Risk assessment for uncertain cash flows: Model ambiguity, discounting ambiguity, and the role of bubbles

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Dynamical setting

We adopt a dynamical setting in order to take into account:

- the timing of payments
- the information released in time

Discrete-time, with finite or infinite time horizon T:

- $T \in \mathbb{N}$, time axis $\mathbb{T} = \{0, 1, ..., T\}$
- $T = \infty$, time axis $\mathbb{T} = \mathbb{N}_0$ or $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$

Multiperiod information structure: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, P)$

$\mathcal{R}^\infty$ = bounded adapted processes on $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in\mathbb{T}}, P)$
= cumulated cash flows, value processes

$\mathcal{R}_t^\infty$ = cash flows from time t on

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Conditional convex risk measures

Def. $\rho_t : \mathcal{R}_t^\infty \to L^\infty(\Omega, \mathcal{F}_t, P)$ is called a **conditional convex risk measure** for processes if for all $X, Y \in \mathcal{R}_t^\infty$:

- Normalization: $\rho_t(0) = 0$
- Monotonicity: $X \leq Y \;\Rightarrow\; \rho_t(X) \geq \rho_t(Y)$
- Conditional convexity: $\forall \lambda \in L^\infty(\Omega, \mathcal{F}_t, P)$, $0 \leq \lambda \leq 1$:
  $\rho_t(\lambda X + (1-\lambda)Y) \leq \lambda \rho_t(X) + (1-\lambda)\rho_t(Y)$
- Conditional cash-invariance:
  $\rho_t(X + m 1_{\{t,t+1,\ldots\}}) = \rho_t(X) - m, \quad m \in L^\infty(\Omega, \mathcal{F}_t, P)$

$\to$ $(\rho_t)_t$ is called **dynamic convex risk measure** for processes

(Cheridito,Delbaen & Kupper 2006)

Acceptance set

An important characterization of a conditional convex risk measure is the **acceptance set**:

$$\mathcal{A}_t = \left\{ X \in \mathcal{R}_t^\infty \,\middle|\, \rho_t(X) \leq 0 \right\}.$$

ρ_t is **uniquely determined** through its acceptance set:

$$\rho_t(X) = \operatorname{ess\,inf} \left\{ Y \in L^\infty(\Omega, \mathcal{F}_t, P) \,\middle|\, X + Y 1_{\{t,t+1,\ldots\}} \in \mathcal{A}_t \right\}.$$

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Product space and optional filtration

- Define the **product space** $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ as: $\bar{\Omega} = \Omega \times \mathbb{T}$,

$$\bar{\mathcal{F}} = \sigma(\{A_t \times \{t\} \mid A_t \in \mathcal{F}_t,\, t \in \mathbb{T}), \quad \bar{P} = P \otimes \mu,$$

where $\mu = (\mu_t)_{t \in \mathbb{T}}$ is some adapted reference process s.t. $\mu_t > 0$ and $\sum_t \mu_t = 1$, and $E_{\bar{P}}[X] := E_P\left[\sum_t X_t \mu_t\right]$

$$\Longrightarrow \qquad \boxed{\mathcal{R}^\infty = L^\infty(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})}$$

- Consider the **optional filtration** $(\bar{\mathcal{F}}_t)_{t \in \mathbb{T}}$ on $(\bar{\Omega}, \bar{\mathcal{F}})$, given by

$$\bar{\mathcal{F}}_t = \sigma\left(\{A_j \times \{j\}, A_t \times \{t, ..\} | A_j \in \mathcal{F}_j, j = 0, .., t-1, A_t \in \mathcal{F}_t\}\right)$$

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Risk measures viewed on the optional filtration

Proposition. There is a **one-to-one correspondence** between

- conditional convex risk measures for processes

$$\rho_t : \mathcal{R}_t^\infty \to L^\infty(\Omega, \mathcal{F}_t, P)$$

- conditional convex risk measures for random variables **on the product space**

$$\bar{\rho}_t : L^\infty(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}) \to L^\infty(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{P})$$

