OSCILLATION THEORY AND RENORMALIZED OSCILLATION THEORY FOR JACOBI OPERATORS

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Abstract. We provide a comprehensive treatment of oscillation theory for Jacobi operators with separated boundary conditions. Our main results are as follows: If u solves the Jacobi equation $(Hu)(n) = a(n)u(n+1) + a(n-1)u(n-1) - b(n)u(n) = \lambda u(n)$, $\lambda \in \mathbb{R}$ (in the weak sense) on an arbitrary interval and satisfies the boundary condition on the left or right, then the dimension of the spectral projection $P_{(-\infty,\lambda)}(H)$ of H equals the number of nodes (i.e., sign flips if $a(n) < 0$) of u. Moreover, we present a reformulation of oscillation theory in terms of Wronskians of solutions, thereby extending the range of applicability for this theory; if $\lambda_{1,2} \in \mathbb{R}$ and if $u_{1,2}$ solve the Jacobi equation $Hu_j = \lambda_j u_j$, $j = 1, 2$ and respectively satisfy the boundary condition on the left/right, then the dimension of the spectral projection $P_{(\lambda_1,\lambda_2)}(H)$ equals the number of nodes of the Wronskian of u_1 and u_2. Furthermore, these results are applied to establish the finiteness of the number of eigenvalues in essential spectral gaps of perturbed periodic Jacobi operators.

1. Introduction

In 1836 Sturm originated the investigations of oscillation properties of solutions of second-order differential and difference equations [32]. Since then numerous extensions have been made. Especially, around 1948, Hartman and others have shown the following in a series of papers ([17], [18], [19]). For a given Sturm–Liouville operator H on $L^2(0,\infty)$, the dimension of the spectral projection $P_{(-\infty,\lambda)}(H)$ equals the number of zeros of certain solutions of $Hu = \lambda u$. Moreover, the dimension of $P_{(\lambda_1,\lambda_2)}(H)$ can be obtained by considering the difference of the number of zeros inside a finite interval $(0, x)$ of two solutions corresponding to their respective spectral parameters λ_1 and λ_2, and performing a limit $x \to \infty$. Only recently it was shown in [13] by F. Gesztesy, B. Simon, and myself that these limits can be avoided by using a renormalized version of oscillation theory, that is, counting zeros of Wronskians of solutions instead.

This naturally raises the question whether similar results hold for second-order difference equations. Despite a variety of literature on this subject (cf., e.g., [1], [3], [6], [7], [12], [14], Sections 14 and 37, [16], [20], [21], [22], [23], [27], [28] and the references therein) only a few things concerning the connections between oscillation properties of solutions and spectra of the corresponding operators appear to be known. In particular, the analogs of the aforementioned theorems seem to be unknown. Moreover, even the analog of the well-known fact that the n-th eigenfunction of a Sturm–Liouville operator (below the essential spectrum) has $n-1$

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nodes is only known in the special case of finite Jacobi operators (i.e., finite tridiagonal matrices) [1], Theorem 4.3.5, [7]. The present paper aims at filling these gaps and provides a complete solution to these problems.

Before we proceed with a more detailed description of our main results, we need to fix some notation. For $I \subseteq \mathbb{Z}$ we denote by $\ell(I)$ the set of $\mathbb{C}$-valued sequences $\{f(n)\}_{n\in I}$. For $M, N \in \mathbb{Z} \cup \{\pm\infty\}$ we abbreviate $\ell(M,N) = \ell(\{n \in \mathbb{Z} | M < n < N\})$ (sometimes we will also write $\ell(N,-\infty)$ instead of $\ell(-\infty,N)$). $\ell^2(I)$ is the Hilbert space of all square-summable sequences with scalar product and norm defined as

$$\langle f,g\rangle = \sum_{n\in I} \overline{f(n)} g(n), \quad \|f\| = \sqrt{\langle f,f\rangle}, \quad f,g \in \ell^2(I). \tag{1.1}$$

Furthermore, $\ell_0(I)$ denotes the set of sequences with only finitely-many values being nonzero, $\ell^1(I)$ the set of summable sequences, $\ell^\infty(I)$ the set of bounded sequences, and $\ell^2_\pm(\mathbb{Z})$ denotes the set of sequences in $\ell(\mathbb{Z})$ which are ℓ^2 near $\pm\infty$.

To set the stage, we shall consider operators on $\ell^2(\mathbb{Z})$ associated with the difference expression

$$(\tau f)(n) = a(n) f(n+1) + a(n-1) f(n-1) - b(n) f(n), \tag{1.2}$$

where $a, b \in \ell(\mathbb{Z})$ and

$$a(n) \in \mathbb{R}\backslash\{0\}, \quad b(n) \in \mathbb{R}, \quad n \in \mathbb{Z}. \tag{1.3}$$

If τ is limit point (*l.p.*) at both $\pm\infty$ (cf., e.g., [1], [2]), then τ gives rise to a unique self-adjoint operator H when defined maximally. Otherwise, we need to fix a boundary condition at each endpoint where τ is limit circle (*l.c.*). Throughout this paper we denote by $u_\pm(z,.)$, $z \in \mathbb{C}$, nontrivial solutions of $\tau u = zu$ which satisfy the boundary condition at $\pm\infty$ (if any) with $u_\pm(z,.) \in \ell^2_\pm(\mathbb{Z})$, respectively. The solution $u_\pm(z,.)$ might not exist for $z \in \mathbb{R}$ (cf. Lemma A.1), but if it exists it is unique up to a constant multiple.

In the sequel a solution of $\tau u = \lambda u$, $\lambda \in \mathbb{R}$, will always mean a real-valued, non-zero solution.

Picking $z_0 \in \mathbb{C}\backslash\mathbb{R}$ we can characterize H by

$$H: \begin{array}{ccc} \mathfrak{D}(H) & \to & \ell^2(\mathbb{Z}) \\ f & \mapsto & \tau f \end{array}, \tag{1.4}$$

where the domain of H is explicitly given by

$$\mathfrak{D}(H) = \{f \in \ell^2(\mathbb{Z}) |\ \tau f \in \ell^2(\mathbb{Z}),\ \lim_{n\to+\infty} W_n(u_+(z_0), f) = 0, \lim_{n\to-\infty} W_n(u_-(z_0), f) = 0\} \tag{1.5}$$

and

$$W_n(f,g) = a(n)\Big(f(n)g(n+1) - f(n+1)g(n)\Big) \tag{1.6}$$

denotes the (modified) Wronskian. By $\sigma(.)$, $\sigma_p(.)$, and $\sigma_{ess}(.)$ we denote the spectrum, point spectrum (i.e., the set of eigenvalues), and essential spectrum of an operator, respectively.

Now, having these preliminaries out of the way, we want to give the reader an intuitive idea of how oscillation theory works. We first need to define what we mean by a node of a real-valued sequence $u \in \ell(\mathbb{Z})$. A point $n \in \mathbb{Z}$, is called a node of u if either

$$u(n) = 0 \quad \text{or} \quad a(n)u(n)u(n+1) > 0. \tag{1.7}$$

In the special case $a(n) < 0$, $n \in \mathbb{Z}$ a node of u is precisely a sign flip of u as one would expect. In the general case, however, one has to take the sign of $a(n)$ into account.

For simplicity we shall assume $a(n) < 0$ (cf. Remark 2.2) and a, b bounded (implying H bounded) for the remainder of this section.

By Lemma A.1 $u_-(\lambda, .)$ can be assumed to be continuous with respect to λ as long as λ is below the essential spectrum of H. In addition, $u_-(\lambda, .)$ can be assumed positive for λ below the spectrum of H and hence has no nodes in this case. Increasing λ one needs to observe three things: (i) Nodes of $u_-(\lambda)$ move to the right (by (2.28)) without colliding; (ii) $u_-(\lambda)$ cannot pick up nodes locally (by (2.8)); (iii) $u_-(\lambda)$ cannot lose nodes at $-\infty$. By (i) and (ii) we infer that $u_-(\lambda)$ can only pick up nodes at $+\infty$. Intuitively this happens if $u_-(\lambda) \in \ell^2(\mathbb{Z})$ (or equivalently, if λ an eigenvalue of H) and hence $\lim_{n\to\infty} u_-(\lambda, n) = 0$. Summarizing, $u_-(\lambda)$ has no nodes below the spectrum of H and picks up one additional node whenever λ is an eigenvalue of H. Since no nodes get lost we are lead to (cf. Theorem 3.7)

$$\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H) = \#(u_-(\lambda)), \tag{1.8}$$

where $\#(u)$ denotes the total number of nodes of u and $P_\Omega(H)$ is the spectral projection of H corresponding to the Borel set $\Omega \subseteq \mathbb{R}$. As a corollary we conclude, as already anticipated, that the n-th eigenfunction (below the essential spectrum) has $n - 1$ nodes.

To obtain the number of eigenvalues between two given values λ_1 and λ_2 it seems natural to consider $\#(u_-(\lambda_2)) - \#(u_-(\lambda_1))$. This gives nothing new below the essential spectrum and otherwise we have $\#(u) = \infty$ for any solution of $\tau u = \lambda u$ with λ above the infimum of the essential spectrum. Hence, a naive use of oscillation theory in the latter case yields $\infty - \infty$. There are two ways to overcome this problem. The first, due to [18] in the case of differential operators, uses a limiting procedure which only works for half-line operators and can be found in Theorem 3.10. The second, due to [13] in the case of differential operators, uses the fact that the nodes of the Wronskian of two solutions u_1, u_2 corresponding to λ_1, λ_2, respectively, essentially counts the additional nodes of u_2 with respect to u_1 (cf. Corollary 4.2). In this sense the Wronskian comes with a built-in renormalization. Moreover, the nodes of Wronskians behave similar to the nodes of solutions and satisfy the above properties (i), (ii), and (iii) as well. Hence, similar techniques apply.

To give rigorous proofs for the indicated results, we first introduce and investigate Prüfer variables in Section 2. They will be our main tool in Section 3 and Section 4 where our major theorems are derived. Section 5 uses the results of Section 3 and 4 to investigate the spectra of short-range perturbations of periodic Jacobi operators. The appendix provides some necessary results from the theory of Jacobi operators.

