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Functional Analytic Methods for Hyperbolic Partial Differential Operators

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Chapter 1

Introduction

This diploma thesis deals with the problem (H)

$$\frac{d^2y}{dt^2} + L(t)y = f, \quad t \in (0,T),$$

with the initial conditions

$$y(0) = y_0 \in V, \quad \frac{dy}{dt}(0) = y_1 \in H,$$

where $f \in L^2(0,T;H)$, $T < \infty$ and $V \subseteq H \subseteq V'$ are Hilbert spaces, all densely embedded. From the operator L there can be constructed an associated sesquilinear form $a(t;\,.\,,\,.\,) : V \times V \to \mathbb{C}$, i.e.

$$a(t;\varphi,\psi) = \big(L(t)\varphi,\psi\big).$$

If this sesquilinear form satisfies certain conditions, i.e. continuity, antisymmetry, V-coercivity and boundedness of the derivative (with resp. to t), there exists a unique solution to the problem (H).
The aim of the thesis at hand is to consider two examples, first the wave equation and second a more general form of a wave equation, and to find out what requirements on the coefficients of the operators have to be made, such that the existence of a solution is guaranteed.
Chapter 2 begins with a short introduction to the theory of Sobolev spaces $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, since these play a decisive role in the theory of partial differential equations and are therefore important for the subject of this thesis. Moreover, a definition of the space of α-Hölder continuous functions ($0 < \alpha \leq 1$), in particular of Lipschitz continuous functions ($\alpha = 1$) is given. Furthermore, definitions of vector-valued distributions are recalled. In Chapter 2 essentially results which are the prerequisites needed in the following chapters are presented.
With Chapter 3 the main part of this diploma thesis begins, which deals with the proof of the existence and uniqueness result mentioned above. It should be noted that Chapter 3 is based heavily on [13]. To prove the existence of a solution to problem (H) the Galerkin method is used.

In Section 3.2 a regularity result is proven, which shows that the problem (H) can be solved in the scope of Sobolev spaces. The classical solvability can then be obtained by applying Sobolev's Lemma.
In Chapter 4 the wave equation

$$\begin{aligned}&\partial_t^2 u - c(t)\partial_x^2 u = 0\\ &u(0,x) = u_0(x), \quad \big(\partial_t u\big)(0,x) = u_1(x)\end{aligned}$$

is considered. To have $a(t;\varphi,\psi)$ satisfy the conditions, which are necessary for the existence of the solution, we will come to the conclusion that the coefficient $c(t)$ has to be Lipschitz continuous and $c(t) \geq c_1 > 0$ for all $t \in (0,T)$. In Section 4.1 it will be also discussed which requirements on c guarantee existence of a solution if we do not restrict to the functional analytic method used in this thesis. Due to [2] it will turn out that c at least has to be Log-Lipschitz continuous.
The Section 4.2 deals with a more general form of a wave equation. Essentially we will find, analogous to the wave equation, that we need the coefficients to be Lipschitz continuous in t.

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Chapter 2

Prerequisites

2.1 Sobolev Spaces

In the analysis of partial differential equations the spaces $H^m(\mathbb{R}^n)$ with $m \in \mathbb{N}$, denoting the spaces of distributions whose derivatives up to the order m are in $L^2(\mathbb{R}^n)$, play an important role. We will start with a definition of the Sobolev spaces $H^s(\mathbb{R}^n)$ with $s \in \mathbb{R}$ and will see, that for $s = m \in \mathbb{N}$ the space $H^s(\mathbb{R}^n)$ coincides with the space $H^m(\mathbb{R}^n)$.

Definition 2.1 *For $s \in \mathbb{R}$ we define*

$$H^s(\mathbb{R}^n) := \left\{ u \in \mathscr{S}'(\mathbb{R}^n) \middle| (1 + |\xi|^2)^{s/2} \hat{u} \in L^2(\mathbb{R}^n) \right\}$$

(with $\hat{u}$ the Fourier transform of u). We furnish $H^s(\mathbb{R}^n)$ with the scalar product

$$(u, v)_s = (2\pi)^{-n} \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi$$

and the associated norm is

$$\|u\|_s = \left((2\pi)^{-n} \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{1/2}.$$

Note that we obviously have $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$. From now on we will use the notation $\lambda(\xi) := (1 + |\xi|^2)^{1/2}$. Moreover we will write H^s instead of $H^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$ and $\mathscr{S}$ instead of $\mathscr{S}(\mathbb{R}^n)$.

Lemma 2.2 *(i) The spaces H^s with $s \in \mathbb{R}$ are Hilbert spaces.*
(ii) Let $u \in H^s$ with $s \in \mathbb{R}$, then

$$\|u\|_s = (2\pi)^{-n/2} \|\lambda^s \hat{u}\|_0$$

and

$$u \in H^s \quad \textit{if and only if} \quad \lambda^s \hat{u} \in L^2.$$

(iii) If $s \geq t$ then $H^s \subseteq H^t$ and the injection is continuous.

Proof. (i) For any given $s \in \mathbb{R}$ we only have to show that H^s is complete. Let therefore $\{u_j\}$ be a Cauchy sequence in H^s, then $\{\lambda^s(\xi)\hat{u_j}\}$ is a Cauchy sequence in L^2. The space of classes of square integrable functions on $\mathbb{R}^n$ is complete and so $\lambda^s(\xi)\hat{u_j} \to v$ in L^2. Hence $v \in \mathscr{S}'$ and $\lambda^{-s}(\xi)v \in \mathscr{S}'$, because $\lambda^{-s}(\xi)$ is of slow growth. Since the Fourier transformation is a continuous isomorphism on $\mathscr{S}$ and the Fourier transform on $\mathscr{S}'$ is defined by duality there exists $u \in \mathscr{S}'$, such that $\lambda^{-s}v = \hat{u}$. Finally we have

$$u \in \mathscr{S}' \quad \text{and} \quad \lambda^s(\xi)\hat{u} \in L^2,$$

i.e. $u \in H^s$ and since

$$\|u_j - u\|_s = \|\lambda^s\hat{u_j} - \lambda^s\hat{u}\|_0 = \|\lambda^s\hat{u_j} - v\|_0 \to 0 \quad \text{as } j \to \infty$$

we have $u_j \to u$ in H^s.
(ii) By definition.
(iii) This follows from

$$\|u\|_t = \|\lambda^t\hat{u}\|_0 = \|\lambda^{t-s}\lambda^s\hat{u}\|_0 \leq \|\lambda^t\hat{u}\|_0 = \|u\|_s.$$

□

Because of Lemma (2.2) (iii) it makes sense to introduce Sobolev spaces of infinite order.

Definition 2.3

$$H^\infty = \bigcap_s H^s \quad \textit{and} \quad H^{-\infty} = \bigcup_s H^s.$$

Observe that we have the following inclusions

$$\mathscr{S} \subset H^\infty \subset H^{-\infty} \subset \mathscr{S}'.$$

To see that the inclusions above are strict observe that for $n = 1$ the function $f(x) = \frac{1}{1+x^2}$ is smooth and square integrable, i.e. $f \in C^\infty \cap L^2$. For all k we have $f^{(k)}(x) = O\big((1+|x|)^{-k-2}\big)$ and therefore $f^{(k)} \in C^\infty \cap L^2$, $\forall k$. Hence $f \in H^\infty(\mathbb{R})$, but $f \notin \mathscr{S}(\mathbb{R})$. So we have shown that the inclusion $\mathscr{S} \subset H^\infty$ is indeed strict.
We find that for $u := \mathscr{F}^{-1}\big(\lambda^{-\frac{(n+1)}{2}-s}\big)$ we have $u \in H^s \backslash H^{s+1}$ for all $s \in \mathbb{R}$. We conclude that $H^\infty \subset H^s$, $\forall s \in \mathbb{R}$, i.e. $H^\infty \subset \bigcup_s H^s = H^{-\infty}$.
By definition we have

$$H^{-\infty} = \{u \in \mathscr{S}' | \exists s \in \mathbb{R} \text{ such that } \lambda^s\hat{u} \in L^2\}.$$

Now assume that $\mathscr{S}' \subseteq H^{-\infty}$. Then for all $u \in \mathscr{S}'$ there would exist $s \in \mathbb{R}$ such that $\lambda^s\hat{u} \in L^2$. But then one would get $\lambda^{-s}(\lambda^s\hat{u}) = \hat{u} \in L^1_{\text{loc}}$, which does not hold for all $u \in \mathscr{S}'$.

Lemma 2.4 *(i) Let $s \in \mathbb{R}$, $u \in \mathscr{S}'$. Then*

$$u \in H^{s+1} \quad \textit{if and only if} \quad u, D_1u, \ldots, D_nu \in H^s \tag{2.1}$$

and

$$\|u\|_{s+1}^2 = \|u\|_s^2 + \sum_{j=1}^{n} \|D_j u\|_s^2. \tag{2.2}$$

(ii) For $m \in \mathbb{N} \cup \{\infty\}$ and $u \in \mathscr{S}'$ we have

$$u \in H^m \quad \textit{if and only if} \quad D^\alpha u \in L^2,\ \forall |\alpha| \leq m \tag{2.3}$$

(where $D_j := -i\partial_j$, $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$).

Proof. (i) This results from

$$\begin{aligned} |\lambda^{s+1}\hat{u}|^2 &= \lambda^2 |\lambda^s \hat{u}|^2 = (1+|\xi|^2)|\lambda^s\hat{u}|^2 = \Big(1 + \sum_{j=1}^{n} \xi_i^2\Big)|\lambda^s \hat{u}|^2 \\ &= |\lambda^s\hat{u}|^2 + \sum_{j=1}^{n} |\lambda^s \xi_j \hat{u}|^2 = |\lambda^s\hat{u}|^2 + \sum_{j=1}^{n} |\lambda^s \widehat{D_j u}|^2, \end{aligned}$$

where we have used the fact that multiplying the Fourier transform of a function by a polynomial corrensponds to the Fourier transform of the derivative of the function.
(ii) We use induction on m: For $m = 0$ we have $H^0 = L^2$ and nothing is to prove. For the inductive step m to $m+1$ use (i). □

Theorem 2.5 (Sobolev's embedding) *Let $m \in \mathbb{N}_0 \cup \{\infty\}$ and $u \in H^s(\mathbb{R}^n)$ with $s > \frac{n}{2} + m$, then*

$$D^\alpha u \in C^0 \cap L^\infty, \quad \forall |\alpha| \leq m.$$

Moreover,

$$\|D^\alpha u\|_{L^\infty} \leq C_{m,s}\|u\|_s. \tag{2.4}$$

Proof. We use induction on k. For $k = 0$ we have $s > \frac{n}{2}$. Then

$$\int_{\mathbb{R}^n} |\hat{u}(\xi)| d\xi = \int_{\mathbb{R}^n} \lambda^s \lambda^{-s} |\hat{u}(\xi)| d\xi \leq \|u\|_s \Big(\int_{\mathbb{R}^n} \frac{d\xi}{(1+|\xi|^2)^s}\Big)^{1/2}. \tag{2.5}$$

Since $2s > n$ the last integral in (2.5) is convergent and thus $\hat{u} \in L^1(\mathbb{R}^n)$. The Fourier transformation on L^1 gives elements, which are bounded ($|\hat{u}(\xi)| \leq \|u\|_{L^1}$) and continuous by the theorem of dominated convergence. Conclusively we get $u \in C^0 \cap L^\infty$. Now we make the inductive step from k to $k+1$. Then we have $s - 1 > \frac{n}{2} + k$ and by Lemma (2.4) (i) $u \in H^s$ is equivalent to u, $D_j u \in H^{s-1}$ for $j = 1, \dots, n$. By applying the induction hypothesis on u, $D_j u \in H^{s-1}$ we obtain $D^\alpha D_j u \in C^0 \cap L^\infty$ for all α with $|\alpha| \leq k$ and finally $D^\beta u \in C^0 \cap L^\infty$ for all β with $|\beta| \leq k+1$. The estimate (2.4) follows from Cauchy Schwarz' inequality. □

Theorem 2.6 (Strengthening Theorem 2.5) *Let $s \in \mathbb{R}$ and $k \in \mathbb{N}$, satisfying $s > \frac{n}{2} + k$. Then $u \in H^s(\mathbb{R}^n)$ implies $u \in \mathscr{B}^k(\mathbb{R}^n) := \{u \in C^k(\mathbb{R}^n) | \lim_{x\to\infty} |D^\alpha u(x)| = 0,\ \forall |\alpha| \leq k\}$.*

Proof. To show the result for $k = 0$ we note that $u \in H^s(\mathbb{R}^n)$ implies $\hat{u} \in L^1(\mathbb{R}^n)$ (take a look at the proof of Theorem 2.5). Since $\hat{u} \in L^1(\mathbb{R}^n) \subseteq \mathscr{S}'(\mathbb{R}^n)$ we get $u = \mathcal{F}^{-1}(\hat{u}) \in \mathscr{B}^0(\mathbb{R}^n)$. Since $\hat{u} \in L^1(\mathbb{R}^n)$, u is given by

$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{u}(\xi) e^{i\xi x} d\xi$$

and satisfies

$$\|u\|_{\mathscr{B}^0(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |u(x)| \leq (2\pi)^{-n} \int_{\mathbb{R}^n} |\hat{u}(\xi)| d\xi \leq (2\pi)^{-n} \Big(\int_{\mathbb{R}^n} \frac{d\xi}{(1+|\xi|^2)^s} \Big)^{1/2} \|u\|_s,$$

where we made use of (2.5).
For the general case we observe that from Lemma 2.4 (i) we get

$$u \in H^s(\mathbb{R}^n) \quad \text{if and only if} \quad D^\alpha u \in H^{s-|\alpha|}(\mathbb{R}^n) \subseteq H^{s-k}(\mathbb{R}^n), \ \forall |\alpha| \leq k.$$

Since $s - k > \frac{n}{2}$ we use the result for $k = 0$ and obtain

$$D^\alpha u \in \mathscr{B}^0(\mathbb{R}^n), \quad \forall |\alpha| \leq k$$

or

$$u \in \mathscr{B}^k(\mathbb{R}^n).$$

□

H^s-distributions are exactly the conjugate linear and continuous forms on H^{-s}, more precisely

Theorem 2.7 (H^s-duality) *(i) Let $u \in H^s$ and $\varphi \in \mathscr{S}$. Then we have $|(u,\varphi)| \leq \|u\|_s \|\varphi\|_{-s}$.*
(ii) Conversely, if $u \in \mathscr{S}'$ and $|(u,\varphi)| \leq C\|\varphi\|_{-s}$, $\forall \varphi \in \mathscr{S}$ we get $u \in H^s$ and $\|u\|_s \leq C$. (By (u,φ) we denote the distributional action of u on φ.)

