ESSENTIAL SELF-ADJOINTNESS OF EVEN-ORDER, STRONGLY SINGULAR, HOMOGENEOUS HALF-LINE DIFFERENTIAL OPERATORS

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Abstract. We consider essential self-adjointness on the space $C_0^\infty((0,\infty))$ of even order, strongly singular, homogeneous differential operators associated with differential expressions of the type

$$\tau_{2n}(c) = (-1)^n \frac{d^{2n}}{dx^{2n}} + \frac{c}{x^{2n}}, \quad x > 0,\ n \in \mathbb{N},\ c \in \mathbb{R},$$

in $L^2((0,\infty); dx)$. While the special case $n = 1$ is classical and it is well-known that $\tau_2(c)\big|_{C_0^\infty((0,\infty))}$ is essentially self-adjoint if and only if $c \geq 3/4$, the case $n \in \mathbb{N}$, $n \geq 2$, is far from obvious. In particular, it is not at all clear from the outset that

$$\begin{aligned} &\textit{there exists } c_n \in \mathbb{R},\ n \in \mathbb{N}, \textit{such that} \\ &\tau_{2n}(c)\big|_{C_0^\infty((0,\infty))} \textit{ is essentially self-adjoint if and only if } c \geq c_n. \end{aligned} \tag{*}$$

As one of the principal results of this paper we indeed establish the existence of c_n, satisfying $c_n \geq (4n-1)!!/2^{2n}$, such that property (*) holds.

In sharp contrast to the analogous lower semiboundedness question,

$$\textit{for which values of } c \textit{ is } \tau_{2n}(c)\big|_{C_0^\infty((0,\infty))} \textit{ bounded from below?},$$

which permits the sharp (and explicit) answer $c \geq [(2n-1)!!]^2/2^{2n}$, $n \in \mathbb{N}$, the answer for (*) is surprisingly complex and involves various aspects of the geometry and analytical theory of polynomials. For completeness we record explicitly,

$$c_1 = 3/4, \quad c_2 = 45, \quad c_3 = 2240\big(214 + 7\sqrt{1009}\big)/27,$$

and remark that c_n is the root of a polynomial of degree $n-1$. We demonstrate that for $n = 6, 7$, c_n are algebraic numbers not expressible as radicals over $\mathbb{Q}$ (and conjecture this is in fact true for general $n \geq 6$).

Contents

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1. Introduction

Consider the $2n$th-order differential expression

$$\tau_{2n}(c) = (-1)^n \frac{d^{2n}}{dx^{2n}} + \frac{c}{x^{2n}}, \quad x \in (0,\infty),\ n \in \mathbb{N},\ c \in \mathbb{R}, \tag{1.1}$$

and introduce the underlying preminimal and symmetric $L^2((0,\infty);dx)$-realization

$$\tau_{2n}(c)\big|_{C_0^\infty((0,\infty))} \tag{1.2}$$

and its closure, the associated minimal operator $T_{2n,min}(c)$ in $L^2((0,\infty);dx)$,

$$T_{2n,min}(c) = \overline{\tau_{2n}(c)\big|_{C_0^\infty((0,\infty))}}. \tag{1.3}$$

The principal question to be posed and answered in this paper is the following:

$$\begin{aligned}&\textit{For which values of } c \in \mathbb{R} \textit{ is } T_{2n,min}(c) \textit{ self-adjoint (equivalently,}\\ &\quad \textit{for which values of } c \in \mathbb{R} \textit{ is } \tau_{2n}(c)\big|_{C_0^\infty((0,\infty))} \textit{ essentially self-adjoint)}\\ &\quad \textit{in } L^2((0,\infty);dx)\textit{?}\end{aligned} \tag{1.4}$$

For the notion of (essentially) self-adjoint Hilbert space operators see, for instance, [26, Sect. V.3], [38, Sect. VIII.2], [41, Sect. 3.2], and [48, Sects. 4.4, 5.3].

In the special case $n = 1$ it is well-known that the precise answer is (see, e.g., [42]),

$$(1.4) \textit{ holds for } n = 1 \textit{ if and only if } c \geq c_1 = 3/4. \tag{1.5}$$

A priori it is not clear at all that this extends to $n \in \mathbb{N}$, $n \geq 2$, that is, it is not obvious from the outset that

$$\begin{aligned}&\textit{there exists } c_n \in \mathbb{R},\ n \in \mathbb{N}, \textit{ such that}\\ &\tau_{2n}(c)\big|_{C_0^\infty((0,\infty))} \textit{ is essentially self-adjoint if and only if } c \geq c_n.\end{aligned} \tag{1.6}$$

Our principal new results, Theorem 4.5 and Corollary 4.7 assert that (1.6) indeed holds for some $c_n \in \mathbb{R}$ satisfying

$$c_n \geq (4n-1)!!/2^{2n}, \quad n \in \mathbb{N}. \tag{1.7}$$

The proof of the existence of c_n in (1.6) (satisfying (1.7)) is surprisingly complex and involves various aspects of the geometry and analytical theory of polynomials. Explicitly, one obtains

$$\begin{aligned}&c_1 = 3/4, \quad c_2 = 45, \quad c_3 = 2240\big(214 + 7\sqrt{1009}\big)/27,\\ &c_4 = 2835\Bigg(13711 + \frac{190309441}{\sqrt[3]{2625188010911 + 1805760\sqrt{-292868607}}}\\ &\qquad + \sqrt[3]{2625188010911 + 1805760\sqrt{-292868607}}\Bigg)\end{aligned} \tag{1.8}$$

and we note that in this context that c_n is the root of a polynomial of degree $n-1$. In addition, we demonstrate that for $n = 6, 7$, c_n are algebraic numbers not expressible as radicals over $\mathbb{Q}$; we conjecture that this actually continues to hold for general $n \geq 6$.

Before explaining some of the strategy behind the proof of the existence of c_n, and for the purpose of comparison and exhibition of a sharp contrast to the essential self-adjointness problem (1.6), we briefly record the precise borderline of semiboundedness of the minimal operator $T_{2n,min}(c)$, which permits a remarkably simple and explicit solution as follows:

$$\begin{aligned}&T_{2n,min}(c) \text{ is bounded from below, and then actually, } T_{2n,min}(c) \geq 0,\ n \in \mathbb{N},\\ &\quad \text{if and only if } c \geq -\frac{[(2n-1)!!]^2}{2^{2n}}.\end{aligned} \tag{1.9}$$

This is a consequence of the sequence of sharp Birman–Hardy–Rellich inequalities, see Birman [5, p. 46] (see also Glazman [16, p. 83–84])

$$\begin{aligned}\int_0^\infty dx\, \big|f^{(n)}(x)\big|^2 \geq \frac{[(2n-1)!!]^2}{2^{2n}} \int_0^\infty dx\, x^{-2n}|f(x)|^2,&\\ f \in C_0^n((0,\infty)),\ n \in \mathbb{N}.&\end{aligned} \tag{1.10}$$

For more details on (1.10) see [15] and the extensive literature cited therein.

Returning to (1.6), our subject at hand, we recall that $\tau_{2n}(c)\big|_{C_0^\infty((0,\infty))}$ is essentially self-adjoint in $L^2((0,\infty);dx)$ if and only if $\tau_{2n}(c)\big|_{C_0^\infty((0,\infty))}$ is in the limit point case at $x=0$ and $x=\infty$. However, since for all $c\in\mathbb{R}$, cx^{-2n} is bounded on (ε,∞) for all $\varepsilon>0$, $\tau_{2n}(c)\big|_{C_0^\infty((0,\infty))}$ is automatically in the limit point case at $x=\infty$ and hence it suffices to exclusively focus on whether or not $\tau_{2n}(c)\big|_{C_0^\infty((0,\infty))}$ is in the limit point case at $x=0$.

In this context one observes that $\tau_{2n}(c)\big|_{C_0^\infty((0,\infty))}$ is said to be in the *limit point case* at an interval endpoint $a\in\{0,\infty\}$ if precisely n solutions of

$$\tau_{2n}(c)y(\mu,\,\cdot\,;c) = \mu y(\mu,\,\cdot\,;c), \quad \mu \in \mathbb{C}\backslash\mathbb{R} \tag{1.11}$$

(i.e., precisely half of the solutions) lie in $L^2(I_a;dx)$, where I_a is an interval of the type $I_0=(0,d)$ if $a=0$, and $I_\infty=(d,\infty)$ if $a=\infty$, for some fixed $d\in(0,\infty)$.

To decide the limit point property of $\tau_{2n}(c)\big|_{C_0^\infty((0,\infty))}$ at $x=0$, one next argues that it is possible to choose $\mu=0$ in (1.11), restricting x to the interval $I_0=(0,d)$, which then leads to a special Euler-type equations which generically has solutions of power-type

$$y_j(0,x;c) = C_j x^{\alpha_j(c)}, \quad 1 \leq j \leq 2n, \tag{1.12}$$

with $\alpha_j(c)$, $1\leq j\leq 2n$, being the solutions of the underlying discriminant or indicial equation,

$$D_{2n}(z;c) = \prod_{j=1}^{2n}[z-(j-1)] + (-1)^n c = 0, \quad z \in \mathbb{C}. \tag{1.13}$$

In exceptional cases, where some of the $\alpha_k(c)$ coincide, (1.12) is replaced by

$$y_k(0,x;c) = C_k x^{\alpha_k(c)} P(\ln(x)), \tag{1.14}$$

where $P(\,\cdot\,)$ is a polynomial of degree at most $2n-1$. Since we are interested in whether or not $y_j(0,x;c)\in L^2((0,d);dx)$ for some $d\in(0,\infty)$, the presence of logarithmic terms is irrelevant and the deciding L^2-criterion for solutions of $\tau_{2n}(c)y(\mu, \cdot\,; c) = 0$ simply becomes

$$\begin{aligned} &\mathrm{Re}(\alpha_j(c)) > -1/2, \ \text{ for } L^2\text{-membership}, \\ &\quad \text{respectively, } \ \mathrm{Re}(\alpha_j(c)) \leq -1/2, \ \text{ for non-}L^2\text{-membership}. \end{aligned} \tag{1.15}$$

In conclusion, to settle the essential self-adjointness problem (1.6) one needs to establish the existence of $c_n \in \mathbb{R}$ such that precisely n roots $\alpha_j(c)$ of $D_{2n}(\,\cdot\,; c) = 0$ satisfy $\mathrm{Re}(\alpha_j(c)) \leq -1/2$ for $c \geq c_n$. (Equivalently, precisely n roots $\alpha_k(c)$ of $D_{2n}(\,\cdot\,; c) = 0$ satisfy $\mathrm{Re}(\alpha_k(c)) > -1/2$ for $c \geq c_n$.)

Turning briefly to the content of each section, we note that Section 2 introduces minimal and maximal operators associated with general differential expressions τ_{2n} of order $2n$, $n \in \mathbb{N}$, in $L^2((0,\infty); dx)$ and reviews the underlying facts on deficiency indices of the minimal operator $T_{2n,min}$, including Kodaira's decomposition principle. Section 3 discusses perturbed Euler differential systems and investigates the underlying deficiency indices for the minimal operator associated with $\tau_{2n}(c)$ in (1.1). In addition, some of the basic theory of first-order systems in the complex domain going back to Fuchs, Frobenius, and Sauvage, in versions championed by Hille and Kneser, is summarized. Moreover, the special examples $\tau_2(c)$ and $\tau_4(c)$ are treated explicitly. Properties of the (real part of the) roots $\alpha_j(c)$ of $D_{2n}(\,\cdot\,; c) = 0$ are the center piece of our principal Section 4, culminating in Theorem 4.5 and Corollary 4.7 which settle the essential self-adjointness problem (1.6). The techniques involved are related to the Grace–Haewood theorem [37, p. 126], the Routh–Hurwitz criterion, and Orlando's formula [12, § XV.7]. Appendix A shows with the help of Galois theory that c_6, c_7 are algebraic numbers that cannot be expressed as radicals over $\mathbb{Q}$; we conjecture this actually remains the case for all c_n, $n \in \mathbb{N}$, $n \geq 6$.

Finally, some remarks on the notation employed: We denote by $\mathbb{C}^{M\times N}$, $M, N \in \mathbb{N}$, the linear space of $M \times N$ matrices with complex-valued entries. I_N represents the identity matrix in $\mathbb{C}^N$. The spectrum of a matrix (or closed operator in a Hilbert space) T is denoted by $\sigma(T)$. The abbreviation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is used.

2. The Deficiency Indices of $T_{2n,min}(c)$

In this section we briefly recall the notions of deficiency indices and limit point, respectively, limit circle cases associated with maximally defined differential operators, generated by formally symmetric differential expressions τ_{2n} on intervals $(a,b) \subseteq \mathbb{R}$, of even order $2n$, $n \in \mathbb{N}$, and then specialize the results to the particular case $\tau_{2n}(c)$ at hand. We will primarily follow [7, Sects. XIII.2, XIII.6], [33, Sects. 17.4, 17.5], [49, Sects. 3, 4] and also refer to [2, § 126], [22], [23], [28], [29], [47, Chs. 2–4] for relevant background material.

Assuming $(a,b) \subseteq \mathbb{R}$ we suppose that

$$\begin{aligned} &p_m, r \text{ are (Lebesgue) measurable and real-valued a.e. on } (a,b), \quad 0 \leq m \leq n, \\ &p_n > 0, \ r > 0 \text{ (Lebesgue) a.e. on } (a,b), \\ &(1/p_n), \ p_m \in L^1_{loc}((a,b); dx), \quad 0 \leq m \leq n-1, \end{aligned} \tag{2.1}$$

and introduce the quasi-derivatives

$$\begin{aligned}
&u^{[0]} = u, \ u^{[m]} = u^{(m)}, \quad 0 \leq m \leq n-1, \\
&u^{[n]} = p_n \big(u^{n-1}\big)', \\
&u^{[n+1]} = -\big(u^{n}\big)' + p_{n-1} u^{n-1}, \\
&u^{[n+j]} = -\big(u^{n+j-1}\big)' + p_{n-j} u^{n-j}, \quad 2 \leq j \leq n-1, \\
&u^{[2n]} = -\big(u^{2n-1}\big)' + p_0 u = r(\tau_{2n} u).
\end{aligned} \tag{2.2}$$

Here the formally symmetric differential expression τ_{2n} of order $2n$ is given by

$$(\tau_{2n} u)(x) = \sum_{m=0}^{n} (-1)^m \big(p_m(x) y^{(m)}(x)\big)^{(m)}, \quad x \in (a,b). \tag{2.3}$$

Given (2.1)–(2.3), the maximal $L^2((a,b);rdx)$-realization (in short, the maximal operator), $T_{2n,max}$, associated with τ_{2n} is then defined by

$$\begin{aligned}
&T_{2n,max} f = \tau_{2n} f, \\
&f \in \operatorname{dom}(T_{2n,max}) = \big\{g \in L^2((a,b);rdx) \,\big|\, g^{[\ell]} \in AC_{loc}((a,b)), \, 0 \leq \ell \leq 2n-1; \\
&\qquad\qquad\qquad\qquad\qquad\qquad \tau_{2n} g \in L^2((a,b);rdx)\big\}.
\end{aligned} \tag{2.4}$$

Introducing the preminimal operator

$$\begin{aligned}
&\dot{T}_{2n,min} f = \tau_{2n} f, \\
&f \in \operatorname{dom}\big(T_{2n,min}\big) = \{g \in \operatorname{dom}(T_{2n,max}) \,|\, \operatorname{supp}(g) \text{ compact}\}
\end{aligned} \tag{2.5}$$

in $L^2((a,b);rdx)$, one can show that $\dot{T}_{2n,min}$ is densely defined, symmetric, and closable. Hence, defining the minimal operator in $L^2((a,b);rdx)$ associated with τ_{2n} as the closure of $\dot{T}_{2n,min}$,

$$T_{2n,min} = \overline{\dot{T}_{2n,min}}, \tag{2.6}$$

one can prove the well-known fact

$$T_{2n,min}^* = T_{2n,max}, \quad T_{2n,max}^* = T_{2n,min}, \tag{2.7}$$

and thus $T_{2n,max}$ is closed. Moreover, if

$$p_m \in C^m((a,b)), \quad 0 \leq m \leq n, \tag{2.8}$$

one can introduce

$$\ddot{T}_{2n,min} = \tau_{2n}\big|_{C_0^\infty((a,b))}, \tag{2.9}$$

and then also obtains

$$\overline{\ddot{T}_{2n,min}} = \overline{\dot{T}_{2n,min}} = T_{2n,min}. \tag{2.10}$$