2. Prüfer Variables

For the rest of this paper we assume for convenience

Hypothesis H. 2.1. *Suppose*

$$a, b \in \ell(\mathbb{Z}), \qquad a(n) < 0, b(n) \in \mathbb{R}. \tag{2.1}$$

Remark 2.2. *Introduce $H_\varepsilon = U_\varepsilon H U_\varepsilon^{-1}$ where $U_\varepsilon = U_\varepsilon^{-1}$ is a unitary operator defined via $(U_\varepsilon f)(n) = \tilde{\varepsilon}(n) f(n)$ with $\tilde{\varepsilon}(n) \in \{+1, -1\}$ and $\tilde{\varepsilon}(n)\tilde{\varepsilon}(n+1) = \varepsilon(n)$.*

Then H_ε is associated with the sequences $a_\varepsilon(n) = \varepsilon(n)a(n)$, $b_\varepsilon(n) = b(n)$, $n \in \mathbb{Z}$ and the case $a(n) \neq 0$ can be easily reduced to the case $a(n) < 0$.

In addition, by a solution of $\tau u = \lambda u$, $\lambda \in \mathbb{R}$, we will always mean a real-valued solution not vanishing identically.

Given a solution $u(\lambda, .)$ of $\tau u = \lambda u$, $\lambda \in \mathbb{R}$, we introduce Prüfer variables $\rho_u(\lambda, .), \theta_u(\lambda, .)$ via

$$u(\lambda, n) = \rho_u(\lambda, n) \sin \theta_u(\lambda, n), \tag{2.2}$$

$$u(\lambda, n+1) = \rho_u(\lambda, n) \cos \theta_u(\lambda, n). \tag{2.3}$$

Notice that the Prüfer angle $\theta_u(\lambda, .n)$ is only defined up to an additive integer multiple of 2π (which depends on n).

Inserting (2.2), (2.3) into $(\tau - \lambda)u = 0$ yields

$$a(n) \cot \theta_u(\lambda, n) + a(n-1) \tan \theta_u(\lambda, n-1) = b(n) + \lambda, \tag{2.4}$$

$$\rho_u(\lambda, n) \sin \theta_u(\lambda, n) = \rho_u(\lambda, n-1) \cos \theta_u(\lambda, n-1). \tag{2.5}$$

Equation (2.4) is a discrete Riccati equation (cf. [21]) for $\cot \theta_u(n)$ and (2.5) can be solved if $\theta_u(n)$ is known provided it is replaced by

$$a(n)\rho_u(\lambda, n) = a(n-1)\rho_u(\lambda, n-1) = 0 \tag{2.6}$$

if $\sin \theta_u(\lambda, n) = \cos \theta_u(\lambda, n-1) = 0$ (use $\tau u = \lambda u$ and (2.8) below). The Wronskian of two solutions $u_{1,2}(\lambda_{1,2}, n)$ reads

$$W_n(u_1(\lambda_1), u_2(\lambda_2)) = a(n)\rho_{u_1}(\lambda_1, n)\rho_{u_2}(\lambda_2, n) \sin(\theta_{u_1}(\lambda_1, n) - \theta_{u_2}(\lambda_2, n)). \tag{2.7}$$

The next lemma considers nodes of solutions and their Wronskians more closely (cf. [23], Lemma 6.1).

Lemma 2.3. *Let $u_{1,2}$ be solutions of $\tau u_{1,2} = \lambda_{1,2} u_{1,2}$ corresponding to $\lambda_1 \neq \lambda_2$, respectively. Then*

$$u_1(n) = 0 \quad \Rightarrow \quad u_1(n-1)u_1(n+1) < 0. \tag{2.8}$$

Moreover, suppose $W_n(u_1, u_2) = 0$ but $W_{n-1}(u_1, u_2)W_{n+1}(u_1, u_2) \neq 0$, then

$$W_{n-1}(u_1, u_2)W_{n+1}(u_1, u_2) < 0. \tag{2.9}$$

Otherwise, if $W_n(u_1, u_2) = W_{n+1}(u_1, u_2) = 0$, then necessarily

$$u_1(n+1) = u_2(n+1) = 0, \quad \textit{and} \quad W_{n-1}(u_1, u_2)W_{n+2}(u_1, u_2) < 0. \tag{2.10}$$

Proof. The fact $u_1(n) = 0$ implies $u_1(n-1)u_1(n+1) \neq 0$ (otherwise u_1 vanishes identically) and $a(n)u_1(n+1) = -a(n-1)u_1(n-1)$ (from $\tau u_1 = \lambda u_1$) shows $u_1(n-1)u_1(n+1) < 0$.

Next, $W_n(u_1, u_2) = 0$ is equivalent to $u_1(n) = cu_2(n)$, $u_1(n+1) = cu_2(n+1)$ for some $c \neq 0$ and from (A.6) we infer

$$W_{n+1}(u_1, u_2) - W_n(u_1, u_2) = (\lambda_2 - \lambda_1)u_1(n+1)u_2(n+1). \tag{2.11}$$

Applying the above formula gives

$$W_{n-1}(u_1, u_2)W_{n+1}(u_1, u_2) = -c^2(\lambda_2 - \lambda_1)^2 u_1(n)^2 u_1(n+1)^2 \tag{2.12}$$

proving the first claim. If $W_n(u_1, u_2)$, $W_{n+1}(u_1, u_2)$ are both zero we must have $u_1(n+1) = u_2(n+1) = 0$ and as before $W_{n-1}(u_1, u_2)W_{n+2}(u_1, u_2) = -(\lambda_2 - \lambda_1)^2 u_1(n)u_1(n+2)u_2(n)u_2(n+2)$. Hence the claim follows from the first part. □

We can make the Prüfer angel $\theta_u(\lambda,.)$ unique by fixing, for instance, $\theta_u(\lambda,0)$ and requiring

$$\llbracket\theta_u(\lambda,n)/\pi\rrbracket \le \llbracket\theta_u(\lambda,n+1)/\pi\rrbracket \le \llbracket\theta_u(\lambda,n)/\pi\rrbracket + 1, \tag{2.13}$$

where

$$\llbracket x\rrbracket = \sup\{n\in\mathbb{Z}\,|\,n<x\}. \tag{2.14}$$

Lemma 2.4. *Let $\Omega\subseteq\mathbb{R}$ be an interval. Suppose $u(\lambda,n)$ is continuous with respect to $\lambda\in\Omega$ and (2.13) holds for one $\lambda_0\in\Omega$. Then it holds for all $\lambda\in\Omega$ if we require $\theta_u(.,n)\in C(\Omega)$.*

Proof. Fix n and set

$$\theta_u(\lambda,n) = k\pi + \delta(\lambda), \quad \theta_u(\lambda,n+1) = k\pi + \Delta(\lambda), \quad k\in\mathbb{Z}, \tag{2.15}$$

where $\delta(\lambda)\in(0,\pi]$, $\Delta(\lambda)\in(0,2\pi]$. If (2.13) should break down then by continuity we must have one of the following cases for some $\lambda_1\in\Omega$. (i) $\delta(\lambda_1)=0$ and $\Delta(\lambda_1)\in(\pi,2\pi)$, (ii) $\delta(\lambda_1)=\pi$ and $\Delta(\lambda_1)\in(0,\pi)$, (iii) $\Delta(\lambda_1)=0$ and $\delta(\lambda_1)\in(0,\pi)$, (iv) $\Delta(\lambda_1)=2\pi$ and $\delta(\lambda_1)\in(0,\pi)$. Abbreviate $R=\rho(\lambda_1,n)\rho(\lambda_1,n+1)$. Case (i) implies $0>\sin(\Delta(\lambda_1))=\cos(k\pi)\sin(k\pi+\Delta(\lambda_1))=R^{-1}u(\lambda_1,n+1)^2>0$, contradicting (i). Case (ii) is similar. Case (iii) implies $\delta(\lambda_1)=\pi/2$ and hence $1=\sin(k\pi+\pi/2)\cos(k\pi)=R^{-1}u(\lambda_1,n)u(\lambda_1,n+2)$ contradicting (2.8). Again, case (iv) is similar. □

Let us call a point $n\in\mathbb{Z}$ a node of a solution u if either $u(n)=0$ or $a(n)u(n)u(n+1)>0$. Then, $\llbracket\theta_u(n)/\pi\rrbracket=\llbracket\theta_u(n+1)/\pi\rrbracket$ implies no node at n. Conversely, if $\llbracket\theta_u(n+1)/\pi\rrbracket=\llbracket\theta_u(n)/\pi\rrbracket+1$, then n is a node by (2.8). Denote by $\#(u)$ the total number of nodes of u and by $\#_{(m,n)}(u)$ the number of nodes of u between m and n. More precisely, we shall say that a node n_0 of u lies between m and n if either $m<n_0<n$ or if $n_0=m$ but $u(m)\neq 0$. Hence we conclude

Lemma 2.5. *Let $m<n$. Then we have for any solution u*

$$\#_{(m,n)}(u) = \llbracket\theta_u(n)/\pi\rrbracket - \lim_{\varepsilon\downarrow 0}\llbracket\theta_u(m)/\pi+\varepsilon\rrbracket \tag{2.16}$$

and

$$\#(u) = \lim_{n\to\infty}\Big(\llbracket\theta_u(n)/\pi\rrbracket - \llbracket\theta_u(-n)/\pi\rrbracket\Big). \tag{2.17}$$

Next, we recall the well-known analog of Sturm's theorem for differential equations and include a proof for the sake of completeness (cf., e.g., [1], [23], Theorem 6.5).

Lemma 2.6. *Let $u_{1,2}$ be solutions of $\tau u=\lambda u$ corresponding to $\lambda_1\le\lambda_2$. Suppose $m<n$ are two consecutive points which are either nodes of u_1 or zeros of $W_.(u_1,u_2)$ (the cases $m=-\infty$ or $n=+\infty$ are allowed if u_1 and u_2 are both in $\ell^2_\pm(\mathbb{Z})$ and $W_{\pm\infty}(u_1,u_2)=0$ respectively) such that u_1 has no further node between m and n. Then u_2 has at least one node between m and $n+1$. Moreover, suppose $m_1<\cdots<m_k$ are consecutive nodes of u_1. Then u_2 has at least $k-1$ nodes between m_1 and m_k. Hence we even have*

$$\#_{(m,n)}(u_2) \ge \#_{(m,n)}(u_1) - 1. \tag{2.18}$$

Proof. Suppose u_2 has no node between m and $n+1$. Hence we may assume (perhaps after flipping signs) that $u_1(j) > 0$ for $m < j < n$, $u_1(n) \geq 0$, and $u_2(j) > 0$ for $m \leq j \leq n$. Moreover, $u_1(m) \leq 0$, $u_1(n+1) < 0$ and $u_2(n+1) \geq 0$ provided m, n are finite. By Green's formula (A.6)

$$0 \leq (\lambda_2 - \lambda_1) \sum_{j=m+1}^{n} u_1(j) u_2(j) = W_n(u_1, u_2) - W_m(u_1, u_2). \tag{2.19}$$

Evaluating the Wronskians shows $W_n(u_1, u_2) < 0$, $W_m(u_1, u_2) > 0$, which is a contradiction.

