)}{h_-(0)}\Psi_-(k)-\frac{h_-(k)}{h_+(0)}\Psi_+(k)\Big)\end{aligned}$$

and similarly in the second case $h_+(0)=h_-(0)=0$ (and thus $h_+'(0)h_-'(0)\ne0$) we have $\Phi_\pm(k)=h_\pm(k)h_\pm'(0)=2\mathrm{i}k\Psi_\pm(k)$ and hence

$$\tilde W(k)=2\mathrm{i}k\Big(\frac{h_+'(k)}{h_-'(0)}\Psi_-(k)-\frac{h_-'(k)}{h_+'(0)}\Psi_+(k)\Big).$$

In summary,

$$\frac{W(k)}{2\mathrm{i}k}=h_-(k)h_+(k)+\begin{cases}\frac{h_+(k)}{h_-(0)}\Psi_-(k)-\frac{h_-(k)}{h_+(0)}\Psi_+(k), & h_+(0)h_-(0)\ne0,\\ \frac{h_+'(k)}{h_-'(0)}\Psi_-(k)-\frac{h_-'(k)}{h_+'(0)}\Psi_+(k), & h_+(0)h_-(0)=0,\end{cases}$$

where the right-hand side is in $\mathcal{A}_1$ by (2.5) and Lemma 2.2. Since $\frac{W(k)}{2\mathrm{i}k}=T(k)^{-1}\ne0$ we conclude $T(k)-1\in\mathcal{A}$. Analogously,

$$\frac{W_\pm(k)}{2\mathrm{i}k}=\begin{cases}\frac{h_\pm(-k)}{h_\mp(0)}\Psi_\mp(k)-\frac{h_\mp(k)}{h_\pm(0)}\Psi_\pm(-k), & h_+(0)h_-(0)\ne0,\\ \frac{h_\pm'(-k)}{h_\mp'(0)}\Psi_\mp(k)-\frac{h_\mp'(k)}{h_\pm'(0)}\Psi_\pm(-k), & h_+(0)h_-(0)=0.\end{cases}$$

where the right-hand side is again in $\mathcal{A}$ and hence $R_\pm(k)=\mp\frac{W_\pm(k)}{2\mathrm{i}k}T(k)\in\mathcal{A}$. □

Finally, we will investigate the function

$$\psi(x,y,k)=h_+(y,k)h_-(x,k)T(k)-1,\qquad y\ge x, \tag{2.18}$$

and $\psi(x,y,k)=\psi(y,x,k)$ for $y<x$. From Theorem 2.1 and formula (2.5) it follows that $\psi(x,y,\cdot)\in\mathcal{A}$.

Lemma 2.3. *The following estimate is valid*

$$\|\psi(x,y,\cdot)\|_{\mathcal{A}}\le C, \tag{2.19}$$

with some constant C, which does not depend on x and y.

Proof. Introduce

$$\sup_{\pm x\geq 0}\left(\pm\int_0^{\pm\infty}|B_\pm(x,y)|dy\right)=C_\pm,$$

which is finite by (2.2). Then

$$\|h_\pm(x,\cdot)\|_{\mathcal{A}_1}\leq 1+C_\pm,\quad \|h_\pm(x,\cdot)-1\|_{\mathcal{A}}\leq C_\pm,\quad \text{for}\quad \pm x\geq 0. \tag{2.20}$$

Now consider the three possibilities (a) $x\leq y\leq 0$, (b) $0\leq x\leq y$, and (c) $x\leq 0\leq y$. In the case (c) the estimate $\|\psi(x,y,\cdot)\|_{\mathcal{A}}\leq C$ follows immediately from (2.20) and Theorem 2.1. In the other two cases we use the scattering relations

$$T(k)f_\pm(x,k)=R_\mp(k)f_\mp(x,k)+f_\mp(x,-k) \tag{2.21}$$

to get the representation

$$\psi(k,x,y)=\begin{cases} h_-(x,k)\left(R_-(k)h_-(y,k)\mathrm{e}^{-2\mathrm{i}yk}+h_-(y,-k)\right)-1 & x\leq y\leq 0,\\ h_+(y,k)\left(R_+(k)h_+(x,k)\mathrm{e}^{2\mathrm{i}xk}+h_+(x,-k)\right)-1 & 0\leq x\leq y.\end{cases} \tag{2.22}$$

Observing that for any function $g(k)\in\mathcal{A}$ and any real s we have $g(k)\mathrm{e}^{\mathrm{i}ks}\in\mathcal{A}$ with the norm independent of s, establishes (2.19). □

3. The Schrödinger equation

Now we are ready to prove the dispersive decay estimate (1.5) for the Schrödinger equation (1.1). For the one-parameter group of (1.1) the spectral theorem implies

$$\mathrm{e}^{-\mathrm{i}tH}P_c=\frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{e}^{-\mathrm{i}t\omega}(\mathcal{R}(\omega+\mathrm{i}0)-\mathcal{R}(\omega-\mathrm{i}0))\,d\omega, \tag{3.1}$$

where $\mathcal{R}(\omega)=(H-\omega)^{-1}$ is the resolvent of the Schrödinger operator H and the limit is understood in the strong sense [23]. Given the Jost solutions we can express the kernel of the resolvent $R(\omega)$ for $\omega=k^2\pm\mathrm{i}0$, $k>0$, as [5, 23]

$$[\mathcal{R}(k^2\pm\mathrm{i}0)](x,y)=-\frac{f_+(y,\pm k)f_-(x,\pm k)}{W(\pm k)}=\mp\frac{f_+(y,\pm k)f_-(x,\pm k)T(\pm k)}{2\mathrm{i}k}$$

for all $x\leq y$ (and the positions of x,y reversed if $x>y$).

Therefore, in the case $x\leq y$, the integral kernel of $\mathrm{e}^{-\mathrm{i}tH}P_{k_0}(H)$ is given by

$$\begin{aligned}[\mathrm{e}^{-\mathrm{i}tH}P_{k_0}](x,y)&=\frac{\mathrm{i}}{\pi}\int_{-k_0}^{k_0}\mathrm{e}^{-\mathrm{i}tk^2}\,\frac{f_+(y,k)f_-(x,k)T(k)}{2\mathrm{i}k}k\,dk\\ &=\frac{1}{2\pi}\int_{-k_0}^{k_0}\mathrm{e}^{-\mathrm{i}(tk^2-|y-x|k)}h_+(y,k)h_-(x,k)T(k)dk,\end{aligned}$$

where $P_{k_0}=P_H([0,k_0^2])$ is the projection onto energies in the interval $[0,k_0^2]$. Taking the limit $k_0\to\infty$ we obtain

$$\begin{aligned}[\mathrm{e}^{-\mathrm{i}tH}P_c](x,y)&=\lim_{k_0\to\infty}[\mathrm{e}^{-\mathrm{i}tH}P_{k_0}](x,y)\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}(tk^2-|y-x|k)}h_+(y,k)h_-(x,k)T(k)dk,\end{aligned} \tag{3.2}$$

where the integral is to be understood as an improper integral for $t\in\mathbb{R}$ (if $\mathrm{Im}(t)<0$ the integral converges absolutely and the limit is of course not needed). In fact, the convergence of the integral for $t\in\mathbb{R}$ will follow from Lemma 3.1 below, and

Lemma 2.3 will imply $|[\mathrm{e}^{-\mathrm{i}tH}P_{k_0}](x,y)| \le C|t|^{-1/2}$. Thus, we can use dominated convergence to conclude that the right-hand side of (3.2) is indeed the kernel of $\mathrm{e}^{-\mathrm{i}tH}P_c$ on $L^1(\mathbb{R})\cap L^2(\mathbb{R})$.