Introducing the Lagrange bracket

$$[u,v]_x = \sum_{j=1}^{n} \big[u^{[j-1]}(x) v^{[2n-j]}(x) - u^{[2n-j]}(x) v^{[j-1]}(x)\big], \quad x \in (a,b), \tag{2.11}$$

one infers for $(d,e) \subset (a,b)$ Lagrange's identity via integrations by parts

$$\int_d^e r(x)dx \left\{ \overline{(\tau_{2n}u)(x)} v(x) - \overline{u(x)} (\tau_{2n}v)(x) \right\} = [\overline{u}, v]_e - [\overline{u}, v]_d = [u,v]_x\big|_{x=d}^e. \quad (2.12)$$

Moreover, if $u(\overline{\mu}, \cdot\,)$ and $v(\mu, \cdot\,)$ are solutions of

$$(\tau_{2n}u(\overline{\mu}, \cdot\,))(x) = \overline{\mu}u(\overline{\mu}, x), \quad (\tau_{2n}v(\mu, \cdot\,))(x) = \mu v(\mu, x), \quad \mu \in \mathbb{C},\ x \in (a,b), \quad (2.13)$$

then

$$\frac{d}{dx}[\overline{u(\overline{\mu}, \cdot\,)}, v(\mu, \cdot\,)]_x = 0, \quad x \in (a,b). \quad (2.14)$$

Finally, we also recall the known fact,

$$\begin{aligned} \operatorname{dom}(T_{2n,min}) = \{g \in \operatorname{dom}(T_{2n,max}) \,|\, &\text{for all } h \in \operatorname{dom}(T_{2n,max}): \\ &[h,g]_a = 0 = [h,g]_b\,\}. \end{aligned} \quad (2.15)$$

In the following, the number of $L^2((a,b); rdx)$-solutions $u(\mu_\pm, \cdot\,)$ of

$$\tau_{2n}u(\mu_\pm, \cdot\,) = \mu_\pm u(\mu_\pm, \cdot\,), \text{ with } \pm\operatorname{Im}(\mu_\pm) > 0, \quad (2.16)$$

is denoted by $n_\pm(T_{2n,min})$ and called the *deficiency indices of* $T_{2n,min}$. This notion is well-defined as $n_\pm(T_{2n,min})$ is known to be constant throughout the open complex upper and lower half-plane. As a result, one typically chooses $\mu_\pm = \pm i$. Since the coefficients of τ_{2n} are real-valued, one obtains by a result of von Neumann [45] that

$$0 \leq n_+(T_{2n,min}) = n_-(T_{2n,min}) \leq 2n. \quad (2.17)$$

Finally, given $d \in (a,b)$, and denoting by $T_{2n,min(max),(a,d)}$ and $T_{2n,min(max),(d,b)}$ the corresponding minimal or maximal operator with the interval (a,b) replaced by (a,d) and (d,b), respectively, where d is now a regular endpoint for $\tau_{2n}\big|_{(a,d)}$ and $\tau_{2n}\big|_{(d,b)}$, one has (cf. [2, p. 483–484])

$$\begin{aligned} &n_+(T_{2n,min,(a,d)}) = n_-(T_{2n,min,(a,d)}), \quad n_+(T_{2n,min,(d,b)}) = n_-(T_{2n,min,(d,b)}), \\ &n \leq n_\pm(T_{2n,min,(a,d)}) \leq 2n, \quad n \leq n_\pm(T_{2n,min,(d,b)}) \leq 2n, \end{aligned} \quad (2.18)$$

and the Kodaira decomposition principle (see, e.g., [7, Corollary XIII.2.26], [33, p. 72])

$$n_\pm(T_{2n,min}) = n_\pm(T_{2n,min,(a,d)}) + n_\pm(T_{2n,min,(d,b)}) - 2n \quad (2.19)$$

holds.

Remark 2.1. Given the fact that $d \in (a,b)$ is a regular endpoint for $\tau_{2n}|_{(a,d)}$ and $\tau_{2n}|_{(d,b)}$, the particular (and extreme) case where

$$n_\pm(T_{2n,min,(a,d)}) = n \text{ (resp., } n_\pm(T_{2n,min,(d,b)}) = n) \quad (2.20)$$

is the precise analog of Weyl's *limit point case* at $x = a$ (resp., $x = b$) in the classical second order case $n = 1$, that is, for $\tau_2|_{(a,d)}$ (resp., $\tau_2|_{(d,b)}$). Hence, we will apply this limit point terminology also in the $2n$th-order context in the following. In particular, if

$$n_\pm(T_{2n,min,(a,d)}) = n = n_\pm(T_{2n,min,(d,b)}), \quad (2.21)$$

then $\tau_{2n}|_{(a,b)}$ is in the limit point case at a and b and (2.19) yields accordingly that

$$n_\pm(T_{2n,min}) = 0 \quad (2.22)$$

in this case. Thus, (2.21), and hence (2.22), is equivalent to

$$T_{2n,min} = T_{2n,max} \text{ is self-adjoint in } L^2((a,b);rdx), \tag{2.23}$$

which in turn is equivalent to

$$\dot{T}_{2n,min} \text{ is essentially self-adjoint in } L^2((a,b);rdx). \tag{2.24}$$

If in addition hypothesis (2.8) holds, then each of (2.21)–(2.24) is also equivalent to

$$\ddot{T}_{2n,min} \text{ is essentially self-adjoint in } L^2((a,b);rdx). \tag{2.25}$$

All other cases, where $1 \leq n_\pm(T_{2n,min}) \leq 2n$, describe various degrees of limit circle cases of τ_{2n}, with $n_\pm(T_{2n,min}) = 2n$ representing the extreme case. ◇

In the bulk of this paper we are particularly interested in the special case where

$$p_n(x) = 1, \quad p_m(x) = 0,\ 1 \leq m \leq n-1, \quad p_0(x) = cx^{-2n}, \quad r(x) = 1, \quad x \in (0,\infty), \tag{2.26}$$

that is, in the concrete example

$$\tau_{2n}(c) = (-1)^n \frac{d^{2n}}{dx^{2n}} + \frac{c}{x^{2n}}, \quad x \in (0,\infty),\ n \in \mathbb{N},\ c \in \mathbb{R}, \tag{2.27}$$

denoting the associated (pre)minimal and maximal operators in $L^2((0,\infty);dx)$ by $T_{2n,min}(c)$, $\dot{T}_{2n,min}(c)$, $\ddot{T}_{2n,min}(c)$, $T_{2n,max}(c)$, etc.

In particular, we are interested in the question,

$$\begin{split}&\text{“for which values of } c \in \mathbb{R} \text{ is } T_{2n,min}(c) \text{ self-adjoint} \\ &\quad \big(\text{resp., } \ddot{T}_{2n,min}(c) \text{ essentially self-adjoint}\big) \text{ in } L^2((0,\infty);dx)?\text{”}\end{split} \tag{2.28}$$

3. Perturbed Euler Differential Systems and Their Deficienciy Indices

In this section we will prove that it suffices to focus on the spectral parameter $\mu = 0$ when trying to determine the number of $L^2((0,d);dx)$-solutions $y(\mu,\,\cdot\,)$ of

$$\begin{split}\tau_{2n}(c)y(\mu,x) = (-1)^n y^{(2n)}(\mu,x) + cx^{-2n}y(\mu,x) = \mu y(\mu,x),& \\ x \in (0,d),\ \mu \in \mathbb{C},\ n \in \mathbb{N},\ c \in \mathbb{R},&\end{split} \tag{3.1}$$

for fixed $d \in (0,\infty)$ (e.g., one could simply choose $d = 1$). In particular, the deficiency indices of the underlying minimal differential operator $T_{2n,min}(c)$ can be determined from the knowledge of the number of $L^2((0,d);dx)$-solutions of $y(0,\,\cdot\,)$, that is, one can reduce (3.1) to the far simpler case $\mu = 0$.

To prove the μ-independence of the number of $L^2((0,d);dx)$-solutions $y(\mu,\,\cdot\,)$ of (3.1), we find it convenient to employ a bit of the celebrated theory of regular singular points (singular points of the first kind) for first-order systems of differential equations in the complex domain, going back to G. Frobenius [9], L. Fuchs [10], [11], and L. Sauvage [39], [40]. The theory is aptly summarized in a number of treatises, we just mention [3, p. 17–36], [6, p. 108–135], [12, 148–164], [17, p. 70–92], [18, p. 105–131], [19], [20, p. 182–198], [21, p. 342–352], [24, p. 356–372, Ch. XVI], [35, Ch. V], [44, Ch. 4], and [46, 216–235].

In the following $\zeta \in \mathbb{C}\backslash\{0\}$ (resp., $\zeta \in D(0;R)\backslash\{0\} = \{\zeta \in \mathbb{C} \,|\, 0 < |\zeta| < R\}$ for some fixed $R \in (0,\infty)$) represents the complex analog of $x \in (0,d)$ in (3.1) and we will study first-order systems of differential equations of the particular form

$$Y'(\zeta) = \zeta^{-1} A(\zeta) Y(\zeta), \tag{3.2}$$

where $Y(\,\cdot\,)$ represents either an $N \times 1$ solution vector or an $N \times N$ solution matrix, $N \in \mathbb{N}$, which generally is multi-valued, and $A(\,\cdot\,)$ is an $N \times N$ entire (resp., analytic in $D(0;R)$) matrix-valued function,

$$A(\zeta) = \sum_{m \in \mathbb{N}_0} A_m \, \zeta^m. \tag{3.3}$$

The very special structure (at most a first-order pole of the coefficient matrix at $z = 0$) of the right-hand side of (3.2) then leads to a rather special structure of solutions as described in the following.

As a warm up we briefly discuss the pure Euler situation where $A(\,\cdot\,)$ is actually a constant matrix $A_0 \in \mathbb{C}^{N \times N}$, that is, we consider

$$Y'(\zeta) = \zeta^{-1} A_0 Y(\zeta), \tag{3.4}$$

with fundamental (typically, many-valued) matrix solutions of the form

$$Y(\zeta) = \zeta^{A_0} C = e^{A_0 \ln(\zeta)} C, \tag{3.5}$$

where $C \in \mathbb{C}^{N \times N}$ is nonsingular (i.e., $\det_{\mathbb{C}^N}(C) \neq 0$). Transforming A_0 into its Jordan normal form $\widehat{A}_0 = T A_0 T^{-1}$ for some nonsingular $T \in \mathbb{C}^{N \times N}$, and setting $\widehat{Y}(\,\cdot\,) = T Y(\,\cdot\,)$ yields

$$\widehat{Y}'(\zeta) = \zeta^{-1} \widehat{A}_0 \widehat{Y}(\zeta), \tag{3.6}$$

hence one can assume without loss of generality that A_0 is in Jordan normal form. In this case A_0 is represented as a block diagonal matrix consisting possibly of a diagonal matrix D and possibly of a number of nontrivial Jordan blocks of varying $r \times r$, $1 \leq r \leq N$, sizes, denoted by $J_r(\alpha_q)$. In particular, if $J_r(\alpha_q)$ is of the form

$$J_r(\alpha_q) = \begin{pmatrix} \alpha_q & 1 & 0 & \cdots & 0 \\ 0 & \alpha_q & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \alpha_q \end{pmatrix}, \quad \alpha_q \in \sigma(A_0), \tag{3.7}$$

then

$$\zeta^{J_r(\alpha_q)} = \zeta^{\alpha_q} \begin{pmatrix} 1 & \ln(\zeta) & [\ln(\zeta)]^2/[2!] & \cdots & [\ln(\zeta)]^{r-1}/[(r-1)!] \\ 0 & 1 & \ln(\zeta) & \cdots & [\ln(\zeta)]^{r-2}/[(r-2)!] \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \ln(\zeta) \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{3.8}$$

explicitly demonstrating the appearance of powers of logarithms of ζ in (3.5) in the case where A_0 has an eigenvalue α_q whose algebraic multiplicity strictly exceeds its geometric one. In particular, the eigenvalues α_q of A_0 are determined via the characteristic equation for A_0, also called the *indicial equation*,

$$D_N(z) = \det_{\mathbb{C}^N}(zI_N - A_0) = 0, \quad z \in \mathbb{C}. \tag{3.9}$$

The general, or perturbed, Euler case (3.2) leads to analogous results as follows.

Theorem 3.1 (Hille [20], p. 192–198, Kneser [27]). *Given the matrix $A(\,\cdot\,) \in \mathbb{C}^{N\times N}$ in* (3.3) *entire* (*resp., analytic in* $D(0;R)$), *the perturbed Euler differential system* (3.2) *has a fundamental set of* (*generally, multi-valued*) *solutions $Y_j \in \mathbb{C}^{N\times 1}$, $j = 1,\dots,N$, of the form,*

$$Y_j(\zeta;q) = \sum_{m\in\mathbb{N}_0} p_{j,m,q}(\ln(\zeta)\, \zeta^{m+\alpha_q}, \quad 1 \leq j \leq N, \tag{3.10}$$

where α_q runs through all distinct eigenvalues of A_0 (*i.e., all elements of $\sigma(A_0)$*), *determined via $D_N(\,\cdot\,) = 0$, and $p_{j,m,q}(\,\cdot\,) \in \mathbb{C}^{N\times 1}$ are polynomials of degree less than or equal to $N-1$. The series in* (3.10) *converges for $0 < |\zeta| < \infty$* (*resp., for $0 < |\zeta| < R$*).

In this context we also refer to Sections 4.3, 4.4, particularly, Theorem 4.11, in Teschl [44], for a succinct treatment of the Frobenius method for first-order systems with a pole structure as in (3.2).

We also note that a fundamental matrix solution of (3.2) can be obtained in analogy to (3.5) in the pure Euler case. In particular, under the spectral hypothesis that

$$\sigma(A_0) \cap \{\sigma(A_0) + \mathbb{Z}\} = \emptyset, \tag{3.11}$$

it was proven by Fuchs [11] (cf. Hille [21, Theorem 9.5.1]) that the perturbed Euler differential system (3.2) has fundamental matrix solutions of the form

$$Y(\zeta) = \sum_{m\in\mathbb{N}_0} C_m\, \zeta^{mI_N + A_0} C, \quad C_0 = I_N,\ C_\ell \in \mathbb{C}^{N\times N},\ \ell \in \mathbb{N}, \tag{3.12}$$

where again $C \in \mathbb{C}^{N\times N}$ is nonsingular.

The case where the spectral assumption (3.11) on A_0 is violated is much more involved[1]. What follows is a shortened description of Hille [21, Theorem 9.5.2], a modified version of Frobenius' method: If (3.11) does not hold, fundamental matrix solutions of the perturbed Euler differential system (3.2) are of the form

$$Y(\zeta) = \sum_{j=0}^{M} [\ln(\zeta)]^j \sum_{m\in\mathbb{N}_0} C_{m,j}\, \zeta^{mI_N + A_0} C, \quad C_{0,0} = [M!] I_N,\ C_{m,j} \in \mathbb{C}^{N\times N}, \tag{3.13}$$

and once again $C \in \mathbb{C}^{N\times N}$ is nonsingular. A characterization of M in (3.13) is possible, see, for instance, [21, p. 342–352].