It remains to prove the last part. We will use induction on k. The case $k = 1$ is trivial and $k = 2$ has already been proven. Denote the nodes of u_2 lower or equal than m_{k+1} by $n_k > n_{k-1} > \cdots$. If $n_k > m_k$ we are done since there are $k-1$ nodes n such that $m_1 \leq n \leq m_k$ by induction hypothesis. Otherwise we can find k_0, $0 \leq k_0 \leq k$ such that $m_j = n_j$ for $1 + k_0 \leq j \leq k$. If $k_0 = 0$ we are clearly done and we can suppose $k_0 \geq 1$. By induction hypothesis it suffices to show that there are $k - k_0$ nodes n of u_2 with $m_{k_0} \leq n \leq m_{k+1}$. By assumption $m_j = n_j$, $1 + k_0 \leq j \leq k$ are the only nodes n of u_2 such that $m_{k_0} \leq n \leq m_{k+1}$. Abbreviate $m = m_{k_0}$, $n = m_{k+1}$ and assume without restriction $u_1(m+1) > 0$, $u_2(m) > 0$. Since the nodes of u_1 and u_2 coincide we infer $0 < \sum_{j=m+1}^{n} u_1(j) u_2(j)$ and we can proceed as in the first part to obtain a contradiction. □

We call τ oscillatory if one solution of $\tau u = 0$ has an infinite number of nodes. In addition, we call τ oscillatory at $\pm\infty$ if one solution of $\tau u = 0$ has an infinite number of nodes near $\pm\infty$. We remark that if one solution of $(\tau - \lambda) u = 0$ has infinitely many nodes so has any other (corresponding to the same λ) by (2.18). Furthermore, $\tau - \lambda_1$ oscillatory implies $\tau - \lambda_2$ oscillatory for all $\lambda_2 > \lambda_1$ (again by (2.18)).

Now we turn to the special solution $s(\lambda, n)$ characterized via the initial conditions $s(\lambda, 0) = 0$, $s(\lambda, 1) = 1$. As in Lemma A.3 we infer

$$W_n(s(\lambda), \dot{s}(\lambda)) = \sum_{j=n+1}^{0} s(\lambda, j)^2, \quad n < -1, \tag{2.20}$$

$$W_n(s(\lambda), \dot{s}(\lambda)) = \sum_{j=1}^{n} s(\lambda, j)^2, \quad n \geq 1. \tag{2.21}$$

Here the dot denotes the derivative with respect to λ. Notice also $W_{-1}(s(\lambda), \dot{s}(\lambda)) = W_0(s(\lambda), \dot{s}(\lambda)) = 0$. Evaluating the above equation using Prüfer variables shows

$$\dot{\theta}_s(\lambda, n) = \frac{\sum_{j=1}^{n} s(\lambda, j)^2}{-a(n) \rho_s(\lambda, n)^2} > 0, \quad n \geq 1, \tag{2.22}$$

$$\dot{\theta}_s(\lambda, n) = \frac{\sum_{j=n+1}^{0} s(\lambda, j)^2}{a(n) \rho_s(\lambda, n)^2} < 0, \quad n < -1. \tag{2.23}$$

Notice, again that $\dot{\theta}_s(\lambda, -1) = \dot{\theta}_s(\lambda, 0) = 0$. Equation (2.22) implies that nodes of $s(\lambda, n)$ for $n \in \mathbb{N}$ move monotonically to the left without colliding (cf., [1] Theorem 4.3.4). In addition, since $s(\lambda, n)$ cannot pick up nodes locally by (2.8), all nodes must enter at ∞ and since $\dot{\theta}_s(\lambda, 0) = 0$ they are trapped inside $(0, \infty)$.

We shall normalize $\theta_s(\lambda,0)=0$ implying $\theta_s(\lambda,-1)=-\pi/2$. Since $s(\lambda,n)$ is a polynomial in λ we easily infer $s(\lambda,n)\gtrless 0$ for fixed $n\gtrless 0$ and λ sufficiently small. This implies

$$-\pi<\theta_s(\lambda,n)<-\pi/2,\ n<-1,\quad 0<\theta_s(\lambda,n)<\pi,\ n\geq 1, \tag{2.24}$$

for fixed n and λ sufficiently small. Moreover, dividing (2.4) by λ and letting $\lambda\to-\infty$ using (2.24) shows

$$\lim_{\lambda\to\pm\infty}\frac{\cot(\theta_s(\lambda,n))^{\pm 1}}{\lambda}=\frac{1}{a(n)},\quad n\begin{smallmatrix}\geq +1\\ <-1\end{smallmatrix} \tag{2.25}$$

and hence

$$\theta_s(\lambda,n)=-\frac{\pi}{2}-\frac{a(n)}{\lambda}+o(\frac{1}{\lambda}),\ n<-1,\quad \theta_s(\lambda,n)=\frac{a(n)}{\lambda}+o(\frac{1}{\lambda}),\ n\geq 1, \tag{2.26}$$

as $\lambda\to-\infty$.

Analogously, let $u_\pm(\lambda,n)$ be solutions of $\tau u=\lambda u$ as in Lemma A.1. Then Lemma A.3 implies

$$\dot\theta_+(\lambda,n) = \frac{\sum_{j=n+1}^{\infty}u_+(\lambda,j)^2}{a(n)\rho_+(\lambda,n)^2}<0, \tag{2.27}$$

$$\dot\theta_-(\lambda,n) = \frac{\sum_{j=-\infty}^{n}u_-(\lambda,j)^2}{-a(n)\rho_-(\lambda,n)^2}>0, \tag{2.28}$$

where we have abbreviated $\rho_{u_\pm}=\rho_\pm$, $\theta_{u_\pm}=\theta_\pm$.

If H is bounded from below we can normalize

$$0<\theta_\mp(\lambda,n)<\pi/2,\quad n\in\mathbb{Z},\quad \lambda<\inf\sigma(H) \tag{2.29}$$

and we get as before

$$\theta_-(\lambda,n)=\frac{a(n)}{\lambda}+o(\frac{1}{\lambda}),\quad \theta_+(\lambda,n)=\frac{\pi}{2}-\frac{a(n)}{\lambda}+o(\frac{1}{\lambda}),\quad n\in\mathbb{Z} \tag{2.30}$$

as $\lambda\to-\infty$.

3. Standard Oscillation Theory

First of all we recall ([13], Lemma 5.2).

Lemma 3.1. *Let H, H_n be self-adjoint operators and $H_n\to H$ in strong resolvent sense as $n\to\infty$. Then*

$$\dim\operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H)\leq\liminf_{n\to\infty}\dim\operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H_n). \tag{3.1}$$

Our first theorem considers half-line operators $H_\pm$ associated with a Dirichlet boundary condition at $n=0$, that is, the following restrictions of H to the subspaces $\ell^2(\pm\mathbb{N})$,

$$\begin{array}{rcl} H_\pm:\ \mathfrak{D}(H_\pm) & \to & \ell^2(\pm\mathbb{N}) \\ f(n) & \mapsto & \begin{cases} a(\begin{smallmatrix}+1\\-2\end{smallmatrix})f(\pm 2)-b(\pm 1)f(\pm 1), & n=\pm 1\\ (\tau f)(n), & n\gtrless\pm 1\end{cases}\end{array}, \tag{3.2}$$

with

$$\mathfrak{D}(H_\pm)=\{f\in\ell^2(\pm\mathbb{N})|\tau f\in\ell^2(\pm\mathbb{N}),\ \lim_{n\to\pm\infty}W_n(u_\pm(z_0),f)=0\}. \tag{3.3}$$

Similarly one defines finite restriction H_{n_1,n_2} to the subspaces $\ell^2(n_1,n_2)$ with Dirichlet boundary conditions at $n = n_1$ and $n = n_2$.

Remark 3.2. *We only consider the case of a Dirichlet boundary condition at $n = 0$ since the operators $H^\beta_{\pm,n_0}$ on $\ell^2(n_0,\pm\infty)$ associated with the general boundary condition*

$$f(n_0+1) + \beta f(n_0) = 0, \qquad \beta \in \mathbb{R} \cup \{\infty\} \tag{3.4}$$

at $n = n_0$ can be reduced to this case by a simple shift and altering the sequence b at one point. More precisely, we have

$$H^0_{+,n_0} = H_{+,n_0+1}, \quad H^\beta_{+,n_0} = H_{+,n_0} - a(n_0)\beta^{-1}\langle\delta_{n_0+1},.\rangle\delta_{n_0+1}, \quad \beta \neq 0, \tag{3.5}$$

and

$$H^\infty_{-,n_0} = H_{-,n_0}, \quad H^\beta_{-,n_0} = H_{-,n_0+1} - a(n_0)\beta\langle\delta_{n_0},.\rangle\delta_{n_0}, \quad \beta \neq \infty, \tag{3.6}$$

where $\delta_{n_0}(n) = 1$ if $n = n_0$ and $\delta_{n_0}(n) = 0$ otherwise. Hence all one has to do is alter the definition of $b(n_0)$ or $b(n_0+1)$. Analogously one defines the corresponding finite operators $H^{\beta_1,\beta_2}_{n_1,n_2}$ which will be used in the next section.

Theorem 3.3. *Let $\lambda \in \mathbb{R}$. Suppose τ is l.p. at $+\infty$ or $\lambda \in \sigma_p(H_+)$. Then*

$$\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_+) = \#_{(0,+\infty)}(s(\lambda)). \tag{3.7}$$

The same theorem holds if $+$ is replaced by $-$.