Lemma 3.1. *Let $\psi(x,y,k)$ be defined by (2.18) and let $\hat\psi(x,y,p)$ be its Fourier transform with respect to k. Then the following representation is valid for $\mathrm{Im}(t)\le 0$:*

$$[\mathrm{e}^{-\mathrm{i}tH}P_c](x,y) = \frac{1}{\sqrt{4\pi \mathrm{i}t}}\left(\mathrm{e}^{-\frac{|x-y|^2}{4\mathrm{i}t}} + \int_{\mathbb{R}} \mathrm{e}^{-\frac{(p+|x-y|)^2}{4\mathrm{i}t}}\hat\psi(x,y,p)dp\right). \tag{3.3}$$

Proof. By (3.2) we have

$$[\mathrm{e}^{-\mathrm{i}tH}P_c](x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(tk^2-|y-x|k)}(1+\psi(x,y,k))dk.$$

Since the first part of the integral is easy to compute we only focus on the second part containing ψ. Using Fubini's theorem this integral is given by

$$\begin{aligned}
&\frac{1}{2\pi}\lim_{k_0\to\infty}\int_{-k_0}^{k_0}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}(tk^2-|y-x|k-kp)}\hat\psi(x,y,p)dp\,dk =\\
&\frac{1}{2\pi}\lim_{k_0\to\infty}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\frac{(p+|y-x|)^2}{4t}}\int_{-k_0}^{k_0}\mathrm{e}^{-\mathrm{i}\frac{(2kt-|y-x|-p)^2}{4t}}dk\,\hat\psi(x,y,p)dp =\\
&\quad = \frac{1}{2\sqrt{4\pi \mathrm{i}t}}\lim_{k_0\to\infty}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\frac{(p+|y-x|)^2}{4t}}\big(\mathrm{erf}(q_+)+\mathrm{erf}(q_-)\big)\hat\psi(x,y,p)dp,
\end{aligned}$$

where $q_\pm = \frac{k_0}{2}\sqrt{4\mathrm{i}t} \pm \mathrm{i}\frac{(p+|x-y|)}{\sqrt{4\mathrm{i}t}}$ and $\mathrm{erf}(z)$ is the error function [19, §7.2]. Using $\mathrm{erf}(z) = 1 + O(\mathrm{e}^{-z^2})$ as $z\to\infty$ with $|\arg(z)| < \frac{3\pi}{4}$ ([19, (7.12.1)]) the claim follows from dominated convergence. □

Proof of Theorem 1.1. Since

$$\|\mathrm{e}^{-\mathrm{i}tH}P_c\|_{L^1\to L^\infty} = \sup_{\|f\|_{L^1}=1,\|g\|_{L^1}=1}\langle f, \mathrm{e}^{-\mathrm{i}tH}P_c g\rangle = \sup_{x,y}|[\mathrm{e}^{-\mathrm{i}tH}P_c](x,y)|$$

the claim follows from Lemmas 3.1 and 2.3. □

In fact we have established the slightly stronger result which covers also the heat semigroup:

Corollary 3.2. *Let $V\in L^1_1(\mathbb{R})$. Then*

$$\|\mathrm{e}^{-\mathrm{i}tH}P_{k_0}\|_{L^1\to L^\infty} \le C|t|^{-1/2}, \quad \mathrm{Im}(t)\le 0,$$

for every $0\le k_0\le\infty$.

Proof. Using the representation for $[\mathrm{e}^{-\mathrm{i}tH}P_{k_0}](x,y)$ from the proof of Lemma 3.1, together with boundedness of $\mathrm{erf}(q_\pm)$ in the region under consideration and (2.19), shows $|[\mathrm{e}^{-\mathrm{i}tH}P_{k_0}](x,y)| \le C|t|^{-1/2}$ as desired. □

4. The Schrödinger equation (non-resonant case)

In this section we consider the non-resonant case and prove the dispersive decay estimate (1.6). We begin by representing the jump of the resolvent across the spectrum as

$$[\mathcal{R}(k^2+\mathrm{i}0)-\mathcal{R}(k^2-\mathrm{i}0)](x,y) = \frac{T(k)f_+(y,k)f_-(x,k)+\overline{T(k)f_+(y,k)f_-(x,k)}}{-2\mathrm{i}k},$$

for $x \le y$ and $k > 0$. The scattering relations (2.21) imply

$$\begin{aligned} f_-(x,k) &= T(-k)f_+(x,-k) - R_-(-k)f_-(x,-k),\\ \overline{f_+(y,k)} &= T(k)f_-(y,k) - R_+(k)f_+(y,k) \end{aligned}$$

and using the consistency relation $T\overline{R}_- + \overline{T}R_+ = 0$ we arrive at the formula (cf. [21, p.13])

$$[\mathcal{R}(k^2+\mathrm{i}0) - \mathcal{R}(k^2-\mathrm{i}0)](x,y) = \frac{|T(k)|^2}{-2\mathrm{i}k}[f_+(y,k)f_+(x,-k) + f_-(y,k)f_-(x,-k)]. \tag{4.1}$$

Inserting this into (3.1) gives

$$\begin{aligned} &[\mathrm{e}^{-\mathrm{i}tH}P_c](x,y) = [\mathcal{K}_+(t)](x,y) + [\mathcal{K}_-(t)](x,y),\\ &[\mathcal{K}_\pm(t)](x,y) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(tk^2 \mp |y-x|k)}|T(k)|^2 h_\pm(y,k)h_\pm(x,-k)dk. \end{aligned}$$

Using integration by parts we get

$$\begin{aligned} [\mathcal{K}_\pm(t)](x,y) = &\pm \frac{|y-x|}{8\pi t}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(tk^2 \mp |y-x|k)}\frac{|T(k)|^2}{k}h_\pm(y,k)h_\pm(x,-k)dk\\ &- \frac{1}{8\pi \mathrm{i}t}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(tk^2 \mp |y-x|k)}\frac{|T(k)|^2}{k^2}h_\pm(y,k)h_\pm(x,-k)dk\\ &+ \frac{1}{8\pi \mathrm{i}t}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(tk^2 \mp |y-x|k)}\frac{\frac{\partial}{\partial k}\Big[|T(k)|^2 h_\pm(y,k)h_\pm(x,-k)\Big]}{k}dk \end{aligned}$$

