

Time-Consistent Mean-Variance Portfolio Selection in Discrete and Continuous Time

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Mean-variance portfolio selection in one period

- Harry Markowitz (Portfolio selection, 1952):
 - ▶ maximise return and minimise risk
 - ▶ return=expectation
 - ▶ risk=variance
- Mean-variance portfolio selection with risk aversion $\gamma > 0$ in one period:

$$U(\vartheta) = E[x + \vartheta^\top \Delta S] - \frac{\gamma}{2} \text{Var}[x + \vartheta^\top \Delta S] = \max_{\vartheta}$$

- Solution is the so-called **mean-variance efficient strategy**, i.e.

$$\tilde{\vartheta} := \frac{1}{\gamma} \text{Cov}[\Delta S | \mathcal{F}_0]^{-1} E[\Delta S | \mathcal{F}_0] =: \hat{\vartheta}.$$

- Question: How does this extend to multi-period or continuous time?

Basic problem

- **Markowitz problem:**

$$U(\vartheta) = E \left[x + \int_0^T \vartheta_u dS_u \right] - \frac{\gamma}{2} \text{Var} \left[x + \int_0^T \vartheta_u dS_u \right] = \max_{(\vartheta_s)_{0 \leq s \leq T}} !$$

- Static: criterion at time 0 determines optimal $\tilde{\vartheta}$ implicitly via $\tilde{g} = \int_0^T \tilde{\vartheta} dS$.
- **Question:** more explicit dynamic description of $\tilde{\vartheta}$ on $[0, T]$ from \tilde{g} ?

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- Dynamic: Use $\tilde{\vartheta}$ on $(0, t]$ and determine optimal strategy on $(t, T]$ via

$$U_t(\vartheta) = E \left[x + \int_0^T \vartheta_u dS_u \middle| \mathcal{F}_t \right] - \frac{\gamma}{2} \text{Var} \left[x + \int_0^T \vartheta_u dS_u \middle| \mathcal{F}_t \right] = \max_{(\vartheta_s)_{t \leq s \leq T}} !$$

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- **Time inconsistent:** this optimal strategy is different from $\tilde{\vartheta}$ on $(t, T]$!
- **Time-consistent mean-variance portfolio selection:**
Find a strategy $\hat{\vartheta}$, which is “optimal” for $U_t(\vartheta)$ and time-consistent.

Previous literature

- Strotz (1955): “choose the best plan among those that [you] will actually follow.” → Recursive approach to time inconsistency for a different problem.

In Markovian models: Deterministic functions, HJB PDEs and verification thm.

- Ekeland et al. (2006): game theoretic formulation for different problems.
- Basak and Chabakauri (2007): results for mean-variance portfolio selection.
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1) How to formulate this and to obtain the solution in a more general model?

Financial market:

- \mathbb{R}^d -valued semimartingale S wlog. $S = S_0 + M + A \in \mathcal{S}^2(P)$.
- $\Theta = \Theta_S := \{\vartheta \in L(S) \mid \int \vartheta dS \in \mathcal{S}^2(P)\} = L^2(M) \cap L^2(A)$.

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2) Rigorous justification of the continuous-time formulation?

Outline

- 1 Discrete time
- 2 Continuous time
- 3 Convergence of solutions

Local mean-variance efficiency in discrete time

- Use $x + \vartheta \cdot S_T := x + \int_0^T \vartheta_u dS_u = x + \sum_{i=1}^T \vartheta_i \Delta S_i$ and suppose $d = 1$.

Definition

A strategy $\hat{\vartheta} \in \Theta$ is **locally mean-variance efficient (LMVE)** if

$$U_{k-1}(\hat{\vartheta}) - U_{k-1}(\hat{\vartheta} + \delta \mathbb{1}_{\{k\}}) \geq 0 \quad P\text{-a.s.}$$

for all $k = 1, \dots, T$ and any $\delta = (\vartheta - \hat{\vartheta}) \in \Theta$.

- Recursive optimisation (Källblad 2008): $\hat{\vartheta} \in \Theta$ is LMVE if and only if

$$\hat{\vartheta}_k = \frac{1}{\gamma} \frac{E[\Delta S_k | \mathcal{F}_{k-1}]}{\text{Var}[\Delta S_k | \mathcal{F}_{k-1}]} - \frac{\text{Cov}[\Delta S_k, \sum_{i=k+1}^T \hat{\vartheta}_i \Delta S_i | \mathcal{F}_{k-1}]}{\text{Var}[\Delta S_k | \mathcal{F}_{k-1}]} = \frac{1}{\gamma} \lambda_k - \xi_k(\hat{\vartheta})$$

for $k = 1, \dots, T$.