The relation is given by

$$\boxed{\bar{\rho}_t(X) = -X_0 1_{\{0\}} - \ldots - X_{t-1} 1_{\{t-1\}} + \rho_t(X) 1_{\{t,t+1,\ldots\}}}$$

Representation of risk measures on random variables

Theorem. For $\rho_t : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{F}_t, P)$ TFAE:

1. ρ_t is continuous from above: $X^n \searrow X \Rightarrow \rho_t(X^n) \nearrow \rho_t(X)$
2. ρ_t has the following **robust representation**:

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t} (E_Q[-X|\mathcal{F}_t] - \alpha_t(Q))$$

where

$$\mathcal{Q}_t = \left\{ Q \ll P \;\middle|\; Q = P|_{\mathcal{F}_t} \right\},$$

and the **minimal penalty function** α_t is given by

$$\alpha_t(Q) = \operatorname*{ess\,sup}_{X \in L^\infty(\mathcal{F})} (E_Q[-X|\mathcal{F}_t] - \rho_t(X))$$

(Detlefsen and Scandolo (2005))

Optional random measures

For any measure $Q \ll_{loc} P$ we introduce:

- the set $\Gamma(Q)$ of **optional random measures** γ on $\mathbb{T}$ which are normalized with respect to Q:

$\gamma = (\gamma_t)_{t \in \mathbb{T}}$ nonnegative adapted process s.t. $\sum_{t \in \mathbb{T}} \gamma_t = 1$ Q-a.s.

with the additional property

$$\gamma_\infty = 0 \;\; Q\text{-a.s. } \;\text{on} \left\{ \lim_{t \to \infty} \frac{dQ}{dP}\bigg|_{\mathcal{F}_t} = \infty \right\} \;\text{if}\;\; \mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$$

Predictable discounting processes

- the set $\mathcal{D}(Q)$ of **predictable discounting processes** D:

$$D = (D_t)_{t\in\mathbb{T}} \text{ predict. non-increasing, } D_0 = 1,\ D_\infty = \lim_{t\to\infty} D_t \quad Q\text{-a.s.}$$

where

$$D_\infty = 0 \quad Q\text{-a.s.} \quad \text{if} \quad \mathbb{T} = \mathbb{N}_0,$$

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- There is a **one-to-one correspondence** between random measures in $\Gamma(Q)$ and predictable discounting in $\mathcal{D}(Q)$:

$$\gamma_t = D_t - D_{t+1}, t < \infty, \quad \gamma_\infty = D_\infty$$

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Decomposition of measures on the optional σ-field

Theorem. For any probability measure $\bar{Q}$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ we have: $\bar{Q} \ll \bar{P}$ if and only if there exist

- a probability measure Q on $(\Omega, \mathcal{F}_T)$, $Q \ll_{loc} P$
- an optional random measure $\gamma \in \Gamma(Q)$ (resp. $D \in \mathcal{D}(Q)$)

such that

$$E_{\bar{Q}}[X] = E_Q\left[\sum_{t \in \mathbb{T}} \gamma_t X_t\right] = E_Q\left[\sum_{t=0}^{T} D_t \Delta X_t\right], \quad X \in \mathcal{R}^\infty$$

(combining the Itô-Watanabe factorization with an extension theorem for standard systems)

In this case we write:

$$\bar{Q} = Q \otimes \gamma = Q \otimes D$$

Robust representation

Theorem. For $\rho_t : \mathcal{R}_t^\infty \to L^\infty(\Omega, \mathcal{F}_t, P)$ TFAE:

1. ρ_t continuous from above: $X_s^n \searrow X_s \, \forall\, s \geq t \Rightarrow \rho_t(X^n) \nearrow \rho_t(X)$
2. ρ_t has the following **robust representation**:

$$\rho_t(X) = \underset{Q \in \mathcal{Q}_t^{\text{loc}}}{\text{ess sup}} \; \underset{D \in \mathcal{D}_t(Q)}{\text{ess sup}} \left(E_Q\Big[-\sum_{s=t}^{T} D_s \Delta X_s \;\Big|\; \mathcal{F}_t \Big] - \alpha_t(Q \otimes D) \right),$$