Proof. (i) Using Parseval's formula and the Cauchy-Schwarz inequality we obtain

$$|(u,\varphi)| = (2\pi)^{-n} |(\hat{u},\hat{\varphi})| = (2\pi)^{-n} |(\lambda^s \hat{u}, \lambda^{-s}\hat{\varphi})| \leq \|u\|_s \|\varphi\|_{-s}$$

by Lemma 2.2 (ii).
(ii) We compute

$$|(\lambda^s \hat{u},\varphi)| = |(\hat{u},\lambda^s\varphi)| = |(\check{u}, \widehat{\lambda^s\varphi})| \leq C\|\widehat{\lambda^s\varphi}\|_{-s} = C(2\pi)^{-n/2} \|\lambda^{-s} \widehat{\widehat{\lambda^s\varphi}}\|_0 = C(2\pi)^{n/2}\|\varphi\|_0.$$

By Riesz' representation theorem we get $\lambda^s \hat{u} \in L^2$ and hence by Lemma 2.2 (ii) $u \in H^s$. □

2.2 Hölder Spaces

Definition 2.8 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $0 < \alpha \leq 1$. We say the function φ is α-Hölder continuous on Ω if there exists a $C > 0$ such that*

$$\frac{|\varphi(x) - \varphi(y)|}{|x-y|^\alpha} \leq C < \infty,$$

for all x, $y \in \Omega$ with $x \neq y$. If $\alpha = 1$ we call φ Lipschitz continuous.
We define for $0 < \alpha \leq 1$

$$C^{0,\alpha}(\Omega) := \Big\{\varphi \in C^0(\Omega) \Big| \sup_{x,y\in\Omega} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^\alpha} < \infty\Big\}$$

and more generally for $l \in \mathbb{N}$

$$C^{l,\alpha}(\Omega) := \Big\{\varphi \in C^l(\Omega) \Big| D^j\varphi \in C^{0,\alpha}(\Omega),\ \forall |j| \leq l\Big\}.$$

We furnish the spaces $C^{l,\alpha}(\Omega)$ with the norm

$$\|\varphi\|_{C^{l,\alpha}} = \sup_{\substack{|s|\leq l \\ x\in\Omega}} |D^s\varphi(x)| + \sup_{\substack{|s|=l \\ x,y\in\Omega \\ x\neq y}} \frac{|D^s\varphi(x) - D^s\varphi(y)|}{|x-y|^\alpha}.$$

By definition it is obvious that for $0 < \alpha \leq \beta \leq 1$ we have $C^{0,\beta} \subseteq C^{0,\alpha}$. Therefore Lipschitz continuity is stronger than Hölder continuity.

Remark 2.9 There is a connection between Sobolev spaces and Hölder continuous functions:
Let $u \in H^s(\mathbb{R}^n)$ with $s \in \mathbb{R}$ and $0 < \alpha < 1$. If $s > \frac{n}{2} + \alpha$ we have $u \in C^{0,\alpha}$.
This result is often known as Sobolev's lemma. The proof can be looked up in [8], p. 207, Prop. 8.6.10.

Note that Lipschitz continuous functions are obviously continuous, whereas the converse in general does not hold. Continuous functions with a bounded first distributional derivative are Lipschitz continuous (on convex sets). However, the concept of Lipschitz continuity is weaker than the notion of continuous differentability. Indeed, look at the example $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$. To make the connection between Lipschitz continuity and differentiability more precise let us define absolutely continuous functions.

Definition 2.10 *Let f be a complex-valued function defined on a subinterval I of $\mathbb{R}$. Suppose that for every $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$\sum_{i=1}^n \big|f(y_i) - f(x_i)\big| < \varepsilon$$

for every finite, pairwise disjoint, family $\big\{(x_i, y_i)\big\}_{i=1}^n$ of open subintervalls of I for which

$$\sum_{i=1}^n (y_i - x_i) < \delta.$$

Then f is said to be absolutely continuous.

One can show that a function f is absolutely continuous on $[a,b] \subseteq \mathbb{R}$ if and only if f has the form

$$f(x) = f(a) + \int_a^x g(t)dt \tag{2.6}$$

for some $g \in L^1[a,b]$. In this case we have $g(x) = f'(x)$ a.e. on (a,b). We will not prove this in the thesis at hand, it is a typical result from measure theory (cf. [7], p. 286). Thus it makes sense to define the space of absolutely continuous functions in the following way

$$AC[a,b] := \big\{ f \in C[a,b] \big| f(x) = f(a) + \int_a^x g(t)dt,\ g \in L^1[a,b] \big\}. \tag{2.7}$$

In some way the definition (2.7) is more feasible and for our aims easier to work with. So if we will later on deal with absolutely continuous functions we will always have the characterizing property (2.7) in mind. However, to show that every Lipschitz continuous function is absolutely continuous we will make use of Definition 2.10.

Lemma 2.11 *Let $f \in C^{0,1}[a,b]$, then $f \in AC[a,b]$.*

Proof. Let f be Lipschitz continuous on $[a,b]$ and C a positive Lipschitz constant. Then for every $\varepsilon > 0$ we can choose $\delta = \frac{\varepsilon}{C}$ such that for every finite, pairwise disjoint, family $\big\{(x_i, y_i)\big\}_{i=1}^n$ of open subintervalls of $[a,b]$ for which

$$\sum_{i=1}^n (y_i - x_i) < \delta$$

we have

$$\sum_{i=1}^n \big| f(y_i) - f(x_i) \big| \leq C \sum_{i=1}^n |y_i - x_i| = C \sum_{i=1}^n (y_i - x_i) < C \cdot \delta = \varepsilon.$$

Hence f is absolutely continuous on $[a,b]$. □

Summarizing our observations above we have that every Lipschitz continuous function is absolutely continuous, and since absolute continuity implies differentiability a.e. also every Lipschitz continuous function is differentiable a.e.
We will now clarify the relation between functions with the Lipschitz property and their distributional derivative. This will be of prime importance in Chapter 4.

Lemma 2.12 *We have $f \in C^{0,1}[a,b]$ if and only if the distributional derivative $f' \in L^\infty[a,b]$.*

Proof. Assume $f \in C^{0,1}[a,b]$, i.e. $\exists C > 0$ such that $|f(t) - f(t')| \leq C|t - t'|$ for all $t, t' \in [a,b]$, then the distributional derivative f' of f is bounded. Indeed, let $\varphi \in \mathscr{D}(a,b)$ be arbitrary, then we have

$$\begin{aligned}
(f', \varphi) &= -(f, \varphi') = -\int_a^b f(t)\varphi'(t)dt = \lim_{h \to 0} \int_a^b f(t) \frac{1}{h} \big(\varphi(t) - \varphi(t+h) \big) dt \\
&= \lim_{h \to 0} \int_a^b \frac{1}{h} \big(f(t) - f(t-h) \big) \varphi(t) dt \leq C \int_a^b \varphi(t) dt.
\end{aligned}$$

Hence

$$|(f', \varphi)| \le C\Big|\int_a^b \varphi(t)dt\Big| \le C\int_a^b |\varphi(t)|dt = C\|\varphi\|_{L^1},$$

and so $f' \in \big(L^1[a,b]\big)' = L^\infty[a,b]$.
Conversely, let $f' \in L^\infty[a,b]$, then $f' \in L^1[a,b]$ and therefore $f \in AC[a,b]$. Hence for all $\varphi \in \mathscr{D}(a,b)$ we obtain that $f \cdot \varphi \in AC[a,b]$ That means $f \cdot \varphi$ is differentiable almost everywhere and we get

$$(f \cdot \varphi)' = f \cdot \varphi' + f' \cdot \varphi.$$

Integration by parts for absolutely continuous functions yields

$$\big(f', \varphi\big) = -\int_a^b f(t)\varphi'(t)dt.$$

Hence the pointwise derivative almost everywhere, i.e. the distributional derivative exists and we obtain

$$|f(t) - f(\tilde t)| = \Big|\int_{\tilde t}^t f'(\eta)d\eta\Big| \le \int_{\tilde t}^t |f'(\eta)|d\eta \le \|f'\|_{L^\infty}|t - \tilde t|.$$

Setting $C := \|f'\|_{L^\infty}$ we obtain $f \in C^{0,1}[a,b]$. □

The following result will be needed in Section 4.2, where we will deal with functions depending on two variables, for example x denoting space and t denoting time. Since Lemma 2.13 and 2.14 will need vector-valued distributions, the reader may recall the definitions in Section 2.3 first.

Lemma 2.13 *If $c \in C^{0,1}([a,b]; L^\infty(\mathbb{R}))$ then $\partial_t c \in L^\infty([a,b] \times \mathbb{R})$.*

Proof. Let $c \in C^{0,1}([a,b]; L^\infty(\mathbb{R}))$, then there exists $C > 0$ such that

$$\forall t_1, t_2 \in [a,b]: \quad \|c(t_1, \,.\,) - c(t_2, \,.\,)\|_{L^\infty(\mathbb{R})} \le C|t_1 - t_2|. \tag{2.8}$$

Now we have to show that $\partial_t c \in L^\infty([a,b] \times \mathbb{R})$. The derivative of a Lipschitz continuous function in general does not exist pointwise in all points. So we consider c as a distribution in $\mathscr{D}'((a,b) \times \mathbb{R})$. Then we get $\forall \varphi \in \mathscr{D}((a,b) \times \mathbb{R})$

$$\begin{aligned}
\big(\partial_t c, \varphi\big) &= -\big(c, \partial_t \varphi\big) = -\int_{(a,b)\times\mathbb{R}} c(t,x)\partial_t\varphi(t,x)d(t,x) = -\int_{\mathbb{R}}\int_a^b c(t,x)\partial_t\varphi(t,x)dtdx \\
&= -\int_{\mathbb{R}} \Big(c(b,x)\varphi(b,x) - c(a,x)\varphi(a,x) - \int_a^b \partial_t c(t,x)\varphi(t,x)dt\Big)dx \\
&= \int_{(a,b)\times\mathbb{R}} \partial_t c(t,x)\varphi(t,x)d(t,x). \qquad (2.9)
\end{aligned}$$

Here we have used integration by parts for absolutely continuous functions. To explain this in a little more details let us, for the sake of clarity, temporarily suppress the dependency on x. Then $c \in C^{0,1}[a,b]$ implies $c \in AC[a,b]$ by Lemma 2.11. Clearly, also $\varphi \in \mathscr{D}(a,b)$ is absolutely continuous. Thus $c \cdot \varphi \in AC[a,b]$. Therefore $c \cdot \varphi$ is differentiable a.e. and we get by a simple calculation

$$(c \cdot \varphi)' = c \cdot \varphi' + c' \cdot \varphi.$$

Finally, integrating implies

$$c(b)\varphi(b) - c(a)\varphi(a) = \int_a^b c(t)\varphi'(t)dt + \int_a^b c'(t)\varphi(t)dt. \tag{2.10}$$

So (2.10) means integration by parts for absolutely continuous functions. Now we rewrite (2.9) in the following way

$$\big(\partial_t c, \varphi\big) = \lim_{h\to 0} \int_{\mathbb{R}\times[a,b]} \frac{c(t+h,x) - c(t,x)}{h} \varphi(t,x) d(t,x) \tag{2.11}$$

and we obtain

$$\begin{aligned} \big|\big(\partial_t c, \varphi\big)\big| &\leq \sup_h \int_{[a,b]\times\mathbb{R}} \frac{|c(t+h,x) - c(t,x)|}{h} |\varphi(t,x)| d(t,x) \\ &\leq \int_{[a,b]\times\mathbb{R}} \frac{\|c(t+h,x) - c(t,x)\|_{L^\infty(\mathbb{R})}}{h} |\varphi(t,x)| d(t,x) \\ &\leq C\|\varphi\|_{L^1([a,b]\times\mathbb{R})}, \end{aligned} \tag{2.12}$$

where we have used the definition of Lipschitz continuity for the last inequality sign. So we have shown

$$\partial_t c \in \big(L^1([a,b]\times\mathbb{R})\big)' \cong L^\infty([a,b]\times\mathbb{R}).$$

□

Lemma 2.14 *The space $L^\infty([a,b]\times\mathbb{R})$ is isomorphic to $L^\infty(a,b;L^\infty(\mathbb{R}))$.*

Proof. Set $X := L^\infty(a,b;L^\infty(\mathbb{R}))$, $Y := L^\infty([a,b]\times\mathbb{R})$ and define $A : X \to Y$ with $\big(A\xi\big)(t,x) := \big(\xi(t)\big)(x)$. A is well defined

$$\big|\big(A\xi\big)(t,x)\big| = \big|\big(\xi(t)\big)(x)\big| \leq \sup_{x\in\mathbb{R}} |\xi(t)| = \|\xi(t)\|_{L^\infty(\mathbb{R})} \leq \|\xi\|_{L^\infty(a,b;L^\infty(\mathbb{R}))}, \tag{2.13}$$

so $A\xi \in Y$. We can also define $B : Y \to X$ with $\big(Bf\big)(t) := f(t, \,.\,)$, $\forall t \in [a,b]$. We have

$$\big\|\big(Bf\big)(t)\big\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^\infty([a,b]\times\mathbb{R})}$$

and

$$\sup_{t\in[a,b]} \big\|\big(Bf\big)(t)\big\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^\infty([a,b]\times\mathbb{R})}. \tag{2.14}$$

Hence $Bf \in X$ and B is also well defined. Now we will show that A is bijective and the corresponding inverse map is B. We have $\forall f \in L^\infty([a,b]\times\mathbb{R})$, $\forall t \in [a,b]$

$$\Big(\big(A\circ B\big)(f)\Big)(t,x) = \Big(\big(A(Bf)\big)(t)\Big)(x) = \big(Bf\big)(t)(x) = f(t,x), \tag{2.15}$$

and hence $(A \circ B)f = f$.
We get $\forall \xi \in L^\infty(a,b;L^\infty(\mathbb{R}))$, $\forall (t,x) \in [a,b] \times \mathbb{R}$

$$\big(B \circ A\big)(\xi)(t,x) = \Big(B\big(A\xi\big)(t)\Big)(x) = \big(A\xi\big)(t,x) = \xi(t)(x), \tag{2.16}$$

and therefore $(B \circ A)\xi = \xi$.
Thus $\big(A \circ B\big) = id_Y$ and $\big(B \circ A\big) = id_X$. Since A, B are linear we have that X and Y are isomorphic. Furthermore, we obtain that this isomorphism is isometric. Indeed, let $\|\,.\,\|$ denote the operator norm, then $\|A\| \leq 1$, $\|B\| \leq 1$. Since $A \circ B = id$, $B \circ A = id$ we get

$$1 = \|A \circ B\| \leq \|A\| \cdot \|B\| \leq 1,$$

and hence $\|A\| \cdot \|B\| = 1$. So finally we conclude $\|A\| = 1$ and $\|B\| = 1$. □

2.3 Vector-valued Distributions resp. Functions

We will give a short review of distributions resp. functions with vector values in a Banach space, since those will be needed frequently later on.

2.3.1 Vector-valued Distributions

Let E be a Banach space, (a,b) an open set of $\mathbb{R}$ and dt the Lebesgue measure over (a,b). The definitions we recall are valid for an arbitrary open set Ω of $\mathbb{R}^n$.

(i) We denote by $L^p(a,b;E)$ $(1 \leq p < \infty)$ the space of classes of functions $t \mapsto f(t) : (a,b) \to E$ such that

$$f \text{ is measurable for } dt \text{ and}$$
$$\|f\|_{L^p(a,b;E)} = \Big(\int_a^b \|f(t)\|_E^p\Big)^{1/p} < \infty.$$

(ii) We denote by $L^\infty(a,b;E)$ the space of classes of functions f from (a,b) into E satisfying

$$f \text{ is measurable for } dt \text{ and}$$
$$f \text{ is bounded a.e. over } (a,b).$$

We set

$$\|f\|_{L^\infty(a,b;E)} = \inf_{\|f(t)\|_E \leq M \text{ a.e.}} M$$

(iii) We call every continuous linear mapping $T : \mathscr{D}(a,b) \to E$ a vectorial distribution over (a,b) with values in E, i.e. for all compact subsets K of $\mathbb{R}$ there exist constants $c > 0$, $m \in \mathbb{N}$ such that

$$\big\|\big(T,\varphi\big)\big\|_E \leq c \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi(x)\|_{L^\infty(K)}, \quad \forall \varphi \in \mathscr{D}(a,b) \text{ with } \operatorname{supp}(\varphi) \subset K.$$

$\mathscr{D}'((a,b);E)$ denotes the space of all such continuous linear mappings of $\mathscr{D}(a,b) \to E$.