Proof. We only carry out the proof for the plus sign (the other part following from reflection). By virtue of (2.22), (2.26), and Lemma 2.5 we infer

$$\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_{0,n}) = [\![\theta_s(\lambda,n)/\pi]\!] = \#_{(0,n)}(s(\lambda)), \quad n > 1, \tag{3.8}$$

since $\lambda \in \sigma(H_{0,n})$ if and only if $\theta_s(\lambda,n) = 0 \bmod \pi$. Let $k = \#(s(\lambda))$ if $\#(s(\lambda)) < \infty$, otherwise the following argument works for arbitrary $k \in \mathbb{N}$. If we pick n so large that k nodes of $s(\lambda)$ are to the left of n we have k eigenvalues $\hat\lambda_1 < \cdots < \hat\lambda_k < \lambda$ of $H_{0,n}$. Taking an arbitrary linear combination $\eta(m) = \sum_{j=1}^k c_j s(\hat\lambda_j, m)$, $c_j \in \mathbb{C}$ for $m < n$ and $\eta(m) = 0$ for $m \geq n$ a straightforward calculation (using orthogonality of $s(\hat\lambda_j)$) yields

$$\langle\eta, H_+\eta\rangle < \lambda\|\eta\|^2. \tag{3.9}$$

Invoking the spectral theorem shows

$$\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_\pm) \geq k. \tag{3.10}$$

For the reversed inequality we can assume $k = \#(s(\lambda)) < \infty$.

We first suppose τ is *l.p.* at $+\infty$. Consider $\tilde H_{0,n} = H_{0,n} \oplus \lambda\mathbb{1}$ on $\ell^2(0,n) \oplus \ell^2(n-1,\infty)$. Then Theorem 9.16.(i) in [33] (take $\ell_0(\mathbb{Z})$ as a core) implies strong resolvent convergence of $\tilde H_{0,n}$ to H_+ as $n \to \infty$ and by Lemma 3.1 we have

$$\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_+) \leq \lim_{n\to\infty} \dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_{0,n}) = k \tag{3.11}$$

completing the proof if τ is *l.p.* at $+\infty$.

Otherwise, that is, if τ is *l.c.* at $+\infty$ (implying that the spectrum of H_+ is purely discrete), λ is an eigenvalue by hypothesis. We first suppose H bounded from below. Hence it suffices to show that the n-th eigenvalue λ_n, $n \in \mathbb{N}$ has at least $n-1$ nodes. This is trivial for $n = 1$. Suppose this is true for λ_n and let m be the largest node of $s(\lambda_n)$. By $\theta_s(\lambda_{n+1},m) > \theta_s(\lambda_n,m)$ we infer that $\theta_s(\lambda_{n+1},m)$

has either more nodes between 0 and m or there is at least one additional node of $\theta_s(\lambda_{n+1}, m)$ larger than m by Lemma 2.6. In the case where H is not bounded from below we can label the eigenvalues λ_n, $n \in \mathbb{Z}$. The same argument as before shows that the eigenfunction corresponding to λ_m has $|m-n|$ nodes more than the one corresponding to λ_n. Letting $m \to -\infty$ shows that the eigenfunction corresponding to λ_n has infinitely many nodes. This completes the proof. □

Remark 3.4. *(i) The l.p. / $\lambda \in \sigma_p(H_+)$ assumption is crucial since we need some information about the boundary condition at $+\infty$.*
(ii) Remark 3.2 implies the following. Let $\lambda \in \mathbb{R}$. Suppose τ is l.p. at $+\infty$ or $\lambda \in \sigma_p(H^{\beta}_{+,n_0})$ and $\beta \neq 0$. Then

$$\dim \mathrm{Ran}\, P_{(-\infty,\lambda)}(H^{\beta}_{+,n_0}) = \#_{(0,+\infty)}(s_\beta(\lambda, ., n_0)), \tag{3.12}$$

where $s_\beta(\lambda, ., n_0)$ is a sequence satisfying $\tau s = \lambda s$ and the boundary condition (3.4). Similar modifications apply to Theorems 3.10, 4.3, and 4.4 below.

As a consequence of Theorem 3.3 we infer

Corollary 3.5. *We have*

$$\dim \mathrm{Ran}\, P_{(-\infty,\lambda)}(H_\pm) < \infty \tag{3.13}$$

if and only if $\tau - \lambda$ is non-oscillatory near $\pm\infty$, respectively, and hence

$$\inf \sigma_{ess}(H_\pm) = \inf\{\lambda \in \mathbb{R} \,|\, (\tau - \lambda) \text{ is oscillatory at } \pm\infty\}. \tag{3.14}$$

Moreover, let $H_\pm$ be bounded from below and $\lambda_1 < \cdots < \lambda_k < \cdots$ be the eigenvalues of $H_\pm$ below the essential spectrum of $H_\pm$. Then the eigenfunction corresponding to λ_k has precisely $k-1$ nodes inside $(0, \pm\infty)$.

We remark that the first part of Corollary 3.5 can be found in [14], Theorem 32 (see also [20]).

Remark 3.6. *Consider the following example*

$$a(n) = -\frac{1}{2},\ n \in \mathbb{N}, \quad b(1) = 1, b(2) = b_2, b(3) = \frac{1}{2}, b(n) = 0,\ n \geq 4. \tag{3.15}$$

The essential spectrum of H_+ is given by $\sigma_{ess}(H_+) = [-1, 1]$ and one might expect that H_+ has no eigenvalues below the essential spectrum if $b_2 \to -\infty$. However, since we have

$$s(-1, 0) = 0, s(-1, 1) = 1, s(-1, 2) = 0, s(-1, n) = -1,\ n \geq 3, \tag{3.16}$$

Theorem 3.3 shows that, independent of $b_2 \in \mathbb{R}$, there is always precisely one eigenvalue below the essential spectrum.

In a similar way we obtain

Theorem 3.7. *Let $\lambda < \inf \sigma_{ess}(H)$. Suppose τ is l.p. at $-\infty$ or $\lambda \in \sigma_p(H)$. Then*

$$\dim \mathrm{Ran}\, P_{(-\infty,\lambda)}(H) = \#(u_+(\lambda)). \tag{3.17}$$

The same theorem holds if l.p. at $-\infty$ and $u_+(\lambda)$ is replaced by l.p. at $+\infty$ and $u_-(\lambda)$.

Proof. Again it suffices to prove the minus case. If H is not bounded from below the same is true for $H_- \oplus H_+$ (which can be embedded into $\ell^2(\mathbb{Z})$ and considered as a finite rank perturbation of H). Hence H_- or H_+ (or both) is not bounded from below implying $\tau - \lambda$ oscillatory near $-\infty$ or $+\infty$ by Corollary 3.5 and we can suppose H bounded from below.

By virtue of (2.28) and (2.30) we infer

$$\dim \mathrm{Ran}\ P_{(-\infty,\lambda)}(H_{-,n}) = [\![\theta_-(\lambda,n)/\pi]\!], \quad n \in \mathbb{Z}. \tag{3.18}$$

We first want to show $[\![\theta_-(\lambda,n)/\pi]\!] = \#_{(-\infty,n)}(u_-(\lambda))$ or equivalently

$$\lim_{n\to\infty} [\![\theta_-(\lambda,n)/\pi]\!] = 0. \tag{3.19}$$

Suppose $\lim_{n\to\infty}[\![\theta_-(\lambda_1,n)/\pi]\!] = k \geq 1$ for some $\lambda_1 \in \mathbb{R}$ (saying that $u_-(.,n)$ loses at least one node at $-\infty$). In this case we can find n such that $\theta_-(\lambda_1,n) > k\pi$ for $m \geq n$. Now pick λ_0 such that $\theta_-(\lambda_0,n) = k\pi$. Then $u_-(\lambda_0,.)$ has a node at n but no node between $-\infty$ and n (by Lemma 2.5). Now apply Lemma 2.6 to $u_-(\lambda_0,.)$, $u_-(\lambda_1,.)$ to obtain a contradiction. The rest follows as in the proof of Theorem 3.3. □

As before we obtain

Corollary 3.8. *We have*

$$\dim \mathrm{Ran}\ P_{(-\infty,\lambda)}(H) < \infty \tag{3.20}$$

if and only if $\tau - \lambda$ is non-oscillatory and hence

$$\inf \sigma_{ess}(H) = \inf\{\lambda \in \mathbb{R} \,|\, (\tau-\lambda) \text{ is oscillatory}\}. \tag{3.21}$$

Furthermore, let H be bounded from below and $\lambda_1 < \cdots < \lambda_k < \dots$ be the eigenvalues of H below the essential spectrum of H. Then the eigenfunction corresponding to λ_k has precisely $k-1$ nodes.

Remark 3.9. *Corresponding results for the projection $P_{(\lambda,\infty)}(H)$ can be obtained from $P_{(\lambda,\infty)}(H) = P_{(-\infty,-\lambda)}(-H)$. In fact, it suffices to change the definition of a node according to $u(n) = 0$ or $a(n)u(n)u(n+1) < 0$ and $P_{(-\infty,\lambda)}(H)$ to $P_{(\lambda,\infty)}(H)$ in all results of this section.*

Now we turn to the analog of [18], Theorem I.

Theorem 3.10. *Let $\lambda_1 < \lambda_2$. Suppose $\tau - \lambda_2$ is oscillatory near $+\infty$ and τ is l.p. at $+\infty$. Then*

$$\dim \mathrm{Ran}\ P_{(\lambda_1,\lambda_2)}(H_+) = \liminf_{n\to+\infty} \Big(\#_{(0,n)}(s(\lambda_2)) - \#_{(0,n)}(s(\lambda_1))\Big). \tag{3.22}$$

The same theorem holds if $+$ is replaced by $-$.