Applying the arguments from the proof of Lemma 3.1 we obtain

$$[\mathcal{K}_\pm(t)](x,y) = \frac{t^{-3/2}}{8\sqrt{\pi \mathrm{i}}}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\frac{(p+|x-y|)^2}{4t}}\sum_{j=1}^{3}\hat{\psi}_j^\pm(x,y,p)dp \tag{4.2}$$

where $\hat{\psi}_j^\pm(x,y,p)$, $j=1,2,3$, are the Fourier transforms of the functions

$$\begin{aligned} \psi_1^\pm(x,y,k) &= \pm|y-x|\frac{|T(k)|^2}{k}h_\pm(y,k)h_\pm(x,-k),\\ \psi_2^\pm(x,y,k) &= \mathrm{i}\frac{|T(k)|^2}{k^2}h_\pm(y,k)h_\pm(x,-k),\\ \psi_3^\pm(x,y,k) &= -\mathrm{i}\frac{\frac{\partial}{\partial k}\Big[|T(k)|^2 h_\pm(y,k)h_\pm(x,-k)\Big]}{k}, \end{aligned}$$

respectively. To estimate their $\mathcal{A}$ norms we first show

Lemma 4.1. *Let $V \in L^1_2$ and $W(0) \neq 0$. Then $T(k)h_\pm(x,k)/k \in \mathcal{A}$, and*

$$\Big\|\frac{T(k)h_\pm(x,k)}{k}\Big\|_{\mathcal{A}} \le C(1+|x|), \quad x \in \mathbb{R}. \tag{4.3}$$

Proof. Since $\frac{T(k)}{k} = \frac{2\mathrm{i}\nu(k)}{\nu(k)W(k)} \in \mathcal{A}$ (recall (2.8)), then for $x \in \mathbb{R}_\pm$ the bound (4.3) follows from (2.20). Consider the case $x \in \mathbb{R}_\mp$. The scattering relations (2.21) imply

$$\begin{aligned} T(k)h_\pm(x,k) =&(R_\mp(k)+1)h_\mp(x,k)\mathrm{e}^{\mp 2\mathrm{i}kx} - (h_\mp(x,k) - h_\mp(x,-k))\mathrm{e}^{\mp 2\mathrm{i}kx}\\ &+ h_\mp(x,-k)(1-\mathrm{e}^{\mp 2\mathrm{i}kx}). \end{aligned} \tag{4.4}$$

Using (2.1) we obtain

$$\frac{h_\mp(x,k)-h_\mp(x,-k)}{k} = \mp\int\limits_0^{\mp\infty} B_\mp(x,r)\frac{\mathrm{e}^{\mp\mathrm{i}kr}-\mathrm{e}^{\pm\mathrm{i}kr}}{k}dr$$

$$= \mathrm{i}\int\limits_0^{\mp\infty} B_\mp(x,r)\int\limits_{-r}^{r}\mathrm{e}^{\mathrm{i}ky}dy\,dr = \mathrm{i}\int\limits_{-\infty}^{\infty}\Big(\int\limits_{\mp|y|}^{\mp\infty} B_\mp(x,r)dr\Big)\mathrm{e}^{\mathrm{i}ky}dy.$$

Next, observe that formula (2.2) implies that if $V\in L^1_2$, then $B_\mp(x,.)\in L^1_1(\mathbb{R}_\mp)$ for any fixed x, and consequently

$$S_\mp(x,y)=\int_y^{\mp\infty}|B_\mp(x,r)|dr\in L^1(\mathbb{R}_\mp).$$

Based on this observation we get

$$\Big\|\frac{h_\mp(x,k)-h_\mp(x,-k)}{k}\Big\|_{\mathcal{A}}\le C,\quad x\in\mathbb{R}_\mp. \tag{4.5}$$

By the same reasons formula (2.3) implies

$$\frac{h'_\mp(0,k)-h'_\mp(0,-k)}{k}\in\mathcal{A}. \tag{4.6}$$

Next, from (2.6) and (2.7) it follows

$$\begin{aligned}\frac{W(k)\mp W_\pm(k)}{k} =& 2\mathrm{i}h_+(k)h_-(k)+\\ &+\frac{h_-(k)h'_+(k)-h'_-(k)h_+(k)\mp h_\mp(k)h'_\pm(-k)\pm h_\pm(-k)h'_\mp(k)}{k}=\\ =&2\mathrm{i}h_+(k)h_-(k)\pm h_\mp(k)\frac{h'_\pm(k)-h'_\pm(-k)}{k}\mp h'_\mp(k)\frac{h_\pm(k)-h_\pm(-k)}{k}.\end{aligned}$$

Applying (2.5), (4.5), (4.6) we get

$$\left(\frac{W(k)\mp W_\pm(k)}{k}-2\mathrm{i}\right)\in\mathcal{A}.$$

As is shown in Theorem 2.1 in the non-resonant case $W^{-1}(k)\in\mathcal{A}$. Thus

$$\frac{R_\pm(k)+1}{k}=\frac{1}{W(k)}\frac{W(k)\mp W_\mp(k)}{k}\in\mathcal{A}. \tag{4.7}$$

Next, $\frac{1-\mathrm{e}^{\mp2\mathrm{i}kx}}{\mathrm{i}k}$ is the Fourier transform of the indicator function of $[0,2x]$, therefore

$$\Big\|\frac{1-\mathrm{e}^{\mp2\mathrm{i}kx}}{k}\Big\|_{\mathcal{A}}\le 2|x|. \tag{4.8}$$

Finally, substituting (4.5), (4.7), and (4.8) into (4.4) we obtain (4.3). □

Since we have already seen the estimate $\|T(k)h_\pm(x,k)\|_{\mathcal{A}}\le C$ in the proof of Theorem 1.1, this lemma immediately implies that

$$\|\hat\psi_j^\pm(x,y,\cdot)\|_{L^1}\le C(1+|x|)(1+|y|),\quad j=1,2. \tag{4.9}$$

To estimate $\|\hat\psi_3^\pm(x,y,\cdot)\|_{L^1}$ we need one more property.