We conclude this overview by specializing the 1st-order $N \times N$ perturbed Euler system (3.2) to the Nth-order scalar case (a special case of which is depicted in (3.1)). Consider the scalar Nth-order differential equation

$$y^{(N)}(\zeta) + b_{N-1}(\zeta) y^{(N-1)}(\zeta) + \cdots + b_1(\zeta) y'(\zeta) + b_0(\zeta) y(\zeta) = 0, \tag{3.14}$$

[1] In fact, we quote Hille [21, p. 344] in this context: "... A number of arguments are available in the literature all of them more or less corny. What I shall give here is not the corniest; ..."

where the coefficients $b_j(\,\cdot\,)$, $0 \leq j \leq N-1$, are of the form

$$b_j(\zeta) = \zeta^{j-N} a_j(\zeta), \quad a_j(\zeta) = \sum_{m \in \mathbb{N}_0} a_{j,m}\, \zeta^m, \tag{3.15}$$

with $a_j(\,\cdot\,)$ entire (resp., analytic in $D(0;R)$). The scalar ODE (3.14) subordinates to the perturbed Euler differential system (3.2) upon identifying $A(\zeta)$ with the $N \times N$ matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & & & 0 \\ 0 & 1 & 1 & 0 & \dots & & & 0 \\ 0 & 0 & 2 & 1 & \dots & & & 0 \\ 0 & 0 & 0 & 3 & \dots & & & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \dots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \dots & & & 1 \\ -a_0(\zeta) & -a_1(\zeta) & -a_2(\zeta) & -a_3(\zeta) & \dots & & & (N-1) - a_{N-1}(\zeta) \end{pmatrix} \tag{3.16}$$

and identifying $Y(\zeta)$ with $(Y_1(\zeta), \dots, Y_N(\zeta))$, where the solutions $Y_j(\,\cdot\,) \in \mathbb{C}^{N \times 1}$ are given by

$$Y_j(\,\cdot\,) = (y_{j,1}(\,\cdot\,), \dots, y_{j,N}(\,\cdot\,))^\top, \quad y_{j,k}(\zeta) = \zeta^{k-1} y_j^{(k-1)}(\zeta), \quad 1 \leq j, k \leq N, \tag{3.17}$$

with $y_j(\,\cdot\,)$, $1 \leq j \leq N$, linearly independent solutions of (3.14). In this scalar context the matrix $A_0 \in \mathbb{C}^{N \times N}$ in (3.3) is thus of the form

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & & & 0 \\ 0 & 1 & 1 & 0 & \dots & & & 0 \\ 0 & 0 & 2 & 1 & \dots & & & 0 \\ 0 & 0 & 0 & 3 & \dots & & & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \dots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \dots & & & 1 \\ -a_{0,0} & -a_{1,0} & -a_{2,0} & -a_{3,0} & \dots & & & (N-1) - a_{N-1,0} \end{pmatrix} \tag{3.18}$$

and hence the eigenvalues α_q of A_0 prominently figuring in the solution (3.10) are determined via the indicial equation (3.9), $D_N(\,\cdot\,) = 0$, where

$$\begin{aligned} D_N(z) &= \det_{\mathbb{C}^N}(zI_N - A_0) \\ &= \sum_{k=0}^{N} a_{N-k,0} \begin{cases} \prod_{r=1}^{N-k} [z - (r-1)], & 0 \leq k \leq N-1, \\ 1, & k = N, \end{cases} \quad a_{N,0} = 1, \ z \in \mathbb{C}. \end{aligned} \tag{3.19}$$

Given these results we can return to the half-line differential expression $\tau_{2n}(c)$ in (3.1), the special case of the scalar case (3.14) with $N = 2n$ and (frequently explicitly indicating the c-dependence of the coefficients)

$$b_j(\zeta; c) = 0, \ 1 \leq j \leq 2n-1, \quad b_0(\zeta; c) = (-1)^n c\, \zeta^{-2n} - (-1)^n \mu, \quad \mu \in \mathbb{C}, \tag{3.20}$$

equivalently,

$$a_j(\zeta;c)=0,\ ,1\leq j\leq 2n-1,\quad a_0(\zeta;c)=(-1)^n c-(-1)^n\mu\,\zeta^{2n},\quad \mu\in\mathbb{C}. \tag{3.21}$$

In this case the indicial equation further reduces to

$$D_{2n}(z;c)=\prod_{j=1}^{2n}[z-(j-1)]+(-1)^n c=0,\quad z\in\mathbb{C}. \tag{3.22}$$

Thus, we can state the following result.

Theorem 3.2. *Let $c\in\mathbb{R}$, $\mu\in\mathbb{C}$. Then for any $d\in(0,\infty)$, the number of $L^2((0,d);dx)$-solutions of $\tau_{2n}(c)y(\mu,\,\cdot\,)=\mu y(\mu;\,\cdot\,)$, denoted by $\#_{L^2}\big(\tau_{2n}(c)|_{(0,d)}\big)$, is independent of μ. In particular,*

$$n\leq \#_{L^2}\big(\tau_{2n}(c)|_{(0,d)}\big)\leq 2n. \tag{3.23}$$

Moreover, the deficiency indices $n_\pm(T_{2n,min}(c))$ (with $T_{2n,min}(c)$ representing the closure of $\tau_{2n}(c)|_{C_0^\infty((0,\infty))}$ in $L^2((0,\infty);dx)$) equal

$$n_\pm(T_{2n,min}(c))=\#_{L^2}\big(\tau_{2n}(c)|_{(0,d)}\big)-n. \tag{3.24}$$

and hence

$$0\leq n_\pm(T_{2n,min}(c))\leq n. \tag{3.25}$$

In particular,

$$\begin{split}&T_{2n,min}(c)\text{ is self-adjoint }\big(\text{equivalently, }\ddot{T}_{2n,min}\text{ is essentially self-adjoint}\big)\\&\quad\text{in }L^2((0,\infty);dx)\text{ if and only if }\ \#_{L^2}\big(\tau_{2n}(c)|_{(0,d)}\big)=n.\end{split} \tag{3.26}$$

Proof. The μ-independence of $\#_{L^2}\big(\tau_{2n}(c)|_{(0,d)}\big)$ follows from the structure of the solutions Y_j in (3.10), the fact that for each $d\in(0,\infty)$, the power x^α lies in $L^2((0,d);dx)$ if and only if $\mathrm{Re}(\alpha)>-1/2$, independently of the presence of any logarithmic factors, and finally that only the spectrum of A_0 determines the powers α_q in (3.10).

Since $c\in\mathbb{R}$, $\tau_{2n}(c)$ possesses an anti-unitary conjugation operator (effected by complex conjugation of elements in $L^2((0,\infty);dx)$) and one obtains by (2.17),

$$n_+(T_{2n,min}(c))=n_-(T_{2n,min}(c)). \tag{3.27}$$

Moreover by a special case of Kodaira's decomposition principle (2.19) for deficiency indices,

$$\begin{aligned}n_\pm(T_{2n,min}(c))&=n_\pm\Big(\tau_{2n}(c)\big|_{C_0^\infty((0,d))}\Big)+n_\pm\Big(\tau_{2n}(c)\big|_{C_0^\infty((d,\infty))}\Big)-2n\\&=n_\pm\Big(\tau_{2n}(c)\big|_{C_0^\infty((0,d))}\Big)-n\\&=\#_{L^2}\big(\tau_{2n}(c)|_{(0,d)}\big)-n,\end{aligned} \tag{3.28}$$

since

$$n_\pm\Big(\tau_{2n}(c)\big|_{C_0^\infty((d,\infty))}\Big)=n. \tag{3.29}$$

Relation (3.29) holds since $\tau_{2n}(c)$ is regular at d and, as x^{-2n} is bounded on the interval $[d,\infty)$ (cf. [33, Sect. 14.7]), $\tau_{2n}(c)$ is in the limit point case at ∞ since $(-1)^n d^{2n}/dx^{2n}$ is in the limit point case at ∞. Moreover, by (2.18),

$$n \leq n_\pm\Big(\tau_{2n}(c)\big|_{C_0^\infty((0,d))}\Big) \leq 2n, \tag{3.30}$$

implying (3.23) and (3.25). □

Remark 3.3. (i) The independence of $\#_{L^2}\big(\tau_{2n}(c)|_{(0,d)}\big)$ with respect to μ permits one to choose the by far simplest situation by taking $\mu = 0$ when counting the number of $L^2((0,d);dx)$-solutions of $\tau_{2n}(c)y(\mu,\,\cdot\,) = \mu y(\mu;\,\cdot\,)$. This in turn grants one to focus on solutions of the simple power-type x^α as in (3.10) (ignoring the possibility of additional logarithmic factors which, however, cannot influence the L^2- or non-L^2-behavior of solutions near $x = 0$). In particular, considering

$$y_\alpha(x) = x^\alpha P(\ln(x)), \quad x \in (0,\infty),\ \alpha \in \mathbb{C}, \tag{3.31}$$

where $P(\,\cdot\,)$ is any polynomial, then for all $d \in (0,\infty)$,

$$y_\alpha(\,\cdot\,) \in L^2((0,d);dx) \ \text{ if and ony if } \ \mathrm{Re}(\alpha) > -1/2. \tag{3.32}$$

Thus, by (3.10), $\mathrm{Re}(\alpha) > -1/2$, respectively, $\mathrm{Re}(\alpha) \leq -1/2$, is the criterion deciding whether or not a particular solution with power-type behavior x^α (again, ignoring possible logarithmic factors) contributes to $\#_{L^2}\big(\tau_{2n}(c)|_{(0,d)}\big)$.

(ii) It will be shown in Corollary 4.8 that any permissible integer value for $\#(\tau_{2n}|_{(0,d)})$ in (3.23) actually is attained for some $c \in \mathbb{R}$. ◇

Remark 3.4. One observes that $D_{2n}(\,\cdot\,;c)$ possesses the symmetry

$$D_{2n}(-(1/2)+n+z) = D_{2n}(-(1/2)+n-z). \tag{3.33}$$

In particular, at $z = 0$ one obtains

$$D_{2n}((-1/2)+n) = (-1)^n\left(\prod_{j=1}^n [j-1/2]^2 + c\right) = (-1)^n\left(\frac{[(2n-1)!!]^2}{2^{2n}} + c\right). \tag{3.34}$$

Consequently, for $c = -[(2n-1)!!]^2/2^{2n}$ one has a double zero at $\alpha = k-(1/2)$ and there are two solutions of the type

$$y_1(0,x,c) = x^{k-(1/2)}, \qquad y_2(0,x,c) = x^{k-(1/2)}\ln(x) \tag{3.35}$$

in this case. ◇

Next, we now recall the special situation $n = 1$ which is explicitly solvable for general spectral parameter μ in terms of Bessel functions as follows:

Example 3.5. *Assuming the case* $n = 1$ *in* (3.1) *we consider*

$$\begin{split} -y''(\mu,x) + cx^{-2}y(\mu,x) = \mu y(\mu,x),& \\ \mu \in \mathbb{C},\ x \in (0,\infty),\ c \in \mathbb{R}.& \end{split} \tag{3.36}$$

The associated characteristic equation

$$D_2(z;c) = z(z-1) - c = 0, \tag{3.37}$$

has the following two complex-valued solutions

$$\begin{aligned} \alpha_1(c) &= (1/2) - \sqrt{c + (1/4)}, \\ \alpha_2(c) &= (1/2) + \sqrt{c + (1/4)}, \end{aligned} \tag{3.38}$$

choosing the principal branch for $[\,\cdot\,]^{1/2}$ *with branch cut* $(-\infty, 0]$*, such that*

$$z^{1/2} = r^{1/2} e^{i\varphi/2}, \quad z = re^{i\varphi}, \quad r, r^{1/2} \in [0, \infty), \ \varphi \in (-\pi, \pi]. \tag{3.39}$$

With this convention in place one checks that for all $c \in \mathbb{R}$*, one has the ordering,*

$$\mathrm{Re}(\alpha_1(c)) \leq 1/2 \leq \mathrm{Re}(\alpha_2(c)). \tag{3.40}$$

(α) **Generic case:** *Suppose* $c \in \mathbb{R}$ *is such that*

$$[\alpha_1(c) - \alpha_2(c)]/2 \notin \mathbb{Z}. \tag{3.41}$$

Then the nonhomogenous differential equation (3.36) *has the following fundamental system of solutions* (*cf.* [1, No. 9.1.49, p. 362])

$$\begin{aligned} y_1(\mu, x; c) &= (\pi/2)\mu^{-\gamma(c)/2} x^{1/2} J_{\gamma(c)}\big(\mu^{1/2} x\big), \\ y_2(\mu, x; c) &= \sin(\pi\gamma(c))\mu^{\gamma(c)/2} x^{1/2} J_{-\gamma(c)}\big(\mu^{1/2} x\big), \\ &\qquad \mu \in \mathbb{C}, \ x \in (0, \infty), \end{aligned} \tag{3.42}$$

where

$$\gamma(c) = \sqrt{c + (1/4)}, \quad \gamma \in [0, \infty), \quad c \in \mathbb{R}, \tag{3.43}$$

(Thus, $\gamma(c) \in \{[0, \infty)\backslash \mathbb{N}_0\} \cup i(0, \infty)$ *in the generic case.)*

(β) **Exceptional Cases:** *Suppose* $c \in \mathbb{R}$ *is such that*

$$[\alpha_1(c) - \alpha_2(c)]/2 \in \mathbb{Z}, \tag{3.44}$$

then

$$c = k^2 - (1/4), \quad k \in \mathbb{N}_0. \tag{3.45}$$

More precisely, for $k \in \mathbb{N}_0$*,*

$$[\alpha_1(c) - \alpha_2(c)]/2 = \pm k \ \text{ if and only if } \ c = k^2 - (1/4). \tag{3.46}$$

Furthermore,

$$\alpha_1(c) = \alpha_2(c) \ \text{ if and only if } \ c = -1/4. \tag{3.47}$$

In the exceptional case, where $\gamma(c) = k \in \mathbb{N}_0$*, one obtains*

$$\begin{aligned} y_1\big(\mu, x; k^2 - (1/2)\big) &= (\pi/2)\mu^{-k/2} x^{1/2} J_k\big(\mu^{1/2} x\big), \\ y_2\big(\mu, x; k^2 - (1/2)\big) &= \mu^{k/2} x^{1/2} \big[- Y_k\big(\mu^{1/2} x\big) + \pi^{-1} \ln(\mu) J_k\big(\mu^{1/2} x\big)\big], \\ &\qquad \mu \in \mathbb{C}, \ x \in (0, \infty), \ c \in \big\{k^2 - (1/4)\big\}_{k \in \mathbb{N}_0}. \end{aligned} \tag{3.48}$$

Here $J_\kappa(\,\cdot\,)$ *represent the standard Bessel functions of order* $\kappa \in \mathbb{C}$ *and first kind, and* $Y_k(\,\cdot\,)$ *denotes the Bessel function of order* $k \in \mathbb{N}_0$ *and second kind* (*see, e.g.,* [1, Ch. 9])*. Moreover, one verifies* (*cf.* [1, p. 360]) *that*

$$W(y_2(\mu, \,\cdot\,, c), y_1(\mu, \,\cdot\,; c)) = 1, \quad \mu \in \mathbb{C}, \ c \in \mathbb{R} \tag{3.49}$$

(here $W(f,g) = fg' - f'g$ denotes the Wronkian of f and g), and that the fundamental system of solutions $y_1(\mu,\cdot;c), y_2(\mu,\cdot,c)$ (3.42), (3.48) *of* (3.36) *is entire with respect to $\mu \in \mathbb{C}$ for fixed $x \in (0,\infty)$, and real-valued for $\mu \in \mathbb{R}$.*

As $\mu \to 0$, the fundamental systems of solutions (3.42), (3.48), *upon disregarding normalization, greatly simplify to*

$$y_1(0,x;c) = x^{\alpha_1(c)}, \quad c \in \mathbb{R}, \quad y_2(0,x;c) = \begin{cases} x^{\alpha_2(c)}, & c \in \mathbb{R}\backslash\{-1/4\}, \\ x^{1/2}\ln(x), & c = -1/4; \end{cases} \quad x \in (0,\infty), \tag{3.50}$$

underscoring once again the advantage of choosing $\mu = 0$.