Proof. As before we only carry out the proof for the plus sign. Abbreviate $\Delta(n) = [\![\theta_s(\lambda_2,n)/\pi]\!] - [\![\theta_s(\lambda_1,n)/\pi]\!] = \#_{(0,n)}(s(\lambda_2)) - \#_{(0,n)}(s(\lambda_1))$. By (3.8) we infer

$$\dim \mathrm{Ran}\ P_{[\lambda_1,\lambda_2)}(H_{0,n}) = \Delta(n), \quad n > 2. \tag{3.23}$$

Let $k = \liminf \Delta(n)$ if $\limsup \Delta(n) < \infty$ and $k \in \mathbb{N}$ otherwise. We claim that there exists a $n \in \mathbb{N}$ such that

$$\dim \mathrm{Ran}\ P_{(\lambda_1,\lambda_2)}(H_{0,n}) \geq k. \tag{3.24}$$

In fact, if $k = \limsup \Delta(n) < \infty$ it follows that $\Delta(n)$ is eventually equal to k and since $\lambda_1 \notin \sigma(H_{0,m}) \cap \sigma(H_{0,m+1})$, $m \in \mathbb{N}$ we are done in this case. Otherwise we can pick n such that $\dim \mathrm{Ran}\, P_{[\lambda_1,\lambda_2)}(H_{0,n}) \ge k+1$. Hence $H_{0,n}$ has at least k eigenvalues $\hat\lambda_j$ with $\lambda_1 < \hat\lambda_1 < \cdots < \hat\lambda_k < \lambda_2$. Again let $\eta(m) = \sum_{j=1}^k c_j s(\hat\lambda_j, n)$, $c_j \in \mathbb{C}$ for $m < n$ and $\eta(m) = 0$ for $n \ge m$ be an arbitrary linear combination. Then

$$\|(H_+ - \frac{\lambda_2+\lambda_1}{2})\eta\| < \frac{\lambda_2-\lambda_1}{2}\|\eta\| \tag{3.25}$$

together with the spectral theorem implies

$$\dim \mathrm{Ran}\, P_{(\lambda_1,\lambda_2)}(H_+) \ge k. \tag{3.26}$$

To prove the second inequality we use that $\tilde H_{0,n} = H_{0,n} \oplus \lambda_2 \mathbb{1}$ converges to H_+ in strong resolvent sense as $n\to\infty$ and proceed as before

$$\dim \mathrm{Ran}\, P_{(\lambda_1,\lambda_2)}(H_+) \le \liminf_{n\to\infty} P_{[\lambda_1,\lambda_2)}(\tilde H_{0,n}) = k \tag{3.27}$$

since $P_{[\lambda_1,\lambda_2)}(\tilde H_{0,n}) = P_{[\lambda_1,\lambda_2)}(H_{0,n})$. □

4. Renormalized Oscillation Theory

The objective of this section is to look at the nodes of the Wronskian of two solutions $u_{1,2}$ corresponding to $\lambda_{1,2}$, respectively. We call $n \in \mathbb{Z}$ a node of the Wronskian if $W_n(u_1,u_2) = 0$ and $W_{n+1}(u_1,u_2) \ne 0$ or if $W_n(u_1,u_2)W_{n+1}(u_1,u_2) < 0$. Again we shall say that a node n_0 of $W(u_1,u_2)$ lies between m and n if either $m < n_0 < n$ or if $n_0 = m$ but $W_{n_0}(u_1,u_2) \ne 0$. We abbreviate

$$\Delta_{u_1,u_2}(n) = (\theta_{u_2}(n) - \theta_{u_1}(n)) \bmod 2\pi. \tag{4.1}$$

and require

$$[\![\Delta_{u_1,u_2}(n)/\pi]\!] \le [\![\Delta_{u_1,u_2}(n+1)/\pi]\!] \le [\![\Delta_{u_1,u_2}(n)/\pi]\!] + 1. \tag{4.2}$$

We shall fix $\lambda_1 \in \mathbb{R}$ and a corresponding solution u_1 and choose a second solution $u(\lambda,n)$ with $\lambda \in [\lambda_1,\lambda_2]$. Now let us consider

$$W_n(u_1,u(\lambda)) = -a(n)\rho_{u_1}(n)\rho_u(\lambda,n)\sin(\Delta_{u_1,u}(\lambda,n)) \tag{4.3}$$

as a function of $\lambda \in [\lambda_1,\lambda_2]$.

Lemma 4.1. *Suppose $\Delta_{u_1,u}(\lambda_1,.)$ satisfies (4.2) then we have*

$$\Delta_{u_1,u}(\lambda,n) = \theta_u(\lambda,n) - \theta_{u_1}(n) \tag{4.4}$$

where $\theta_u(\lambda,.)$, $\theta_{u_1}(.)$ both satisfy (2.13). That is, $\Delta_{u_1,u}(.,n) \in C[\lambda_1,\lambda_2]$ and (4.2) holds for all $\Delta_{u_1,u}(\lambda,.)$ with $\lambda \in [\lambda_1,\lambda_2]$. In particular, the second inequality in (2.13) is attained if and only if n is a node of $W_.(u_1,u(\lambda))$. Moreover, denote by $\#_{(m,n)}W(u_1,u_2)$ the total number of nodes of $W_.(u_1,u_2)$ between m and n. Then

$$\#_{(m,n)}W(u_1,u_2) = [\![\Delta_{u_1,u_2}(n)/\pi]\!] - \lim_{\varepsilon\downarrow 0}[\![\Delta_{u_1,u_2}(m)/\pi + \varepsilon]\!] \tag{4.5}$$

and

$$\#W(u_1,u_2) = \#_{(-\infty,\infty)}W(u_1,u_2) = \lim_{n\to\infty}\Big([\![\Delta_{u_1,u_2}(n)/\pi]\!] - [\![\Delta_{u_1,u_2}(-n)/\pi]\!]\Big). \tag{4.6}$$

Proof. We fix n and set

$$\Delta_{u_1,u}(\lambda,n) = k\pi + \delta(\lambda), \quad \Delta_{u_1,u}(\lambda,n+1) = k\pi + \Delta(\lambda), \tag{4.7}$$

where $k \in \mathbb{Z}, \delta(\lambda_1) \in (0,\pi]$ and $\Delta(\lambda_1) \in (0,2\pi]$. Clearly (4.4) holds for $\lambda = \lambda_1$ since $W_.(u_1, u(\lambda_1))$ is constant. If (4.2) should break down we must have one of the following cases for some $\lambda_0 \ge \lambda_1$. (i) $\delta(\lambda_0) = 0$, $\Delta(\lambda_0) \in (\pi, 2\pi]$, or (ii) $\delta(\lambda_0) = \pi$, $\Delta(\lambda_0) \in (0,\pi]$, or (iii) $\Delta(\lambda_0) = 2\pi$, $\delta(\lambda_0) \in (\pi,\pi]$, or (iv) $\Delta(\lambda_0) = 0$, $\delta(\lambda_0) \in (\pi,\pi]$. For notational convenience let us set $\delta = \delta(\lambda_0), \Delta = \Delta(\lambda_0)$ and $\theta_{u_1}(n) = \theta_1(n), \theta_u(\lambda_0,n) = \theta_2(n)$. Furthermore, we can assume $\theta_{1,2}(n) = k_{1,2}\pi + \delta_{1,2}$, $\theta_{1,2}(n+1) = k_{1,2}\pi + \Delta_{1,2}$ with $k_{1,2} \in \mathbb{Z}, \delta_{1,2} \in (0,\pi]$ and $\Delta_{1,2} \in (0,2\pi]$.

Suppose (i). Then

$$W_{n+1}(u_1, u(\lambda_0)) = (\lambda_0 - \lambda_1) u_1(n+1) u(\lambda_0, n+1). \tag{4.8}$$

Inserting Prüfer variables shows

$$\sin(\Delta_2 - \Delta_1) = \rho \cos^2(\delta_1) \ge 0 \tag{4.9}$$

for some $\rho > 0$ since $\delta = 0$ implies $\delta_1 = \delta_2$. Moreover, $k = (k_2 - k_1) \bmod 2$ and $k\pi + \Delta = (k_2 - k_1)\pi + \Delta_2 - \Delta_1$ implies $\Delta = (\Delta_2 - \Delta_1) \bmod 2\pi$. Hence we have $\sin\Delta \ge 0$ and $\Delta \in (\pi, 2\pi]$ implies $\Delta = 2\pi$. But this says $\delta_1 = \delta_2 = \pi/2$ and $\Delta_1 = \Delta_2 = \pi$. Since we have at least $\delta(\lambda_2 - \varepsilon) > 0$ and hence $\delta_2(\lambda_2 - \varepsilon) > \pi/2$, $\Delta_2(\lambda_2 - \varepsilon) > \pi$ for $\varepsilon > 0$ sufficiently small. Thus from $\Delta(\lambda_2 - \varepsilon) \in (\pi, 2\pi)$ we get

$$0 > \sin\Delta(\lambda_2 - \varepsilon) = \sin(\Delta_2(\lambda_2 - \varepsilon) - \pi) > 0, \tag{4.10}$$

contradicting (i).

Suppose (ii). Again by (4.8) we have $\sin(\Delta_2 - \Delta_1) \ge 0$ since $\delta_1 = \delta_2$. But now $(k+1) = (k_1 - k_2) \bmod 2$. Furthermore, $\sin(\Delta_2 - \Delta_1) = -\sin(\Delta) \ge 0$ says $\Delta = \pi$ since $\Delta \in (0,\pi]$. Again this implies $\delta_1 = \delta_2 = \pi/2$ and $\Delta_1 = \Delta_2 = \pi$. But since $\delta(\lambda)$ increases/decreases precisely if $\Delta(\lambda)$ increases/decreases for λ near λ_0 (4.2) stays valid.

Suppose (iii) or (iv). Then

$$W_n(u_1, u(\lambda_0)) = -(\lambda_0 - \lambda_1) u_1(n+1) u(\lambda_0, n+1). \tag{4.11}$$

Inserting Prüfer variables gives

$$\sin(\delta_2 - \delta_1) = -\rho \sin(\Delta_1)\sin(\Delta_2) \tag{4.12}$$

for some $\rho > 0$. We first assume $\delta_2 > \delta_1$. In this case we infer $k = (k_2 - k_1) \bmod 2$ implying $\Delta_2 - \Delta_1 = 0 \bmod 2\pi$ contradicting (4.12). Next assume $\delta_2 \le \delta_1$. Then we obtain $(k+1) = (k_2 - k_1) \bmod 2$ implying $\Delta_2 - \Delta_1 = \pi \bmod 2\pi$ and hence $\sin(\delta_2 - \delta_1) \ge 0$ from (4.12). Thus we get $\delta_1 = \delta_2 = \pi/2$ $\Delta_1 = \Delta_2 = \pi$, and hence $\Delta_2 - \Delta_1 = 0 \bmod 2\pi$ contradicting (iii), (iv). This settles (4.4).