$\nearrow$ model ambiguity (for $Q \in \mathcal{Q}_t^{\text{loc}}$)

$\nwarrow$ discounting ambiguity (for $D \in \mathcal{D}_t(Q)$)

where

$$\mathcal{Q}_t^{\text{loc}} = \{Q \ll_{loc} P : Q = P|_{\mathcal{F}_t}\}, \; \mathcal{D}_t(Q) = \{D \in \mathcal{D}(Q) : D_s = 1 \; s \leq t\}$$

and the **minimal penalty function** α_t is given by:

$$\alpha_t(Q \otimes D) = \underset{X \in \mathcal{R}_t^\infty}{\text{Q-ess sup}} \left(E_Q\Big[-\sum_{s \geq t} \frac{\gamma_s}{D_t} X_s \;\Big|\; \mathcal{F}_t \Big] - \rho_t(X) \right)$$

Time consistency

$X \in \mathcal{R}^\infty \;\to\; (\rho_t(X))_t$ describes the evolution of risk over time.

Question: How should risk measurement be updated as more information becomes available?

Def. $(\rho_t)_t$ is called **(strongly) time consistent** if for all $t \geq 0$

$$X_t = Y_t \text{ and } \rho_{t+1}(X) \leq \rho_{t+1}(Y) \;\Rightarrow\; \rho_t(X) \leq \rho_t(Y)$$

An equivalent characterization is **recursiveness**:

$$\rho_t(X) = \rho_t(X_t 1_{\{t\}} - \rho_{t+1}(X) 1_{\{t+1,\dots\}}) \quad \forall\, t \geq 0$$

Remark. $(\rho_t)_t$ on $\mathcal{R}^\infty$ is time consistent $\iff$ the corresponding $(\bar{\rho}_t)_t$ on $L^\infty(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ is time consistent

Supermartingale properties

Proposition. Let $(\rho_t)_t$ on $\mathcal{R}^\infty$ be continuous from above and **time consistent**. Then, $\forall\, \bar{Q} = Q \otimes D \ll \bar{P}$ such that $\alpha_0(Q \otimes D) < \infty$,

- the discounted penalty process $\quad (D_t \alpha_t(\bar{Q}))_{t \in \mathbb{T} \cap \mathbb{N}_0}$
- the 'global risk' process of $X \in \mathcal{R}^\infty$

$$D_t(\rho_t(X - X_t) + \alpha_t(\bar{Q})) - \sum_{s=0}^{t} D_s \Delta X_s, \quad t \in \mathbb{T} \cap \mathbb{N}_0$$

are Q-supermartingales.

Appearance of bubbles in the dynamic penalization

Riesz decomposition of the discounted penalty process:

$$D_t\alpha_t(\bar{Q}) = \underbrace{E_Q\Big[\sum_{k=t}^{T-1} D_k\alpha_{k,k+1}(\bar{Q})|\mathcal{F}_t\Big]}_{\text{“\textit{fundamental penalization}”}} + \underbrace{\lim_{s\to\infty} E_Q[D_s\alpha_s(\bar{Q})|\mathcal{F}_t]}_{\text{“\textit{bubble}”}} \quad Q\text{-a.s.}$$

↓

breakdown of asymptotic safety

where $\alpha_{k,k+1}$ is the ‘one-step’ penalty function, i.e., the penalty function of ρ_k restricted to the ‘one-step’ processes

→ Bubbles reflect an excessive neglect of models which may be relevant for the risk assessment

Asymptotic safety

Consider $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$, and fix a model $\bar{Q}$ s.t. $\alpha_0(\bar{Q}) < \infty$.