One observes that in accordance with (1.9) *(see also* (1.10)*) and Remark 3.4, the logarithmic case in* (3.50) *occurs at $c = -1/4$, that is, precisely at the borderline of semiboundedness of $T_{min,2}(c)$.*

Thus, determining whether or not $\operatorname{Re}(\alpha_j(c) > -1/2$*, $j = 1,2$, one concludes that*

$$\#_{L^2}\big(\tau_2(c)|_{(0,d)}\big) = \begin{cases} 1, & \text{if} \quad c \geq 3/4, \\ 2, & \text{if} \quad c < 3/4. \end{cases} \tag{3.51}$$

Remark 3.6. In view of the next example, where $n = 2$, in fact, in view of the general case $n \in \mathbb{N}$, it might be interesting to rewrite the Bessel function solutions in the case $n = 1$ in terms of the corresponding generalized hypergeometric function and Meijer's G-function as follows: In the generic case, where $c \in \mathbb{R}$ is such that $[\alpha_1(c) - \alpha_2(c)]/2 \notin \mathbb{Z}$, the nonhomogenous differential equation (3.36) has the following fundamental system of solutions

$$\begin{aligned} y_1(\mu,x;c) &= x^{\alpha_1(c)}\, {}_0F_1\left({\atop 1+\frac{\alpha_1(c)-\alpha_2(c)}{2}} \,\middle|\, -\frac{\mu x^2}{4} \right), \\ y_2(\mu,x;c) &= x^{\alpha_2(c)}\, {}_0F_1\left({\atop 1+\frac{\alpha_2(c)-\alpha_1(c)}{2}} \,\middle|\, -\frac{\mu x^2}{4} \right), \\ &\qquad \mu \in \mathbb{C},\ x \in (0,\infty). \end{aligned} \tag{3.52}$$

Here ${}_0F_1\left({\atop b_1} \,\middle|\, \cdot \right)$ represents the generalized hypergeometric function given by

$${}_0F_1\left({\atop b_1} \,\middle|\, \zeta \right) = \sum_{k \in \mathbb{N}_0} \frac{\zeta^k}{(b_1)_k k!}, \quad b_1 \in \mathbb{C}\backslash\{-\mathbb{N}_0\},\ \zeta \in \mathbb{C}, \tag{3.53}$$

with $(a)_k$ denoting Pochhammer's symbol,

$$(a)_0 = 1, \quad (a)_k = \prod_{j=0}^{k-1}(a+j) = \Gamma(a+k)/\Gamma(a), \quad k \in \mathbb{N},\ a \in \mathbb{C}. \tag{3.54}$$

In particular, ${}_0F_1\left({\atop b_1} \,\middle|\, \zeta \right)$ is entire in $\zeta \in \mathbb{C}$ and

$${}_0F_1\left({\atop b_1} \,\middle|\, \zeta \right) \underset{\zeta \to 0}{=} 1 + O(\zeta). \tag{3.55}$$

In the exceptional case, where $\gamma(c) = k \in \mathbb{N}_0$, one obtains

$$\begin{aligned} y_1\big(\mu, x; k^2 - (1/2)\big) &= x^{k+(1/2)}\, {}_0F_1\left(\begin{matrix} \\ 1+k \end{matrix} \,\middle|\, -\frac{\mu x^2}{4} \right), \\ y_2\big(\mu, x; k^2 - (1/2)\big) &= \Gamma(k+1) 2^k \mu^{-k/2} x^{1/2} G^{2,0}_{0,2}\left(\begin{matrix} \\ k/2; -k/2 \end{matrix} \,\middle|\, -\frac{\mu x^2}{4} \right) \\ &\quad + \big[\pi(-1)^{k+1} i^{k+1} + \ln(\mu)\big] x^{k+(1/2)}\, {}_0F_1\left(\begin{matrix} \\ 1+k \end{matrix} \,\middle|\, -\frac{\mu x^2}{4} \right), \\ &\qquad \mu \in \mathbb{C},\ x \in (0,\infty),\ c \in \big\{k^2 - (1/4)\big\}_{k \in \mathbb{N}_0}. \end{aligned} \tag{3.56}$$

Here Meijer's G-function, $G^{2,0}_{0,2}\Big(\begin{smallmatrix} \\ c_1, c_2 \end{smallmatrix} \,\Big|\, \cdot \Big)$, is given by a Mellin–Barnes-type integral,

$$G^{2,0}_{0,2}\Big(\begin{matrix} \\ c_1, c_2 \end{matrix} \,\Big|\, \zeta \Big) = \frac{1}{2\pi i} \int_{\mathcal{C}} ds\, \zeta^s \Gamma(c_1 - s) \Gamma(c_2 - s), \tag{3.57}$$

where $\mathcal{C}$ is a contour beginning and ending at $+\infty$ encircling all poles of $\Gamma(c_j - s)$, $j = 1, 2$, once in negative orientation, and the left-hand side of (3.57) is defined as the (absolutely convergent) sum of residues of the right-hand side. The exceptional case where c_1 and c_2 differ by an integer is treated by a limiting argument. (For more details see [13].) $\diamond$

For details on generalized hypergeometric functions and Meijer's G-function we refer, for instance, to [4], [8, Ch. IV, Sects. 5.3–5.6], [30, Ch. V], [31, Ch. V], and [34, Ch. 16], [36, Sect. 8.2].

Example 3.7. *Assuming the case $n = 2$ in (3.1) we consider*

$$\begin{aligned} y''''(\mu, x) + c x^{-4} y(\mu, x) &= \mu y(\mu, x), \\ x \in (0,\infty),\ \mu \in \mathbb{C},\ c \in \mathbb{R}. \end{aligned} \tag{3.58}$$

The associated characteristic equation

$$D_4(z; c) = z(z-1)(z-2)(z-3) - c = 0, \quad z \in \mathbb{C},\ c \in \mathbb{R}, \tag{3.59}$$

has the following four complex-valued solutions,

$$\begin{aligned} \alpha_1(c) &= \Big[3 - \sqrt{5 + 4\sqrt{1-c}}\Big] \Big/ 2, \\ \alpha_2(c) &= \Big[3 - \sqrt{5 - 4\sqrt{1-c}}\Big] \Big/ 2, \\ \alpha_3(c) &= \Big[3 + \sqrt{5 - 4\sqrt{1-c}}\Big] \Big/ 2, \\ \alpha_4(c) &= \Big[3 + \sqrt{5 + 4\sqrt{1-c}}\Big] \Big/ 2; \quad c \in \mathbb{R}, \end{aligned} \tag{3.60}$$

employing the principal branch (3.39) *for* $[\,\cdot\,]^{1/2}$. *With this convention, one checks that for all $c \in \mathbb{R}$, one has*

$$\mathrm{Re}(\alpha_1(c)) \leq \mathrm{Re}(\alpha_2(c)) \leq 3/2 \leq \mathrm{Re}(\alpha_3(c)) \leq \mathrm{Re}(\alpha_4(c)). \tag{3.61}$$

(α) **Generic case:** *Suppose $c \in \mathbb{R}$ is such that*

$$[\alpha_j(c) - \alpha_{j'}(c)]/4 \notin \mathbb{Z}, \text{ for all } 1 \leq j, j' \leq 4, \ j \neq j'. \tag{3.62}$$

Then the nonhomogenous differential equation (3.58) *has the following fundamental system of solutions,*

$$\begin{aligned}
y_1(\mu, x; c) &= x^{\alpha_1(c)} \, {}_0F_3\left({}_{1+\frac{\alpha_1(c)-\alpha_2(c)}{4}, 1+\frac{\alpha_1(c)-\alpha_3(c)}{4}, 1+\frac{\alpha_1(c)-\alpha_4(c)}{4}} \, \middle| \, \frac{\mu x^4}{256} \right), \\
y_2(\mu, x; c) &= x^{\alpha_2(c)} \, {}_0F_3\left({}_{1+\frac{\alpha_2(c)-\alpha_1(c)}{4}, 1+\frac{\alpha_2(c)-\alpha_3(c)}{4}, 1+\frac{\alpha_2(c)-\alpha_4(c)}{4}} \, \middle| \, \frac{\mu x^4}{256} \right), \\
y_3(\mu, x; c) &= x^{\alpha_3(c)} \, {}_0F_3\left({}_{1+\frac{\alpha_3(c)-\alpha_1(c)}{4}, 1+\frac{\alpha_3(c)-\alpha_2(c)}{4}, 1+\frac{\alpha_3(c)-\alpha_4(c)}{4}} \, \middle| \, \frac{\mu x^4}{256} \right), \\
y_4(\mu, x; c) &= x^{\alpha_4(c)} \, {}_0F_3\left({}_{1+\frac{\alpha_4(c)-\alpha_1(c)}{4}, 1+\frac{\alpha_4(c)-\alpha_2(c)}{4}, 1+\frac{\alpha_4(c)-\alpha_3(c)}{4}} \, \middle| \, \frac{\mu x^4}{256} \right); \\
&\qquad \mu \in \mathbb{C}, \ x \in (0, \infty).
\end{aligned} \tag{3.63}$$

Asymptotically,

$$y_j(\mu, x; c) \underset{x \downarrow 0}{=} x^{\alpha_j(c)}[1 + O(x)], \quad 1 \leq j \leq 4, \tag{3.64}$$

and thus, the four functions are indeed linearly independent.

Here ${}_0F_3\left({}_{b_1, b_2, b_3} \, \middle| \, \cdot \right)$ *represents the generalized hypergeometric function given by*

$${}_0F_3\left({}_{b_1, b_2, b_3} \, \middle| \, \zeta \right) = \sum_{k \in \mathbb{N}_0} \frac{\zeta^k}{(b_1)_k (b_2)_k (b_3)_k k!}, \quad b_1, b_2, b_3 \in \mathbb{C}\backslash\{-\mathbb{N}_0\}, \ \zeta \in \mathbb{C}. \tag{3.65}$$

Again, ${}_0F_3\left({}_{b_1, b_2, b_3} \, \middle| \, \zeta \right)$ *is entire in $\zeta \in \mathbb{C}$ and*

$${}_0F_3\left({}_{b_1, b_2, b_3} \, \middle| \, \zeta \right) \underset{\zeta \to 0}{=} 1 + O(\zeta). \tag{3.66}$$

That these functions are in fact solutions of (3.58) *can be confirmed by direct verification using the differential equation for generalized hypergeometric functions.*

(β) **Exceptional Cases:** *Suppose $c \in \mathbb{R}$ is such that*

$$[\alpha_j(c) - \alpha_{j'}(c)]/4 \in \mathbb{Z} \text{ for some } 1 \leq j, j' \leq 4, \ j \neq j', \tag{3.67}$$

then

$$\text{either } c = 1 - 20k^2 + 64k^4, \ \text{ or, } \ c = -(9/16) + 10k^2 - 16k^4, \quad k \in \mathbb{N}_0. \tag{3.68}$$

More precisely, for $k \in \mathbb{N}_0$,

$$\begin{aligned}
&[\alpha_1(c) - \alpha_2(c)]/4 = \pm k \ \textit{ implies } \ c = 1 - 20k^2 + 64k^4, \\
&[\alpha_1(c) - \alpha_3(c)]/4 = \pm k \ \textit{ implies } \ c = 1 - 20k^2 + 64k^4, \\
&(\alpha_1(c) - \alpha_4(c)]/4 = \pm k \ \textit{ implies } \ c = -(9/16) + 10k^2 - 16k^4, \\
&[\alpha_2(c) - \alpha_3(c)]/4 = \pm k \ \textit{ implies } \ c = -(9/16) + 10k^2 - 16k^4, \\
&(\alpha_2(c) - \alpha_4(c)]/4 = \pm k \ \textit{ implies } \ c = 1 - 20k^2 + 64k^4, \\
&[\alpha_3(c) - \alpha_4(c)]/4 = \pm k \ \textit{ implies } \ c = 1 - 20k^2 + 64k^4.
\end{aligned} \tag{3.69}$$

Furthermore,

$$\alpha_1(c) = \alpha_2(c) \ \textit{ if and only if } \ \alpha_3(c) = \alpha_4(c) \ \textit{ if and only if } \ c = 1 \tag{3.70}$$

and

$$\alpha_2(c) = \alpha_3(c) \ \textit{ if and only if } \ c = -9/16. \tag{3.71}$$

If $c = 1$, *then*

$$\alpha_1(1) = \alpha_2(1) = \big[3 - \sqrt{5}\,\big]/2, \quad \alpha_3(1) = \alpha_4(1) = \big[3 + \sqrt{5}\,\big]/2, \tag{3.72}$$

and a fundamental system of solutions is given by,

$$\begin{aligned}
y_1(\mu, x; 1) &= x^{[3-\sqrt{5}]/2}\, {}_0F_3\left({}_{1,1-\frac{\sqrt{5}}{4},1-\frac{\sqrt{5}}{4}} \,\middle|\, \frac{\mu x^4}{256} \right), \\
y_2(\mu, x; 1) &= G^{2,0}_{0,4}\left({}_{\frac{3-\sqrt{5}}{8},\frac{3-\sqrt{5}}{8};\frac{3+\sqrt{5}}{8},\frac{3+\sqrt{5}}{8}} \,\middle|\, \frac{\mu x^4}{256} \right), \\
y_3(\mu, x; 1) &= x^{[3+\sqrt{5}]/2}\, {}_0F_3\left({}_{1,1+\frac{\sqrt{5}}{4},1+\frac{\sqrt{5}}{4}} \,\middle|\, \frac{\mu x^4}{256} \right), \\
y_4(\mu, x; 1) &= G^{2,0}_{0,4}\left({}_{\frac{3+\sqrt{5}}{8},\frac{3+\sqrt{5}}{8};\frac{3-\sqrt{5}}{8},\frac{3-\sqrt{5}}{8}} \,\middle|\, \frac{\mu x^4}{256} \right); \\
&\qquad\qquad\qquad\qquad\qquad \mu \in \mathbb{C}, \ x \in (0, \infty).
\end{aligned} \tag{3.73}$$

Asymptotically,

$$\begin{aligned}
y_2(\mu, x; 1) &\underset{x\downarrow 0}{=} c_2 x^{[3-\sqrt{5}]/2} \ln(x)[1 + O(x)], \\
y_4(\mu, x; 1) &\underset{x\downarrow 0}{=} c_4 x^{[3+\sqrt{5}]/2} \ln(x)[1 + O(x)].
\end{aligned} \tag{3.74}$$

Here Meijer's G-function, $G^{2,0}_{0,4}\left({}_{c_1,c_2;c_3,c_4} \,\middle|\, \cdot \right)$, *is again given by a Mellin–Barnes-type integral,*

$$G^{2,0}_{0,4}\left({}_{c_1,c_2;c_3,c_4} \,\middle|\, \zeta \right) = \frac{1}{2\pi i} \int_{\mathcal{C}} ds\, \zeta^s \frac{\Gamma(c_1 - s)\Gamma(c_2 - s)}{\Gamma(1 - c_3 + s)\Gamma(1 - c_4 + s)}, \tag{3.75}$$

where $\mathcal{C}$ *is a contour beginning and ending at* $+\infty$ *encircling all poles of* $\Gamma(c_j - \cdot)$, $j = 1, 2$, *once in negative orientation, and the left-hand side of* (3.75) *is defined as the (absolutely convergent) sum of residues of the right-hand side. The exceptional case where* c_1 *and* c_2 *differ by an integer is once more treated by a limiting argument.*