(iv) Let $k \in \mathbb{N}$. By the Sobolev space $H^k(a,b;E)$ we denote the space of all measurable functions (or distributions) $t \mapsto f(t) : (a,b) \to E$ such that

$$\frac{d^j f}{dt^j} \in L^2(a,b;E) \quad \text{for } 0 \leq j \leq k.$$

The norm in $H^k(a,b;E)$ is given by

$$\|f\|_k^2 = \sum_{j=0}^{k} \int_0^T \left\| \frac{d^j f(t)}{dt^j} \right\|_E^2 dt.$$

Note that for $k = 0$ the space $H^0(a,b;E)$ coincides with $L^2(a,b;E)$.

2.3.2 Vector-valued Functions

Let E be a Banach space and I an subinterval of $\mathbb{R}$.

(i) By $C(I;E)$ we denote the space of all continuous functions $f : I \to E$ such that for all $t_2 \in I$ we have

$$\forall \varepsilon > 0\, \exists \delta > 0 \text{ such that } \forall t_1 \in I \text{ with } |t_1 - t_2| < \delta \text{ we have } \|f(t_1) - f(t_2)\|_E < \varepsilon.$$

(ii) Let $k \in \mathbb{N}$, then we denote by $C^k(I;E)$ the space of all functions $f : I \to E$, whose derivatives up to the order k exist and are continuous as defined in (i).

(iii) Let $0 < \alpha \leq 1$. Then $C^{0,\alpha}(I;E)$ denotes the space of all continuous functions $f : I \to E$ such that

$$\sup_{\substack{t_1 \neq t_2 \\ t_1, t_2 \in I}} \frac{\|f(t_1) - f(t_2)\|_E}{|t_1 - t_2|^\alpha} < \infty.$$

Chapter 3

Hyperbolic Operators

3.1 Existence and Uniqueness of the Solution

Let V and H be Hilbert spaces. Suppose V is densly embedded into H, V separable, and we identify H with its dual H'. If V' denotes the dual of V we have

$$V \subseteq H \subseteq V',$$

where the embedding of each space in the following is continuous, injectiv and the image of the embedding is dense in the following space.

3.1.1 Formulation of Problem (H)

For $t \in [0, T]$, $0 < T < \infty$, let the sesquilinear form $a(t;\ .\,,\,.\,) : V \times V \to \mathbb{C}$ be a continuous sesquilinear form, i.e. there exists $c > 0$ such that

$$|a(t; \varphi, \psi)| \leq c \|\varphi\|_V \|\psi\|_V, \quad \forall \psi, \varphi \in V \tag{3.1}$$

where c does not depend on t.
It follows that, for each $t \in [0, T]$ the sesquilinear form $a(t; \varphi, \psi)$ defines a continuous linear operator $L(t) : V \to V'$ with

$$a(t; \varphi, \psi) = \big(L(t)\varphi, \psi\big)_H. \tag{3.2}$$

Moreover, we make the following hypotheses:
The sesquilinear form $a(t; \varphi, \psi)$ should be antisymmetric

$$a(t; \varphi, \psi) = \overline{a(t; \psi, \varphi)} \qquad \forall \varphi, \psi \in V. \tag{3.3}$$

A further condition is the V-coercivity, i.e. there exist constants $k_0 \geq 0$, $\alpha > 0$ such that

$$a(t; \varphi, \varphi) + k_0 \|\varphi\|_H^2 \geq \alpha \|\varphi\|_V^2 \qquad \forall t \in [0, T],\ \forall \varphi \in V, \tag{3.4}$$

(note that $a(t;\varphi,\varphi)$ is real by (3.3)).
Moreover, we require the function $t \mapsto a(t;\varphi,\psi)$ (φ,ψ fixed) to be C^1 from $[0,T]$ to $\mathbb{C}$, i.e.

$$a(\,.\,;\varphi,\psi) \in C^1[0,T], \quad \forall \varphi,\psi \in V,$$

such that

$$\Big|\frac{d}{dt}a(t;\varphi,\psi)\Big| \leq c\|\varphi\|_V\|\psi\|_V, \quad \forall t \in [0,T], \tag{3.5}$$

where c again does not depend on t.
We will also consider a weaker condition than (3.5): For all φ, $\psi \in V$ the function $t \mapsto a(t;\varphi,\psi)$ (φ, ψ fixed) should be Lipschitz continuous, i.e.

$$a(\,.\,,\varphi,\psi) \in C^{0,1}[0,T], \quad \forall \varphi,\psi \in V.$$

In light of Lemma 2.12 we see that this holds if and only if the distributional derivative is bounded

$$\Big|\frac{d}{dt}a(t;\varphi,\psi)\Big| \leq c\|\varphi\|_V\|\psi\|_V \tag{3.6}$$

for almost every $t \in [0,T]$.

We consider the problem (H):
Let $f \in L^2(0,T;H)$, $T < \infty$ be given, and the initial values

$$y_0 \in V, \quad y_1 \in H.$$

Then we want to find a solution $y(t,x)$ such that

$$y \in L^2(0,T;V), \quad \frac{dy}{dt} \in L^2(0,T;H),$$

and such that the following holds (in V')

$$\begin{aligned} &\frac{d^2y}{dt^2} + L(t)y = f \quad \text{for} \quad t \in (0,T), \\ &y(0) = y_0, \qquad \frac{dy}{dt}(0) = y_1. \end{aligned} \tag{3.7}$$

Note that we have

$$\frac{dy}{dt} : [0,T] \to V' \text{ continuous}, \quad \text{and} \quad y : [0,T] \to H \text{ continuous},$$

(cf. [13], Remark 29.1). Hence we have a priori

$$y(0) = y_0 \in H, \quad \text{and} \quad \frac{dy}{dt}(0) = y_1 \in V'.$$

From the proof of Theorem 3.1 (more precisely, from the proof of the existence of a solution of problem (H)) we will get that the intitial data can even be fulfilled in the following sense

$$y(0) = y_0 \in V, \quad \text{and} \quad \frac{dy}{dt}(0) = y_1 \in H.$$

We read (3.7) in V' or equivalently in a weak sense, i.e.

$$\Big(\frac{d^2y}{dt^2}, v\Big)_H + \big(L(t)y, v\big)_H = \big(f, v\big)_H \qquad \forall v \in V. \tag{3.8}$$

Note that from now on we will always use the notation $(\, , \,)_H$ instead of $(\, , \,)_{(V',V)}$ or $(\, , \,)_{(H,H)}$. So one should always be aware of which duality, the (V', V)– or (H, H)–duality, is meant. Our aim is to prove the following result:

Theorem 3.1 *Let $T < \infty$ and the conditions (3.1)–(3.4) and (3.5) be satisfied. Then the problem (H)*

$$\frac{d^2y}{dt^2} + L(t)y = f \quad \text{for} \quad t \in (0, T)$$

with the initial conditions

$$y(0) = y_0 \in V, \quad \frac{dy}{dt}(0) = y_1 \in H,$$

and $f \in L^2(0, T; H)$ has a unique solution $y \in L^2(0, T; V)$.
The map

$$\{f, y_0, y_1\} \mapsto \{y, \frac{dy}{dt}\}$$

is linear and continuous

$$L^2(0, T; H) \times V \times H \longrightarrow L^2(0, T; V) \times L^2(0, T; H).$$

It should be mentioned that Theorem 3.1 also holds if the weaker condition (3.6) is satisfied instead of (3.5) (cf. Remark 3.11).

In Section 3.1.2 we will show some integral inequalities like for example Gronwall's lemma. Moreover, we will recall an important functional analytic result about weak compactness. In 3.1.3 we will prove the uniqueness and in 3.1.4 the existence of the solution of the problem (H). Finally, in Section 3.1.5 we show that the solution, if it exists, depends continuously on the initial data y_0, y_1 and on the inhomogeneity f.

3.1.2 Preparations

Integral Inequalities

Lemma 3.2 *Let g, v, $w \in C[0, T]$, and $0 \le h \in L^1(0, T)$. Assume that the following inequalities hold in $[0, T]$*

$$v(t) \le g(t) + \int_0^t h(\tau)v(\tau)d\tau, \qquad w(t) \ge g(t) + \int_0^t h(\tau)w(\tau)d\tau, \tag{3.9}$$

where for each $t \in [0, T]$ equalitiy in the above inequalities holds at most in one equation. Then

$$v(t) < w(t), \quad \forall t \in [0, T]. \tag{3.10}$$

Proof. For $t = 0$ it follows from (3.9) that $v(0) \leq g(0)$, $g(0) \leq w(0)$, where equality can not hold in both equations at the same time, so $v(0) < w(0)$. Assume (3.10) is not true, then there exists $t_0 \in (0, T]$ such that $v(t_0) = w(t_0)$ and $v(t) < w(t)$ for $0 \leq t \leq t_0$. Since $h(t) \geq 0$ it follows that

$$h(\tau)v(\tau) \leq h(\tau)w(\tau) \quad \text{for} \quad 0 \leq \tau \leq t_0.$$

Then by (3.9) we obtain

$$v(t_0) \leq g(t_0) + \int_0^{t_0} h(\tau)v(\tau)d\tau \leq g(t_0) + \int_0^{t_0} h(\tau)w(\tau)d\tau \leq w(t_0),$$

where one of the outermost inequalities is strict. That means we get $v(t_0) < w(t_0)$, which is a contradiction to $v(t_0) = w(t_0)$. □

Lemma 3.3 (Gronwall) *Let $g(t)$, $v(t) \in C[0, T]$, and $0 \leq h(t) \in L_1(0, T)$. Moreover in $[0, T]$ the following inequality should be satisfied*

$$v(t) \leq g(t) + \int_0^t h(\tau)v(\tau)d\tau. \tag{3.11}$$

Then in $[0, T]$

$$v(t) \leq g(t) + \int_0^t g(\tau)h(\tau)e^{H(t)-H(\tau)}d\tau \tag{3.12}$$

where $H(t) = \int_0^t h(\tau)d\tau$. Moreover if $g(t) \in C^1$ we have the estimate

$$v(t) \leq e^{H(t)}\Big(g(0) + \int_0^t g'(\tau)e^{-H(\tau)}d\tau\Big). \tag{3.13}$$

Proof. First we observe that for every $\bar{g} \in C[0, T]$

$$w(t) := \bar{g}(t) + \int_0^t \bar{g}(\tau)h(\tau)e^{H(t)-H(\tau)}d\tau \tag{3.14}$$

solves the integral equation

$$w(t) = \bar{g}(t) + \int_0^t h(\tau)w(\tau)d\tau. \tag{3.15}$$

Indeed, differentiating (3.14) gives

$$\begin{aligned} w'(t) &= \bar{g}'(t) + \bar{g}(t)h(t) + \int_0^t \bar{g}(\tau)h(\tau)e^{H(t)-H(\tau)}d\tau h(t) = \bar{g}'(t) + h(t)w(t) && (3.16)\\ w(0) &= \bar{g}(0), && (3.17) \end{aligned}$$

which clearly is equivalent to (3.15). If we choose $\bar{g} > g$, then (3.15) becomes a strict inequality

$$w(t) > g(t) + \int_0^t h(\tau)w(\tau)d\tau. \tag{3.18}$$

Considering (3.11) and (3.18) we see that the conditions of Lemma 3.2 are satisfied and we obtain the result

$$v(t) < w(t) \qquad \forall t \in [0,T].$$

(3.12) follows by taking the limit $\bar{g} \to g$ and by integrating by parts we get (3.13). $\square$

Lemma 3.4 *Let $v \in C[0,T]$, $v \geq 0$. If v additionally satisfies the following estimate*

$$v(t) \leq c \int_0^t v(\tau)d\tau \quad \textit{for} \quad t \in [0,T],$$

where $c \geq 0$, then $v \equiv 0$.

Proof. Set $g(t) = 0$ and $h(t) = c$, then we have by Gronwall $v(t) \leq 0$, so together with our condition $v(t) \geq 0$ we get $v \equiv 0$. $\square$

Weak Compactness

We will need a typical functional analytical result about weak compactness. It will be used first in step 3 of the existence proof of a solution to the problem (H) and second to show that the solution of problem (H), if it exists, continuously depends on the intitial data. Therefore we review definitions and results, which should be well known from any lecture course or book on functional analysis.

Definition 3.5 *Let H be a Hilbert space.*
(i) A sequence h_n is called to converge weakly to h if

$$\big(g, h_n\big)_H \to \big(g, h\big)_H \textit{ for every } g \in H.$$

One writes

$$h_n \rightharpoonup h \tag{3.19}$$

in this situation.
(ii) A sequence h_n is called weak Cauchy sequence if $\big(g, h_n\big)_H$ is a Cauchy for every $g \in H$.

Obviously $h_n \to h$ implies $h_n \rightharpoonup h$. Note that the weak limit is unique, since $\big(g, h_n\big)_H \to \big(g, h\big)_H$ and $\big(g, h_n\big)_H \to \big(g, \tilde{h}\big)_H$ implies $\big(g, (h - \tilde{h})\big)_H = 0$.

Lemma 3.6 *Let H be a Hilbert space. Then*
(i) $h_n \rightharpoonup h$ implies $\|h\| \leq \liminf \|h_n\|$.
(ii) every weak Cauchy sequence h_n is bounded: $\exists K$ such that $\forall n$ $\|h_n\| \leq K$.
(iii) every weak Cauchy sequence converges weakly.
(iv) for a weakly convergent sequence $h_n \rightharpoonup h$ we have:

$$h_n \to h \quad \textit{if and only if} \quad \limsup \|h_n\| \leq \|h\|.$$

Proof. (i) Note that

$$\|h\|^2 = \big(h,h\big)_H = \liminf \big(h,h_n\big)_H \leq \|h\| \liminf \|h_n\|.$$

(ii) Suppose h_n is a weak Cauchy sequence. Then for every g there exists N_0 such that for $N \geq N_0$ and all $n \geq N_0$ we have

$$\big|\big(g,h_n\big)_H\big| \leq \big|\big(g,h_N\big)_H\big| + \big|\big(g,h_n-h_N\big)_H\big| \leq \big|\big(g,h_N\big)_H\big| + 1 =: K(g).$$

Hence (h_n) is a family of pointwise bounded linear functionals on H and we get that

$$\|h_n\| = \big\|\big(h_n,\,.\,\big)_H\big\| \leq K,$$

i.e. (h_n) is uniformly bounded with respect to the operator norm.
(iii) Suppose h_n is a weak Cauchy sequence. We choose an orthonormal basis g_m and define $a_m := \lim_{n\to\infty}\big(g_m,h_n\big)_H$. Since $h_n = \sum_m \big(g_m,h_n\big)_H g_m$ we obtain

$$\sum_m \big|\big(g_m,h_n\big)_H\big|^2 = \|h_n\|^2 \leq K^2 \quad \text{for all } n.$$

This yields for all N

$$\sum_{m=1}^N |a_m|^2 = \lim_{n\to\infty}\sum_{m=1}^N \big|\big(g_m,h_n\big)_H\big|^2 = \lim_{n\to\infty}\|h_n\|^2 \leq K^2.$$

Therefore $(a_m) \in l^2$ and finally $h := \sum_m a_m g_m \in H$. h is the weak limit of the weak Cauchy sequence h_n, since we have that

$$\big(h_n,g_m\big)_H \to \big(h,g_m\big)_H$$

and that g_m is an orthonormal basis.
(iv) Let h_n be weakly convergent, i.e. $h_n \rightharpoonup h$. Let $\limsup\|h_n\| \leq \|h\|$, then we get by (i) that $\lim\|h_n\| = \|h\|$ and hence

$$\begin{aligned}\|h-h_n\|^2 &= \|h\|^2 - \big(h,h_n\big)_H - \overline{\big(h,h_n\big)_H} + \|h_n\|^2 \\ &= \|h\|^2 - 2\mathrm{Re}\big(h,h_n\big)_H + \|h_n\|^2 \to 0.\end{aligned}$$