Lemma 4.2. *Let $V \in L^1_2$ and $W(0) \neq 0$. Then $\frac{\partial}{\partial k}(T(k)h_\pm(x,k)) \in \mathcal{A}$ with*

$$\Big\| \frac{\partial}{\partial k}(T(k)h_\pm(x,k))\Big\|_{\mathcal{A}} \le C(1+|x|), \quad x \in \mathbb{R}.$$

Proof. The representation (2.1) and the bounds (2.2)–(2.3) imply

$$\frac{\partial}{\partial k} h_\pm(x,k) = \dot h_\pm(x,k) \in \mathcal{A}, \quad \frac{\partial}{\partial k} h'_\pm(x,k) \in \mathcal{A} \quad \text{if } V \in L^1_2 \tag{4.10}$$

with

$$\|\frac{\partial}{\partial k} h'_\pm(x,\cdot)\|_{\mathcal{A}} + \|\dot h_\pm(x,\cdot)\|_{\mathcal{A}} \le C, \quad x \in \mathbb{R}_\pm. \tag{4.11}$$

Therefore $\frac{d}{dk}W_\pm(k) := \dot W_\pm(k) \in \mathcal{A}$. Further, from (2.6) and (4.10) it follows that $\nu(k)\dot W(k) \in \mathcal{A}$, where $\nu(k)$ is defined by (2.8). Since in the nonresonant case $(\nu(k)W(k))^{-1} \in \mathcal{A}_1$ and $W^{-1}(k) \in \mathcal{A}$ then

$$\dot T(k) = \frac{1}{W(k)}\Big(2\mathrm{i} - \dot W(k) T(k)\Big) \in \mathcal{A}, \quad \dot R_\pm(k) \in \mathcal{A}. \tag{4.12}$$

Thus for $x \in \mathbb{R}_\pm$ the statement of the Lemma is evident in view of (2.20), (4.12) and (4.11). To get it for $x \in \mathbb{R}_\mp$ we use (4.12), (4.11), and again the scattering relations (2.21) which gives

$$\frac{\partial}{\partial k}(T(k)h_\pm(x,k)) = \mathrm{e}^{\mp 2ikx}\left(\frac{\partial}{\partial k}(R_\mp(k)h_\mp(x,k)) \mp 2\mathrm{i}xR_\mp(k)h_\mp(x,k)\right) + \dot h_\mp(x,-k).$$

□

As pointed out in the proof of Theorem 1.1 the estimate $\|T(k)h_\pm(x,k)\|_{\mathcal{A}_1} \le C$ is valid for $x \in \mathbb{R}$. This and Lemma 4.2 imply

$$\|\hat\psi_3^\pm(x,y,\cdot)\|_{L^1} \le C(1+|x|)(1+|y|). \tag{4.13}$$

Finally, combining (4.2), (4.9), (4.13) and Lemma 3.1 we obtain

$$|[\mathcal{K}_\pm(t)](x,y)| \le Ct^{-3/2}(1+|x|)(1+|y|), \quad t \ge 1,$$

which shows (1.6) and finishes the proof of Theorem 1.2.

5. The Klein–Gordon equation

In this section we prove the estimate (1.10) for the Klein–Gordon equation (1.3). We estimate the low-energy and high-energy components of the solution separately. Equation (1.10) will immediately follow from the two theorems below.

Theorem 5.1. *Assume $V \in L^1_1(\mathbb{R})$. Then for any smooth function ζ with bounded support the following decay holds*

$$\big\|\mathrm{e}^{-\mathrm{i}t\mathbf{H}}\mathbf{P}_c\,\zeta(\mathbf{H}^2)\big\|_{L^1\to L^\infty} = \mathcal{O}(t^{-1/2}), \quad t \to \infty.$$

Theorem 5.2. *Assume $V \in L^1_1(\mathbb{R})$ and let $\xi(x)$ be a smooth function such that $\xi(x) = 0$ for $x \le m^2+1$ and $\xi(x) = 1$ for $x \ge m^2+2$. Then*

$$\big\|[\mathrm{e}^{-\mathrm{i}t\mathbf{H}}]^{12}\,\xi(\mathbf{H}^2)\big\|_{H^{\frac12,1}\to L^\infty} = \mathcal{O}(t^{-1/2}), \quad t \to \infty.$$

As a consequence of (1.10) we get

Corollary 5.3. *Assume $V \in L^1_1(\mathbb{R})$. Then (1.12) holds for any $p \in [2,\infty]$. Namely*

$$\|[\mathrm{e}^{-\mathrm{i}t\mathbf{H}}\mathbf{P}_c]^{12}\|_{B^{\frac12-\frac3p}_{p',p'}\to L^p} = \mathcal{O}(t^{-\frac12+\frac1p}), \quad t\to\infty, \quad \frac{1}{p'}+\frac{1}{p} = 1. \tag{5.1}$$

Proof. Recall that the Klein–Gordon equation preserves the energy

$$\|\dot{\psi}\|_{L^2}^2 + \langle\psi, H\psi\rangle_{L^2} + m^2\|\psi\|_{L^2}^2.$$

Since $[\mathrm{e}^{-\mathrm{i}t\mathbf{H}}\mathbf{P}_c]^{12}\pi_0$ corresponds to the initial condition $(\psi(0),\dot{\psi}(0)) = (0,\pi_0)$ with $\pi_0 = P_c(H)\pi_0$ we obtain the estimate $\langle\psi, H\psi\rangle_{L^2} + m^2\|\psi\|_{L^2}^2 \le \|\pi_0\|_{L^2}^2$ in this case. Moreover, since for $V \in L^1$ the multiplication operator V is relatively form bounded with bound 0 with respect to $H_0 = -\frac{d^2}{dx^2}$ ([23, Lemma 9.33]), the graph norms of H and H_0 are equivalent, and we obtain $\|\psi\|_{H^1} \le C\|\pi_0\|_{L^2}$. Hence by duality we also get

$$\|[\mathrm{e}^{-\mathrm{i}t\mathbf{H}}\mathbf{P}_c]^{12}\|_{H^{-1}\to L^2} = \mathcal{O}(1), \quad t\to\infty, \quad H^{-1} = H^{-1,2} \tag{5.2}$$

Since $H^{-1} = B^{-1}_{2,2}$ due to [24, Theorem 2.3.2 (d)], real interpolation between (1.10) and (5.2) gives (5.1) . □

5.1. **Low-energy decay.** Here we prove Theorem 5.1. We will need a small variant of the van der Corput lemma which is of independent interest.

Lemma 5.4. *Consider the oscillatory integral*

$$I(t) = \int_a^b \mathrm{e}^{\mathrm{i}t\phi(k)} f(k)dk,$$

where $\phi(k)$ is real-valued function. If $\phi''(k) \ne 0$ in [a,b] and $f \in \mathcal{A}_1$, then

$$|I(t)| \le C_2[t \min_{a\le k\le b}|\phi''(k)|]^{-1/2}\|f\|_{\mathcal{A}_1}, \quad t \ge 1.$$

where $C_2 \le 2^{8/3}$ is the optimal constant from the van der Corput lemma.