If $c = 1 - 20k^2 + 64k^4$, $k \in \mathbb{N}$, *then*

$$\begin{aligned}
\alpha_1\big(1-20k^2+64k^4\big) &= \Big[3-4k-\sqrt{5-16k^2}\,\Big]\Big/2,\\
\alpha_2\big(1-20k^2+64k^4\big) &= \Big[3-4k+\sqrt{5-16k^2}\,\Big]\Big/2,\\
\alpha_3\big(1-20k^2+64k^4\big) &= \Big[3+4k-\sqrt{5-16k^2}\,\Big]\Big/2,\\
\alpha_4\big(1-20k^2+64k^4\big) &= \Big[3+4k+\sqrt{5-16k^2}\,\Big]\Big/2,
\end{aligned} \tag{3.76}$$

and a fundamental system of solutions is given by,

$$\begin{aligned}
&y_1\big(\mu,x;1-20k^2+64k^4\big)\\
&\quad = G^{2,0}_{0,4}\left({\atop \frac{3-4k-\sqrt{5-16k^2}}{8}, \frac{3+4k-\sqrt{5-16k^2}}{8}; \frac{3-4k+\sqrt{5-16k^2}}{8}, \frac{3+4k+\sqrt{5-16k^2}}{8}} \,\middle|\, \frac{\mu x^4}{256}\right),\\
&y_2\big(\mu,x;1-20k^2+64k^4\big)\\
&\quad = G^{2,0}_{0,4}\left({\atop \frac{3-4k+\sqrt{5-16k^2}}{8}, \frac{3+4k+\sqrt{5-16k^2}}{8}; \frac{3-4k-\sqrt{5-16k^2}}{8}, \frac{3+4k-\sqrt{5-16k^2}}{8}} \,\middle|\, \frac{\mu x^4}{256}\right),\\
&y_3\big(\mu,x;1-20k^2+64k^4\big)\\
&\quad = x^{[(3+4k)-\sqrt{5-16k^2}\,]/2}\, {}_0F_3\left({\atop 1+k, 1+k-\frac{\sqrt{5-16k^2}}{4}, 1-\frac{\sqrt{5-16k^2}}{4}} \,\middle|\, \frac{\mu x^4}{256}\right),\\
&y_4\big(\mu,x;1-20k^2+64k^4\big)\\
&\quad = x^{[(3+4k)+\sqrt{5-16k^2}\,]/2}\, {}_0F_3\left({\atop 1+k, 1+k+\frac{\sqrt{5-16k^2}}{4}, 1+\frac{\sqrt{5-16k^2}}{4}} \,\middle|\, \frac{\mu x^4}{256}\right);\\
&\hspace{20em} \mu \in \mathbb{C},\ x \in (0,\infty).
\end{aligned} \tag{3.77}$$

Asymptotically,

$$\begin{aligned}
y_1\big(\mu,x;1-20k^2+64k^4\big) &\underset{x\downarrow 0}{=} x^{[(3-4k)-\sqrt{5-16k^2}\,]/2}\ln(x)[1+O(x)],\\
y_2\big(\mu,x;1-20k^2+64k^4\big) &\underset{x\downarrow 0}{=} x^{[(3-4k)+\sqrt{5-16k^2}\,]/2}\ln(x)[1+O(x)].
\end{aligned} \tag{3.78}$$

If $c = -9/16$, *then*

$$\begin{aligned}
&\alpha_1(-9/16) = \big[3-\sqrt{10}\,\big]/2,\\
&\alpha_2(-9/16) = \alpha_3(-9/16) = 3/2,\\
&\alpha_4(-9/16) = \big[3+\sqrt{10}\,\big]/2,
\end{aligned} \tag{3.79}$$

and a fundamental system of solutions is given by,

$$\begin{aligned}
y_1(\mu,x;-9/16) &= x^{[3-\sqrt{10}]/2}\,{}_0F_3\left(\begin{array}{c} \\ 1-\frac{\sqrt{10}}{4},1-\frac{\sqrt{10}}{8},1-\frac{\sqrt{10}}{8} \end{array} \middle| \, \frac{\mu x^4}{256} \right), \\
y_2(\mu,x;-9/16) &= x^{3/2}\,{}_0F_3\left(\begin{array}{c} \\ 1,1-\frac{\sqrt{10}}{8},1+\frac{\sqrt{10}}{8} \end{array} \middle| \, \frac{\mu x^4}{256} \right), \\
y_3(\mu,x;-9/16) &= G^{2,0}_{0,4}\left(\begin{array}{c} \\ \frac{3}{8},\frac{3}{8};\frac{3-\sqrt{10}}{8},\frac{3+\sqrt{10}}{8} \end{array} \middle| \, \frac{\mu x^4}{256} \right), \\
y_4(\mu,x;-9/16) &= x^{[3+\sqrt{10}]/2}\,{}_0F_3\left(\begin{array}{c} \\ 1+\frac{\sqrt{10}}{4},1+\frac{\sqrt{10}}{8},1+\frac{\sqrt{10}}{8} \end{array} \middle| \, \frac{\mu x^4}{256} \right); \\
&\qquad \mu\in\mathbb{C},\ x\in(0,\infty).
\end{aligned} \tag{3.80}$$

Asymptotically,

$$y_3(\mu,x,-9/16) \underset{x\downarrow 0}{=} c_3 x^{3/2}\ln(x)[1+O(x)]. \tag{3.81}$$

One observes that the case $c=-9/16$*, is again precisely the borderline of semi-boundedness of* $T_{min,4}(c)$ *again in accordance with* (1.9) *(see also* (1.10)*) and Remark 3.4.*

If $c=-(9/16)+10k^2-16k^4$*,* $k\in\mathbb{N}$*, then*

$$\begin{aligned}
\alpha_1\big(-(9/16)+10k^2-16k^4\big) &= (3-4k)/2, \\
\alpha_2\big(-(9/16)+10k^2-16k^4\big) &= \Big[3-\sqrt{10-16k^2}\,\Big]\Big/2, \\
\alpha_3\big(-(9/16)+10k^2-16k^4\big) &= \Big[3+\sqrt{10-16k^2}\,\Big]\Big/2, \\
\alpha_4\big(-(9/16)+10k^2-16k^4\big) &= (3+4k)/2,
\end{aligned} \tag{3.82}$$

and a fundamental system of solutions is given by,

$$\begin{aligned}
& y_1\big(\mu,x;-(9/16)+10k^2-16k^4\big) \\
&\quad = G^{2,0}_{0,4}\left(\begin{array}{c} \\ \frac{3-4k}{8},\frac{3+4k}{8};\frac{3-\sqrt{10-16k^2}}{8},\frac{3+\sqrt{10-16k^2}}{8} \end{array} \middle| \, \frac{\mu x^4}{256} \right), \\
& y_2\big(\mu,x;-(9/16)+10k^2-16k^4\big) \\
&\quad = x^{[3-\sqrt{10-16k^2}\,]/2}\,{}_0F_3\left(\begin{array}{c} \\ \frac{8-2\sqrt{10-16k^2}}{8},\frac{8-4k-\sqrt{10-16k^2}}{8},\frac{8+4k-\sqrt{10-16k^2}}{8} \end{array} \middle| \, \frac{\mu x^4}{256} \right), \\
& y_3\big(\mu,x;-(9/16)+10k^2-16k^4\big) \\
&\quad = x^{[3+\sqrt{10-16k^2}\,]/2}\,{}_0F_3\left(\begin{array}{c} \\ \frac{8+2\sqrt{10-16k^2}}{8},\frac{8-4k+\sqrt{10-16k^2}}{8},\frac{8+4k+\sqrt{10-16k^2}}{8} \end{array} \middle| \, \frac{\mu x^4}{256} \right), \\
& y_4\big(\mu,x;-(9/16)+10k^2-16k^4\big) \\
&\quad = x^{(3+4k)/2}\,{}_0F_3\left(\begin{array}{c} \\ 1+k,\frac{8+4k-\sqrt{10-16k^2}}{8},\frac{8+4k+\sqrt{10-16k^2}}{8} \end{array} \middle| \, \frac{\mu x^4}{256} \right); \\
&\qquad \mu\in\mathbb{C},\ x\in(0,\infty).
\end{aligned} \tag{3.83}$$

Asymptotically,

$$y_1(x) \underset{x\downarrow 0}{=} c_1 x^{(3-4k)/2}[1+O(x)] + c_2 x^{(3+4k)/2}\ln(x)[1+O(x)]. \tag{3.84}$$

Once more, as $\mu \to 0$*, the fundamental system of solutions of* (3.58) *considerably simplifies to*

$$\begin{aligned} &y_1(0,x;c) = x^{\alpha_1(c)}, \quad y_2(0,x;c) = x^{\alpha_2(c)}, \\ &y_3(0,x;c) = x^{\alpha_3(c)}, \quad y_4(0,x;c) = x^{\alpha_4(c)}; \quad c \in \mathbb{R}\backslash\{1,-9/16\}, \end{aligned} \tag{3.85}$$

$$\begin{aligned} &y_1(0,x;1) = x^{[3-\sqrt{5}]/2}, \quad y_2(0,x;1) = x^{[3-\sqrt{5}]/2}\ln(x), \\ &y_3(0,x;1) = x^{[3+\sqrt{5}]/2}, \quad y_4(0,x;1) = x^{[3+\sqrt{5}]/2}\ln(x), \quad c=1, \end{aligned} \tag{3.86}$$

$$\begin{aligned} &y_1(0,x;-9/16) = x^{[3-\sqrt{10}]/2}, \quad y_3(0,x;-9/16) = x^{3/2}, \\ &y_3(0,x;-9/16) = x^{3/2}\ln(x), \quad y_4(0,x;-9/16) = x^{[3+\sqrt{10}]/2}, \quad c=-9/16; \\ &\qquad x \in (0,\infty). \end{aligned} \tag{3.87}$$

By inspection, one verifies that $\tau_4(c) y_j(0,\cdot\,;c) = 0$*,* $1 \leq j \leq 4$*. Alternatively, one can apply the theory of nth-order Euler differential equations as presented, for instance, in* [6, p. 122–123].

Thus, determining whether or not $\mathrm{Re}(\alpha_j(c) > -1/2$*,* $1 \leq j \leq 4$*, one concludes that*

$$\#_{L^2}\big(\tau_4(c)|_{(0,d)}\big) = \begin{cases} 2, & \text{if} \quad c \geq 45, \\ 4, & \text{if} \quad -(7!!)/2^4 \leq c < 45, \\ 3, & \text{if} \quad c < -(7!!)/2^4. \end{cases} \tag{3.88}$$

(Explicitly, $(7!!)/2^4 = 105/16$*.)*

Without going into further details we note that also the higher-order examples $n \in \mathbb{N}$, $n \geq 3$, can be explicitly solved in terms generalized hypergeometric functions and Meijer's G-function (this will be discussed in [13]).

4. On the Real Part of the Roots of $D_{2n}(\,\cdot\,;c)$, $c \in \mathbb{R}$

For $n \in \mathbb{N}$ and $c \in \mathbb{R}$, let $D_{2n}(\,\cdot\,;c)$ be the polynomial given by (3.22) and note that all of its coefficients are real. The goal of this section is to determine how many of the roots of $D_{2n}(\,\cdot\,;c)$ have real part $> -1/2$. Results of this sort are typically approached by using the Routh–Hurwitz criterion. We propose a different approach here, even though Hurwitz's ideas still play a central role.

Let us begin by fixing some notation. For $c \in \mathbb{R}$, let the roots of $D_{2n}(\,\cdot\,;c) = 0$ be denoted $\alpha_j(c)$, $j = 1,\dots,2n$. By the continuous dependence of the roots of a polynomial on the coefficients (see [32, Theorem (1.4)]), we may choose our labelling such that each $\alpha_j(c)$ is a continuous function of c and

$$\mathrm{Re}(\alpha_1(c)) \leq \mathrm{Re}(\alpha_2(c)) \leq \cdots \leq \mathrm{Re}(\alpha_n(c)) \leq \cdots \leq \mathrm{Re}(\alpha_{2n}(c)), \quad c \in \mathbb{R}. \tag{4.1}$$

Note that $\mathrm{Re}(\alpha_j(0)) = \alpha_j(0) = j-1$ for $j = 1,\dots,2n$. The fact that

$$D_{2n}(\,\cdot\,;0) \text{ has } 2n \text{ distinct } \textit{real} \text{ roots } > -1/2 \tag{4.2}$$

will be of crucial importance in all that follows.

Example 4.1. *Figure 1 shows the graphs of the the real parts of the roots of* $D_6(\,\cdot\,;c)$ *as functions of* $c \in \mathbb{R}$. *The scale for the* x*-axis has been chosen such that* $x = c^{1/6}$ *for* $c > 0$ *and* $x = \operatorname{sgn}(c)|c|^{1/6}$ *for* $c < 0$. *The dotted lines show the graphs of the real parts of the roots of* $(\,\cdot\,)^6 - c = 0$ *as functions of* c. *One notes that these dotted lines are straight lines precisely because of our special choice of scale for the* x*-axis. Furthermore, as* $c \to \pm\infty$, *the graph of each function* $\operatorname{Re}(\alpha_j(c))$ *approaches one of these straight lines asymptotically. One observes that for* $c \ll 0$, *one has* $\operatorname{Re}(\alpha_1(c)) = \operatorname{Re}(\alpha_2(c)) < \operatorname{Re}(\alpha_3(c)) = \operatorname{Re}(\alpha_4(c)) < \operatorname{Re}(\alpha_5(c)) = \operatorname{Re}(\alpha_6(c))$. *Similarly, for* $c \gg 0$, *one infers that* $\operatorname{Re}(\alpha_1(c)) < \operatorname{Re}(\alpha_2(c)) = \operatorname{Re}(\alpha_3(c)) < \operatorname{Re}(\alpha_4(c)) = \operatorname{Re}(\alpha_5(c)) < \operatorname{Re}(\alpha_6(c))$.

As will be shown later, we have

$$
\begin{aligned}
\operatorname{Re}(\alpha_1(c)) \leq -\frac{1}{2} \quad &\textit{iff} \quad c \leq \frac{2240\left(214 - 7\sqrt{1009}\right)}{27} \approx -693.0 \\
&\textit{or} \quad c \geq \frac{10395}{64} \approx 162.4, \\
\operatorname{Re}(\alpha_2(c)) \leq -\frac{1}{2} \quad &\textit{iff} \quad c \leq \frac{2240\left(214 - 7\sqrt{1009}\right)}{27} \approx -693.0 \\
&\textit{or} \quad c \geq \frac{2240\left(214 + 7\sqrt{1009}\right)}{27} \approx 36201.2, \\
\operatorname{Re}(\alpha_3(c)) \leq -\frac{1}{2} \quad &\textit{iff} \quad c \geq \frac{2240\left(214 + 7\sqrt{1009}\right)}{27} \approx 36201.2,
\end{aligned} \tag{4.3}
$$

where the algebraic numbers on the right are roots of the quadratic equation $27c^2 - 958720c - 677376000 = 0$. *If* $j \in \{4,5,6\}$, *then* $\operatorname{Re}(\alpha_j(c)) > -1/2$ *for all* $c \in \mathbb{R}$.

The proof of our main result, Theorem 4.5, concerning the real parts of the roots of $D_{2n}(\,\cdot\,;c)$, $c \in \mathbb{R}$, will depend on three lemmas. The first lemma states that for any $c \in \mathbb{R}$, the polynomial $D_{2n}(\,\cdot\,;c)$ cannot have more than two roots (counting multiplicity) having the same real part. More precisely, we have the following result:

Lemma 4.2. *For* $j, j' \in \{1, 2, \dots, 2n\}$ *and* $c \in \mathbb{R}$,

$$\operatorname{Re}(\alpha_j(c)) = \operatorname{Re}(\alpha_{j'}(c)) \ \textit{ implies } \ |j - j'| \leq 1, \tag{4.4}$$

Furthermore, if $\operatorname{Re}(\alpha_j(c)) = \operatorname{Re}(\alpha_{j'}(c))$ *and* $|j - j'| = 1$, *then* $\alpha_j(c), \alpha_{j'}(c) \notin \mathbb{R}$ *and* $\overline{\alpha_j(c)} = \alpha_{j'}(c)$.