Def. $(\rho_t)_{t\in\mathbb{N}_0}$ on $\mathcal{R}^\infty$ is called **asymptotically safe** under the model $\bar{Q} = Q \otimes D$ if for any $X \in \mathcal{R}^\infty$

$$\rho_\infty(X) := \lim_{t\to\infty} \rho_t(X) \geq -X_\infty \quad Q\text{-a.s. on } \{D_\infty > 0\}$$

Theorem. For $(\rho_t)_{t\in\mathbb{N}_0}$ time consistent and continuous from above, TFAE:

- $(\rho_t)_{t\in\mathbb{N}_0}$ is asymptotically safe under $\bar{Q}$;
- the model $\bar{Q}$ has no bubble, i.e., the martingale in the Riesz decomposition of $(D_t\alpha_t(\bar{Q}))_{t\in\mathbb{N}_0}$ vanishes.

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A maximal inequality for the capital requirements

Risk evaluation of $X \in \mathcal{R}^\infty$ at time t, using the specific model Q and the specific discounting process D:

$$F_t^{Q,D}(X) := E_Q\Big[-\sum_{s\geq t}\frac{\gamma_s}{D_t}X_s \mid \mathcal{F}_t\Big] - \alpha_t(Q\otimes\gamma) \quad \text{on } \{D_t > 0\}$$

The following maximal inequality for the excess of the required capital $\rho_t(X)$ over the risk evaluation $F_t^{Q,D}(X)$ holds $\forall c > 0$:

$$Q\Big(\sup_{t\in\mathbb{T}\cap\mathbb{N}_0}\Big\{D_t\Big(\rho_t(X) - F_t^{Q,D}(X)\Big)\Big\} \geq c\Big) \leq \frac{\rho_0(X) - F_0^{Q,D}(X)}{c}$$

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Cash additivity and subadditivity

Def. A conditional convex risk measure for processes ρ_t is called

- **cash subadditive** if for all $s > t$

$$\rho_t(X + m1_{\{s,s+1,\dots\}}) \geq \rho_t(X) - m \qquad \forall m \in L^\infty_+(\mathcal{F}_t)$$
$$(\text{resp. } \leq \quad \forall m \in L^\infty_-(\mathcal{F}_t))$$

  (El Karoui & Ravanelli (2009))

- **cash additive at** s, for some $s > t$, if

$$\rho_t(X + m1_{\{s,s+1,\dots\}}) = \rho_t(X) - m \qquad \forall m \in L^\infty(\mathcal{F}_t)$$

Remark. By monotonicity and cash-invariance every conditional convex risk measure for processes is cash subadditive

Proposition. Let $\rho_t : \mathcal{R}_t^\infty \to L_t^\infty$ be continuous from above. Then

- ρ_t is **cash additive at time** $s > t \iff$ there is no discounting up to time s: $\forall\, \bar{Q} = Q \otimes D$ s.t. $\alpha_t(\bar{Q}) < \infty$

$$D_t = D_{t+1} = \cdots = D_s = 1 \quad Q\text{-a.s.}$$

- if $T < \infty$ or $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$, ρ_t is **cash additive at all times** $s > t \iff$ it reduces to a risk measure for random variables:

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t} \left(E_Q[-X_T \mid \mathcal{F}_t] - \alpha_t(Q) \right)$$

- if $\mathbb{T} = \mathbb{N}_0$, ρ_t **cannot** be **cash additive at all times** $s > t$

Calibration to ZCB

Let be given in the market:

- $(B_t)_{t=0,\dots,T}$, $B_t > 0\ \forall t$, money market account;
- zero coupon bonds for all maturities are available, with $B_{t,k}$ price at time t of a ZCB paying 1 at maturity k.