Proof. Writing $f(k) = c + \int_{\mathbb{R}} e^{\mathrm{i}ky}\hat{g}(y)dy$ we have

$$I(t) = \int_{\mathbb{R}} \hat{g}(y) I_{y/t}(t)dy + cI_0(t), \quad I_v(t) = \int_a^b \mathrm{e}^{\mathrm{i}t(\phi(k)+vk)}dk.$$

By the van der Corput lemma

$$|I_v(t)| \le C_2[t \min_{a\le k\le b}|\phi''(k)|]^{-1/2}, \quad t\ge 1,$$

where $C_2 \le 2^{8/3}$ (cf. [20]) and the claim follows from the definition of the norm in $\mathcal{A}_1$. □

Note that the analogous lemma extends to higher derivatives and to unbounded intervals (where the integral has to be understood as an improper Riemann integral).

The resolvent $\mathbf{R}(\omega)$ of the operator (1.4) associated with the Klein-Gordon equation (1.2) can be expressed in terms of the resolvent of the Schrödinger operator $\mathcal{R}(\omega) = (H-\omega)^{-1}$ as

$$\mathbf{R}(\omega) = \begin{pmatrix} 0 & 0 \\ -\mathrm{i} & 0 \end{pmatrix} + \begin{pmatrix} \omega & \mathrm{i} \\ -\mathrm{i}\omega^2 & \omega \end{pmatrix} \mathcal{R}(\omega^2 - m^2).$$

For $\mathrm{e}^{-\mathrm{i}t\mathbf{H}}\mathbf{P}_c\,\zeta(\mathbf{H}^2)$ the spectral representation of type (3.1) holds:

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}t\mathbf{H}}\mathbf{P}_c\,\zeta(\mathbf{H}^2) &= \frac{1}{2\pi\mathrm{i}}\int\limits_\Gamma \mathrm{e}^{-\mathrm{i}t\omega}\zeta(\omega^2)(\mathbf{R}(\omega+\mathrm{i}0)-\mathbf{R}(\omega-\mathrm{i}0))\,d\omega\\ &= \frac{1}{2\pi\mathrm{i}}\int_\Gamma \mathrm{e}^{-\mathrm{i}\omega t}\zeta(\omega^2)\begin{pmatrix}\omega & \mathrm{i}\\ -\mathrm{i}\omega^2 & \omega\end{pmatrix}\big(\mathcal{R}((\omega+\mathrm{i}0)^2-m^2)-\mathcal{R}((\omega-\mathrm{i}0)^2-m^2)\big)\,d\omega,\end{aligned} \tag{5.3}$$

where $\Gamma=(-\infty,-m)\cup(m,\infty)$. Denote

$$\mathcal{M}_t(k)=\begin{pmatrix}\cos(t\sqrt{k^2+m^2}) & \dfrac{\sin(t\sqrt{k^2+m^2})}{\sqrt{k^2+m^2}}\\ -\sqrt{k^2+m^2}\sin(t\sqrt{k^2+m^2}) & \cos(t\sqrt{k^2+m^2})\end{pmatrix},$$

then (5.3) can be rewritten as

$$[\mathrm{e}^{-\mathrm{i}t\mathbf{H}}\mathbf{P}_c\,\zeta(\mathbf{H}^2)](x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{M}_t(k)\,\mathrm{e}^{\mathrm{i}|y-x|k}\zeta(k^2+m^2)\,(\psi(x,y,k)+1)dk, \tag{5.4}$$

where the function $\psi(x,y,k)$ is defined by (2.18). We obtain oscillatory integrals with the phase functions $\phi_\pm(k)=\pm\sqrt{k^2+m^2}-vk$, where $v=\frac{|y-x|}{t}$. The second derivative of $\phi_\pm(k)$ satisfies

$$|\phi_\pm''(k)|=\frac{m^2}{\sqrt{(k^2+m^2)^3}}\geq C(m,\zeta),\quad (k^2+m^2)\in\operatorname{supp}\zeta.$$

Since $(k^2+m^2)^{j/2}\zeta(k^2+m^2)\in\mathcal{A}$ for $j=-1,0,1$ and $\|\psi(x,y,k)\|_{\mathcal{A}}\leq C$ due to (2.19), then Lemma 5.4 implies

$$\max_{x,y\in\mathbb{R}}|[\mathrm{e}^{-\mathrm{i}t\mathbf{H}}\mathbf{P}_c\,\zeta(\mathbf{H}^2)](x,y)|\leq Ct^{-1/2},\quad t\geq 1.$$

5.2. **High-energy decay.** Here we prove Theorem 5.2. Our proof is based on the following version of [15, Lemma 2]:

Lemma 5.5. *Let $\eta(k)$, $k\geq 1$, be a smooth function such that $|\eta^{(j)}(k)|\leq k^{-j}$ for $j=0,1$. Then for any $g(k)\in\mathcal{A}_1$, $\alpha>3/2$ and $t\geq 1$*

$$\sup_{p\in\mathbb{R}}\Big|\int_1^\infty \eta(k)\frac{\mathrm{e}^{\pm\mathrm{i}t\sqrt{k^2+m^2}+\mathrm{i}kp}}{k^\alpha}g(k)dk\Big|\leq C\|g\|_{\mathcal{A}_1}t^{-1/2}. \tag{5.5}$$

Moreover,

$$\sup_{p\in\mathbb{R}}\Big|\int_1^\infty \eta(k)\frac{\mathrm{e}^{\pm\mathrm{i}t\sqrt{k^2+m^2}+\mathrm{i}kp}}{k^{3/2}}dk\Big|\leq Ct^{-1/2}. \tag{5.6}$$

Here the constants C depend on the parameters m and α only.

Proof. Consider "+" case and set $v=-p/t$. To prove (5.5) we have to estimate the oscillatory integral

$$I_\alpha(t)=\int_1^\infty k^{-\alpha}\eta(k)\mathrm{e}^{\mathrm{i}t\phi(k)}g(k)dk$$

with the phase function $\phi(k)=\sqrt{k^2+m^2}-vk$. Split the integral according to

$$I_\alpha(t)=I_\alpha^1(t)+I_\alpha^2(t)=\int_1^t+\int_t^\infty.$$

Since $\|g\|_\infty \le \|g\|_{\mathcal{A}_1}$ we obtain

$$|I_\alpha^2(t)| \le \|g\|_{\mathcal{A}_1} \int_t^\infty k^{-\alpha} dk \le C\|g\|_{\mathcal{A}_1} t^{1-\alpha}. \tag{5.7}$$

To estimate $I_\alpha^1(t)$ we abbreviate

$$\Psi(k,t) = \int_1^k \mathrm{e}^{\mathrm{i}t\phi(\tau)} g(\tau) d\tau.$$

Since

$$\min_{1\le\tau\le k} \phi''(\tau) = \phi''(k) = \frac{m^2}{(\sqrt{k^2+m^2})^3} \ge \frac{C}{k^3}$$

Lemma 5.4 implies

$$|\Psi(k,t)| \le C\|g\|_{\mathcal{A}_1} t^{-1/2} k^{3/2}. \tag{5.8}$$

Integrating $I_\alpha^1(t)$ by parts we get

$$|I_\alpha^1(t)| \le |\Psi(t,t)| t^{-\alpha} + \int_1^t |\Psi(k,t)||\Lambda(k)| dk,$$

where $\Lambda(k) = \frac{k\eta'(k)-\alpha\eta(k)}{k^{\alpha+1}}$ is a smooth bounded function and $\Lambda(k) = O(k^{-\alpha-1})$ as $k\to\infty$. By (5.8)

$$|I_\alpha^1(t)| \le C\|g\|_{\mathcal{A}_1}\left(t^{1-\alpha} + (1+\alpha)t^{-1/2}\int_1^t k^{1/2-\alpha}dk\right) \le C\|g\|_{\mathcal{A}_1} t^{-1/2}.$$

Together with (5.7) this proves (5.5).