Proof. Let $c \in \mathbb{R}$ and note that

$$\frac{d}{dz} D_{2n}(z;c) = \frac{d}{dz}\left(D_{2n}(z;0) + (-1)^n c\right) = \frac{d}{dz} D_{2n}(z;0), \quad z \in \mathbb{C}. \tag{4.5}$$

In particular, $D_{2n}(\,\cdot\,;c)$ and $D_{2n}(\,\cdot\,;0)$ have the same critical points. By (4.2)

$$\text{all of the roots of the derivative of } D_{2n}(\,\cdot\,;0) \text{ are real and simple,} \tag{4.6}$$

and hence it follows that $D_{2n}(\,\cdot\,;c)$ does not have real roots of multiplicity greater than two. Moreover, since $c \in \mathbb{R}$, all roots of $D_{2n}(\,\cdot\,;c)$ are real or complex conjugates. Arguing by contradiction, suppose the polynomial $D_{2n}(\,\cdot\,;c)$ has more than

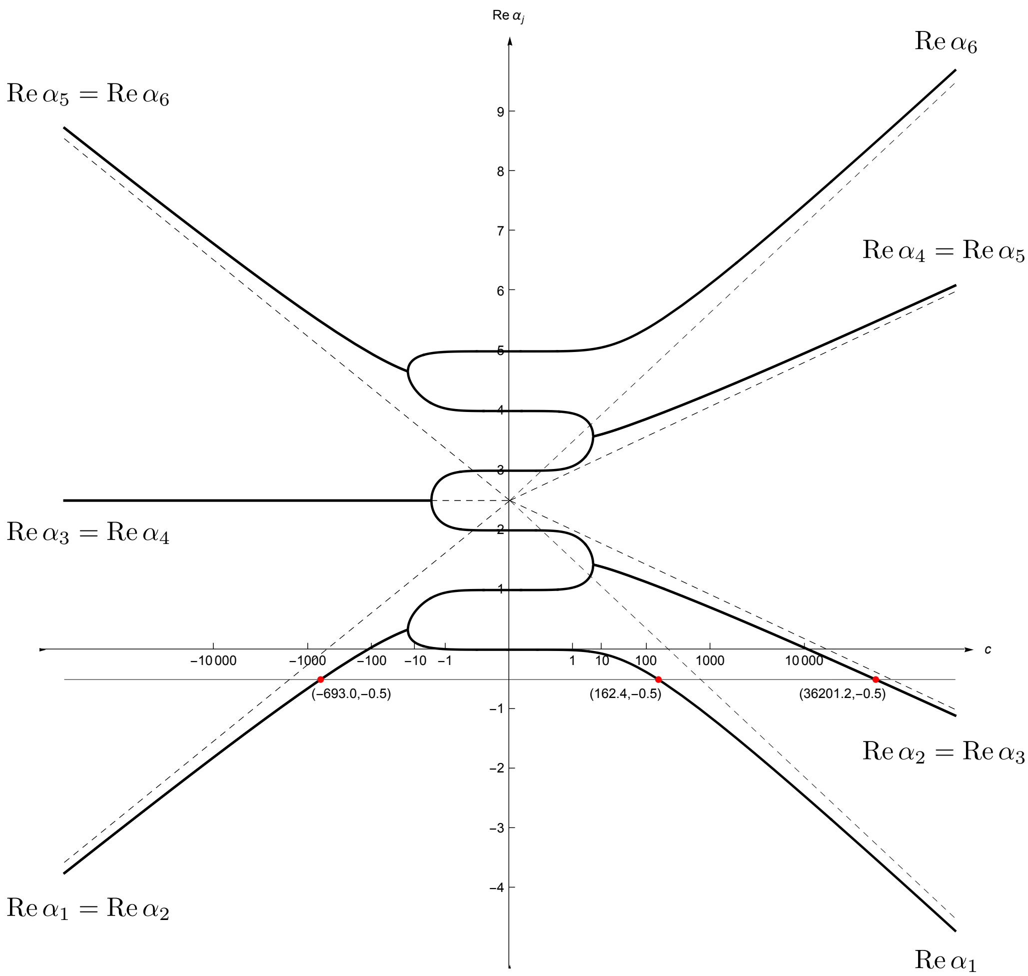


FIGURE 1. Graphs of the the real parts of the roots of $D_6(\,\cdot\,;c)$ as functions of $c \in \mathbb{R}$.

two roots (counting multiplicity) having the same real part. Then

$$\begin{gathered}\text{there exist two roots } z_1, z_2 \in \mathbb{C} \text{ of } D_{2n}(\,\cdot\,;c) \text{ such that} \\ \operatorname{Re}(z_1) = \operatorname{Re}(z_2) \text{ and } 0 \leq \operatorname{Im}(z_1) < \operatorname{Im}(z_2).\end{gathered} \tag{4.7}$$

We now use the Grace–Heawood theorem to obtain a contradiction. More precisely, we use the following corollary of (the proof of) the Grace–Heawood theorem, which is stated on page 126 of [37] as a "Supplement":

If $z_1, z_2 \in \mathbb{C}$ are two distinct roots of a complex polynomial of degree ≥ 2, then neither of the two closed half-planes whose boundary is the perpendicular bisector of the line segment $[z_1, z_2]$ is devoid of any critical points of the polynomial.

When applied to the two roots z_1, z_2 of $D_{2n}(\,\cdot\,; c)$ as in the claim, this leads to a contradiction as follows. Note that the perpendicular bisector of the line segment $[z_1, z_2]$ in our situation is of the form $\{z \in \mathbb{C} \mid \operatorname{Im}(z) = y_0\}$, where $y_0 := [\operatorname{Im}(z_1) + \operatorname{Im}(z_2)]/2 > 0$. Now recall that by (4.6) all the critical points of $D_{2n}(\,\cdot\,; c)$ are real. Thus, the closed half-plane $\{z \in \mathbb{C} \,|\, \operatorname{Im}(z) \geq y_0\}$ would be devoid of any critical points of $D_{2n}(\,\cdot\,; c)$. This is the desired contradiction. □

The second lemma is concerned with the asymptotic behavior of the real parts of the roots of $D_{2n}(\,\cdot\,; c)$ as $c \to \pm\infty$.

Lemma 4.3. *For $j \in \{1, 2, \dots, 2n\}$ and $c \in \mathbb{R}$,*

$$\lim_{c\to+\infty} \operatorname{Re}(\alpha_j(c)) = \begin{cases} -\infty, & 1 \leq j \leq n, \\ +\infty, & n+1 \leq j \leq 2n, \end{cases} \tag{4.8}$$

and

$$\lim_{c\to-\infty} \operatorname{Re}(\alpha_j(c)) = \begin{cases} -\infty, & 1 \leq j \leq n-1, \\ n-(1/2), & n \leq j \leq n+1, \\ +\infty, & n+2 \leq j \leq 2n. \end{cases} \tag{4.9}$$

Proof. For the purpose of this proof, let $f(\cdot)$ be the polynomial given by

$$f(z) := D_{2n}(z + (n-(1/2)); 0), \quad z \in \mathbb{C}. \tag{4.10}$$

The half-integer $n-(1/2)$ is the center of mass of the roots of $D_{2n}(\,\cdot\,; 0)$ and hence the center of mass of the roots of $f(\,\cdot\,)$ is 0. In other words,

$$\text{if we write } f(z) = \textstyle\sum_{j=0}^{2n} a_j z^j, \text{ then } a_{2n-1} = 0. \tag{4.11}$$

For $z_0 \in \mathbb{C}$, it will be convenient to define polynomials $f(\,\cdot\,; z_0)$ and $g(\,\cdot\,; z_0)$ by

$$f(z; z_0) := f(z) - z_0^{2n}, \quad g(z; z_0) := z^{2n} - z_0^{2n}, \quad z \in \mathbb{C}. \tag{4.12}$$

One notes that if $z_0^{2n} = (-1)^{n-1}c$, then $f(z; z_0) = D_{2n}(z-(1/2); c)$ for all $z \in \mathbb{C}$.

Next, let $\varepsilon > 0$. We claim that there exists a real number $R > 0$ such that if $|z_0| > r$, then the polynomial $f(\cdot; z_0)$ has a unique root in the open disc $U(z_0; \varepsilon) := \{z \in \mathbb{C} \,|\, |z - z_0| < \varepsilon\}$. Notice that $g(\cdot; z_0)$ has a unique root in $U(z_0; \varepsilon)$, namely z_0, as long as $|z_0|$ is sufficiently large. Thus, one can use Rouché's theorem as follows. Let $M := \max\{|a_{2n-2}|, \dots, |a_1|, |a_0|\}$. If $|z_0| \geq 1 + \varepsilon$ and $z \in \partial U(z_0; \varepsilon)$, then $1 \leq |z| \leq |z_0| + \varepsilon$ and hence (keeping in mind (4.11))

$$\begin{aligned} |f(z; z_0) - g(z; z_0)| &= |a_{2n-2} z^{n-2} + \ldots + a_1 z + a_0| \\ &\leq |a_{2n-2}||z|^{2n-2} + \cdots + |a_1||z| + |a_0| \\ &\leq M(|z|^{2n-2} + \cdots + |z| + 1) \\ &\leq (2n-1)M|z|^{2n-2} \\ &\leq (2n-1)M(|z_0| + \varepsilon)^{2n-2}. \end{aligned} \tag{4.13}$$

Furthermore, if $|z_0| \geq 1+\varepsilon$, then the minimum of $|g(\cdot; z_0)|$ on the boundary $\partial U(z_0;\varepsilon)$ is attained at $z = (|z_0| - \varepsilon)z_0/|z_0|$ and hence for every $z \in \partial U(z_0;\varepsilon)$ one has

$$\begin{aligned} |g(z;z_0)| &= |z^{2n} - z_0^{2n}| \geq |(|z_0| - \varepsilon)^{2n} - |z_0|^{2n}| \\ &= \varepsilon\, |(|z_0| - \varepsilon)^{2n-1} + \cdots + (|z_0| - \varepsilon) + 1|. \end{aligned} \tag{4.14}$$

One notes that if $|z_0|$ is sufficiently large, then

$$\varepsilon\, [(|z_0| - \varepsilon)^{2n-1} + \cdots + (|z_0| - \varepsilon) + 1] > (2n-1)M(|z_0| + \varepsilon)^{2n-2} \tag{4.15}$$

since the left-hand side is a polynomial in $|z_0|$ of degree $2n-1$ (with positive leading coefficient) and the right-hand side is a polynomial in $|z_0|$ of degree $2n-2$ (with positive leading coefficient.) Therefore, if $|z_0|$ is sufficiently large, then

$$|g(z;z_0)| > |f(z;z_0) - g(z;z_0)| \quad \text{for every } z \in \partial U(z_0;\varepsilon) \tag{4.16}$$

and hence, by Rouché's theorem, $f(\,\cdot\,; z_0)$ and $g(\,\cdot\,; z_0)$ have the same number of roots (counted with multiplicity) in $U(z_0;\varepsilon)$. It follows that there exists some $R > 0$ such that if $|z_0| > R$, then $f(\,\cdot\,; z_0)$ has a unique root in the open disc $U(z_0;\varepsilon)$.

We can now complete the proof of Lemma 4.3. For $c \in \mathbb{R}$, let the roots of

$$[z - (n - (1/2))]^{2n} + (-1)^n c = 0, \quad z \in \mathbb{C}, \tag{4.17}$$

be denoted $\beta_j(c)$, $j = 1, \dots, 2n$. One can choose a labelling such that

$$\mathrm{Re}(\beta_1(c)) \leq \mathrm{Re}(\beta_2(c)) \leq \cdots \leq \mathrm{Re}(\beta_n(c)) \leq \cdots \leq \mathrm{Re}(\beta_{2n}(c)), \quad c \in \mathbb{R}. \tag{4.18}$$

There is a statement analogous to Lemma 4.2 for the roots $\beta_j(c)$, $j = 1, \dots, 2n$. In light of this, there is a "canonical" labeling for both the roots $\alpha_j(c)$ and $\beta_j(c)$ such that if $1 \leq j < 2n$ and $\mathrm{Re}(\alpha_j(c)) = \mathrm{Re}(\alpha_{j+1}(c))$ [resp. $\mathrm{Re}(\beta_j(c)) = \mathrm{Re}(\beta_{j+1}(c))$], then $\mathrm{Im}(\alpha_j(c)) < \mathrm{Im}(\alpha_{j+1}(c))$ [resp. $\mathrm{Im}(\beta_j(c)) < \mathrm{Im}(\beta_{j+1}(c))$]. The roots of (4.17) are trivial to determine and a straightforward (but somewhat tedious) analysis shows that the asymptotic behavior of $\mathrm{Re}(\beta_j(c))$ as $c \to \pm\infty$ is given by (4.8) and (4.9), respectively, with $\alpha_j(c)$ replaced by $\beta_j(c)$, $j = 1, 2 \dots, 2n$.

Now for $\varepsilon > 0$ and $|c| \gg 0$, by the Rouché argument from above applied to $z_0 = \beta_j(c)$,

$$|\alpha_j(c) - \beta_j(c)| < \varepsilon, \quad j = 1, 2 \dots, 2n. \tag{4.19}$$

Therefore, the asymptotic behavior of $\mathrm{Re}(\beta_j(c))$ as $c \to \pm\infty$ is given by (4.8) and (4.9), respectively. ☐

Finally, the last lemma is related to the Routh–Hurwitz criterion, adapted to our situation. This takes some preparation. For $c \in \mathbb{R}$, one first expands $D_{2n}(z-(1/2); c)$ as a polynomial in z,

$$D_{2n}(z - (1/2); c) = q_{2n} z^{2n} + q_{2n-1} z^{2n-1} + \cdots + q_1 z + \big[q_0 + (-1)^n c\big], \tag{4.20}$$

and then considers the associated $(2n \times 2n)$ Hurwitz matrix,

$$H_{2n}(c) := \begin{pmatrix} q_{2n-1} & q_{2n-3} & q_{2n-5} & \cdots & 0 & 0 & 0 \\ q_{2n} & q_{2n-2} & q_{2n-4} & \ddots & \vdots & \vdots & \vdots \\ 0 & q_{2n-1} & q_{2n-3} & \ddots & \vdots & \vdots & \vdots \\ \vdots & q_{2n} & q_{2n-2} & & 0 & \vdots & \vdots \\ \vdots & 0 & q_{2n-1} & & q_0+(-1)^n c & \vdots & \vdots \\ \vdots & \vdots & q_{2n} & & q_1 & 0 & \vdots \\ \vdots & \vdots & 0 & & q_2 & q_0+(-1)^n c & \vdots \\ \vdots & \vdots & \vdots & & q_3 & q_1 & 0 \\ 0 & 0 & 0 & \cdots & q_4 & q_2 & q_0+(-1)^n c \end{pmatrix}. \tag{4.21}$$

One notes that $q_j \in \mathbb{Q}$ for all $j \in \{0, 1, \dots, 2n\}$. Furthermore, observe that c only occurs in the even rows. This implies that the function $\det(H_{2n}(\,\cdot\,))$ is a polynomial of degree n with rational coefficients. By Laplace expansion along the last column,

$$\det(H_{2n}(c)) = [q_0 + (-1)^n c]\, h_{n-1}(c), \tag{4.22}$$

where $h_{n-1}(\,\cdot\,)$ is a polynomial of degree $n-1$ with rational coefficients. There is a simple closed expression for q_0, which is reminiscent of the expression on the right-hand side of (1.9):

$$q_0 = \frac{(4n-1)!!}{2^{2n}}. \tag{4.23}$$

Formula (4.23) is easily proved by induction using that

$$q_0 = D_{2n}(-1/2; 0) = \prod_{j=1}^{2n} [j - (1/2)]. \tag{4.24}$$

Lemma 4.4. *For $j \in \{1, 2, \dots, 2n\}$ and $c \in \mathbb{R}$, if $\mathrm{Re}(\alpha_j(c)) = -1/2$, then*

$$\det(H_{2n}(c)) = 0, \tag{4.25}$$

that is,

$$c = (-1)^{n-1} q_0, \quad or, \quad h_{n-1}(c) = 0, \tag{4.26}$$

where $h_{n-1}(\,\cdot\,)$ is given by (4.22).