For the converse there is nothing to prove. □

Theorem 3.7 (Weak Compactness) *Let H be a Hilbert space and (h_n) be a bounded sequence in H, i.e. $\|h_n\| \leq K$, $n \in \mathbb{N}$ for some constant K. Then there exists a subsequence, which converges weakly to an element $h \in H$, and we have $\|h\| \leq K$.*

Proof. Let $M := \mathrm{span}\big\{h_n \big| n \in \mathbb{N}\big\}$, then M is a closed linear subspace of H and we can apply the projection theorem: every $h \in H$ can be uniquely written as $h = h_\| + h_\perp$ with $h_\| \in M$ and $h_\perp \in M^\perp$, where $M^\perp$ denotes the orthogonal complement of M, i.e. $M^\perp := \big\{g\big|\big(f,g\big)_H = 0,\ \forall f \in M\big\}$. More precisely, since M is closed it is a Hilbert space and has an orthonormal basis $\{m_j\}$. Then we set $h_\| = \sum_j \big(m_j,h\big)_H m_j$. Obviously, for

$g \in \overline{M}^{\perp}$ the scalar product $\big(g, h_n\big)_H = 0$ converges. Thus it suffices to show the result for $g \in \overline{M}$. By Cauchy-Schwarz' inequality we get

$$\left|\big(h_n, h_1\big)_H\right| \le K^2, \quad n \in \mathbb{N}.$$

Hence there exists a subsequence $(h_{n_1(j)})_{j\in\mathbb{N}}$ such that

$$\big(h_{n_1(j)}, h_1\big)_H \text{ converges as } j \to \infty \text{ and } \left|\big(h_{n_1(j)}, h_1\big)_H\right| \le K^2 \text{ for all } j.$$

Considering the scalar product $\big(h_{n_1(j)}, h_2\big)_H$ we obtain by the same procedure as above a subsequence $(h_{n_2(j)})_{j\in\mathbb{N}}$ of $(h_{n_1(j)})_{j\in\mathbb{N}}$, where

$$\big(h_{n_2(j)}, h_2\big)_H \text{ converges as } j \to \infty \text{ and } \left|\big(h_{n_2(j)}, h_2\big)_H\right| \le K^2 \text{ for all } j.$$

Successively, for all $k \in \mathbb{N}$ we get a subsequence $(h_{n_{k+1}(j)})_{j\in\mathbb{N}}$ of $(h_{n_k(j)})_{j\in\mathbb{N}}$ such that

$$\big(h_{n_{k+1}(j)}, h_{k+1}\big)_H \text{ converges as } j \to \infty \text{ and } \left|\big(h_{n_{k+1}(j)}, h_{k+1}\big)_H\right| \le K^2 \text{ for all } j.$$

Choosing the diagonal sequence $(h_{n_k(k)})_{k\in\mathbb{N}}$ we have that $\big(h_{n_k(k)}, h_m\big)_H$ converges for all $m \in \mathbb{N}$. Thus $\big(h_{n_k(k)}, g\big)_H$ converges for all $g \in M$. Let $g \in \overline{M}$ and choose $\varepsilon > 0$ arbitrary. Then there exists $f \in S$ with $\|f - g\|_H \le \varepsilon/4K$ and there is $N_\varepsilon \in \mathbb{N}$ such that

$$\left|\big(h_{n_k(k)} - h_{n_m(m)}, f\big)_H\right| \le \frac{\varepsilon}{2}, \quad \forall k, m \ge N_\varepsilon.$$

Finally we get for $k, m \ge N_\varepsilon$

$$\begin{aligned} \left|\big(h_{n_k(k)} - h_{n_m(m)}, g\big)_H\right| &\le \left|\big(h_{n_k(k)} - h_{n_m(m)}, f\big)_H\right| + \left|\big(h_{n_k(k)} - h_{n_m(m)}, g - f\big)_H\right| \\ &\le \frac{\varepsilon}{2} + \|h_{n_k(k)} - h_{n_m(m)}\|_H \|g - f\|_H \le \frac{\varepsilon}{2} + 2K\frac{\varepsilon}{4K} = \varepsilon. \end{aligned}$$

In summary $l(g) = \lim_{k\to\infty} \big(h_{n_k(k)}, g\big)_H$ exists for all $g \in H$ by Lemma 3.6 (iii). Since l is linear and continuous by

$$|l(g)| = \lim_{k\to\infty} \left|\big(h_{n_k(k)}, g\big)_H\right| \le \limsup_{k\to\infty} \|h_{n_k(k)}\|_H \|g\|_H \le K\|g\|$$

we can apply Riesz' representation theorem, which yields that there exists $h \in H$ with

$$l(g) = \big(g, h\big)_H \quad \text{and} \quad \|h\| = \|l\| \le K.$$

□

3.1.3 Uniqueness of the Solution of Problem (H)

Theorem 3.8 *Let the conditions (3.1)–(3.4) and (3.5) be satisfied and let* $f \in L^2(0, T; H)$, $T < \infty$ *be given, and the initial values*

$$y_0 \in V, \quad y_1 \in H.$$

If there exists a solution $y(t,x)$ such that

$$y \in L^2(0,T;V), \quad \frac{dy}{dt} \in L^2(0,T;H)$$

and such that the following holds

$$\frac{d^2y}{dt^2} + L(t)y = f \quad \text{for} \quad t \in (0,T),$$
$$y(0) = y_0, \quad \frac{dy}{dt}(0) = y_1,$$

then the solution is unique.

Proof. Let y be a solution of (H) with $y_0 = 0$, $y_1 = 0$ and $f = 0$. We have to show that $y \equiv 0$.
Let $s \in (0,T)$ and set

$$\psi(t) := \begin{cases} -\int_s^t y(\sigma)d\sigma & t < s \\ 0 & t \geq s. \end{cases}$$

Then from (3.8) we get by our conditions

$$\int_0^T \Big(\frac{d^2y}{dt^2} + L(t)y(t), \psi\Big)_H dt = 0 \tag{3.20}$$

From the relation

$$\frac{d}{dt}\big(u(t), v(t)\big)_H = \big(u'(t), v(t)\big)_H + \big(u(t), v'(t)\big)_H, \tag{3.21}$$

which holds for all $u \in AC([0,T];V')$ and $v \in AC([0,T];V)$ we get

$$\frac{d}{dt}\big(y'(t), \psi(t)\big)_H = \big(y''(t), \psi(t)\big)_H + \big(y'(t), \psi'(t)\big)_H, \tag{3.22}$$

since $\frac{d^2y}{dt^2} \in L^2(0,T;V')$ and therefore $y' \in AC([0,T];V')$. Using (3.22) and integration by parts we can rewrite (3.20)

$$\begin{aligned}
&\int_0^T \Big(\frac{d^2y}{dt^2} + L(t)y(t), \psi\Big)_H dt \\
&= \int_0^T \Big(\big(y''(t), \psi(t)\big)_H + \big(L(t)y(t), \psi(t)\big)_H\Big) dt \\
&= \int_0^T \Big(\big(y''(t), \psi(t)\big)_H + a\big(t; y(t), \psi(t)\big)\Big) dt \\
&= \int_0^T \frac{d}{dt}\big(y'(t), \psi(t)\big)_H dt + \int_0^T \Big(a\big(t; y(t), \psi(t)\big) - \big(y'(t), \psi'(t)\big)_H\Big) dt \\
&= \big(y'(T), \psi(T)\big)_H - \big(y_1, \psi(0)\big)_H + \int_0^T \Big(a\big(t; y(t), \psi(t)\big) - \big(y'(t), \psi'(t)\big)_H\Big) dt \\
&= \int_0^T \Big(a\big(t; y(t), \psi(t)\big) - \big(y'(t), \psi'(t)\big)_H\Big) dt
\end{aligned}$$

For the last equality we have used that $y_1 = 0$ and by the definition of ψ that $\psi(T) = 0$. The above calculation together with (3.20) finally implies

$$\int_0^T \Big(a\big(t; y(t), \psi(t)\big) - \big(y'(t), \psi'(t)\big)_H \Big) dt = 0 \tag{3.23}$$

That means

$$\mathrm{Re}\bigg(\int_0^s \Big(a\big(t; \psi'(t), \psi(t)\big) - \big(y'(t), y(t)\big)_H \Big) dt \bigg) = 0 \tag{3.24}$$

We set

$$\frac{d}{dt} a(t; u, v) = a'(t; u, v), \qquad \forall u, v \in V.$$

Then we obtain for all u, $v \in C^1([0,T]; V)$

$$\frac{d}{dt} a\big(t; u(t), v(t)\big) = a'\big(t; u(t), v(t)\big) + a\big(t; u'(t), v(t)\big) + a\big(t; u(t), v'(t)\big). \tag{3.25}$$

So considering (3.25) and condition (3.3) we have

$$\begin{aligned} \frac{d}{dt} a\big(t; \psi(t), \psi(t)\big) &= a'\big(t; \psi(t), \psi(t)\big) + a\big(t; \psi'(t), \psi(t)\big) + a\big(t; \psi(t), \psi'(t)\big) \\ &= a'\big(t; \psi(t), \psi(t)\big) + a\big(t; \psi'(t), \psi(t)\big) + \overline{a\big(t; \psi'(t), \psi(t)\big)} \\ &= a'\big(t; \psi(t), \psi(t)\big) + 2 \ \mathrm{Re} \ a\big(t; \psi'(t), \psi(t)\big) \end{aligned} \tag{3.26}$$

By (3.26) and (3.24) we get

$$\begin{aligned} &\int_0^s \Big(\frac{d}{dt} a\big(t; \psi(t), \psi(t)\big) - a'\big(t; \psi(t), \psi(t)\big) - \frac{d}{dt}\big(y(t), y(t)\big)_H \Big) dt \\ &= \int_0^s \Big(2 \ \mathrm{Re} \ a\big(t; \psi'(t), \psi(t)\big) - \frac{d}{dt}\big(y(t), y(t)\big)_H \Big) dt \\ &= \int_0^s \Big(2 \ \mathrm{Re} \ a\big(t; \psi'(t), \psi(t)\big) - \big(y'(t), y(t)\big)_H - \big(y(t), y'(t)\big)_H \Big) dt \\ &= 2 \ \mathrm{Re} \ \int_0^s \Big(a\big(t; \psi'(t), \psi(t)\big) - \big(y'(t), y(t)\big)_H \Big) dt = 0. \end{aligned}$$

Therefore we have

$$\int_0^s \Big(\frac{d}{dt} a\big(t; \psi(t), \psi(t)\big) - a'\big(t; \psi(t), \psi(t)\big) - \frac{d}{dt}\big(y(t), y(t)\big)_H \Big) dt = 0.$$

This implies

$$a\big(0; \psi(0), \psi(0)\big) + \|y(s)\|_H^2 = - \int_0^s a'\big(t; \psi(t), \psi(t)\big) dt, \tag{3.27}$$

since $\psi(s) = 0$ and our intitial condition $y_0 = 0$. Because of condition (3.4) for our sesquilinear form a there exist constants $k_0 \geq 0$ and $\alpha > 0$ such that

$$\alpha \|\psi(0)\|_V^2 - k_0 \|\psi(0)\|_H^2 \leq a\big(0; \psi(0), \psi(0)\big). \tag{3.28}$$

Adding $\|y(s)\|_H^2$ to both sides and using (3.27) we have

$$\begin{aligned}\alpha\|\psi(0)\|_V^2 - k_0\|\psi(0)\|_H^2 + \|y(s)\|_H^2 &\leq a\big(0;\psi(0),\psi(0)\big) + \|y(s)\|_H^2 \\ &= -\int_0^s a'\big(t;\psi(t),\psi(t)\big)\,dt \\ &\leq \int_0^s \big|a'\big(t;\psi(t),\psi(t)\big)\big|\,dt \\ &\leq c\int_0^s \|\psi(t)\|_V^2 dt,\end{aligned}$$

where the last inequality follows from our condition (3.5). So we get

$$\alpha\|\psi(0)\|_V^2 + \|y(s)\|_H^2 \leq c\int_0^s \|\psi(t)\|_V^2 dt + k_0\|\psi(0)\|_H^2 \tag{3.29}$$

or

$$\|\psi(0)\|_V^2 + \|y(s)\|_H^2 \leq c_1\Big(\int_0^s \|\psi(t)\|_V^2 dt + \|\psi(0)\|_H^2\Big). \tag{3.30}$$

Define $w(t) := \int_0^t y(\sigma)d\sigma$, so $\psi(t) = w(s) - w(t)$ and $\psi(0) = w(s)$. Then we can rewrite (3.30)

$$\begin{aligned}\|w(s)\|_V^2 + \|y(s)\|_H^2 \;&\leq\; c_1\Big(\int_0^s \|w(t)-w(s)\|_V^2 dt + \|w(s)\|_H^2\Big) \\ &\leq\; c_1\Big(\int_0^s \|w(t)\|_V^2 dt + \int_0^s \|w(s)\|_V^2 dt + \int_0^s \|y(t)\|_H^2 dt\Big)\end{aligned}$$

and finally

$$(1-2c_1 s)\|w(s)\|_V^2 + \|y(s)\|_H^2 \leq c_2\int_0^s \Big(\|w(t)\|_V^2 + \|y(t)\|_H^2\Big)dt. \tag{3.31}$$

Now set $s_0 = \frac{1}{4c_1}$, then we obtain for $0 \leq s \leq s_0$

$$\|w(s)\|_V^2 + \|y(s)\|_H^2 \leq c_3\int_0^s \Big(\|w(t)\|_V^2 + \|y(t)\|_H^2\Big)dt. \tag{3.32}$$

Let $v(s) := \|w(s)\|_V^2 + \|y(s)\|_H^2$, then $v(s) \in C[0,s_0]$ and $v(s) \geq 0$. So we get from (3.32) by Lemma 3.4 that

$$\|w(s)\|_V^2 + \|y(s)\|_H^2 = 0 \qquad \text{for } s \in [0,s_0] \tag{3.33}$$

or

$$\Big\|\int_0^t y(\sigma)d\sigma\Big\|_V^2 + \|y(s)\|_H^2 = 0 \qquad \text{for } s \in [0,s_0]. \tag{3.34}$$

That means $y \equiv 0$ on $[0,s_0]$. Analoguously one can show that $y \equiv 0$ on $[s_0, 2s_0]$, etc. So we have proven that

$$y \equiv 0 \quad \text{on } [0,T].$$

This implies the uniqueness of the solution. Indeed let $\tilde{y_1}$, $\tilde{y_2}$ be two solutions of problem (H), then for $i = 1, 2$

$$\Big(\frac{d^2}{dt^2} + L(t)\Big)\tilde{y_i} = f \qquad t \in (0, T)$$

with initial conditions

$$\tilde{y_i}(0) = y_0, \qquad \frac{d\tilde{y_i}}{dt}(0) = y_1.$$

By subtracting the two equations for $i = 1$ resp. $i = 2$, we get

$$\Big(\frac{d^2}{dt^2} + L(t)\Big)(\tilde{y_1} - \tilde{y_2}) = 0, \qquad t \in [0, T]$$

$$(\tilde{y_1} - \tilde{y_2})(0) = 0, \qquad \frac{d}{dt}(\tilde{y_1} - \tilde{y_2})(0) = 0,$$

and applying the above result yields $\tilde{y_1} = \tilde{y_2}$, hence uniqueness. □

3.1.4 Existence of a Solution of Problem (H)

Theorem 3.9 *Let the conditions (3.1)–(3.4) and (3.5) be satisfied. Then the problem* (H)

$$\frac{d^2y}{dt^2} + L(t)y = f \quad \textit{for} \quad t \in (0, T)$$

with the initial conditions

$$y(0) = y_0 \in V, \quad \frac{dy}{dt}(0) = y_1 \in H,$$

and $f \in L^2(0, T; H), T < \infty$ *has a solution (which is unique by Theorem 3.8).*

To prove this result, i.e. to show existence of a solution of problem (H), we make use of the Galerkin method. In the following we will first give the definition of an approximation of a Hilbert space, a so-called Galerkin approximation, and second describe what is meant by a Galerkin method.