Next we turn to (5.6). Since (5.7) is valid for $\alpha = 3/2$ and $g(k) = 1$ this follows from Lemma A.1. □

To prove Theorem 5.2 we have to show that for any function smooth function $f \in C_0^\infty$ with compact support

$$\left\|[\mathrm{e}^{-\mathrm{i}t\mathbf{H}}]^{12}\,\xi(\mathbf{H^2})f\right\|_{L^\infty} \le Ct^{-1/2}\|f\|_{H^{\frac{1}{2},1}}, \quad t\ge 1. \tag{5.9}$$

The kernel of the resolvent of the free Schrödinger operator reads ([23, §7.4])

$$[\mathcal{R}_0(k^2 \pm \mathrm{i}0)](x,y) = \pm\mathrm{i}\mathrm{e}^{\pm\mathrm{i}k|x-y|}/(2k), \quad k>0.$$

Substituting the second resolvent identity $\mathcal{R}(\lambda) = \mathcal{R}_0(\lambda) - \mathcal{R}_0(\lambda)V\mathcal{R}(\lambda)$ into the 12 entry of (5.4), and taking into account that $\xi(x) = 0$ for $x \le m^2+1$, we obtain

$$[\mathrm{e}^{-\mathrm{i}t\mathbf{H}}]^{12}\,\xi(\mathbf{H}^2) = \mathbf{K}_0(t) + \mathbf{K}_1(t),$$

where the kernels of the operators $\mathbf{K}_0(t)$ and $\mathbf{K}_1(t)$ read

$$[\mathbf{K}_0(t)](x,y) = \frac{1}{2\pi}\int_{|k|\ge1} \xi(k^2+m^2)\frac{\sin(t\sqrt{k^2+m^2})}{\sqrt{k^2+m^2}}\mathrm{e}^{\mathrm{i}k(x-y)}dk, \tag{5.10}$$

$$\begin{aligned}[\mathbf{K}_1(t)](x,y) = &\frac{\mathrm{i}}{4\pi}\int_{\mathbb{R}} V(z)\left(\int_{|k|\ge1}\xi(k^2+m^2)\right.\\ &\left.\times\frac{\sin(t\sqrt{k^2+m^2})}{\sqrt{k^2+m^2}}\frac{\mathrm{e}^{\mathrm{i}k(|x-z|+|z-y|)}}{k}(\psi(y,z,k)+1)dk\right)dz.\end{aligned} \tag{5.11}$$

Note that the derivative $\xi'(x)$ has support inside the set $[m^2+1, m^2+2]$. Therefore the function

$$\eta(k) := \frac{\mathrm{i}}{4\pi}\xi(k^2+m^2)\frac{k}{\sqrt{k^2+m^2}} \tag{5.12}$$

satisfies the conditions of Lemma 5.5. Applying this lemma with $\alpha = 2$, $g(k) = \psi(y,z,k)+1$, $p = |x-z|+|z-y|$, and taking into account (2.19), we get

$$\|\mathbf{K}_1(t)f\|_{L^\infty} \le Ct^{-1/2}\|f\|_{L^1} \le C|t|^{-1/2}\|f\|_{H^{\frac{1}{2},1}}, \quad t\ge 1, \tag{5.13}$$

since $H^{\frac{1}{2},1}\subset L^1$ due to [3, Theorems 6.2.3]. It remains to get an estimate of type (5.9) for $\mathbf{K}_0(t)$.

Lemma 5.6. *Assume $V\in L^1_1$. Then*

$$\|\mathbf{K}_0(t)f\|_{L^\infty} \le C|t|^{-1/2}\|f\|_{H^{\frac{1}{2},1}}, \quad |t|\ge 1.$$

Proof. For any $f\in C_0^\infty$ (5.10) implies

$$\int_{\mathbb{R}}[\mathbf{K}_0(t)](x,y)f(y)dy = \sum_{\mp}\int_{|k|\ge 1}\frac{\eta(k)}{k(1+k^2)^{1/4}}\mathrm{e}^{\pm t\sqrt{k^2+m^2}+\mathrm{i}kx}(1+k^2)^{1/4}\hat{f}(k)dk,$$

where $\eta(k)$ is defined by (5.12). Denote $g=\mathcal{J}_{\frac{1}{2}}f$. By definition (1.9) we have

$$\|g\|_{L^1} = \|f\|_{H^{\frac{1}{2},1}}. \tag{5.14}$$

Thus

$$\|\mathbf{K}_0(t)f\|_{L^\infty} \le C\int_{\mathbb{R}}|g(y)|\sup_{x,y\in\mathbb{R}}\Big|\int_1^\infty \eta(k)\frac{\mathrm{e}^{\pm\mathrm{i}t\sqrt{k^2+m^2}+\mathrm{i}k(x-y)}}{k^{3/2}}dk\Big|dy,$$

which together with (5.14) and (5.6) implies the required estimate for $\mathbf{K}_0$. □

Together with (5.13) this establishes Theorem 5.2. This finishes the proof of Theorem 1.3 (i).

6. The Klein–Gordon equation (non-resonant case)

Suppose that the operator H in (1.4) has no resonance at 0. To prove Theorem 1.3 (ii) consider first the low-energy part of the solution. Using the representation (4.1) we rewrite (5.4) as follows

$$\begin{aligned}&[\mathrm{e}^{-\mathrm{i}t\mathbf{H}}\mathbf{P}_c\,\zeta(\mathbf{H}^2)](x,y)\\ &\quad=\sum_{\sigma_1,\sigma_2\in\{\pm\}}\frac{1}{2\pi}\int_{-\infty}^{\infty}A_{\sigma_1}(k)\mathrm{e}^{\mathrm{i}t\sqrt{k^2+m^2}}\,\mathrm{e}^{\mathrm{i}|y-x|k}\zeta(k^2+m^2)\,\mathcal{T}_{\sigma_2}(x,y,k)dk\end{aligned}$$

where

$$A_\pm(k) = \begin{pmatrix} 1 & \mp\dfrac{\mathrm{i}}{\sqrt{k^2+m^2}}\\ \pm\mathrm{i}\sqrt{k^2+m^2} & 1\end{pmatrix},$$

and $\mathcal{T}_\pm(x,y,k) = |T(k)|^2 f_\pm(k)f_\pm(-k)$. Applying integration by parts we obtain for the summand $[\mathrm{e}^{-\mathrm{i}t\mathbf{H}}\mathbf{P}_c\,\zeta(\mathbf{H}^2)]_{++}(x,y)$ with A_+ and $\mathcal{T}_+$:

$$\begin{aligned}&[\mathrm{e}^{-\mathrm{i}t\mathbf{H}}\mathbf{P}_c\,\zeta(\mathbf{H}^2)]_{++}(x,y)\\ &=-\frac{1}{2\pi\mathrm{i}t}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}t\sqrt{k^2+m^2}}\frac{\partial}{\partial k}\Big[\mathrm{e}^{\mathrm{i}|y-x|k}\zeta(k^2+m^2)\frac{\sqrt{k^2+m^2}}{k}A_+(k)\,\mathcal{T}_+(x,y,k)\Big]dk.\end{aligned}$$