Suppose that ρ_t, continuous from above, satisfies the following **calibration condition**:

$$\rho_t\Big(\lambda_t \frac{B_t}{B_k} 1_{\{k,k+1,\dots\}}\Big) = -\lambda_t B_{t,k} \quad \forall \lambda_t \in L^\infty(\mathcal{F}_t),\ k \geq t.$$

Then ρ_t is cash additive at time k if and only if

$$E_Q\Big[\frac{B_t}{B_k}\,\Big|\,\mathcal{F}_t\Big] = B_{t,k} \quad \forall Q : \exists D \text{ with } \alpha_t(Q \otimes D) < \infty$$

$\longrightarrow$ "no arbitrage" condition

Calibration to ZCB

In particular, if

- $(B_t)_{t=0,\dots,T}$ is predictable
- $(\rho_t)_{t=0,\dots,T}$ is time consistent

then ρ_t reduces to a convex risk measure on random variables $\forall t$.

That is, **discounting ambiguity is completely resolved** and we are only left with model ambiguity.

$\rightarrow$ the time value of the money is completely determined by the term structure specified by the prices of zero coupon bonds

Example: The Swiss Solvency Test

Swiss FOPI $\rightarrow$ SST for the determination of the solvency capital requirement for an insurance company.

Target capital (TC) = 1-year risk capital (ES) + risk margin (M)

$C \in \mathcal{R}^\infty$: risk-bearing capital = assets − liabilities

$ES = \mathbf{ES}_\alpha(\mathbf{\Delta C_1}) = \mathbf{C_0} + \mathbf{ES}_\alpha(\mathbf{C_1})$ capital necessary for the risks emanating within a one year time horizon (currently $\alpha = 1\%$)

$M =$ cost of future regulatory capital for the whole run-off of the in-force portfolio

$= \beta \sum_{\mathbf{s=2}}^{\mathbf{T}} \mathbf{ES}_\alpha(\mathbf{\Delta C_s})$

where

$\beta =$ cost-of-capital rate (currently 6%)

Example: The Swiss Solvency Test

$$TC = C_0 + ES_\alpha(C_1) + \beta \sum_{s=2}^{T} ES_\alpha(\Delta C_s) = C_0 + \rho_{SST}(C)$$

$\rightarrow$ ρ_{SST} is a multiperiod “risk measure” which is cash-invariant and convex, but NOT monotone.

- Filipović & Vogelpoth (2007) propose a less conservative version (the best monotone approximation) of ρ_{SST}:

$$\underline{\rho_{SST}}(C) := \min_{K \leq C} \rho_{SST}(K) = (1-\beta) ES_\alpha(C_1) + \beta ES_\alpha(C_T)$$

$\rightarrow$ $\underline{\rho_{SST}}$ is a convex risk measure on processes with representation over pairs (Q, γ) s.t.

$$\gamma_1 + \gamma_T = 1 \;\; Q\text{-a.s.} \quad \text{and} \quad E_Q[\gamma_T] = \beta$$

Worst stopping

Let $\psi_t : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{F}_t, P)$ be a conditional convex risk measure on random variables.

Θ_t = set of all stopping times valued in $\{t, t+1, ...\}$

Then $\rho_t : \mathcal{R}_t^\infty \to L^\infty(\Omega, \mathcal{F}_t, P)$ defined by the worst stopping of $(\psi_t(X_s))_{s \geq t}$:

$$\rho_t(X) := \operatorname*{ess\,sup}_{\tau \in \Theta_t} \psi_t(X_\tau)$$

is a **convex risk measure on processes** (Cheridito & Kupper (2006)), with representation over the set of optional random measures

$$\left\{ (1_{\{\tau = s\}})_{s=t,t+1,...} | \tau \in \Theta_t \right\}$$

Entropic risk measure for processes

On the product space the **conditional entropic risk measure** $\bar{\rho}_t : L^\infty(\bar{\Omega}, \bar{\mathcal{F}}_T, \bar{P}) \to L^\infty(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{P})$ is defined by

$$\bar{\rho}_t(X) = \frac{1}{R_t} \cdot \log E_{\bar{P}} \left[e^{-R_t \cdot X} \mid \bar{\mathcal{F}}_t \right]$$

with risk aversion parameter $R_t = (r_0, ..., r_{t-1}, r_t, ..., r_t)$, $r_s > 0$ and $\mathcal{F}_s$-measurable, for all $s = 0, ..., t$.