Furthermore, if $\Delta(\lambda) \in (0,\pi]$ we have no node at n since $\delta(\lambda) = \pi$ implies $\Delta(\lambda) = \pi$ by (ii). Conversely, if $\Delta(\lambda) \in (\pi, 2\pi]$ we have a node at n since $\Delta(\lambda) = 2\pi$ is impossible by (iii). The rest being straightforward. □

Equations (2.16), (4.4), and (4.5) imply

Corollary 4.2. *Let $\lambda_1 \le \lambda_2$ and suppose $u_{1,2}$ satisfy $\tau u_{1,2} = \lambda_{1,2} u_{1,2}$, respectively. Then we have*

$$|\#_{(n,m)} W(u_1,u_2) - (\#_{(n,m)}(u_2) - \#_{(n,m)}(u_1))| \le 2 \tag{4.13}$$

Now we come to a renormalized version of Theorem 3.10. We first need the result for a finite interval.

Theorem 4.3. *Fix* $n_1 < n_2$ *and* $\lambda_1 < \lambda_2$. *Then*

$$\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H_{n_1,n_2}) = \#_{(n_1,n_2)} W(s(\lambda_1,.,n_1), s(\lambda_2,.,n_2)). \tag{4.14}$$

Proof. We abbreviate

$$\Delta(\lambda,n) = \Delta_{s(\lambda_1,.,n_1),s(\lambda,.,n_2)}(n) \tag{4.15}$$

and normalize (perhaps after flipping the sign of $s(\lambda_1,.,n_1)$) $\Delta(\lambda_1,n) \in (0,\pi]$. From (2.22) we infer

$$\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H_{n_1,n_2}) = -\lim_{\varepsilon\downarrow 0} [\![\Delta(\lambda_2,n_1)/\pi + \varepsilon]\!] \tag{4.16}$$

since $\lambda \in \sigma(H_{n_1,n_2})$ is equivalent to $\Delta(\lambda,n_1) = 0 \bmod \pi$. Using (4.5) completes the proof. □

Theorem 4.4. *Fix* $\lambda_1 < \lambda_2$ *and suppose* τ *is in the l.p. case near* $+\infty$ *or* $\lambda_2 \in \sigma_p(H_+)$. *Then*

$$\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H_+) = \#_{(0,+\infty)} W(s(\lambda_1), s(\lambda_2)). \tag{4.17}$$

The same theorem holds if $+$ *is replaced by* $-$.

Proof. Again we only prove the result for H_+ and set $k = \#_{(0,\infty)}W(s(\lambda_1),s(\lambda_2))$ provided this number is finite and $k \in \mathbb{N}$ otherwise. We abbreviate

$$\Delta(\lambda,n) = \Delta_{s(\lambda_1),s(\lambda)}(n) \tag{4.18}$$

and normalize $\Delta(\lambda_1,n) = 0$ implying $\Delta(\lambda,n) > 0$ for $\lambda > \lambda_1$. Hence if we chose n so large that all k nodes are to the left of n we have

$$\Delta(\lambda,n) > k\pi. \tag{4.19}$$

Thus we can find $\lambda_1 < \hat\lambda_1 < \cdots < \hat\lambda_k < \lambda_2$ with $\Delta(\hat\lambda_j,n) = j\pi$. Now define

$$\eta_j(m) = \begin{cases} s(\hat\lambda_j,m) - \rho_j s(\lambda_1,m) & m \le n \\ 0 & m \ge n \end{cases}, \tag{4.20}$$

where $\rho_j \neq 0$ is chosen such that $s(\hat\lambda_j,m) = \rho_j s(\lambda_1,m)$ for $m = n, n+1$. Furthermore observe that

$$\tau\eta_j(m) = \begin{cases} \hat\lambda_j s(\hat\lambda_j,m) - \lambda_1 \rho_1 s(\lambda_1,m) & m \le n \\ 0 & m \ge n \end{cases} \tag{4.21}$$

and that $s(\lambda_1,m)$, $s(\hat\lambda_j,.)$, $1 \le j \le k$ are orthogonal on $1,\dots,n$. Next, let $\eta = \sum_{j=1}^k c_j\eta_j$, $c_j \in \mathbb{C}$ be an arbitrary linear combination, then a short calculation verifies

$$\|(H_+ - \frac{\lambda_2+\lambda_1}{2})\eta\| < \frac{\lambda_2-\lambda_1}{2}\|\eta\|. \tag{4.22}$$

And invoking the spectral theorem gives

$$\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H_+) \ge k. \tag{4.23}$$

To prove the reversed inequality is only necessary if $\#_{(0,\infty)}W(s(\lambda_1),s(\lambda_2)) < \infty$. In this case we look at $H_{0,n}^{\infty,\beta}$ with $\beta = s(\lambda_2,n+1)/s(\lambda_2,n)$. By Theorem 4.3 and Remark 3.4 (ii) we have

$$\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(\tilde H_{0,n}^{\infty,\beta}) = \#_{(0,n)} W(s(\lambda_1), s(\lambda_2)). \tag{4.24}$$

Now use strong resolvent convergence of $\tilde{H}_{0,n}^{\infty,\beta} = H_{0,n}^{\infty,\beta} \oplus \lambda_1 \mathbb{1}$ to H_+ (due to our *l.p.* / $\lambda_2 \in \sigma_p(H_+)$ assumption) as $n \to \infty$ to obtain

$$\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H_+) \le \liminf_{n\to\infty} \dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(\tilde{H}_{0,n}^{\infty,\beta}) = k \tag{4.25}$$

completing the proof. □

As a consequence we infer.

Corollary 4.5. *Let $u_{1,2}$ satisfy $\tau u_{1,2} = \lambda_{1,2} u_{1,2}$. Then*

$$\#_{(0,\pm\infty)} W(u_1, u_2) < \infty \quad \Leftrightarrow \quad \dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H_\pm) < \infty. \tag{4.26}$$

Proof. By Corollary 4.2 the result does not depend on the choice of $u_{1,2}$. Since the proof of (4.23) does not use the *l.p.* / $\lambda_2 \in \sigma_p(H_+)$ assumption the first direction follows. Conversely, we can replace the sequence β in (4.25) by a sequence $\hat{\beta}$ such that $\tilde{H}_{0,n}^{\infty,\hat{\beta}}$ converges to H_+. Since we have

$$|\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(\tilde{H}_{0,n}^{\infty,\hat{\beta}}) - \dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(\tilde{H}_{0,n}^{\infty,\beta})| \le 1 \tag{4.27}$$

the corollary is proven. □

Finally we turn to our main result for Jacobi operators H on $\mathbb{Z}$. We emphasize that to date, Theorem 4.6 appears to be the only oscillation theoretic result concerning the number of eigenvalues in essential spectral gaps of Jacobi operators on $\mathbb{Z}$.

Theorem 4.6. *Fix $\lambda_1 < \lambda_2$ and suppose $[\lambda_1, \lambda_2] \cap \sigma_{ess}(H) = \emptyset$. Then*

$$\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H) = \# W(u_\mp(\lambda_1), u_\pm(\lambda_2)). \tag{4.28}$$

In addition, if τ is l.p. at $+\infty$ we even have

$$\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H) = \# W(u_+(\lambda_1), u_+(\lambda_2)). \tag{4.29}$$

The same result holds if $+$ is replaced by $-$.

Proof. Since the proof is similar to the proof of Theorem 4.4 we shall only outline the first part. Let $k = \# W(u_+(\lambda_1), u_-(\lambda_2))$ if this number is finite and $k \in \mathbb{N}$ else. Pick $n > 0$ so large that all zeros of the Wronskian are between $-n$ and n. We abbreviate

$$\Delta(\lambda, n) = \Delta_{u_+(\lambda_1), u_-(\lambda)}(n) \tag{4.30}$$

and normalize $\Delta(\lambda_1, n) \in [0, \pi)$ implying $\Delta(\lambda, n) > 0$ for $\lambda > \lambda_1$. Hence if we chose $n \in \mathbb{N}$ so large that all k nodes are between $-n$ and n we can assume

$$\Delta(\lambda, n) > k\pi. \tag{4.31}$$

Thus we can find $\lambda_1 < \hat{\lambda}_1 < \cdots < \hat{\lambda}_k < \lambda_2$ with $\Delta(\hat{\lambda}_j, n) = 0 \bmod \pi$. Now define

$$\eta_j(m) = \begin{cases} u_-(\hat{\lambda}_j, m) & m \le n \\ \rho_j u_+(\lambda_1, m) & m \ge n \end{cases}, \tag{4.32}$$

where $\rho_j \neq 0$ is chosen such that $u_-(\hat{\lambda}_j, m) = \rho_j u_+(\lambda_1, m)$ for $m = n, n+1$. Now proceed as in the previous theorems. □

Again, we infer as a consequence.

Corollary 4.7. *Let $u_{1,2}$ satisfy $\tau u_{1,2} = \lambda_{1,2} u_{1,2}$. Then*

$$\#W(u_1, u_2) < \infty \quad \Leftrightarrow \quad \dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H) < \infty. \tag{4.33}$$

Proof. Follows from Corollaries 4.2, 4.5, and $\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H) < \infty$ if and only if $(\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H_-) + \dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H_+)) < \infty$. □

Remark 4.8. *The most general three-term recurrence relation*

$$\tilde{\tau} f(n) = \tilde{a}(n) f(n+1) - \tilde{b}(n) f(n) + \tilde{c}(n) f(n-1), \tag{4.34}$$

with $\tilde{a}(n)\tilde{c}(n+1) > 0$, can be transformed to a Jacobi recurrence relation as follows. First we symmetrise $\tilde{\tau}$ via

$$\tilde{\tau} f(n) = \frac{1}{w(n)} \Big(c(n) f(n+1) + c(n-1) f(n-1) - d(n) f(n) \Big), \tag{4.35}$$

where

$$w(n) = \begin{cases} \prod\limits_{j=n_0}^{n-1} \frac{\tilde{a}(j)}{\tilde{c}(j+1)} & \text{for } n > n_0 \\ 1 & \text{for } n = n_0 \\ \prod\limits_{j=n}^{n_0-1} \frac{\tilde{c}(j+1)}{\tilde{a}(j)} & \text{for } n < n_0 \end{cases} > 0, \tag{4.36}$$

$$c(n) = w(n)\tilde{a}(n) = w(n+1)\tilde{c}(n+1), \quad d(n) = w(n)\tilde{b}(n). \tag{4.37}$$