Using the same arguments as in the proof of Theorem 1.2 (see Section 4) we obtain

$$|[\mathrm{e}^{-\mathrm{i}t\mathbf{H}}\mathbf{P}_c\,\zeta(\mathbf{H}^2)]_{++}(x,y)| \le Ct^{-3/2}(1+|x|)(1+|y|), \quad t\ge 1$$

and hence

$$\big\|[\mathrm{e}^{-\mathrm{i}t\mathbf{H}}\mathbf{P}_c]_{++}\zeta(\mathbf{H}^2)\big\|_{L^1_1\to L^\infty_{-1}} = \mathcal{O}(|t|^{-3/2}), \quad t\to\infty.$$

The other summands can been estimated similarly, and we get

$$\big\|[\mathrm{e}^{-\mathrm{i}t\mathbf{H}}\mathbf{P}_c]\zeta(\mathbf{H}^2)\big\|_{L^1_1\to L^\infty_{-1}} = \mathcal{O}(t^{-3/2}), \quad t\to\infty. \tag{6.1}$$

It remains to consider the high-energy part. To simplify notations we denote $c(k,t) := \cos(t\sqrt{k^2+m^2})$ and $\chi(k) := \xi(k^2+m^2)$. Applying integration by parts to (5.10) and (5.11) we get

$$\begin{aligned}
[\mathbf{K}_0(t)](x,y) &= \frac{1}{2\pi t}\int\limits_{|k|\ge 1} c(k,t)\frac{\partial}{\partial k}\frac{\chi(k)\mathrm{e}^{\mathrm{i}k(x-y)}}{k}dk,\\
[\mathbf{K}_1(t)](x,y) &= \frac{\mathrm{i}}{4\pi t}\int\limits_{\mathbb{R}} V(z)\int\limits_{|k|\ge 1} c(k,t)\frac{\partial}{\partial k}\frac{\chi(k)\mathrm{e}^{\mathrm{i}k(|x-z|+|z-y|)}(1+\psi(y,z,k))}{k^2}dk\,dz.
\end{aligned} \tag{6.2}$$

To estimate (6.2), recall that $\|\frac{\partial}{\partial k}\psi(z,y,k)\|_{\mathcal{A}} \le C(1+|z|)(1+|y|)$ by (4.9) and (4.13). Moreover, $|x-z|+|z-y| \le (1+|x|)(1+|y|)(1+2|z|)$. Thus the $\mathcal{A}_1$ norm of the derivative with respect to k in the integrand of (6.2) can be estimated by $C(1+|x|)(1+|y|)(1+|z|)$. Moreover, $\chi'(k)$ is a smooth function with a finite support. Applying Lemma 5.5 to the integral with respect to k in (6.2) and taking into account that $|V(z)|\in L^1_1(\mathbb{R})$ we come to the estimate

$$[\mathbf{K}_1(t)](x,y)| \le Ct^{-3/2}(1+|x|)(1+|y|), \quad t\ge 1. \tag{6.3}$$

Further,

$$\begin{aligned}
[\mathbf{K}_0(t)](x,y) &= \frac{1}{2\pi t}\int_{|k|\ge 1} c(k,t)\chi'(k)k^{-1}\mathrm{e}^{\mathrm{i}k(x-y)}dk\\
&\quad - \frac{1}{2\pi t}\int_{|k|\ge 1} c(k,t)\chi(k)\mathrm{e}^{\mathrm{i}k(x-y)}k^{-2}dk\\
&\quad + \frac{\mathrm{i}}{2\pi t}\int_{|k|\ge 1} (x-y)c(k,t)\chi(k)\mathrm{e}^{\mathrm{i}k(x-y)}k^{-1}dk\\
&= [\mathbf{K}_{01}(t)](x,y) + [\mathbf{K}_{02}(t)](x,y) + [\mathbf{K}_{03}(t)](x,y).
\end{aligned}$$

Lemma 5.5 applied to $\mathbf{K}_{01}$ and $\mathbf{K}_{02}$ implies

$$\|\mathbf{K}_{0j}(t)f\|_{L^\infty} \le Ct^{-3/2}\|f\|_{L^1}, \quad j=1,2, \quad t\ge 1. \tag{6.4}$$

It remains to estimate $\mathbf{K}_{03}$. Denote $g = \mathcal{J}_{\frac{1}{2}} f$, then we have $\hat{f}(k) = (1+k^2)^{-\frac{1}{4}}\hat{g}(k)$. Using formulas $\mathcal{F}[\cdot f(\cdot)] = \hat{f}'$ and $\hat{f}'(k) = (1+k^2)^{-\frac{1}{4}}\hat{g}'(k) - \frac{k}{2}(1+k^2)^{-\frac{5}{4}}\hat{g}(k)$ we get

$$\begin{aligned}\int_{\mathbb{R}}[\mathbf{K}_{03}(t)](x,y)f(y)dy = \frac{\mathrm{i}x}{4\pi t}\sum_{\pm}\int\limits_{|k|\geq 1}\frac{\chi(k)}{k(1+k^2)^{\frac{1}{4}}}\mathrm{e}^{\pm t\sqrt{k^2+m^2}+\mathrm{i}kx}(1+k^2)^{\frac{1}{4}}\hat{f}(k)dk\\ +\frac{\mathrm{i}}{4\pi t}\sum_{\pm}\int\limits_{|k|\geq 1}\frac{\chi(k)}{k(1+k^2)^{\frac{1}{4}}}\mathrm{e}^{\pm t\sqrt{k^2+m^2}+\mathrm{i}kx}\hat{g}'(k)dk\\ -\frac{\mathrm{i}}{8\pi t}\sum_{\pm}\int\limits_{|k|\geq 1}\frac{\chi(k)}{(1+k^2)^{\frac{5}{4}}}\mathrm{e}^{\pm t\sqrt{k^2+m^2}+\mathrm{i}kx}\hat{g}(k)dk.\end{aligned}$$

Procceding as in the proof of Lemma 5.6 we get

$$\|(1+|\cdot|)^{-1}\mathbf{K}_{03}(t)f\|_{L^\infty} \leq Ct^{-3/2}\Big(\|g\|_{L^1}+\||\cdot|g\|_{L^1}\Big) \leq Ct^{-3/2}\Big(\|f\|_{H^{\frac{1}{2},1}}+\|f\|_{H_1^{\frac{1}{2},1}}\Big),$$

The last estimate and (6.4) then imply

$$\|\mathbf{K}_0(t)\|_{H_1^{\frac{1}{2},1}\to L^\infty_{-1}} \leq Ct^{-3/2},$$

Together with (6.3) this gives

$$\|[\mathrm{e}^{-\mathrm{i}t\mathbf{H}}]^{12}\xi(\mathbf{H}^2)f\|_{L^\infty_{-1}} \leq C|t|^{-3/2}\|f\|_{H_1^{\frac{1}{2},1}}.$$

Combining this with (6.1) completes the proof of Theorem 1.3 ii).