Definition 3.10 *Let* V *be a separable Hilbert space and* $\{V_m\}_{m\in\mathbb{N}}$ *a family of finite dimensional vector spaces satisfying the axioms:*

$$V_m \subset V \ (\dim V_m < \infty) \tag{3.35}$$

$$V_m \to V \ \textit{when } m \to \infty \textit{ in the following sense:} \tag{3.36}$$

there exists $\tilde{V}$ *a dense subspace of* V*, such that, for all* $v \in \tilde{V}$*, we can find a sequence* $\{v_m\}_{m\in\mathbb{N}}$ *satisfying:*
for all m *we have* $v_m \in V_m$ *and* $v_m \to v$ *in* V *as* $m \to \infty$.
We call the space V_m *the Galerkin approximation of order* m *of* V.

To establish existence of a solution of the problem (H) in a space of functions constructed over a separable Hilbert space V, we suppose that we know that the solution z of the problem (H) is unique (which we have already shown in Section 3.1.3). Then the first step is to make a choice of a Galerkin approximation V_m of V, and second define a so-called approximate problem (H_m) in the finite dimensional space V_m, which has a unique solution y_m.
The procedure of study will be the following:
(i) Step 1: **Approximate problem** (H_m): We formulate the approximate problem (H_m) and define its solution y_m.
(ii) Step 2: **A priori estimates**: We establish some estimates on y_m and y'_m, from which we obtain that y_m resp. y'_m are bounded in $L^2(0,T;V)$ resp. $L^2(0,T;H)$.
(iii) Step 3: **Passage to the limit**: By using the result of weak compactness, which we derived in Section 3.1.2, it is possible to extract from $(y_m)_{m\in\mathbb{N}}$ a subsequence $(y_{m_\eta})_{\eta\in\mathbb{N}}$ which has a weak limit in $L^2(0,T;V)$ and the derivative $(y'_{m_\eta})_{\eta\in\mathbb{N}}$ converges weakly in $L^2(0,T;H)$. Let z be the limit obtained from $(y_{m_\eta})_{\eta\in\mathbb{N}}$ as $\eta\to\infty$.
(iv) Step 4: z **is the solution of problem** (H): We will prove that z is the solution of problem (H), therefore the solution sought from uniqueness.

Proof.
Step 1: **Approximate problem** (H_m):
Let $\{V_m\}_{m\in\mathbb{N}}$ be a family of finite dimensional vector subspaces of V choosen as in Definition 3.10. Set $d_m = \dim V_m$ and let $\{w_{jm}\}$, $j = 1,\ldots,d_m$ be a basis of V_m. Set

$$
\begin{aligned}
y_{0m} &= \sum_{j=1}^{d_m} \xi^0_{jm} w_{jm},\\
y_{1m} &= \sum_{j=1}^{d_m} \xi^1_{jm} w_{jm},
\end{aligned}
$$

where $y_{0m}\to y_0$ in V for $m\to\infty$ and $y_{1m}\to y_1$ in H for $m\to\infty$.
The approximate problem (H_m) is then: Find a sequence $y_m(t)$ in the form

$$y_m(t) = \sum_{j=1}^{d_m} g_{jm}(t) w_{jm},$$

which approximates the prospective solution such that

$$\frac{d^2}{dt^2} y_m(t) + L(t) y_m(t) = f(t) \tag{3.37}$$

and

$$y_m(0) = y_{0m}, \qquad y'_m(0) = y_{1m}.$$

So applying the inner products $\big(\,.\,, w_{jm}\big)_H$ to equation (3.37) we get a system of d_m linear ordinary differential equations

$$\frac{d^2}{dt^2}\big(y_m(t), w_{jm}\big)_H + a\big(t; y_m(t), w_{jm}\big) = \big(f(t), w_{jm}\big)_H, \quad \text{for } 1\le j\le d_m, \tag{3.38}$$

$$y_m(0) = y_{0m}, \qquad y_m'(0) = y_{1m}.$$

This system has a unique solution.

Step 2: **A priori estimates**:
From equation (3.37) we obtain

$$\Big(y_m''(t), \sum_{j=1}^{d_m} g_{jm}'(t) w_{jm}\Big)_H + a\Big(t; y_m(t), \sum_{j=1}^{d_m} g_{jm}'(t) w_{jm}\Big) = \Big(f(t), \sum_{j=1}^{d_m} g_{jm}'(t) w_{jm}\Big)_H$$

hence

$$\big(y_m''(t), y_m'(t)\big)_H + a\big(t; y_m(t), y_m'(t)\big) = \big(f(t), y_m'(t)\big)_H. \tag{3.39}$$

As in equation (3.21) we now have the following relation

$$\frac{d}{dt}\big(y_m'(t), y_m'(t)\big)_H = \big(y_m''(t), y_m'(t)\big)_H + \big(y_m'(t), y_m''(t)\big)_H \tag{3.40}$$

and therefore

$$\begin{aligned}
\big(y_m''(t), y_m'(t)\big)_H &= \frac{d}{dt}\big(y_m'(t), y_m'(t)\big)_H - \big(y_m'(t), y_m''(t)\big)_H \\
&= \frac{d}{dt}\big(y_m'(t), y_m'(t)\big)_H - \overline{\big(y_m''(t), y_m'(t)\big)_H} \\
&= \frac{d}{dt}\|y_m'(t)\|_H^2 + \overline{a\big(t; y_m(t), y_m'(t)\big)} - \overline{\big(f(t), y_m'(t)\big)_H},
\end{aligned} \tag{3.41}$$

where the last equality follows from (3.39). Considering (3.39) and (3.41) we obtain

$$\begin{aligned}
&\frac{d}{dt}\|y_m'(t)\|_H^2 + a\big(t; y_m(t), y_m'(t)\big) + a\big(t; y_m'(t), y_m(t)\big) = 2\ \mathrm{Re}\big(f(t), y_m'(t)\big)_H \\
&\frac{d}{dt}\|y_m'(t)\|_H^2 + \frac{d}{dt}a\big(t; y_m(t), y_m(t)\big) - a'\big(t; y_m(t), y_m(t)\big) = 2\ \mathrm{Re}\big(f(t), y_m'(t)\big)_H \\
&\frac{d}{dt}\left\{\|y_m'(t)\|_H^2 + a\big(t; y_m(t), y_m(t)\big)\right\} = a'\big(t; y_m(t), y_m(t)\big) + 2\ \mathrm{Re}\big(f(t), y_m'(t)\big)_H
\end{aligned} \tag{3.42}$$

Integration of (3.42) yields

$$\begin{aligned}
&a\big(t; y_m(t), y_m(t)\big) + \|y_m'(t)\|_H^2 - a\big(0; y_m(0), y_m(0)\big) - \|y_m'(0)\|_H^2 = \\
&= \int_0^t a'\big(\xi; y_m(\xi), y_m(\xi)\big) d\xi + 2\ \mathrm{Re} \int_0^t \big(f(\xi), y_m'(\xi)\big)_H d\xi
\end{aligned} \tag{3.43}$$

and hence

$$\begin{aligned}
&a\big(t; y_m(t), y_m(t)\big) + \|y_m'(t)\|_H^2 = \\
&a\big(0; y_{0m}, y_{0m}\big) + \|y_{1m}\|_H^2 + \int_0^t a'\big(\xi; y_m(\xi), y_m(\xi)\big) d\xi + 2\ \mathrm{Re} \int_0^t \big(f(\xi), y_m'(\xi)\big)_H d\xi
\end{aligned} \tag{3.44}$$

For the right-hand side (RHS) of (3.44) we have the following estimate

$$
\begin{aligned}
|\mathrm{RHS}| &\leq c_1\|y_{0m}\|_V^2 + \|y_{1m}\|_H^2 + c_2\int_0^t \|y_m(\xi)\|_V^2 d\xi + 2\int_0^t \|f(\xi)\|_H \|y_m'(\xi)\|_H d\xi \\
&\leq c_1\|y_{0m}\|_V^2 + \|y_{1m}\|_H^2 + c_2\int_0^t \|y_m(\xi)\|_V^2 d\xi + \int_0^t \|f(\xi)\|_H^2 d\xi + \int_0^t \|y_m'(\xi)\|_H^2 d\xi
\end{aligned}
$$

where we have used condition (3.1), (3.5) and $2|ab| \leq a^2 + b^2$. For the left-hand side (LHS) of (3.44) we get

$$
\mathrm{LHS} \geq \alpha\|y_m(t)\|_V^2 - k_0\|y_m(t)\|_H^2 + \|y_m'(t)\|_H^2
$$

by the condititon (3.4). Using $y_m(t) = y_{0m} + \int_0^t y_m'(\xi)d\xi$ and the Cauchy-Schwarz inequality we calculate

$$
\begin{aligned}
&\|y_m(t)\|_H^2 = \big(y_m(t), y_m(t)\big)_H \\
&\quad = \|y_{0m}\|_H^2 + 2\ \mathrm{Re}\int_0^t \big(y_{0m}, y_m'(\xi)\big)_H d\xi + \Big(\int_0^t y_m'(\xi)d\xi, \int_0^t y_m'(\xi)d\xi\Big)_H \\
&\quad = \|y_{0m}\|_H^2 + 2\ \mathrm{Re}\int_0^t \big(y_{0m}, y_m'(\xi)\big)_H d\xi + \int_0^t\int_0^t \big(y_m'(\xi), y_m'(\eta)\big) d\xi d\eta \\
&\quad \leq \|y_{0m}\|_H^2 + \int_0^t \|y_{0m}\|_H^2 d\xi + \int_0^t \|y_m'(\xi)\|_H^2 d\xi + \int_0^t\int_0^t \|y_m'(\xi)\|_H \|y_m'(\eta)\|_H d\xi d\eta \\
&\quad \leq \|y_{0m}\|_H^2 + \int_0^t \|y_{0m}\|_H^2 d\xi + \int_0^t \|y_m'(\xi)\|_H^2 d\xi + \frac{1}{2}\int_0^t\int_0^t \big(\|y_m'(\xi)\|_H^2 + \|y_m'(\eta)\|_H^2\big) d\xi d\eta \\
&\quad \leq \|y_{0m}\|_H^2 + t\|y_{0m}\|_H^2 + \int_0^t \|y_m'(\xi)\|_H^2 d\xi + t\int_0^t \|y_m'(\xi)\|_H^2 d\xi \\
&\quad = (1+t)\big(\|y_{0m}\|_H^2 + \int_0^t \|y_m'(\xi)\|_H^2 d\xi\big).
\end{aligned}
$$

So finally we have

$$
\|y_m(t)\|_H^2 \leq (1+t)\Big(\|y_{0m}\|_H^2 + \int_0^t \|y_m'(\xi)\|_H^2 d\xi\Big). \tag{3.45}
$$

Using the estimate for the (LHS) of (3.44) we get

$$
\alpha\|y_m(t)\|_V^2 - k_0\|y_m(t)\|_H^2 + \|y_m'(t)\|_H^2 \leq \mathrm{LHS} = \mathrm{RHS} \leq |\mathrm{RHS}|
$$

and therefore

$$
\alpha\|y_m(t)\|_V^2 + \|y_m'(t)\|_H^2 \leq |\mathrm{RHS}| + k_0\|y_m(t)\|_H^2. \tag{3.46}
$$

Next we will use (3.45) and the estimate of the (RHS) of (3.44)

$$\begin{aligned}
&\min\{1,\alpha\}\Big(\|y_m(t)\|_V^2 + \|y_m'(t)\|_H^2\Big)\\
&\le \alpha\|y_m(t)\|_V^2 + \|y_m'(t)\|_H^2 \le |\text{RHS}| + k_0\|y_m(t)\|_H^2\\
&\le |\text{RHS}| + k_0(1+t)\Big(\|y_{0m}\|_H^2 + \int_0^t \|y_m'(\xi)\|_H^2 d\xi\Big)\\
&\le |\text{RHS}| + \widetilde{k_0}(1+t)\|y_{0m}\|_V^2 + k_0(1+t)\int_0^t \|y_m'(\xi)\|_H^2 d\xi\\
&\le \big(c_1 + \widetilde{k_0}(1+t)\big)\|y_{0m}\|_V^2 + \|y_{1m}\|_H^2 + \int_0^t \|f(\xi)\|_H^2 d\xi + c_2\int_0^t \|y_m(\xi)\|_V^2 d\xi\\
&\quad + \big(1 + k_0(1+t)\big)\int_0^t \|y_m'(\xi)\|_H^2 d\xi\\
&\le \gamma_t\|y_{0m}\|_V^2 + \|y_{1m}\|_H^2 + \int_0^t \|f(\xi)\|_H^2 d\xi\\
&\quad + \max\{c_2, 1 + k_0 + k_0 t\}\int_0^t \Big(\|y_m'(\xi)\|_H^2 + \|y_m(\xi)\|_V^2\Big) d\xi,
\end{aligned} \tag{3.47}$$

where we have set $\gamma_t := c_1 + \widetilde{k_0}(1+t)$. Note that in the calculation above we used the fact that $\|\,.\,\|_H \le c\|\,.\,\|_V$ for some constant c and hence $\widetilde{k_0} = c \cdot k_0$.
Now define $w_m(t) := \|y_m'(t)\|_H^2 + \|y_m(t)\|_V^2$ and divide equation (3.47) by $\min\{1,\alpha\}$, then we obtain

$$w_m(t) \le C_1\Big(\|y_{0m}\|_V^2 + \|y_{1m}\|_H^2 + \int_0^t \|f(\xi)\|_H^2 d\xi\Big) + C_2\int_0^t w_m(\xi)d\xi. \tag{3.48}$$

Applying Gronwall's Lemma on (3.48) we get

$$w_m(t) \le C, \qquad \text{with } C = c_3\Big(\|y_0\|_V^2 + \|y_1\|_H^2 + \varepsilon + \int_0^T \|f(\xi)\|_H^2 d\xi\Big). \tag{3.49}$$

Therefore (y_m) is bounded in $L^2(0,T;V)$ and $\left(\frac{dy_m}{dt}\right)$ is bounded in $L^2(0,T;H)$.