The natural Hilbert space for $\tilde{\tau}$ is the weighted space $\ell^2(\mathbb{Z}, w)$ with scalar product

$$\langle f, g \rangle = \sum_{n \in \mathbb{Z}} w(n) \overline{f(n)} g(n), \qquad f, g \in \ell^2(\mathbb{Z}, w). \tag{4.38}$$

Let $\tilde{H}$ be a self-adjoint operator associated with $\tilde{\tau}$ in $\ell^2(\mathbb{Z}, w)$. Then the unitary operator

$$U: \begin{array}{ccc} \ell^2(\mathbb{Z}, w) & \to & \ell^2(\mathbb{Z}) \\ u(n) & \mapsto & \sqrt{w(n)} u(n) \end{array} \tag{4.39}$$

transforms $\tilde{H}$ into a Jacobi operator $H = U \tilde{H} U^{-1}$ in $\ell^2(\mathbb{Z})$ associated with the sequences

$$a(n) = \frac{c(n)}{\sqrt{w(n) w(n+1)}} = \operatorname{sgn}(\tilde{a}(n)) \sqrt{\tilde{a}(n) \tilde{c}(n+1)}, \tag{4.40}$$

$$b(n) = \frac{d(n)}{w(n)} = \tilde{b}(n). \tag{4.41}$$

In addition we infer

$$\begin{aligned} & c(n)\Big(f(n) g(n+1) - f(n+1) g(n) \Big) = \\ & \quad a(n)\Big((Uf)(n)(Ug)(n+1) - (Uf)(n+1)(Ug)(n) \Big). \end{aligned} \tag{4.42}$$

Hence all results derived for Jacobi operator thus far apply to generalized Jacobi operators of the type $\tilde{H}$ as well.

5. Applications

One important class of Jacobi operators are periodic ones (cf., e.g., [4], Appendix B, [25], [26]). Instead of periodic operators themselves we are interested in short-range perturbations of these operators. In fact, we are going to prove the analog of the Theorem by Rofe-Beketov ([29], see also [11]) about the finiteness of the number of eigenvalues in essential spectral gaps of the perturbed Hill operator. Since constant coefficients a, b are a special case of periodic ones our results contain results from scattering theory (cf., e.g., [5], [15]).

To set the stage, we first recall some basic facts from the theory of periodic operators. Let H_p be a Jacobi operator associated with periodic sequences $a_p < 0, b_p$, that is,

$$a_p(n+N) = a_p(n), \qquad b_p(n+N) = b_p(n), \tag{5.1}$$

for some fixed $N \in \mathbb{N}$. The spectrum of H_p is purely absolutely continuous and consists of a finite number of gaps, that is,

$$\sigma(H_p) = \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}], \qquad g \in \mathbb{N}_0, \tag{5.2}$$

with $E_0 < E_1 < \cdots < E_{2g+1}$ and $g \le N-1$. Moreover, Floquet theory implies the existence of solutions $u_{p,\pm}(z,.)$ of $\tau_p u = zu$, $z \in \mathbb{C}$ (τ_p the difference expression corresponding to H_p) satisfying

$$u_{p,\pm}(z, n+N) = m^{\pm}(z) u_{p,\pm}(z,n), \tag{5.3}$$

where $m^{\pm}(z) \in \mathbb{C}$ are called Floquet multipliers. $m^{\pm}(z)$ satisfy $m^+(z)m^-(z) = 1$, $m^{\pm}(z)^2 = 1$ for $z \in \{E_j\}_{j=0}^{2g+1}$, $|m^{\pm}(z)| = 1$ for $z \in \sigma(H_p)$, and $|m^+(z)| < 1$ for $z \in \mathbb{C}\backslash\sigma(H_p)$. (This says in particular, that $u_{p,\pm}(z,.)$ are bounded for $z \in \sigma(H_p)$ and linearly independent for $z \in \mathbb{C}\backslash\{E_j\}_{j=0}^{2g+1}$.)

We are going to study perturbations H of H_p associated with sequences a, b satisfying $a(n) \to a_p(n)$ and $b(n) \to b_p(n)$ as $|n| \to \infty$. Clearly, H and H_p are both bounded and hence defined on the whole of $\ell^2(\mathbb{Z})$. In fact, we have

$$\sigma(H) \subseteq [\underline{c}, \overline{c}], \tag{5.4}$$

where $\underline{c} = \inf_{n\in\mathbb{Z}}(b(n) + a(n-1) + a(n))$ and $\overline{c} = \sup_{n\in\mathbb{Z}}(b(n) - a(n-1) - a(n))$. Using this notation our theorem reads:

Theorem 5.1. *Suppose a_p, b_p are given periodic sequences and H_p is the corresponding Jacobi operator. Let H be a perturbation of H_p such that*

$$\sum_{n\in\mathbb{Z}} |n(a(n) - a_p(n))| < \infty, \quad \sum_{n\in\mathbb{Z}} |n(b(n) - b_p(n))| < \infty. \tag{5.5}$$

Then we have $\sigma_{ess}(H) = \sigma(H_p)$, the point spectrum of H is finite and confined to the spectral gaps of H_p, that is, $\sigma_p(H) \subset \mathbb{R}\backslash\sigma(H_p)$. Furthermore, the essential spectrum of H_p is purely absolutely continuous.

For the proof we will need the following lemma the proof of which is elementary.

Lemma 5.2. *The Volterra sum equation*

$$f(n) = g(n) + \sum_{m=n+1}^{\infty} K(n,m) f(m), \tag{5.6}$$

with

$$|K(n,m)| \le \hat{K}(n,m), \quad \hat{K}(n+1,m) \le \hat{K}(n,m), \quad \hat{K}(n,.) \in \ell^1(0,\infty), \tag{5.7}$$

has for $g \in \ell^\infty(0,\infty)$ *a unique solution* $f \in \ell^\infty(0,\infty)$*, fulfilling the estimate*

$$|f(n)| \le \Big(\sup_{m>n} |g(m)| \Big) \exp\Big(\sum_{m=n+1}^{\infty} \hat{K}(n,m) \Big). \tag{5.8}$$

Proof. (of Theorem 5.1) The fact that $H - H_p$ is compact implies $\sigma_{ess}(H) = \sigma_{ess}(H_p)$. To prove the remaining claims it suffices to show the existence of solutions $u_\pm(\lambda,.)$ of $\tau u = \lambda u$ for $\lambda \in \sigma(H_p)$ satisfying

$$\lim_{n\to\pm\infty} |u_\pm(\lambda,n) - u_{p,\pm}(\lambda,n)| = 0. \tag{5.9}$$

In fact, since $u_\pm(\lambda,.)$, $\lambda \in \sigma(H_p)$ are bounded and do not vanish near $\pm\infty$, there are no eigenvalues in the essential spectrum of H and invoking the principal of subordinacy (cf., [30], [31]) shows that the essential spectrum of H is purely absolutely continuous. Moreover, (5.9) with $\lambda = E_0$ implies that $H - E_0$ is non-oscillatory since we can assume (perhaps after flipping signs) $u_{p,\pm}(E_0,n) \ge \varepsilon > 0$, $n \in \mathbb{Z}$ and by Corollary 3.8 there are only finitely many eigenvalues below E_0. Similarly, (using Remark 3.9) there are only finitely many eigenvalues above E_{2g+1}. Applying Corollary 4.7 in each gap (E_{2j-1}, E_{2j}), $1 \le j \le g$ shows that the number of eigenvalues in each gap is finite as well.

It remains to show (5.9). Suppose $u_+(\lambda,.)$, $\lambda \in \sigma(H_p)$ satisfies (disregarding summability for a moment)

$$u_+(\lambda,n) = \frac{a_p(n)}{a(n)} u_{p,+}(\lambda,n) - \sum_{m=n+1}^{\infty} \frac{a_p(n)}{a(n)} K(\lambda,n,m) u_+(\lambda,m), \tag{5.10}$$

with

$$\begin{aligned} K(\lambda,n,m) = {}& \frac{s_p(\lambda,n,m-1)}{a_p(m-1)}(a(m-1) - a_p(m-1)) \\ & + \frac{s_p(\lambda,n,m+1)}{a_p(m+1)}(a(m) - a_p(m)) - \frac{s_p(\lambda,n,m)}{a_p(m)}(b(m) - b_p(m)), \end{aligned} \tag{5.11}$$

where $s_p(\lambda,.,m)$ is the solution of of $\tau_p u = zu$ satisfying the initial conditions $s_p(z,m,m) = 0$ and $s_p(z,m+1,m) = 1$. Then $u_+(\lambda,.)$ fulfills $\tau u = \lambda u$ and (5.9). Hence if we can apply Lemma 5.2 we are done. To do this we need an estimate for $K(\lambda,n,m)$ which again follows from Floquet theory

$$|s_p(\lambda,n,m)| \le M|n-m|, \qquad \lambda \in \sigma(H_p), \tag{5.12}$$

for some suitable constant $M > 0$. □

As pointed out to the author by J. Geronimo, the above theorem in the case of H_+ can also be obtained combining Lemma 9 and Theorem 4 of [10]. The theorems for H and H_+ are equivalent since $H_- \oplus b(0) \oplus H_+$ and H differ by a finite rank operator. Alternatively, one could also invoke the Birman-Schwinger principle (cf., [8], [9], [11]). However, the proof given here has the advantage of being rather short and transparent. In addition, the idea of proof applies to much general scattering situations (where H_p is not necessarily periodic) as long as sufficient information about the spectrum of H_p and the asymptotic behavior of (weak) solutions of H_p

and H is available. The reader should also compare [14], Section 67 and [24] where special cases of Theorem 5.1 are considered.

As anticipated, specializing to the case $a_p(n) = -1/2$, $b_p(n) = 0$, we obtain a corresponding result for the free scattering case.

Corollary 5.3. *([15]) Suppose*

$$\sum_{n\in\mathbb{Z}} |n(1+2a(n))| < \infty, \quad \sum_{n\in\mathbb{Z}} |n\, b(n)| < \infty. \tag{5.13}$$

Then we have

$$\sigma_{ess}(H) = [-1,1], \quad \sigma_p(H) \subseteq [\underline{c}, -1) \cup (1, \overline{c}]. \tag{5.14}$$

Moreover, the essential spectrum of H is purely absolutely continuous and the point spectrum of H is finite.