- The corresponding conditional convex risk measure for processes $\rho_t : \mathcal{R}_t^\infty \to L^\infty(\Omega, \mathcal{F}_t, P)$ takes the form

$$\begin{aligned} \rho_t(X) &= \rho_t^{P,r_t}\Big(-\frac{1}{r_t} \log \Big(\sum_{s \geq t} e^{-r_t X_s} \mu_s^t \Big) \Big) \\ &= \rho_t^{P,r_t} \left(-\rho_t^{\mu(\omega), r_t(\omega)} (X_\cdot(\omega)) \right), \end{aligned}$$

  where ρ_t^{P,r_t} is the usual conditional entropic risk measure on random variables with risk aversion parameter r_t and $\rho^{\mu,r}$ is its analogous "with respect to time".

Entropic risk measure for processes

On the product space the **conditional entropic risk measure** $\bar{\rho}_t : L^\infty(\bar{\Omega}, \bar{\mathcal{F}}_T, \bar{P}) \to L^\infty(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{P})$ is defined by

$$\bar{\rho}_t(X) = \frac{1}{R_t} \cdot \log E_{\bar{P}} \left[e^{-R_t \cdot X} \mid \bar{\mathcal{F}}_t \right]$$

with risk aversion parameter $R_t = (r_0, ..., r_{t-1}, r_t, ..., r_t)$, $r_s > 0$ and $\mathcal{F}_s$-measurable, for all $s = 0, ..., t$.

- The corresponding conditional convex risk measure for processes $\rho_t : \mathcal{R}_t^\infty \to L^\infty(\Omega, \mathcal{F}_t, P)$ takes the form

$$\begin{aligned} \rho_t(X) &= \rho_t^{P,r_t}\Big(-\frac{1}{r_t}\log\Big(\sum_{s\geq t} e^{-r_t X_s}\mu_s^t\Big)\Big) \\ &= \rho_t^{P,r_t}\left(-\rho_t^{\mu(\omega),r_t(\omega)}\left(X_\cdot(\omega)\right)\right), \end{aligned}$$

  where ρ_t^{P,r_t} is the usual conditional entropic risk measure on random variables with risk aversion parameter r_t and $\rho^{\mu,r}$ is its analogous "with respect to time".

Average Value at Risk for processes

On the product space the **conditional Average Value at Risk** at level $\Lambda_t = (\lambda_0, ..., \lambda_{t-1}, \lambda_t, ..., \lambda_t)$, $0 < \lambda_s \leq 1$, $\lambda_s \in L^\infty(\mathcal{F}_s) \,\forall\, s$ is

$$\bar{\rho}_t(X) = \operatorname{ess\,sup}\{E_{\bar{Q}}[-X|\bar{\mathcal{F}}_t] \mid \bar{Q} \in \bar{\mathcal{Q}}_t, d\bar{Q}/d\bar{P} \leq \Lambda_t^{-1}\}$$

- The corresponding conditional convex risk measure for processes $\rho_t : \mathcal{R}_t^\infty \to L^\infty(\Omega, \mathcal{F}_t, P)$ takes the form

$$\rho_t(X) = \operatorname{ess\,sup}\left\{E_Q\left[-\sum_{s\geq t} X_s\gamma_s \,\Big|\, \mathcal{F}_t\right] : \frac{\gamma_s M_s}{\mu_s^t} \leq \frac{1}{\lambda_t} \forall s \geq t\right\}$$

On the product space the **conditional Average Value at Risk** at level $\Lambda_t = (\lambda_0, ..., \lambda_{t-1}, \lambda_t, ..., \lambda_t)$, $0 < \lambda_s \leq 1$, $\lambda_s \in L^\infty(\mathcal{F}_s) \, \forall \, s$ is

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Thank you for your attention

and

Happy birthday Walter!