Proof. Note that the roots of the polynomial (4.20) are just the roots of $D_{2n}(\,\cdot\,; c)$ shifted by $1/2$, that is, roots of the polynomial (4.20) are $\alpha_j(c) + (1/2)$, where $j \in \{1, 2, \dots, 2n\}$. It then follows from Orlando's formula (see [12, § XV.7]) that

$$h_{n-1}(c) = \prod_{1 \le j_1 < j_2 \le 2n} \{[\alpha_{j_1}(c) + (1/2)] + [\alpha_{j_2}(c) + (1/2)]\}. \tag{4.27}$$

Next, let $j \in \{1, 2, \dots, 2n\}$ and $c \in \mathbb{R}$ such that $\mathrm{Re}(\alpha_j(c)) = -1/2$. First suppose $\alpha_j(c) \in \mathbb{R}$. Then $\alpha_j(c) = -1/2$ and $D_{2n}(-1/2; c) = D_{2n}(-1/2; 0) + (-1)^n c = 0$, which implies that $c = (-1)^{n-1} q_0$. Next suppose $\alpha_j(c) \notin \mathbb{R}$. By Lemma 4.2, there exists some $j' \in \{1, 2, \dots, 2n\}$, $j \ne j'$, such that $\alpha_{j'}(c) = \overline{\alpha_j(c)}$. Then $[\alpha_j(c) + (1/2)] + [\alpha_{j'}(c) + (1/2)] = 0$ and hence $h_{n-1}(c) = 0$ by (4.27). □

We now have all the necessary ingredients to prove the main result of this section, Theorem 4.5. In this context we will use the floor and ceiling notation: One recalls that for $n \in \mathbb{N}$, $\lfloor n/2 \rfloor$ denotes the greatest integer less than or equal to $n/2$; similarly, $\lceil n/2 \rceil$ denotes the least integer greater than or equal $n/2$. Thus, for $n \in \mathbb{N}$, one has

$$\lceil n/2 \rceil = \begin{cases} \lfloor n/2 \rfloor + 1 = (n+1)/2 & \text{if } n \text{ is odd,} \\ \lfloor n/2 \rfloor = n/2 & \text{if } n \text{ is even.} \end{cases} \tag{4.28}$$

Recalling Remark 3.3 (i), one obtains for $c \in \mathbb{R}$, $d \in (0,\infty)$,

$$\#\left(\tau_{2n}(c)|_{(0,d)}\right) = \text{the number of } j \in \{1,2,\dots,2n\} \text{ such that } \operatorname{Re}(\alpha_j(c)) > -1/2. \tag{4.29}$$

Theorem 4.5. (i) *For every $n \in \mathbb{N}$, $n \geq 2$, there exist n real constants*

$$c_n^{(1)} < c_n^{(2)} < \cdots < c_n^{(n)} \tag{4.30}$$

such that the following items (a)–(c) hold:

(a) *For $c \in \mathbb{R}$, $d \in (0,\infty)$, one has*

$$\#\left(\tau_{2n}(c)|_{(0,d)}\right) = \begin{cases} n, & \text{if } \ c \geq c_n^{(n)}, \\ n + 2(n-k), & \text{if } \ c_n^{(k)} \leq c < c_n^{(k+1)} \text{ and } \lfloor n/2 \rfloor < k \leq n-1, \\ 2n, & \text{if } \ c_n^{(k)} < c < c_n^{(k+1)} \text{ and } k = \lfloor n/2 \rfloor, \\ n + 2k + 1, & \text{if } \ c_n^{(k)} < c \leq c_n^{(k+1)} \text{ and } 1 \leq k < \lfloor n/2 \rfloor, \\ n+1, & \text{if } \ c \leq c_n^{(1)}. \end{cases} \tag{4.31}$$

(b) *The constant $c_n^{(\lceil n/2 \rceil)}$ is given by the formula*

$$c_n^{(\lceil n/2 \rceil)} = (-1)^{n-1} \frac{(4n-1)!!}{2^{2n}}. \tag{4.32}$$

(c) *The constants $c_n^{(1)}, c_n^{(2)}, \dots c_n^{(\lceil n/2 \rceil - 1)}, c_n^{(\lceil n/2 \rceil + 1)}, \dots, c_n^{(n)}$ are the roots of the polynomial $h_{n-1}(\,\cdot\,)$ of degree $n-1$ with rational coefficients. In addition,*

$$c_n^{(n)} \geq \frac{(4n-1)!!}{2^{2n}} \underset{n\to\infty}{=} 2^{1/2}(2/e)^n n^{2n}[1 + O(1/n)]. \tag{4.33}$$

(ii) *For $n = 1$ one obtains*

$$\#\left(\tau_2(c)|_{(0,d)}\right) = \begin{cases} 1, & \text{if } \ c \geq 3/4, \\ 2, & \text{if } \ c < 3/4. \end{cases} \tag{4.34}$$

Proof. (i) The constants $c_n^{(1)}, \dots, c_n^{(n)}$ will turn out to be the roots of the polynomial $\det(H_{2n}(\,\cdot\,))$ of degree n given by (4.21). However, it is not clear, a priori, that $\det(H_{2n}(\,\cdot\,))$ has n distinct real roots. For that reason, we will have to define our constants differently.

Next, we recall that the polynomial $D_{2n}(\,\cdot\,;0)$ has $2n$ distinct real roots, namely the non-negative integers $\alpha_j(0) = j-1$, where $j \in \{1,2\dots,2n\}$. In particular, $\mathrm{Re}(\alpha_j(0)) > -1/2$ for all $j \in \{1,2\dots,2n\}$. By Lemma 4.3, if $1 \leq j \leq n-1$, one has $\lim_{c\to-\infty}\mathrm{Re}(\alpha_j(c)) = -\infty$ and hence $\{c<0\,|\,\mathrm{Re}(\alpha_j(c)) = -1/2\}$ is nonempty by continuity; similarly, if $1 \leq j \leq n$, then $\lim_{c\to\infty}\mathrm{Re}(\alpha_j(c)) = -\infty$ and hence $\{c>0\,|\,\mathrm{Re}(\alpha_j(c)) = -1/2\}$ is nonempty by continuity. Now, for $1 \leq k \leq n$, define

$$c_n^{(k)} := \begin{cases} \min\{c\in\mathbb{R}\,|\,\mathrm{Re}(\alpha_{n-2k+1}(c)) = -1/2\} & \text{if } 1 \leq k \leq \lfloor n/2\rfloor \\ \max\{c\in\mathbb{R}\,|\,\mathrm{Re}(\alpha_{2(k-\lfloor n/2\rfloor)-1}(c)) = -1/2\} & \text{if } \lfloor n/2\rfloor < k \leq n. \end{cases} \tag{4.35}$$

One notes that if $1 \leq k \leq \lfloor n/2\rfloor$, then $1 \leq n-2k+1 \leq n-1$ and $c_n^{(k)} < 0$; similarly, if $\lfloor n/2\rfloor < k \leq n$, then $1 \leq 2(k-\lfloor n/2\rfloor)-1 \leq n$ and $c_n^{(k)} > 0$. By (4.1), we then obtain

$$c_n^{(1)} \leq c_n^{(2)} \leq \cdots \leq c_n^{(\lfloor n/2\rfloor)} < 0 < c_n^{(\lfloor n/2\rfloor+1)} \leq \cdots \leq c_n^{(n-1)} \leq c_n^{(n)} \tag{4.36}$$

Next we use Lemma 4.2 to show that all the inequalities in (4.36) are strict. Suppose $c_n^{(k)} = c_n^{(k+1)}$ for some $1 \leq k \leq \lfloor n/2\rfloor - 1$. Then $\mathrm{Re}(\alpha_{n-2k+1}(c_n^{(k)})) = \mathrm{Re}(\alpha_{n-2k-1}(c_n^{(k)}))$ and since $|(n-2k+1)-(n-2k-1)| = 2 > 1$, this contradicts (4.4). The same argument also yields a contradiction if $c_n^{(k)} = c_n^{(k+1)}$ for some $\lfloor n/2\rfloor < k \leq n-1$. Therefore, all the inequalities in (4.36) are strict.

We can say a bit more about the constants $c_n^{(\lfloor n/2\rfloor)}$ and $c_n^{(\lfloor n/2\rfloor+1)}$.

Claim 4.6. We have

$$c_n^{(\lfloor n/2\rfloor)} \leq -q_0 < 0 < q_0 \leq c_n^{(\lfloor n/2\rfloor+1)}, \tag{4.37}$$

where $q_0 = D_{2n}(-1/2;0)$ as in (4.24).

By (4.36) and the discussion leading up to it, the claim follows if we show that for $c \in \mathbb{R}$, the polynomial $D_{2n}(\,\cdot\,;c)$ has no roots with real part equal $-1/2$ if $|c| < q_0$. To prove the latter, we will use a simple argument due to Tallis and Gordon [43, Theorem 1(a)]. Consider the polynomial $f(\,\cdot\,)$ given by $f(z) := D_{2n}(z-(1/2);0)$, $z \in \mathbb{C}$. By (3.22),

$$f(z) = \prod_{j=1}^{2n}[(z-(1/2))-(j-1)] = \prod_{j=1}^{2n}[z-(j-(1/2))], \quad z \in \mathbb{C}. \tag{4.38}$$

Note that $D_{2n}(\,\cdot\,;c)$ has a root with real part equal $-1/2$ if and only if $f(\,\cdot\,)+(-1)^n c$ has a root on the imaginary axis. Suppose $f(ib)+(-1)^n c = 0$ for some $b \in \mathbb{R}$. Then

$$|c| = |f(ib)| = \prod_{j=1}^{2n}|ib-(j-(1/2))| \geq \prod_{j=1}^{2n}[j-(1/2)] = q_0, \tag{4.39}$$

which proves Claim 4.6.

Combining (4.36) and (4.37) implies

$$c_n^{(n)} \geq q_0 = \frac{(4n-1)!!}{2^{2n}} = \frac{\Gamma(4n)}{2^{4n-1}\Gamma(2n)}. \tag{4.40}$$

Stirling's formula (see, e.g., [1, No. 6.1.37]),

$$\Gamma(z) \underset{\substack{z\to\infty \\ |\arg(z)|<\pi}}{=} (2\pi)^{1/2} e^{-z} z^{z-(1/2)}[1+O(1/z)], \tag{4.41}$$

then yields (4.33).

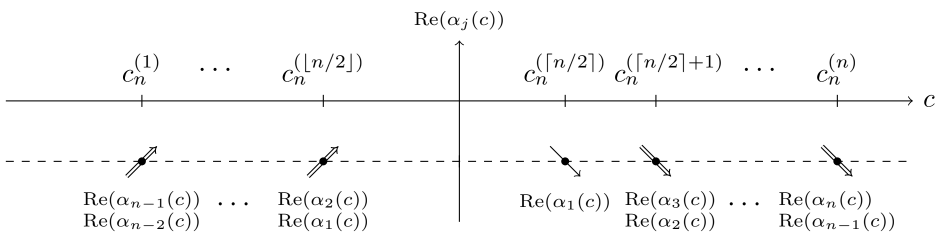
By Lemma 4.4, $\det\big(H_{2n}\big(c_n^{(k)}\big)\big) = 0$ for every $1 \leq k \leq n$. Since the constants $c_n^{(k)}$ are distinct and since $\det(H_{2n}(\,\cdot\,))$ is a polynomial of degree n, the polynomial $\det(H_{2n}(\,\cdot\,))$ does not have any other roots. Furthermore, one of the constants $c_n^{(k)}$ must by equal to $(-1)^{n-1}q_0$ and the other $n-1$ constants must be the roots of the polynomial $h_{n-1}(\,\cdot\,)$, see (4.22). If n is odd, then $(-1)^{n-1}q_0 = q_0 > 0$ and it follows from (4.37) and (4.36) that $c_n^{(\lfloor n/2\rfloor+1)} = q_0$; similarly, if n is even, then $(-1)^{n-1}q_0 = -q_0 < 0$ and it follows from (4.37) and (4.36) that $c_n^{(\lfloor n/2\rfloor)} = -q_0$. In either case, in light of (4.28), we have $c_n^{(\lceil n/2\rceil)} = (-1)^{n-1}q_0$. Thus, recalling the formula for q_0 from (4.23), we obtain (4.32). This completes the proof of parts (b) and (c) of Theorem 4.5.

Before we prove part (a), we recall that by the continuity argument given in the first paragraph of this proof, for every $1 \leq j \leq n-1$, there exists some $c < 0$ such that $\mathrm{Re}(\alpha_j(c)) = -1/2$. By our observations above, this c must be one of the constants $c_n^{(k)}$ with $1 \leq k \leq \lfloor n/2\rfloor$. Similarly, for every $1 \leq j \leq n$, there exists some $c > 0$ such that $\mathrm{Re}(\alpha_j(c)) = -1/2$ and, by our observations above, this c must be one of the constants $c_n^{(k)}$ with $\lfloor n/2\rfloor + 1 \leq k \leq n$.

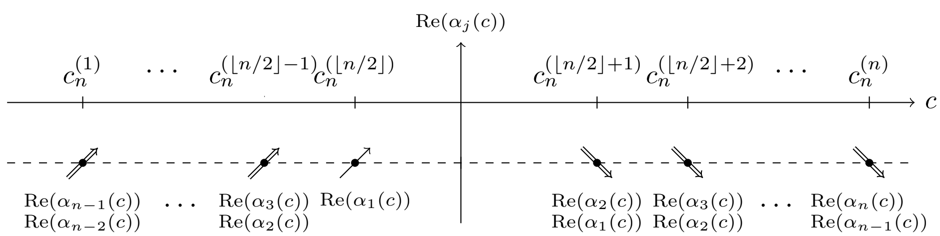
We will now prove part (a) in the case when n is odd. Then $n-1$ is even and $n-1 = 2\lfloor n/2\rfloor = 2(n-\lceil n/2\rceil)$. By Lemma 4.2 and since $n-1 = 2\lfloor n/2\rfloor$, for every $1 \leq k \leq \lfloor n/2\rfloor$, there are exactly two distinct $j, j' \in \{1, 2\dots, n-1\}$ such that $\mathrm{Re}\,\big(\alpha_j\big(c_n^{(k)}\big)\big) = \mathrm{Re}\,\big(\alpha_{j'}\big(c_n^{(k)}\big)\big)$. Furthermore, $c_n^{(\lfloor n/2\rfloor+1)} = c_n^{(\lceil n/2\rceil)} = q_0$ and $\alpha_1\big(c_n^{(\lceil n/2\rceil)}\big) = -1/2 \in \mathbb{R}$. By Lemma 4.2 and since $n-1 = 2(n-\lceil n/2\rceil)$, for every $\lceil n/2\rceil + 1 \leq k \leq n$, there are exactly two distinct $j, j' \in \{2, 3\dots, n\}$ such that $\mathrm{Re}\,\big(\alpha_j\big(c_n^{(k)}\big)\big) = \mathrm{Re}\,\big(\alpha_{j'}\big(c_n^{(k)}\big)\big)$. The resulting situation is summarized in Figure 2(a). We now use Figure 2(a) to understand how the value of $\#\,\big(\tau_{2n}(c)|_{(0,d)}\big)$ changes with $c \in \mathbb{R}$. For $c \leq c_n^{(1)}$, Figure 2(b) shows that $\mathrm{Re}(\alpha_j(c)) > -1/2$ if and only if $n \leq j \leq 2n$. Therefore, $\#\,\big(\tau_{2n}(c)|_{(0,d)}\big) = n+1$ for $c \leq c_n^{(1)}$. As c increases beyond $c_n^{(1)}$, the value of $\#\,\big(\tau_{2n}(c)|_{(0,d)}\big)$ jumps from $n+1$ to $n+3$ since for $c_n^{(1)} < c \leq c_n^{(2)}$, $\mathrm{Re}(\alpha_j(c)) > -1/2$ if and only if $n-2 \leq j \leq 2n$ (assuming that $n \geq 3$). As c increases more, the value of $\#\,\big(\tau_{2n}(c)|_{(0,d)}\big)$ increases by 2 each time c crosses one of the constants $c_n^{(k)}$ until c reaches $c_n^{(\lfloor n/2\rfloor)}$), when the value $\#\,\big(\tau_{2n}(c)|_{(0,d)}\big)$ only increases by 1 from $2n-1$ to $2n$. From then on, the value of $\#\,\big(\tau_{2n}(c)|_{(0,d)}\big)$ starts decreasing by 2 each time c moves beyond one of the constants $c_n^{(k)}$ until, finally, c passes $c_n^{(n)}$, and we have $\#\,\big(\tau_{2n}(c)|_{(0,d)}\big) = n$ since for $c \geq c_n^{(n)}$, $\mathrm{Re}(\alpha_j(c)) > -1/2$ if and only if $n+1 \leq j \leq 2n$. The result is the piecewise-formula for $\#\,\big(\tau_{2n}(c)|_{(0,d)}\big)$ stated in part (a).

In the case when n is even, the argument is, mutatis mutandis, the same. The situation is summarized in Figure 2(b). The result is the same piecewise-formula for $\#\big(\tau_{2n}(c)|_{(0,d)}\big)$ stated in part (a).