Corollary 5.3 is stated in [15] (for the case $a_p(n) = 1$ – but Remark 2.2 plus a scaling transform takes care of that). In addition, explicit bounds on the number of eigenvalues can be found in [8], [9].

Appendix A. Some useful lemmas

This appendix provides some useful results from the theory of Jacobi operators. Most of these results are either standard or easy consequences of well-known facts (cf., e.g., [1], [2]).

Denote by $s(z,n)$ and $c(z,n)$ the solutions of $\tau u = zu$ corresponding to the initial conditions $s(z,0) = c(z,1) = 0$, $s(z,1) = c(z,0) = 1$.

Lemma A.1. *Let $\lambda_0 < \lambda_1$ be such that $[\lambda_0, \lambda_1] \cap \sigma_{ess}(H_+) = \emptyset$. Then there exists a solution $u_+(z,.) \in \ell^2_+(\mathbb{Z})$ of $\tau u = zu$ satisfying the boundary condition of H at $+\infty$ (if any) which is holomorphic with respect to z for $z \in \mathbb{C}\backslash((-\infty,\lambda_0] \cup [\lambda_1, \infty))$. Explicitly, we can set*

$$u_+(z,n) = \Big(\prod_{\mu\in\sigma(H_+)\cap[\lambda_0,\lambda_1]} (z-\mu)\Big)\Big(a(0)^{-1}c(z,n) - m_+(z)s(z,n)\Big), \tag{A.1}$$

where $m_+(z) = \langle \delta_1, (H_+ - z)^{-1}\delta_1\rangle$ is one of the Weyl m-functions of H. Clearly, $u_+(z,.) \not\equiv 0$ and $\overline{u_+(z,.)} = u_+(\overline{z},.)$.

Similarly, $[\lambda_0, \lambda_1] \cap \sigma_{ess}(H_-) = \emptyset$ implies the existence of a solution $u_-(z,.) \in \ell_-(\mathbb{Z})$ fulfilling the boundary condition of H at $-\infty$ (if any) and, as a function of z, satisfies the same conditions as $u_+(z,.)$.

Lemma A.2. *Suppose $a(n) < 0$ and let $\lambda < \inf \sigma(H)$. Then we can assume*

$$u_\pm(\lambda,n) > 0, \quad n \in \mathbb{Z}, \tag{A.2}$$

$$n\, s(\lambda,n) > 0, \quad n \in \mathbb{Z}\backslash\{0\}. \tag{A.3}$$

The solutions $u_\pm(\lambda,.)$ are called principal solutions of $(H-\lambda)u = 0$ near $\pm\infty$ in [16].

Proof. From $(H-\lambda) > 0$ one infers $(H_{+,n} - \lambda) > 0$ and hence

$$0 < \langle \delta_{n+1}, (H_{+,n} - \lambda)^{-1}\delta_{n+1}\rangle = \frac{u_+(\lambda, n+1)}{-a(n)u_+(\lambda,n)} \tag{A.4}$$

showing that $u_+(\lambda)$ can be chosen to be positive. Furthermore, for $n > 0$ we obtain

$$0 < \langle \delta_n, (H_+ - \lambda)^{-1}\delta_n\rangle = \frac{u_+(\lambda, n)s(\lambda, n)}{-a(0)u_+(\lambda, 0)} \tag{A.5}$$

implying $s(\lambda, n) > 0$ for $n > 0$. Similarly one proves the remaining results. □

Let $u_\pm(z, n)$ are solutions of $\tau u = zu$ as in Lemma A.1. Then Green's formula

$$\sum_{j=m}^{n} \Big(f(\tau g) - (\tau f)g\Big)(j) = W_n(f, g) - W_{m-1}(f, g). \tag{A.6}$$

implies

$$W_n(u_+(z), u_+(\tilde{z})) = (z - \tilde{z}) \sum_{j=n+1}^{\infty} u_+(z, j)u_+(\tilde{z}, j) \tag{A.7}$$

and furthermore,

$$\begin{aligned} W_n(u_+(z), \dot{u}_+(z)) &= \lim_{\tilde{z}\to z} W_n(u_+(z), \frac{u_+(z) - u_+(\tilde{z})}{z - \tilde{z}}) \\ &= \sum_{j=n+1}^{\infty} u_+(z, j)^2. \end{aligned} \tag{A.8}$$

Here the dot denotes the derivative with respect to z. An analogous result holds for $u_-(z, n)$. Interchanging limit and summation can be justified using (cf. Remark 3.2)

$$u_+(\tilde{z}, j) = const(\tilde{z})(H^\beta_{+,n-1} - \tilde{z})^{-1}\delta_n(j) \quad \text{for } j \le n \tag{A.9}$$

(with β such that $z \notin \sigma(H^\beta_{+,n-1})$) and the first resolvent identity. Summarizing (compare [1], Theorem 4.2.2):

Lemma A.3. *Let $u_\pm(z, n)$ be solutions of $\tau u = zu$ as in Lemma A.1. Then we have*

$$W_n(u_\pm(z), \dot{u}_\pm(z)) = \begin{cases} -\sum\limits_{j=n+1}^{\infty} u_+(z, j)^2 \\ \sum\limits_{j=-\infty}^{n} u_-(z, j)^2 \end{cases}. \tag{A.10}$$

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References

[1] F. Atkinson, *Discrete and Continuous Boundary Problems*, Academic Press, New York, 1964.

[2] J. Berezanskii, *Expansions in Eigenfunctions of Self-adjoint Operators*, Transl. Math. Monographs, vol. 17, Amer. Math. Soc., Providence, R. I., 1968.

[3] M. Bôcher, *Leçons de Méthodes de Sturm*, Gauthier-Villars, Paris, 1917.

[4] W. Bulla, F. Gesztesy, H. Holden, and G. Teschl *Algebro-Geometric Quasi-Periodic Finite-Gap Solutions of the Toda and Kac-van Moerbeke Hierarchies*, preprint (1995).

[5] K. M. Case and M. Kac, *A discrete version of the inverse scattering problem*, J. Math. Phys. **14**, 594–603 (1973).

[6] Y. Domshlak, *Sturmian comparison method on oscillation study for discrete difference equations. I,II*, Diff. Int. Eq. **7**, 571–582, 583-592, (1994).

[7] T. Fort, *Finite Differences and Difference Equations in the Real Domain*, Oxford University Press, London, 1948.

[8] J. S. Geronimo, *On the spectra of infinite–dimensional Jacobi matrixes*, J. App. Th., **53**, 251–265 (1988).

[9] J. S. Geronimo, *An upper bound on the number of eigenvalues of an infinite–dimensional Jacobi matrix*, J. Math. Phys., **23**, 917 1982.

[10] J. S. Geronimo and W. Van Assche, *Orthogonal polynomials with asymptotically periodic recurrence coefficients*, J. App. Th., **46**, 251–283 (1986).

[11] F. Gesztesy and B. Simon, *A short proof of Zheludev's theorem*, Trans. Am. Math. Soc. **335**, 329–340 (1993).

[12] F. Gesztesy and Z. Zhao, *Critical and subcritical Jacobi operators defined as Friedrichs extensions*, J. Diff. Eqs. **103**, 68–93 (1993).

[13] F. Gesztesy, B. Simon, and G. Teschl, *Zeros of the Wronskian and renormalized oscillation Theory*, Am. J. Math. **118** 571–594 (1996).

[14] I. M. Glazman, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators*, I.P.S.T., Jerusalem (1965).

[15] G. S. Guseinov, *The inverse problem of scattering theory for a second-order difference equation on the whole axis*, Soviet Math. Dokl., **17**, 1684–1688 (1976).

[16] P. Hartman, *Difference equations: Disconjugacy, principal solutions, Green's functions, complete monotonicity*, Trans. Amer. Math. Soc., **246**, 1–30 (1978).

[17] P. Hartman, *Differential equations with non-oscillatory eigenfunctions*, Duke Math. J., **15**, 697–709 (1948).

[18] P. Hartman, *A characterization of the spectra of one-dimensional wave equations*, Am. J. Math., **71**, 915–920 (1949).

[19] P. Hartman and C.R. Putnam, *The least cluster point of the spectrum of boundary value problems*, Am. J. Math. **70**, 849–855 (1948).

[20] D. B. Hinton and R. T. Lewis, *Spectral analysis of second order difference equations*, J. Math. Anal. Appl. **63**, 421–438 (1978).

[21] J. W. Hooker and W. T. Patula, *Riccati type transformations for second-order linear difference equations*, J. Math. Anal. Appl. **82**, 451–462 (1981).

[22] J. W. Hooker, M. K. Kwong and W. T. Patula, *Oscillatory second-order linear difference equations and Riccati equations*, SIAM J. Math. Anal., **18**, 54–63 (1987).

[23] W. G. Kelley and A. C. Peterson, *Difference Equations*, Academic Press, San Diego, 1991.

[24] M. Klaus, *On bound states of the infinite harmonic crystal*, Hel. Phys. Acta **51**, 793-803 (1978).

[25] I. M. Krichever, *Algebro-geometric spectral theory of the Schrödinger difference operator and the Peierls model*, Soviet Math. Dokl. **26**, 194-198 (1982).

[26] P. van Moerbeke, *The spectrum of Jacobi Matrices*, Inv. Math. **37**, 45–81 (1976).

[27] W. T. Patula, *Growth and oscillation properties of second order linear difference equations*, SIAM J. Math. Anal. **6**, 55–61 (1979).

[28] W. T. Patula, *Growth, oscillation and comparison theorems for second order linear difference equations*, SIAM J. Math. Anal. **6**, 1272–1279 (1979).

[29] F. S. Rofe-Beketov, *A test for the finiteness of the number of discrete levels introduced into gaps of a continuous spectrum by perturbations of a periodic potential*, Soviet Math. Dokl. **5**, 689-692 (1964).

[30] B. Simon, *Bounded eigenfunctions and absolutely continuous spectra for one–dimensional Schrödinger operators*, Proc. Amer. Math. Soc. **124**, 3361–3369 (1996).

[31] G. Stolz, *Spectral Theory for slowly oscillating potentials I. Jacobi Matrices*, Man. Math. **84**, 245–260 (1994).

[32] J.C.F. Sturm, *Mémoire sur les équations différentielles linéaires du second ordre*, J. Math. Pures Appl., **1**, 106–186 (1836).

[33] J. Weidmann, *Linear Operators in Hilbert Spaces*, Springer, New York, 1980.

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