Appendix A. A decay estimate

The following is [4, Lem. 6.7]) which is an adapted version of [15, Lemma 2]. We include a proof here for the sake of completeness.

Lemma A.1 ([4, 15]). *Let $\Lambda(k)$, $k \geq 0$, be a smooth function such that $\Lambda(k) = O(k^{-5/2})$ as $k \to \infty$ and let*

$$\Psi(k,t) := \int_0^k \mathrm{e}^{\mathrm{i}t\phi(\tau)}d\tau, \quad t \geq 1, \quad k \geq 0,$$

where $\phi(\tau) = \sqrt{\tau^2+1} + v\tau$ with $v \in \mathbb{R}$. Then the following estimate is valid uniformly with respect to v

$$J(t) := \int_1^t |\Psi(k,t)\Lambda(k)|dk \leq Ct^{-1/2}. \tag{A.1}$$

Proof. For simplicity we refer to the van der Corput lemma for the first, second derivative as vdC-1, vdC-2, respectively (see [20, Corollary 5 and Lemma 7]). First of all, we observe the second derivative of the phase function $\phi(\tau)$ admits the estimate

$$\min_{0\leq\tau\leq k}\phi''(\tau) = \min_{0\leq\tau\leq k}(1+\tau^2)^{-3/2} = (1+k^2)^{-3/2}. \tag{A.2}$$

Hence, vdC-2 implies

$$|\Psi(k,t)| \leq Ct^{-1/2}(k+1)^{3/2}, \quad k \geq 0, \quad t \geq 1. \tag{A.3}$$

The first derivative of the phase function $\phi(\tau)$ is an increasing function, satisfying the following estimates

$$|\phi'(\tau)| \geq \begin{cases} 2^{-1/2}, & v \geq 0,\ \tau \geq 1, \\ \frac{1}{2}(\tau^2+1)^{-1}, & v \leq -1,\ \tau \geq 0. \end{cases} \tag{A.4}$$

For $v \in (-1,0)$ the function ϕ' has a zero at $\tau_0 = -v(1-v^2)^{-1/2}$. We study these three regions for v separately. For $v \geq 0$ and $k \geq 1$ we use vdC-1 and (A.4) to get the bound $|\Psi(k,t) - \Psi(1,t)| \leq Ct^{-1}$. Since $|\Psi(1,t)| \leq Ct^{-1/2}$ by (A.3) then for $v \geq 0$ (A.1) follows immediately. Similarly, for $v \leq -1$ from (A.4) and (A.3) it follows

$$|\Psi(k,t)| \leq Ck^2t^{-1} + |\Psi(1,t)| \leq C(k^2t^{-1} + t^{-1/2}), \quad k \geq 1. \tag{A.5}$$

Thus, (A.1) holds in this case also.

It remains to consider the case $v \in (-1,0)$, or equivalently $\tau_0 \in (0,\infty)$. In particular, will estimate $J(t)$ in terms of τ_0 rather than v. By monotonicity of ϕ' and (A.2) for all $\tau \in (0, \tau_0/2]$ we get

$$|\phi'(\tau)| = \frac{\tau_0}{\sqrt{\tau_0^2+1}} - \frac{\tau}{\sqrt{\tau^2+1}} \geq \phi'(2\tau) - \phi'(\tau) \geq \phi''(2\tau)\tau \geq \frac{C}{\tau^2}.$$

Thus for $\tau_0/2 \geq 1$ we obtain similarly as in (A.5)

$$|\Psi(k,t)| \leq C(k^2t^{-1} + t^{-1/2}), \quad 1 \leq k \leq \tau_0/2. \tag{A.6}$$

Furthermore, for all $\tau \geq 2\tau_0$ we obtain

$$\phi'(\tau) = \frac{\tau}{\sqrt{\tau^2+1}} - \frac{\tau_0}{\sqrt{\tau_0^2+1}} \geq \phi'(\tau) - \phi'(\tau/2) \geq \phi''(\tau)\tau/2 \geq \frac{C}{\tau^2}.$$

Therefore

$$|\Psi(k,t)| \leq C(k^2t^{-1} + t^{-1/2}), \quad \max\{1, 2\tau_0\} \leq k. \tag{A.7}$$

Moreover, (A.3) implies

$$\int_{\tau_0/2}^{2\tau_0} |\Psi(k)\Lambda(k)| dk \leq Ct^{-1/2}, \tag{A.8}$$

since $\int_{y/2}^{2y} k^{-1}dk$ does not depend on $y > 0$. For the same reasons we also have the estimate $J(t) - J(t/4) \leq Ct^{-1/2}$. Moreover in the case $4 \leq t \leq 2\tau_0$ it follows from (A.6) that $J(t/4) \leq Ct^{-1/2}$ and we obtain (A.1) for $4 \leq t \leq 2\tau_0$. (Note that in the case $1 \leq t \leq 4$ (A.1) holds for any $\tau_0 \in (0,\infty)$.)

Next consider $1 \leq 2\tau_0 \leq t$. If, additionally, $\tau_0/2 \leq 1$, then

$$J(t) \leq \int_{\tau_0/2}^{2\tau_0} |\Psi(k)\Lambda(k)| dk + \int_{2\tau_0}^{t} |\Psi(k)\Lambda(k)| dk \leq Ct^{-1/2}$$

by (A.8) and (A.7). In the case $1 \leq \tau_0/2 \leq 2\tau_0 \leq t$ we get

$$J(t) \leq \int_1^{\tau_0/2} |\Psi(k)\Lambda(k)| dk + \int_{\tau_0/2}^{2\tau_0} |\Psi(k)\Lambda(k)| dk + \int_{2\tau_0}^{t} |\Psi(k)\Lambda(k)| dk \leq Ct^{-1/2}$$

by (A.6), (A.8), and (A.7). Finally, in the case $2\tau_0 \leq 1$ equation (A.1) follows from (A.7). $\square$

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