Step 3: **Passage to the limit**:
Now we construct a subsequence (y_{m_η}) of (y_m) such that

$$y_{m_\eta} \rightharpoonup z \quad \text{in } L^2(0,T;V), \quad \text{for } \eta \to \infty,$$

and

$$\frac{dy_{m_\eta}}{dt} \rightharpoonup \tilde{z} \quad \text{in } L^2(0,T;H), \quad \text{for } \eta \to \infty.$$

(the existence of such a subsequence (y_{m_η}) and such elements z, $\tilde{z}$ follows from Theorem 3.7). Therefore we obtain for all $\varphi \in \mathscr{D}(0,T)$

$$\begin{aligned}
\Big(\frac{dy_{m_\eta}}{dt}(t), \varphi \otimes v\Big)_H &= -\int_0^T \big(y_{m_\eta}(t), v\big)_H \varphi'(t)dt\\
&= \int_0^T \big(y_{m_\eta}'(t), v\big)_H \varphi(t)dt \to \int_0^T \big(\tilde{z}(t), v\big)_H \varphi(t)dt \quad \text{as } j \to \infty,
\end{aligned} \tag{3.50}$$

and

$$\big(\frac{dz}{dt}, \varphi \otimes v\big)_H = \lim_{\eta\to\infty} \big(\frac{dy_{m_\eta}}{dt}, \varphi \otimes v\big)_H. \tag{3.51}$$

From (3.50) and (3.51) we deduce

$$\tilde{z} = \frac{dz}{dt}.$$

Consider $\varphi \in \mathscr{D}(0,T)$ and $v \in V$. From Definition 3.10 we obtain the existence of a sequence $\{v_m\}_{m\in\mathbb{N}}$, $v_m \in V_m$ for all m such that $v_m \to v$ strongly in V. We define

$$\begin{aligned} &\psi_m(t) = \varphi \otimes v_m \quad \text{(i.e. } \psi_m(t) = \varphi(t)v_m), \\ &\psi(t) = \varphi \otimes v \quad \text{(i.e. } \psi(t) = \varphi(t)v). \end{aligned}$$

We have

$$\psi_m \to \psi \text{ in } L^2(0,T;V) \text{ strongly}, \tag{3.52}$$

$$\psi_m' \to \psi' \text{ in } L^2(0,T;V) \text{ strongly}. \tag{3.53}$$

From (3.38) we deduce

$$\int_0^T \Big(\big(\frac{d^2}{dt^2} y_{m_\eta}(t), \psi_{m_\eta}(t)\big)_H + a\big(t; y_{m_\eta}(t), \psi_{m_\eta}(t)\big)\Big)dt = \int_0^T \big(f(t), \psi_{m_\eta}(t)\big)_H dt. \tag{3.54}$$

After integrating the first term by parts and noting that the boundary terms vanish because of the compact support of φ we obtain

$$-\int_0^T \Big(\big(y_{m_\eta}'(t), v_{m_\eta}\big)_H \varphi'(t) - a\big(t; y_{m_\eta}(t), \psi_{m_\eta}(t)\big)\Big)dt = \int_0^T \big(f(t), \psi_{m_\eta}(t)\big)_H dt. \tag{3.55}$$

We apply our result from step 2, that means all integrands are bounded and by taking (3.52) and (3.53) into account we get the following limit for $\eta \to \infty$

$$-\int_0^T \big(z'(t), v\big)_H \varphi'(t)dt + \int_0^T a(t; z(t), \psi(t))dt = \int_0^T \big(f(t), \psi(t)\big)_H dt. \tag{3.56}$$

We interpret the first term as a distributional action of $\big(z''(\,.\,), v\big)_H \in \mathscr{D}'(0,T)$ on $\varphi \in \mathscr{D}(0,T)$. In this sense we can rewrite (3.56) in the following way

$$\int_0^T \Big(\big(\frac{d^2}{dt^2} z(t), \psi(t)\big)_H + a\big(t; z(t), \psi(t)\big)\Big)dt = \int_0^T \big(f(t), \psi(t)\big)_H dt, \tag{3.57}$$

which is (3.8).
Next we will show that $z(0) = y_0$. Choose $\varphi \in C^1[0,T]$ such that $\varphi(T) = 0$. Note that

$$\int_0^T \Big(\frac{dy_{m_\eta}(t)}{dt}, \psi_{m_\eta}(t)\Big)_H dt = -\big(y_{0m_\eta}, \psi_{m_\eta}(0)\big)_H - \int_0^T \big(y_{m_\eta}(t), \psi_{m_\eta}'(t)\big)_H dt, \tag{3.58}$$

where the boundary term for T vanishes because of $\varphi(T) = 0$. Then for $\eta \to \infty$ we deduce

$$\int_0^T \big(z'(t), \psi(t)\big)_H dt = -\big(y_0, v\big)_H \varphi(0) - \int_0^T \big(z(t), \psi'(t)\big)_H dt. \tag{3.59}$$

This passage to the limit is permissible because of our result we derived in step 2. Moreover, we have

$$\int_0^T \big(z'(t), \psi(t)\big)_H dt = -\big(z(0), v\big)_H \varphi(0) - \int_0^T \big(z(t), \psi'(t)\big)_H dt. \tag{3.60}$$

By comparison of (3.59) and (3.60) we finally obtain

$$\big(y_0, v\big)_H = \big(z(0), v\big), \quad \forall v \in V,$$

which implies $z(0) = y_0$. To finish the proof it is only left to verify that $z'(0) = y_1$. So let again $\varphi \in C^1[0,T]$ with $\varphi(T) = 0$. Then set

$$\begin{aligned} \psi_m(t) &= \varphi(t) v_m, \\ \psi(t) &= \varphi(t) v, \end{aligned}$$

where $v_m \in V$ such that $v_m \to v$ strongly in V. From (3.38) we deduce

$$\begin{aligned} \int_0^T \big(f(t), \psi_{m_\eta}(t)\big)_H dt &= \int_0^T \varphi(t) \frac{d^2}{dt^2} \big(y_{m_\eta}, v_{m_\eta}\big)_H dt + \int_0^T a\big(t; y_{m_\eta}(t), \psi_{m_\eta}(t)\big) dt \\ = -\varphi(0)\big(y'_{m_\eta}(0), v_{m_\eta}\big)_H &- \int_0^T \Big(\big(y'_{m_\eta}(t), \psi'_{m_\eta}(t)\big)_H - a\big(t; y_{m_\eta}(t), \psi_{m_\eta}(t)\big)\Big) dt. \end{aligned}$$

Therefore

$$\begin{aligned} &-\int_0^T \Big(\big(y'_{m_\eta}(t), \psi'_{m_\eta}(t)\big)_H - a\big(t; y_{m_\eta}(t), \psi_{m_\eta}(t)\big)\Big) dt \\ &= \int_0^T \big(f(t), \psi_{m_\eta}(t)\big)_H dt + \big(y_{1m_\eta}, \psi_{m_\eta}(0)\big)_H. \end{aligned}$$

All integrands are bounded by step 2 and converge pointwise, so we can apply the theorem of dominated convergence and finally get for $\eta \to \infty$ the following limit

$$-\int_0^T \Big(\big(z'(t), \psi'(t)\big)_H - a\big(t; z(t), \psi(t)\big)\Big) dt = \int_0^T \big(f(t), \psi(t)\big)_H dt + \big(y_1, \psi(0)\big)_H. \tag{3.61}$$

We calculate the first term in (3.61) again by integration by parts

$$\begin{aligned} -\int_0^T \big(z'(t), v\big)_H \varphi'(t) dt &= -\int_0^T \frac{d}{dt}\big(z(t), v\big)_H \varphi'(t) dt = \\ = \big(z'(0), v\big)_H \varphi(0) &+ \int_0^T \frac{d^2}{dt^2}\big(z(t), v\big)_H \varphi(t) dt. \end{aligned} \tag{3.62}$$

Plugging (3.62) into (3.61) we obtain

$$\begin{aligned} &\int_0^T \Big(\frac{d^2}{dt^2} z(t) + L(t) z(t), \psi(t)\Big)_H dt + \big(z'(0), \psi(0)\big)_H = \\ &= \int_0^T \big(f(t), \psi(t)\big)_H dt + \big(y_1, \psi(0)\big)_H. \end{aligned} \tag{3.63}$$

Comparing the last equation with (3.8) it follows that for $\varphi \in C^1[0,T]$ with $\varphi(T) = 0$ arbitrary we get

$$\big(z'(0), v\big)_H \varphi(0) = \big(y_1, v\big)_H \varphi(0), \quad \forall v \in V, \quad \text{i.e.} \quad z'(0) = y_1.$$

So z is a solution of the problem (H). □

Remark 3.11 Upon inspecting the proof of existence and uniqueness of a solution of the problem (H) we find that the proof also works if the sesquilinear form $a(t;\varphi,\psi)$ fulfills the weaker condition (3.6) instead of (3.5). Note that a generalized version of the main theorem of calculus holds for functions in L^1_{loc} (by (3.6) the distributional derivative of $a(t;\varphi,\psi)$ is in L^∞ and hence L^1_{loc}). To adapt the proof one has to assume occasionally that equations hold for almost all t. It should be mentioned that also Theorem 3.12 and Theorem 3.13 still hold if we replace condition (3.5) by (3.6).

3.1.5 Continuity of the Solution with Respect to the Data

Theorem 3.12 *If $y \in L^2(0,T;V)$ is the solution to*

$$\frac{d^2y}{dt^2} + L(t)y = f \quad \textit{for} \quad t \in (0,T)$$

with the initial conditions

$$y(0) = y_0 \in V, \quad \frac{dy}{dt}(0) = y_1 \in H,$$

and $f \in L^2(0,T;H)$, $T < \infty$, then the map

$$\{f, y_0, y_1\} \mapsto \{y, \frac{dy}{dt}\}$$

is linear and continuous

$$L^2(0,T;H) \times V \times H \longrightarrow L^2(0,T;V) \times L^2(0,T;H).$$

Proof. To show that the solution depends continuously on the data we integrate (3.49) with respect to t

$$\int_0^T \|y_m'(t)\|_H^2 dt + \int_0^T \|y_m(t)\|_V^2 dt \le c_3 T\Big(\|y_0\|_V^2 + \|y_1\|_H^2 + \int_0^t \|f(\xi)\|_H^2 d\xi\Big)$$

and by Theorem 3.7 we conclude that

$$\int_0^T \|z'(t)\|_H^2 dt + \int_0^T \|z(t)\|_V^2 dt \le c_3 T\Big(\|y_0\|_V^2 + \|y_1\|_H^2 + \int_0^t \|f(\xi)\|_H^2 d\xi\Big).$$

□

3.2 Regularity of the Solution

Now we will prove a regularity result, which shows that we are able to solve our problem (H) within the scope of Sobolev spaces. We first assume that the operator L and the corresponding sesquilinear form $a(\varphi,\psi)$ do not depend on t.

Theorem 3.13 *We consider the hyperbolic equation*

$$\frac{d^2y(t)}{dt^2} + Ly(t) = f(t) \quad \text{in } (0,T), \tag{3.64}$$

with the initial conditions

$$y(0) = y_0, \qquad y'(0) = y_1. \tag{3.65}$$

Expect of the preconditions we had in Section 3.1 we assume that

$$f \in H^{k-1}(0,T;H), \quad k \geq 1, \tag{3.66}$$

$$\frac{d^jy(0)}{dt^j} \in V, \quad j = 0,\dots,k-1, \qquad \frac{d^ky(0)}{dt^k} \in H. \tag{3.67}$$

Then the solution of (3.64) and (3.65) satisfies

$$y \in H^{k-1}(0,T;V), \quad \frac{d^ky(t)}{dt^k} \in L^2(0,T;H), \quad \frac{d^{k+1}y(t)}{dt^{k+1}} \in L^2(0,T;V'). \tag{3.68}$$

Proof. Induction with respect to k. For $k = 1$ we have

$$f \in L^2(0,T;H), \quad y(0) = y_0 \in V, \quad y'(0) = y_1 \in H,$$

which are exactly the conditions of the problem (H). So by Theorem 3.1 we obtain

$$y \in L^2(0,T;V), \quad y_t \in L^2(0,T;H), \quad y_{tt} \in L^2(0,T;V'). \tag{3.69}$$

We will only show the inductive step to $k = 2$ in detail (the general inductive step from $k-1$ to k can be shown analogously). By formaly differentiating (3.64) one gets

$$y_{ttt} + Ly_t = f_t(t), \tag{3.70}$$

and (by (3.66) and (3.67))

$$y_t(0) = y_1 \in V, \quad y_{tt}(0) = f(0) - Ly_0 \in H, \quad f_t \in L^2(0,T;H). \tag{3.71}$$

We consider the initial value problem

$$v_{tt} + Lv = f_t, \quad v(0) = y_1 \in V, \quad v_t(0) = f(0) - Ly_0 \in H, \tag{3.72}$$

and want to show that $v = y_t$.
By Theorem 3.1 we know that (3.72) has a unique solution v, which satisfies

$$v \in L^2(0,T;V), \quad v_t \in L^2(0,T;H), \quad v_{tt} \in L^2(0,T;V'). \tag{3.73}$$

Now define $w(t) := y(0) + \int_0^t v(\tau)d\tau$. Then it follows that

$$\begin{aligned} &w \in C([0,T];V) \subset L^2(0,T;V), \quad w_t = v \in L^2(0,T;V) \subset L^2(0,T;H), \\ &w_{tt} = v_t \in L^2(0,T;H) \subset L^2(0,T;V'), \end{aligned} \tag{3.74}$$

and (by looking at (3.72))

$$w(0) = y(0) = y_0, \quad w_t(0) = v(0) = y_1. \tag{3.75}$$

Integrating (3.72) yields for all $t \in (0,T)$

$$v_t(t) - v_t(0) = -\int_0^t Lv(\tau)d\tau + f(t) - f(0) = -Ly_0 - \int_0^t Lv(\tau)d\tau + f(t) - v_t(0),$$

where we have used the intitial conditions of (3.72) for the last equality. Since

$$\int_0^t Lv(\tau)d\tau = \int_0^t Lw_\tau d\tau = Lw(t) - Ly(0), \quad \forall t \in (0,T),$$

we get

$$w_{tt}(t) = v_t(t) = -Lw(t) + f(t) \quad \text{in } (0,T). \tag{3.76}$$

Now we substract equation (3.64) from (3.76) and obtain

$$(w-y)_{tt}(t) + L(w-y)(t) = 0, \quad \text{in } (0,T), \tag{3.77}$$

and (because of (3.75))

$$(w-y)(0) = 0, \quad (w-y)_t(0) = 0.$$

Because of (3.74) and (3.69) we can apply the existence and uniqueness result (Theorem 3.1) on (3.77). This finally implies

$$w = y \quad \text{in } [0,T], \quad \text{or} \quad v = w_t = y_t.$$

$\square$

An inspection of the above proof shows that the result also holds for operators $L(t)$, which depend on t, if the following is satisfied

$$\begin{aligned} &\frac{d}{dt}L(t) : V \to H \quad \text{and,} \\ &\frac{d^{k-1}}{dt^{k-1}}L(t) : V \to H \quad \text{are continuous.} \end{aligned}$$

Chapter 4

Examples

In this chapter we will look at some typical examples of linear hyperbolic differential equations with nonconstant coefficients. The aim is to find out, what conditions on the coefficients have to be made, that the existence and uniqueness result of Section 3.1 can be applied and therefore guarantees the solvability of the equations considered.

4.1 The Wave Equation

We consider the following

Example 4.1

$$\partial_t^2 u - c(t)\partial_x^2 u = 0 \tag{4.1}$$

$$u(0,x) = u_0(x), \quad \big(\partial_t u\big)(0,x) = u_1(x). \tag{4.2}$$

We choose $V = H^1$, $H = L^2$ and $V' = H^{-1}$. From the operator $L(t) = -c(t)\frac{d^2}{dx^2}$ we can construct the appropriate sequilinear form

$$a(t;\varphi,\psi) = \big(L(t)\varphi,\psi\big)_H = \int_{-\infty}^{\infty} c(t)\varphi'(x)\overline{\psi'(x)}dx. \tag{4.3}$$

To guarantee the existence (and uniqueness) of the problem (4.1) with the intitial conditions (4.2) the sesquilinear form $a(t;\varphi,\psi)$ has to satisfy the conditions (3.1)–(3.5). Therefore we have to check, what requirements on the coefficient $c(t)$ have to be made that these conditions hold. More precisely, we are searching for requirements on $c(t)$, which are essential to ensure solvability by Theorem 3.1. Since the proof of Theorem 3.1 also works if the sesquilinear form $a(t;\varphi,\psi)$ satisfies the weaker condition (3.6) instead of (3.5) (cf. Remark 3.11), it will be sufficient that $a(t;\varphi,\psi)$ fulfills (3.1)–(3.4) and (3.6).

The operator $L(t) = -c(t)\frac{d^2}{dx^2}$ is linear and continuous from V to V' for all $t \in (0,T)$ if $c \in C(0,T)$. Hence we just have to check the continuity, antisymmetry, V-coercivity and boundedness of the distributional derivative of $a(t;\varphi,\psi)$.