(ii) This has been discussed in Example 3.5. □



(a) n odd



(b) n even

FIGURE 2. The constants $c_n^{(k)}$

Corollary 4.7. *For every $n \in \mathbb{N}$, there exists a positive constant $c_n \in \mathbb{R}$ such that*

$$\big\{c \in \mathbb{R} \,\big|\, \#\big(\tau_{2n}(c)|_{(0,d)}\big) = n\big\} = [c_n, \infty), \tag{4.42}$$

and thus,

$$\begin{split} &T_{2n,min}(c) \text{ is self-adjoint } \big(\text{equivalently, } \ddot{T}_{2n,min} \text{ is essentially self-adjoint}\big) \\ &\quad \text{in } L^2((0,\infty);dx) \text{ if and only if } \ c \geq c_n. \end{split} \tag{4.43}$$

In addition,

$$c_1 = 3/4, \quad c_n = c_n^{(n)} \geq \frac{(4n-1)!!}{2^{2n}}, \; n \in \mathbb{N}, \; n \geq 2 \tag{4.44}$$

(see (4.30), (4.31), *and* (4.40)*).*

Put differently, Corollary 4.7 asserts there exist no "islands" (i.e., intervals or its degeneration to points) of non-essential self-adjointness for $\tau_{2n}(c)\big|_{C_0^\infty((0,\infty))}$ for $c \geq c_n$.

We explicitly record the following exact expressions:

$$c_1 = 3/4,$$
$$c_2 = 45,$$

$$
\begin{aligned}
c_3 &= 2240\big(214 + 7\sqrt{1009}\big)/27 \approx 36201.1645283357,\\
c_4 &= 2835\Bigg(13711 + \frac{190309441}{\sqrt[3]{2625188010911 + 1805760\sqrt{-292868607}}}\\
&\qquad + \sqrt[3]{2625188010911 + 1805760\sqrt{-292868607}}\Bigg) \qquad (4.45)\\
&= 38870685 + 5670\sqrt{\frac{292868607}{127}}\sin\left(\frac{1}{3}\tan^{-1}\left(\frac{9\sqrt{292868607}}{466120}\right)\right)\\
&\qquad + \frac{876128400}{\sqrt{127}}\cos\left(\frac{1}{3}\tan^{-1}\left(\frac{9\sqrt{292868607}}{466120}\right)\right)\\
&\approx 117089256.9368802.
\end{aligned}
$$

Corollary 4.8. *For every $n \in \mathbb{N}$ and every $m \in \{n, n+1, \cdots, 2n\}$, there exists some $c \in \mathbb{R}$ such that $\#\left(\tau_{2n}(c)|_{(0,d)}\right) = m$.*

Proof. By Theorem 4.5, as c increases from $c \ll 0$ to $c \gg 0$, $\#\left(\tau_{2n}(c)|_{(0,d)}\right)$ takes on the values

$$n+1, n+3, \dots, 2n-2, 2n, 2n-1, 2n-3, \dots, n+2, n, \quad \text{if } n \text{ is odd}, \qquad (4.46)$$

and

$$n+1, n+3, \dots, 2n-3, 2n-1, 2n, 2n-2, \dots, n+2, n, \quad \text{if } n \text{ is even}. \qquad (4.47)$$

In either case, $\#\left(\tau_{2n}(c)|_{(0,d)}\right)$ takes on all integer values from n to $2n$. ☐

In particular, Corollary 4.8 proves that every possible integer in the interval $[n, 2n]$ in (3.23) is attained for some $c \in \mathbb{R}$.

Example 4.9. *If $n = 3$, then $q_0 = 10395/64$ and*

$$
h_2(c) = \begin{vmatrix}
18 & 435 & 4881/8 & 0 & 0\\
-1 & -505/4 & -12139/16 & c - 10395/64 & 0\\
0 & 18 & 435 & 4881/8 & 0\\
0 & -1 & -505/4 & -12139/16 & c - 10395/64\\
0 & 0 & 18 & 435 & 4881/8
\end{vmatrix}
$$

$$= -5832c^2 + 207083520c + 146313216000, \quad c \in \mathbb{R}. \qquad (4.48)$$

The roots of $h_2(\,\cdot\,)$ are $2240\left(214 \pm 7\sqrt{1009}\right)/27$. Therefore, by Theorem 4.5 one finds

$$
\#\left(\tau_6(c)|_{(0,d)}\right) = \begin{cases}
3, & \text{if } \ 2240\left(214 + 7\sqrt{1009}\right)/27 \le c;\\
5, & \text{if } \ 10395/64 \le c < 2240\left(214 + 7\sqrt{1009}\right)/27;\\
6, & \text{if } \ 2240\left(214 - 7\sqrt{1009}\right)/27 < c < 10395/64;\\
4, & \text{if } \ c \le 2240\left(214 - 7\sqrt{1009}\right)/27.
\end{cases} \qquad (4.49)
$$

Appendix A. Some Conjectures

In this section, when dealing with polynomials, we will view them as elements in a polynomial ring as in abstract algebra. We will review some standard notational conventions and basic results. Let X be an indeterminate (formal symbol). We denote by $\mathbb{Z}[X]$ (resp. $\mathbb{Q}[X]$) the ring of polynomials in the indeterminate X with coefficients in $\mathbb{Z}$ (resp. $\mathbb{Q}$). A polynomial $f(X) \in \mathbb{Q}[X]$ is called irreducible, if it has postive degree and it cannot be written as a product $f(X) = g(X)h(X)$, where $g(X), h(X) \in \mathbb{Q}[X]$ are polynomials of degree strictly less than the degree of $f(X)$.

Conjecture A.1. *For $n \in \mathbb{N}$, $n \geq 2$, the polynomial*

$$g_{n-1}(X) := \frac{(-1)^{\lfloor n/2 \rfloor}}{(2n^2)^n}\, h_{n-1}(X) \tag{A.1}$$

is a monic irreducible polynomial in $\mathbb{Q}[X]$ of degree $n-1$ with Galois group S_{n-1}. In particular, for $n \geq 6$, the constants $c_n^{(1)}, c_n^{(2)}, \dots c_n^{(\lceil n/2 \rceil - 1}, c_n^{(\lceil n/2 \rceil + 1)}, \dots, c_n^{(n)}$ are algebraic numbers that are not expressible in radicals over $\mathbb{Q}$.

Proof for $n = 5$. We have

$$\begin{aligned} g_4(X) = X^4 &- 5237598744576X^3/5 - 34774240217244108191170 56X^2/3125 \\ &+ 29338631588882233803951612887 04X/125 \\ &+ 246639641224100448713004224731938816/55. \end{aligned} \tag{A.2}$$

Let $\widetilde{g}_4(X) := (3125)^4\, g_4(X/3125)$. Then $\widetilde{g}_4(X)$ is a monic polynomial of degree 4 with integer coefficients. Reducing the coefficient of modulo 19, one obtains

$$\widetilde{g}_4(X) \equiv X^4 + 11X^3 + 3X^2 + 11X + 15 \mod 19. \tag{A.3}$$

It is easy to check that $X^4 + 11X^3 + 3X^2 + 11X + 15$ is irreducible modulo 19. By Gauss' lemma, it follows that $g_4(X)$ is irreducible over $\mathbb{Q}$. $\square$

Proof for $n = 6$. We have

$$\begin{aligned} g_5(X) = X^5 &- 15354318108567042605X^4/729 \\ &- 333441081709503846926848000000X^3/3 \\ &+ 498340439140956743662843159904256000000 0X^2 \\ &+ 8770826733513986444066497798757941248000000000000X \\ &- 2088913117666248881257824386993081779822264320000000000000 \end{aligned} \tag{A.4}$$

Let $\widetilde{g}_5(X) := (729)^5\, g_5(X/729)$. Then $\widetilde{g}_5(X)$ is a monic polynomial of degree 5 with integer coefficients. Note that $g_5(X)$ is irreducible over $\mathbb{Q}$ if and only if $\widetilde{g}_5(X)$ is irreducible over $\mathbb{Q}$. Furthermore, the Galois group of $g_5(X)$ is isomorphic to the Galois group of $\widetilde{g}_5(X)$. To prove the irreducibility and to compute the Galois group, we reduce the coefficients of $\widetilde{g}_5(X)$ modulo the primes 23 and 109:

$$\widetilde{g}_5(X) \equiv X^5 + 5X^4 + 11X^3 + 7X^2 + 13X + 16 \mod 23, \tag{A.5}$$

$$\widetilde{g}_5(X) \equiv (X^2+38X+24)(X+42)(X+41)(x+11) \mod 109. \tag{A.6}$$

It is easy to check that $X^5+5X^4+11X^3+7X^2+13X+16$ is irreducible modulo 23. Therefore, the polynomial $\widetilde{g}_5(X)$ is irreducible over $\mathbb{Z}$ and also over $\mathbb{Q}$ by Gauss' lemma. It also follows, by a theorem due to Dedekind (see [25, Thm. 4.37]), that the Galois group of the polynomial $\widetilde{g}_5(X)$ contains a 5-cycle. Since the reduction of $\widetilde{g}_5(X)$ modulo 109 is the product an irreducible quadratic polynomial and three linear polynomials, Dedekind's theorem implies that the Galois group of $\widetilde{g}_5(X)$ contains a transposition (2-cycle). A subgroup of S_5 that contains a transposition and a 5-cycle is S_5. Since S_5 is not a solvable group, Galois' theorem then implies that $g_5(X)$ is not solvable and hence c_6 cannot be written in terms of radicals. □

Proof for $n=7$. The least common denominator of the coefficients of $g_6(X)$ turns out to be 823543. Let $\widetilde{g}_6(X) := (823543)^6\, g_6(X/823543)$. Then $\widetilde{p}_6(X)$ is a monic polynomial with integer coefficients and factorizations of $\widetilde{g}_6(X)$ modulo the primes 37, 43, and 89 are

$$\widetilde{g}_6(X) \equiv (X^6+4X^5+25X^4+20X^3+16X^2+34X+8) \mod 37, \tag{A.7}$$

$$\widetilde{g}_6(X) \equiv (X^2+15X+5)(X+27)(X+20)(X+19)(X+9) \mod 43, \tag{A.8}$$

$$\widetilde{g}_6(X) \equiv (X^5+46X^4+4X^3+23X^2+46X+50)(X+30) \mod 89. \tag{A.9}$$

The factorization $\widetilde{g}_6(X)$ modulo 37 reveals that $\widetilde{g}_6(X)$ is irreducible over $\mathbb{Q}$. □

The same idea can be used to prove the conjecture for larger n. The following table shows what primes are used to verify the conjecture for $4 \le n \le 12$.

n	Smallest prime needed to prove existence of		
	$(n-1)$-cycle	$(n-2)$-cycle	2-cycle
4	23	13	13
5	19	17	71
6	23	47	109
7	37	89	43
8	67	29	8089
9	179	47	7639
10	43	167	11519
11	59	41	2651743
12	53	67	19419221

TABLE 1.

We conclude with a vexing open conjecture:

Conjecture A.2. *We have* $\big($*recalling* $c_n = c_n^{(n)}\big)$

$$c_n \underset{n\to\infty}{\sim} \big(2n^2/\pi\big)^{2n}. \tag{A.10}$$

Sketch of the underlying idea. By (4.17), one infers that

$$[\mathrm{Re}(\beta_n(c)) - (n-(1/2)) + i\,\mathrm{Im}(\beta_n(c))]^{2n} = e^{i\pi} e^{i(\pi/2)2n} \big[c^{1/(2n)}\big]^{2n}, \tag{A.11}$$

and hence,

$$\mathrm{Re}(\beta_n(c)) - (n-(1/2)) = -\sin(\pi/(2n))c^{1/(2n)}. \tag{A.12}$$

Observing that $\mathrm{Re}(\alpha_n(c_n)) \approx \mathrm{Re}(\beta_n(c_n))$ for $c \gg 0$, one arrives at

$$\mathrm{Re}(\alpha_n(c)) \approx (n-(1/2)) - c^{1/(2n)} \sin(\pi/(2n)) \ \text{ for } c \gg 0. \tag{A.13}$$

Finally, recalling that $\mathrm{Re}(\alpha_n(c_n)) = -1/2$, and assuming that c_n increases sufficiently rapidly with increasing n,

$$(c_n)^{1/(2n)} \sin(\pi/(2n)) \approx n, \ \text{ for } n \gg 0. \tag{A.14}$$

Therefore, we *expect* (see Table 2 below) that

$$c_n \underset{n\to\infty}{\sim} \big(2n^2/\pi\big)^{2n}. \tag{A.15}$$

TABLE 2. Asymptotic behavior of c_n and $(2n^2/\pi)^{2n}$

n	c_n	$(2n^2/\pi)^{2n}$
1	$3/4$	0.40529
2	45	42.0495
3	36201.2	35378.2
4	1.17089×10^{8}	1.15878×10^{8}
5	1.04858×10^{12}	1.04280×10^{12}
6	2.10674×10^{16}	2.09987×10^{16}
7	8.27892×10^{20}	8.26165×10^{20}
8	5.77530×10^{25}	5.76715×10^{25}
9	6.65283×10^{30}	6.64619×10^{30}
10	1.19652×10^{36}	1.19565×10^{36}
11	3.21278×10^{41}	3.21100×10^{41}
12	1.24167×10^{47}	1.24115×10^{47}
13	6.70013×10^{52}	6.69788×10^{52}
14	4.91961×10^{58}	4.91828×10^{58}
15	4.80811×10^{64}	4.80706×10^{64}
16	6.13651×10^{70}	6.13540×10^{70}
17	1.00581×10^{77}	1.00566×10^{77}
18	2.08622×10^{83}	2.08595×10^{83}
19	5.40462×10^{89}	5.40404×10^{89}
20	1.72840×10^{96}	1.72824×10^{96}
21	6.75182×10^{102}	6.75128×10^{102}
22	3.19118×10^{109}	3.19096×10^{109}
23	1.80914×10^{116}	1.80903×10^{116}
24	1.22053×10^{123}	1.22046×10^{123}
25	9.72809×10^{129}	9.72763×10^{129}
26	9.09940×10^{136}	9.09902×10^{136}
27	9.92726×10^{143}	9.92689×10^{143}
28	1.25603×10^{151}	1.25599×10^{151}
29	1.83328×10^{158}	1.83322×10^{158}
30	3.07164×10^{165}	3.07155×10^{165}
31	5.88069×10^{172}	5.88055×10^{172}
32	1.28096×10^{180}	1.28093×10^{180}
33	3.16182×10^{187}	3.16175×10^{187}
34	8.81027×10^{194}	8.81010×10^{194}
35	2.76148×10^{202}	2.76143×10^{202}
36	9.70367×10^{209}	9.70351×10^{209}
37	3.81058×10^{217}	3.81052×10^{217}
38	1.66725×10^{225}	1.66723×10^{225}
39	8.10464×10^{232}	8.10454×10^{232}

The following Table 3 seems to suggest that for every $n \in \mathbb{N}$,

$$2n^2/\pi < \big(c_n^{(n)}\big)^{1/(2n)} < n/\sin(\pi/(2n)). \tag{A.16}$$

n	$2n^2/\pi$	$(c_n)^{1/2n}$	$n/\sin(\pi/(2n))$
1	0.6366198	0.8660254	1.0000000
2	2.5464791	2.5900201	2.8284271
3	5.7295780	5.7515790	6.0000000
4	10.185916	10.199165	10.452504
5	15.915494	15.924294	16.180340
6	22.918312	22.924559	23.182220
7	31.194369	31.199023	31.457714
8	40.743665	40.747262	41.006647
9	51.566202	51.569062	51.828934
10	63.661977	63.664305	63.924532
11	77.030992	77.032923	77.293416
12	91.673247	91.674874	91.935571
13	107.58874	107.59013	107.85099
14	124.77748	124.77868	125.03966
15	143.23945	143.24050	143.50158
16	162.97466	162.97558	163.23676
17	183.98311	183.98393	184.24517
18	206.26481	206.26554	206.52684
19	229.81974	229.82039	230.08175
20	254.64791	254.64850	254.90990
21	280.74932	280.74986	281.01129
22	308.12397	308.12446	308.38593
23	336.77186	336.77231	337.03380
24	366.69299	366.69340	366.95492
25	397.88736	397.88774	398.14928

TABLE 3.

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