Continuity

Assume that the sesquilinear form $a(t;\varphi,\psi)$ is continuous in t, i.e. $c \in C(0,T)$. Suppose in addition that $c \in L^\infty(0,T)$, then

$$\begin{aligned} |a(t;\varphi,\psi)| &= \left|c(t)\int_{-\infty}^{\infty} \varphi'(x)\overline{\psi'(x)}dx\right| \leq |c(t)|\int_{-\infty}^{\infty} |\varphi'(x)\overline{\psi'(x)}|dx \\ &\leq |c(t)|\,\|\varphi'\|_0\|\overline{\psi'}\|_0 \leq c_0\|\varphi'\|_0\|\psi'\|_0 \\ &\leq c_0\|\varphi\|_1\|\psi\|_1, \end{aligned}$$

where the last inequality follows from (2.2). So for $a(t;\varphi,\psi)$ to be continuous, i.e. to have the condition (3.1) fulfilled it is sufficient that the coefficient $c(t)$ of the wave equation satisfies

$$|c(t)| \leq c_0 \quad \forall t \in (0,T), \text{ i.e. } \|c(t)\|_{L^\infty(0,T)} \leq c_0. \tag{4.4}$$

Conversely, continuity of $a(t;\varphi,\psi)$ implies $c \in L^\infty(0,T)$. Indeed, suppose $a(t;\varphi,\psi)$ satisfies

$$|a(t;\varphi,\psi)| \leq c_0\|\varphi\|_1\|\psi\|_1, \quad \forall\varphi,\, \psi \in H^1.$$

Setting $\psi = \varphi \in \mathscr{D}$ we obtain

$$\|\varphi\|_1^2 c_0 \geq |a(t;\varphi,\varphi)| = |c(t)|\int_{-\infty}^{\infty} |\varphi'(t)|^2 dt = |c(t)|\,\|\varphi'\|_0^2 \geq |c(t)|\|\varphi\|_1^2\tilde{c},$$

where we have used Poincaré's inequality. Hence

$$c \in L^\infty(0,T).$$

So condition (3.1) is fulfilled if and only if there exists c_0 such that $\|c(t)\|_{L^\infty(0,T)} \leq c_0$.

Antisymmetry

$$\begin{aligned} \overline{a(t;\psi,\varphi)} &= \overline{\int_{-\infty}^{\infty} c(t)\psi'(x)\overline{\varphi'(x)}dx} = \int_{-\infty}^{\infty} \overline{c(t)\psi'(x)}\varphi'(x)dx \\ &= \int_{-\infty}^{\infty} c(t)\varphi'(x)\overline{\psi'(x)}dx = a(t;\varphi,\psi), \end{aligned}$$

holds if and only if $c(t)$ is real valued.

V-Coercivity

$$\begin{aligned} a(t;\varphi,\varphi) + k_0\|\varphi\|_0^2 &= c(t)\int_{-\infty}^{\infty} |\varphi'(x)|^2 dx + k_0\|\varphi\|_0^2 \\ &\geq c_1\|\varphi'\|_0^2 + k_0\|\varphi\|_0^2 \geq \min\{c_1,k_0\}\left(\|\varphi\|_0^2 + \|\varphi'\|_0^2\right) \\ &= \min\{c_1,k_0\}\|\varphi\|_1^2, \end{aligned}$$

where we have used (2.2) for the last equality sign.
Hence we can choose $k_0 \geq 0$ arbitrary and then $\alpha > 0$ such that $\alpha \leq \min\{c_1, k_0\}$. So we finally get

$$a(t; \varphi, \varphi) + k_0 \|\varphi\|_0^2 \geq \alpha \|\varphi\|_1^2,$$

which is exactly condition (3.4). Here we necessarily had to make the assumption that $c(t)$ stays away from 0 for all $t \in (0, T)$, i.e.

$$\exists c_1 > 0 \text{ such that } c(t) \geq c_1 \quad \forall t \in (0, T). \tag{4.5}$$

If conversely there exists $k_0 \geq 0$ and $\alpha > 0$ such that

$$a(t; \varphi, \varphi) + k_0 \|\varphi\|_0^2 \geq \alpha \|\varphi\|_1^2$$

holds for all $t \in [0, T]$ and $\varphi \in H^1$ we have that c stays away from zero, i.e. there exists $c_1 > 0$ such that $c(t) \geq c_1$ for all $t \in (0, T)$. Indeed, let $k_0 \geq 0$ and $\alpha > 0$ be given such that $a(t; \varphi, \psi)$ is V-coercive and suppose that c comes arbitrary close to 0. Then consider $\varphi_n = \sin(nx)e^{-x^2}$, choose n_0 such that $k_0\|\varphi_{n_0}\|_0^2 < \alpha\|\varphi_{n_0}\|_1^2$ and set $\varepsilon = \alpha\|\varphi_{n_0}\|_1^2 - k_0\|\varphi_{n_0}\|_0^2 > 0$. Since we assumed that c can be arbitrary close to 0 we can choose $t \in [0, T]$ such that $c(t) < \frac{\varepsilon}{\|\varphi_{n_0}'\|_0^2}$, but this gives a contradiction to the V-coercivity of $a(t; \varphi, \psi)$.
So condition (3.4) is fulfilled if and only if

$$\exists c_1 > 0 \text{ such that } c(t) \geq c_1 \quad \forall t \in (0, T).$$

Boundedness of the distributional derivative

If c is Lipschitz continuous in t, so is the sesquilinear form $a(t; \varphi, \psi)$ and we get

$$\begin{aligned} \left|\frac{d}{dt} a(t; \varphi, \psi)\right| &= \left|\frac{d}{dt} \int_{-\infty}^{\infty} c(t)\varphi'(x)\overline{\psi'(x)}dx\right| \leq |c'(t)| \int_{-\infty}^{\infty} |\varphi'(x)\overline{\psi'(x)}|dx \\ &\leq \|c'\|_{L^\infty} \|\varphi'\|_0 \|\psi'\|_0 \leq \|c'\|_{L^\infty} \|\varphi\|_1 \|\psi\|_1, \end{aligned}$$

where we have used (2.2). Note that $\|c'\|_{L^\infty} < \infty$ by Lemma 2.12 (c' denotes the distributional derivative of c). Hence for having the condition (3.6) be satisfied, it suffices that c is Lipschitz continuous, i.e.

$$c(t) \in C^{0,1}(0, T). \tag{4.6}$$

Conversely, assume that (3.6) holds, i.e. there exists $c > 0$ such that

$$\left|\frac{d}{dt} a(t; \varphi, \psi)\right| \leq c\|\varphi\|_1 \|\psi\|_1, \quad \forall \varphi,\ \psi \in H^1 \text{ and for almost every } t \in (0, T).$$

Setting $\psi = \varphi \in \mathscr{D}$ yields

$$c\|\varphi\|_1^2 \geq \left|\frac{d}{dt} \int_{-\infty}^{\infty} c(t)\varphi'(x)\overline{\varphi'(x)}dx\right| = |c'(t)| \int_{-\infty}^{\infty} |\varphi'(x)|^2 dx \geq |c'(t)|\, \|\varphi\|_1^2 \tilde{c},$$

for almost every $t \in (0, T)$. Here we used again Poincaré's inequality. Hence $c' \in L^\infty(0, T)$, i.e. $c \in C^{0,1}(0, T)$ by Lemma 2.12.

So condition (3.6) is fulfilled if and only if $c \in C^{0,1}(0,T)$.

In summary we made the following conditions to have the requirements of Theorem 3.1 be satisfied:

$$c \in C^{0,1}((0,T),\mathbb{R}) \quad \text{and} \quad \exists c_0,\, c_1 :\ 0 < c_1 \leq c(t) \leq c_0 \quad \forall t \in (0,T). \tag{4.7}$$

One might ask oneself if a requirement on the coefficient $c(t)$ weaker than Lipschitz continuity would also be sufficient to guarantee existence of a solution of the problem (4.1) with the initial conditions (4.2). If we use the functional analytic method presented in the thesis at hand we can not reduce our requirements because of Lemma 2.12. More precisely, to ensure the boundedness of the distributional derivative of the sesquilinear form $a(t;\varphi,\psi)$ it's essential to require that c' is bounded, but this is equivalent to $c \in C^{0,1}((0,T))$ by Lemma 2.12. Now let's forget of the method used in this thesis, then the question is, whether it is possible to ensure (maybe with other methods) the existence of a solution with weaker conditions such as Hölder continuity or if the Lipschitz continuity of $c(t)$ is essential in general. This question can be answered by the following result derived by F. Colombini and S. Spagnolo.

Theorem 4.2 *There exists a function $c(t)$ such that*

$$\frac{1}{2} \leq c(t) \leq \frac{3}{2} \tag{4.8}$$

$$c(t) \in C^{0,\alpha}(\mathbb{R}) \quad \textit{for all } \alpha < 1 \tag{4.9}$$

and two C^∞ functions $u_0(x)$, $u_1(x)$ for which the problem

$$\partial_t^2 u - c(t)\partial_x^2 u = 0 \tag{4.10}$$

$$u(0,x) = u_0(x), \quad \big(\partial_t u\big)(0,x) = u_1(x) \tag{4.11}$$

has no solution $u(t,x)$ in $C^2([-r,r],\mathscr{D}'(\bar{x}-r,\bar{x}+r))$, for any $\bar{x} \in \mathbb{R}$ and $r > 0$.

The consequence of Theorem 4.2 is, that in general α-Hölder continuity of the coefficient is not sufficient to ensure the existence of a solution. That means if we only required $c(t)$ to be Hölder instead of Lipschitz continuous we could construct a counter example, i.e a coefficient $c \in C^{0,\alpha}(\mathbb{R})$ with $\alpha < 1$, where our problem has no distribution-solution on any strip $\{|t| < r\}$. The basis of the proof of Theorem 4.2 is the construction of such a particular coefficient. We leave this theorem without proof here, because the methods used in the latter are far from the subject of this thesis. In the main F. Colombini and S. Spagnolo use results from the theory of ordinary differential equations and Fourier series to prove their theorem. For more details the proof can be looked up in [3].
So there exists a counter example showing that one-dimensional wave equations with α-Hölder continuous coefficients are not well-posed. To the interested reader it may seem obvious that the next question one can ask is if there is a condition on the coefficient, which is stronger than α-Hölder continuity but weaker than Lipschitz continuity, such that

the existence of a solution to the Cauchy problem (4.1) with (4.2) is still guaranteed. In [2] F. Colombini and N. Lerner go beyond the classical well-posedness result for hyperbolic operators with Lipschitz continuous coefficients. First of all we will need the following definition.

Definition 4.3 *For a real-valued function $c \in L^\infty(\mathbb{R}^n)$ to be Log-Lipschitz (LL for short) means that there exists $k > 0$ such that*

$$|c(x_1) - c(x_2)| \leq k|x_1 - x_2| \big| \log |x_1 - x_2| \big|, \tag{4.12}$$

whenever $|x_1 - x_2|$ is small (say for $|x_1 - x_2| \leq \frac{1}{2}$).
We set

$$\|c\|_{LL} = \sup_{x \in \mathbb{R}^n} |c(x)| + \sup_{\substack{0 < |x_1 - x_2| \leq \frac{1}{2} \\ x_i \in \mathbb{R}^n}} \frac{|c(x_1) - c(x_2)|}{|x_1 - x_2| \big| \log |x_1 - x_2| \big|} \tag{4.13}$$

and define the set of Log-Lipschitz (LL) functions c such that $\|c\|_{LL} < \infty$.

If we denote by Λ the space of functions c satisfying

$$\exists k > 0 \text{ such that } |c(t) - c(s)| \leq k|t - s| \big| \log |t - s| \big| \varphi(t - s), \tag{4.14}$$

with $\varphi(r) = \log |\log r|$, we have

$$C^{0,1} \subset LL \subset \Lambda \subset C^{1-0} := \bigcap_{\alpha < 1} C^{0,\alpha}.$$

Theorem 4.2 gives an example of a nonsolvable strictly hyperbolic equation with C^{1-0}-coefficients. Moreover, in [1] counter examples are presented about nonuniqueness for the Cauchy problems for strictly hyperbolic equations with C^{1-0}-coefficients.
F. Colombini and N. Lerner showed that the class Λ is still too large to expect the existence of a solution. More precisely, consider the initial value problem

$$\partial_t^2 u - c(t) \partial_x^2 u = 0, \quad t \in \mathbb{R}, \, x \in \mathbb{R} \tag{4.15}$$

$$u(0, x) = u_0(x), \quad \big(\partial_t u\big)(0, x) = u_1(x). \tag{4.16}$$

Then for any function φ, with $\varphi(r) \to \infty$ as $r \to 0^+$, one can find c satisfying

$$\exists k > 0 \text{ such that } |c(t) - c(s)| \leq k|t - s| \big| \log |t - s| \big| \varphi(t - s), \tag{4.17}$$

such that the problem (4.15), (4.16) is not well-posed.
Anyway, what about wave operators with LL coefficients? There are two different types of results proved in [2]. First, consider wave equations in divergence form

$$P \equiv \frac{\partial^2}{\partial t^2} - \sum_{1 \leq i,j \leq n} \frac{\partial}{\partial x_i} c_{ij}(t, x) \frac{\partial}{\partial x_j}, \tag{4.18}$$

with c_{ij} real-valued satisfying

$$c_{ij} = c_{ji} \quad \text{and} \tag{4.19}$$
$$\sum_{1\le i,j\le n} c_{ij}\xi_i\xi_j \ge \delta|\xi|^2,\ \delta > 0 \text{ for any } \xi \in \mathbb{R}^n. \tag{4.20}$$

Moreover, $c_{ij} \in LL$, i.e. $c_{ij} \in L^\infty$ and

$$\exists k > 0 \text{ such that } |c_{ij}(y_1) - c_{ij}(y_2)| \le k|y_1 - y_2|\big|\log|y_1 - y_2|\big|, \tag{4.21}$$

whenever $|y_1-y_2| \le \frac{1}{2}$, where $y_i \in \mathbb{R}^{1+n} = \mathbb{R}_t\times\mathbb{R}^n_x$. Colombini and Lerner show that some energy estimates hold under these conditions, which allow them to prove well-posedness results for the Cauchy problem P.
Second, consider a wave operator in $\mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n_x$ of the form

$$L \equiv \frac{\partial^2}{\partial t^2} - \sum_{1\le i,j\le n} c_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}, \tag{4.22}$$

where L is strictly hyperbolic, i.e. c_{ij} satisfy (4.19) and (4.20). Assume that c_{ij} are LL in the time variable t, smooth in the space variables x. So $c_{ij}(t, .)$ are smooth functions such that

$$\exists k > 0 \text{ such that } \sup_x |c_{ij}(t,x) - c_{ij}(s,x)| \le k|t-s|\big|\log|t-s|\big|, \tag{4.23}$$

when $|t-s| \le \frac{1}{2}$.
For instance consider the initial value problem

$$\begin{aligned} Lu &= 0 \\ u(0,x) &= u_0(x), \\ \big(\partial_t u\big)(0,x) &= u_1(x), \end{aligned}$$

with smooth u_0, u_1. Then one can find a unique solution $u(t,x)$ depending continuously on the data u_0, u_1 such that

$$\partial_t^2 u(t,x) \in LL(\mathbb{R}_+; C^\infty(\mathbb{R}^n_x)).$$

For our Example 4.1 this means that Log-Lipschitz continuity of the coefficient $c(t)$ comes up as the natural treshold beyond which no well-posedness could be expected.

4.2 A Generalization of the Wave Equation

Now we look at a more general form of a wave equation

Example 4.4

$$\partial_t^2 u - \partial_x\big(c(t,x)\partial_x u\big) + b(t,x)\partial_x u + a(t,x)u = 0 \tag{4.24}$$

$$u(0,x) = u_0(x), \quad \big(\partial_t u\big)(0,x) = u_1(x). \tag{4.25}$$

We choose again $V = H^1$, $H = L^2$ and $V' = H^{-1}$. As in the case of Example 4.1 we get the sesquilinear form

$$\begin{aligned} a(t;\varphi,\psi) \;&=\; \big(L(t)\varphi,\psi\big)_H = \Big(-\partial_x\big(c(t,x)\varphi'(x)\big) + b(t,x)\varphi'(x) + a(t,x)\varphi(x),\psi\Big)_{L^2} \\ &= \int_{-\infty}^{\infty}\big(c(t,x)\varphi'(x)\overline{\psi'(x)} + b(t,x)\varphi'(x)\overline{\psi(x)} + a(t,x)\varphi(x)\overline{\psi(x)}\big)dx. \end{aligned} \tag{4.26}$$

As above we now have to check, what requirements on the coefficients $a(t,x)$, $b(t,x)$ and $c(t,x)$ we have to make that the sesquilinear form $a(t,\varphi,\psi)$ satisfies the conditions (3.1)–(3.4) and (3.6).

The operator $L(t)$, which corresponds to the sesquilinear form $a(t;\varphi,\psi)$ is obviously linear and continuous from V to V' for all $t \in [0,T]$. Analogously to Example 4.1 we now check the continuity, antisymmetry, V-coercivity and boundedness of the distributional derivative of $a(t;\varphi,\psi)$.

Continuity

$$\begin{aligned} |a(t;\varphi,\psi)| \;&\le\; \Big|\int_{-\infty}^{\infty}\big(c(t,x)\varphi'(x)\overline{\psi'(x)} + b(t,x)\varphi'(x)\overline{\psi(x)} + a(t,x)\varphi(x)\overline{\psi(x)}\big)dx\Big| \\ &\le \|c(t,\,.\,)\|_{L^\infty(\mathbb{R})}\int_{-\infty}^{\infty}|\varphi'(x)\overline{\psi'(x)}|dx + \|b(t,\,.\,)\|_{L^\infty(\mathbb{R})}\int_{-\infty}^{\infty}|\varphi'(x)\overline{\psi(x)}|dx \\ &\quad + \|a(t,\,.\,)\|_{L^\infty(\mathbb{R})}\int_{-\infty}^{\infty}|\varphi(x)\overline{\psi(x)}|dx \\ &\le c_0\|\varphi'\|_0\|\overline{\psi'}\|_0 + b_0\|\varphi'\|_0\|\overline{\psi}\|_0 + a_0\|\varphi\|_0\|\overline{\psi}\|_0 \\ &\le (c_0 + b_0 + a_0)\|\varphi\|_1\|\psi\|_1 \\ &\le 3\max\{c_0,b_0,a_0\}\|\varphi\|_1\|\psi\|_1, \end{aligned}$$

where we have used (2.2) and the Cauchy-Schwarz inequality. Hence to have the condition 3.1 fulfilled we required the coefficients $a(t,x)$, $b(t,x)$ and $c(t,x)$ to satisfy the following:

$$\|c(t,\,.\,)\|_{L^\infty(\mathbb{R})} \le c_0,\ \|b(t,\,.\,)\|_{L^\infty(\mathbb{R})} \le b_0,\ \|a(t,\,.\,)\|_{L^\infty(\mathbb{R})} \le a_0, \quad \forall t \in [0,T]. \tag{4.27}$$

Antisymmetry

$$
\begin{aligned}
\overline{a(t;\psi,\varphi)} &= \int_{-\infty}^{\infty} \left(\overline{c(t,x)\psi'(x)\overline{\varphi'(x)}} + \overline{b(t,x)\psi'(x)\overline{\varphi(x)}} + \overline{a(t,x)\psi(x)\overline{\varphi(x)}}\right) dx \\
&= \int_{-\infty}^{\infty} \left(c(t,x)\overline{\psi'(x)}\varphi'(x) + b(t,x)\overline{\psi'(x)}\varphi(x) + a(t,x)\overline{\psi(x)}\varphi(x)\right) dx \\
&= a(t;\varphi,\psi),
\end{aligned}
$$

where we had to require that $a(t,x)$, $b(t,x)$ and $c(t,x)$ are real valued.

V-Coercivity

$$
\begin{aligned}
a(t;\varphi,\varphi) &= \int_{-\infty}^{\infty} \left(c(t,x)|\varphi'(x)|^2 + b(t,x)\varphi'(x)\overline{\varphi(x)} + a(t,x)|\varphi(x)|^2\right) dx \\
&\geq c_2\|\varphi'\|_0^2 - \|b(t,\,.\,)\|_{L^\infty(\mathbb{R})}\|\varphi'\|_0\|\overline{\varphi}\|_0 - \|a(t,\,.\,)\|_{L^\infty(\mathbb{R})}\|\varphi\|_0^2 \\
&\geq c_2\|\varphi'\|_0^2 - b_2\|\varphi'\|_0\|\varphi\|_0 - a_2\|\varphi\|_0^2 \\
&\geq c_2\|\varphi'\|_0^2 - \frac{b_2}{2}\left(\|\varphi'\|_0^2 + \|\varphi\|_0^2\right) - a_2\|\varphi\|_0^2.
\end{aligned}
$$

So we finally get

$$
a(t;\varphi,\varphi) \geq \left(c_2 - \frac{b_2}{2}\right)\|\varphi'\|_0^2 - \left(\frac{b_2}{2} + a_2\right)\|\varphi\|_0^2. \tag{4.28}
$$

Note that by considering (2.2) the condition (3.4) is equivalent to

$$
\exists k_0 \geq 0,\ \alpha > 0: \quad a(t;\varphi,\varphi) \geq \alpha\|\varphi'\|_H^2 + (\alpha - k_0)\|\varphi\|_H^2, \quad \forall t \in [0,T],\ \forall \varphi \in V.
$$

Hence if we choose a_2, b_2 and c_2 suitable we have that $\alpha := \left(c_2 - \frac{b_2}{2}\right) > 0$ and $k_0 := \left(\alpha + \frac{b_2}{2} + a_2\right) \geq 0$.
So for the sesquilinear form $a(t;\varphi,\psi)$ to satisfy the V-Coercivity we made the following demands:

$$
c(t,x) \geq c_2, \quad \forall (t,x) \in [0,T]\times\mathbb{R}, \tag{4.29}
$$

$$
\|b(t,\,.\,)\|_{L^\infty(\mathbb{R})} \leq b_2,\ \|a(t,\,.\,)\|_{L^\infty(\mathbb{R})} \leq a_2, \quad \forall t \in [0,T] \tag{4.30}
$$

where the constants $a_2 > 0$, $b_2 > 0$ and c_2 have to satisfy

$$
\left(c_2 - \frac{b_2}{2}\right) > 0, \quad c_2 + a_2 \geq 0. \tag{4.31}
$$

Boundedness of the distributional derivative

$$
\begin{aligned}
\Big|\frac{d}{dt}a(t;\varphi,\psi)\Big| &= \Big|\int_{-\infty}^{\infty}\Big(\frac{\partial c(t,x)}{\partial t}\varphi'(x)\overline{\psi'(x)}+\frac{\partial b(t,x)}{\partial t}\varphi'(x)\overline{\psi(x)}+\frac{\partial a(t,x)}{\partial t}\varphi(x)\overline{\psi(x)}\Big)dx\Big| \\
&\leq \|c_t(t,\,.\,)\|_{L^\infty(\mathbb{R})}\|\varphi'\|_0\|\overline{\psi'}\|_0+\|b_t(t,\,.\,)\|_{L^\infty(\mathbb{R})}\|\varphi'\|_0\|\overline{\psi}\|_0 \\
&\quad +\|a_t(t,\,.\,)\|_{L^\infty(\mathbb{R})}\|\varphi\|_0\|\overline{\psi}\|_0 \\
&\leq (c_1+b_1+a_1)\|\varphi\|_1\|\psi\|_1 \\
&\leq 3\max\{c_1,b_1,a_1\}\|\varphi\|_1\|\psi\|_1,
\end{aligned}
$$

by using again (2.2). For this inequality to be fulfilled we made the following demands on the coefficients:

$$\|c_t(t,\,.\,)\|_{L^\infty(\mathbb{R})}\leq c_1,\ \|b_t(t,\,.\,)\|_{L^\infty(\mathbb{R})}\leq b_1,\ \|a_t(t,\,.\,)\|_{L^\infty(\mathbb{R})}\leq a_1,\quad \forall t\in[0,T] \tag{4.32}$$

and

$$c(t),\ b(t),\ a(t)\in C^{0,1}[0,T]. \tag{4.33}$$

In summary we required the coefficients $a(t,x)$, $b(t,x)$ and $c(t,x)$ to satisfy the following conditions to ensure the existence of a solution of the problem (4.24) with the initial conditions (4.25):

$$a,\ b,\ c\in C^{0,1}([0,T];L^\infty(\mathbb{R})), \tag{4.34}$$

$$\|a(t,\,.\,)\|_{L^\infty(\mathbb{R})}\leq a_3,\ \|b(t,\,.\,)\|_{L^\infty(\mathbb{R})}\leq b_3,\ \|c(t,\,.\,)\|_{L^\infty(\mathbb{R})}\leq c_3,\quad \forall t\in[0,T], \tag{4.35}$$

$$\|a_t(t,\,.\,)\|_{L^\infty(\mathbb{R})}\leq a_1,\ \|b_t(t,\,.\,)\|_{L^\infty(\mathbb{R})}\leq b_1,\ \|c_t(t,\,.\,)\|_{L^\infty(\mathbb{R})}\leq c_1,\quad \forall t\in[0,T], \tag{4.36}$$

$$a,\ b,\ c:[0,T]\times\mathbb{R}\to\mathbb{R}, \tag{4.37}$$

$$c(t,x)\geq c_2,\quad \forall(t,x)\in[0,T]\times\mathbb{R}, \tag{4.38}$$

$$\Big(c_2-\frac{b_3}{2}\Big)>0,\quad c_2+a_3\geq 0. \tag{4.39}$$

Now we will argue how we can reduce the above requirements. More precisely, we know by Lemma 2.13 (and Lemma 2.14) that (4.34) implies $\partial_t a,\ \partial_t b,\ \partial_t c\in L^\infty(a,b;L^\infty(\mathbb{R}))$. Then we have for $a(t,x)$ that

$$\|\partial_t a(t,\,.\,)\|_{L^\infty(\mathbb{R})}\leq\|\partial_t a\|_{L^\infty(0,T;L^\infty)}\leq A.$$

Now taking the sup with respect to t we get

$$\|\partial_t a(t,\,.\,)\|_{L^\infty(\mathbb{R})}\leq A,\quad \forall t\in[0,T].$$

This works analoguously for $b(t,x)$ and $c(t,x)$. Hence we have shown that (4.34) already implies (4.36). So we can simplify the conditions (4.34)–(4.39). To guarantee existence of a solution of the problem (4.24) it's sufficient that the coefficients $a(t,x)$, $b(t,x)$ and $c(t,x)$ fulfill

$$a,\ b,\ c\in C^{0,1}([0,T];L^\infty(\mathbb{R})),$$

which implies

$$\|a(t,\,.\,)\|_{L^\infty(\mathbb{R})} \le a_3,\ \ \|b(t,\,.\,)\|_{L^\infty(\mathbb{R})} \le b_3,\ \ \|c(t,\,.\,)\|_{L^\infty(\mathbb{R})} \le c_3, \quad \forall t \in [0,T].$$

Furthermore we require

$$\begin{aligned}
&a,\ b,\ c : [0,T] \times \mathbb{R} \to \mathbb{R},\\
&c(t,x) \ge c_2, \quad \forall (t,x) \in [0,T] \times \mathbb{R},\\
&\left(c_2 - \frac{b_3}{2}\right) > 0, \quad c_2 + a_3 \ge 0.
\end{aligned}$$

Table of Notation

Ω	usually denotes an open subset of $\mathbb{R}^n$
K	usually denotes a compact subset of $\mathbb{R}^n$
I	usually denotes an subinterval of $\mathbb{R}$
$\hat{f}$	the Fourier transform of f
$\mathscr{F}^{-1}f$	the inverse Fourier transform of f
$f \otimes g$	the tensor product of f and g
$L^p(\Omega)$, $1 \leq p \in \mathbb{R}$	the space of classes of measurable functions on Ω such that $x \mapsto \|f(x)\|^p$ is integrable on Ω
$L^p(a, b; E)$	cf. Section 2.3.1 (i)
L^2	the space of all square integrable functions
L^1	the space of all integrable functions
L^1_{loc}	the space of all locally integrable functions
$L^\infty(\Omega)$	the space of classes of measurable functions on Ω such that $x \mapsto \|f(x)\|$ is essentially bounded
$L^\infty(a, b; E)$	cf. Section 2.3.1 (ii)
$C^k(\Omega)$, $k \in \mathbb{N}$	the space of functions, whose derivatives up to the order k exist and are continuous on Ω
$C(I; E)$	cf. Section 2.3.2 (i)
$C^k(I; E)$, $k \in \mathbb{N}$	cf. Section 2.3.2 (ii)
$C^\infty(\Omega)$	the space of smooth functions on Ω
$AC(I)$	the space of functions $f : I \subseteq \mathbb{R} \to \mathbb{C}$, which are absolutely continuous
$C^{0,\alpha}(\Omega)$, $0 < \alpha \leq 1$	the space of Hölder continuous functions on Ω, i.e. the space of all $\varphi \in C^0(\Omega)$ such that $$\sup_{x,y \in \Omega} \frac{\|\varphi(x) - \varphi(y)\|}{\|x - y\|^\alpha} < \infty$$
$C^{0,\alpha}(I; E)$	cf. Section 2.3.2 (iii)

$C^{l,\alpha}(\Omega)$, $0 < \alpha \leq 1$, $l \in \mathbb{N}$	the space of all $\varphi \in C^l(\Omega)$ such that $$D^j\varphi \in C^{0,\alpha}(\Omega),\ \forall \lvert j\rvert \leq l$$
$C^{0,1}(\Omega)$	the space of Lipschitz continuous functions on Ω
LL	the space of Log-Lipschitz continuous functions, i.e. the space of all real-valued functions $f \in L^\infty(\mathbb{R}^n)$ such that $$\exists k > 0 :\ \lvert c(x_1) - c(x_2)\rvert \leq k\lvert x_1 - x_2\rvert\,\big\lvert \log\lvert x_1 - x_2\rvert\big\rvert$$
$\mathscr{D}(\Omega)$	the space of those functions of $\mathbb{C}^\infty(\Omega)$ which have compact support in Ω
$\mathscr{S}(\mathbb{R}^n)$	the space of rapidly decreasing functions of $C^\infty(\mathbb{R}^n)$ such that all derivatives satisfy $$\forall k \in \mathbb{N},\ \forall \alpha \in \mathbb{N}^n :\ \lvert x\rvert^k\lvert D^\alpha f(x)\rvert \to 0 \text{ as } \lvert x\rvert \to \infty$$
$\mathscr{D}'(\Omega)$	the space of distributions on Ω, i.e. the set of continuous linear forms on $\mathscr{D}(\Omega)$
$\mathscr{D}'((a,b);E)$	cf. Section 2.3.1 (iii)
$\mathscr{S}'(\mathbb{R}^n)$	the space of tempered distributions on $\mathbb{R}^n$, i.e. the set of continuous linear forms on $\mathscr{S}(\mathbb{R}^n)$
$H^m(\mathbb{R}^n)$, $m \in \mathbb{N}$	the space of functions $f \in L^2(\mathbb{R}^n)$ such that $$D^\alpha f \in L^2(\mathbb{R}^n),\ \forall \alpha \in \mathbb{N}^n,\ \lvert\alpha\rvert \leq m$$
$H^k(a,b;E)$, $k \in \mathbb{N}$	cf. Section 2.3.1 (iv)
$H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$	the space of distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $$(1 + \lvert\xi^2\rvert)^{s/2}\hat{u} \in L^2(\mathbb{R}^n)$$

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