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$$\begin{aligned} \hat{\vartheta}_k &= \frac{1}{\gamma} \frac{E[\Delta S_k | \mathcal{F}_{k-1}]}{\text{Var}[\Delta S_k | \mathcal{F}_{k-1}]} - \frac{\text{Cov}[\Delta S_k, \sum_{i=k+1}^T \hat{\vartheta}_i \Delta S_i | \mathcal{F}_{k-1}]}{\text{Var}[\Delta S_k | \mathcal{F}_{k-1}]} = \frac{1}{\gamma} \lambda_k - \xi_k(\hat{\vartheta}) \\ &= \frac{1}{\gamma} \frac{\Delta A_k}{E[(\Delta M_k)^2 | \mathcal{F}_{k-1}]} - \frac{E[\Delta M_k E[\sum_{i=k+1}^T \hat{\vartheta}_i \Delta S_i | \mathcal{F}_k] | \mathcal{F}_{k-1}]}{E[(\Delta M_k)^2 | \mathcal{F}_{k-1}]} \end{aligned}$$

for $k = 1, \dots, T$.

Structure condition and mean-variance tradeoff process

- S satisfies the **structure condition (SC)**, i.e. there exists a predictable process λ such that

$$A_k = \sum_{i=1}^k \lambda_i E [(\Delta M_i)^2 | \mathcal{F}_{i-1}] = \sum_{i=1}^k \lambda_i \Delta \langle M \rangle_i$$

for $k = 0, \dots, T$ and the **mean-variance tradeoff process (MVT)**

$$K_k := \sum_{i=1}^k \frac{(E[\Delta S_i | \mathcal{F}_{i-1}])^2}{\text{Var}[\Delta S_i | \mathcal{F}_{i-1}]} = \sum_{i=1}^k \lambda_i^2 \Delta \langle M \rangle_i = \sum_{i=1}^k \lambda_i \Delta A_i$$

for $k = 0, \dots, T$ is finite valued, i.e. $\lambda \in L^2_{loc}(M)$.

- If the LMVE strategy $\hat{\vartheta}$ exists, then $\lambda \in L^2(M)$, i.e. $K_T \in L^1(P)$.
- **Comments:** 1) SC and MVT also appear naturally in other quadratic optimisation problems in mathematical finance; see Schweizer (2001).
2) No arbitrage condition: $A \ll \langle M \rangle$.

Expected future gains

- For each $\vartheta \in \Theta$, define the **expected future gains** $Z(\vartheta)$ and the square integrable martingale $Y(\vartheta)$ by

$$\begin{aligned} Z_k(\vartheta) &:= E \left[\sum_{i=k+1}^T \vartheta_i \Delta S_i \middle| \mathcal{F}_k \right] = E \left[\sum_{i=1}^T \vartheta_i \Delta A_i \middle| \mathcal{F}_k \right] - \sum_{i=1}^k \vartheta_i \Delta A_i \\ &=: Y_k(\vartheta) - \sum_{i=1}^k \vartheta_i \Delta A_i \\ &= Y_0(\vartheta) + \sum_{i=1}^k \xi_i(\vartheta) \Delta M_i + L_k(\vartheta) - \sum_{i=1}^k \vartheta_i \Delta A_i \end{aligned}$$

for $k = 0, 1, \dots, T$ inserting the **GKW decomposition** of $Y(\vartheta)$.

Lemma

The LMVE strategy $\hat{\vartheta}$ exists if and only if

1) S satisfies (SC) with $\lambda \in L^2(M)$, i.e. $K_T \in L^1(P)$, and 2) $\hat{\vartheta} = \frac{1}{\gamma} \lambda - \xi(\hat{\vartheta})$.

Global description of $\xi(\hat{\vartheta})$ via FS decomposition

- Combining both representations we obtain

$$\begin{aligned} Y_T(\hat{\vartheta}) &= Y_0(\hat{\vartheta}) + \sum_{i=1}^T \xi_i(\hat{\vartheta}) \Delta M_i + L_T(\hat{\vartheta}) \\ &= \sum_{i=1}^T \hat{\vartheta}_i \Delta A_i = \sum_{i=1}^T \left(\frac{1}{\gamma} \lambda_i - \xi_i(\hat{\vartheta}) \right) \Delta A_i \\ \frac{1}{\gamma} K_T &= \frac{1}{\gamma} \sum_{i=1}^T \lambda_i \Delta A_i = Y_0(\hat{\vartheta}) + \sum_{i=1}^T \xi_i(\hat{\vartheta}) \Delta S_i + L_T(\hat{\vartheta}) \end{aligned} \quad (1)$$

- (1) is almost the **Föllmer–Schweizer (FS) decomposition** of $\frac{1}{\gamma} K_T$.
- The integrand $\xi(\hat{\vartheta}) =: \frac{1}{\gamma} \hat{\xi}$ in the FS decomposition yields the **locally risk-minimising strategy** for the contingent claim $\frac{1}{\gamma} K_T$.
- Global description:** $\hat{\vartheta} \in \Theta$ exists iff SC, (1) and $\hat{\vartheta} = \frac{1}{\gamma}(\lambda - \hat{\xi})$.

Continuous time setting

- Increasing, integrable, predictable process B called “operational time” such that: $A = a \cdot B$, $\langle M, M \rangle = \tilde{c}^M \cdot B$ and $a = \tilde{c}^M \lambda + \eta$ with $\eta \in \text{Ker}(\tilde{c}^M)$.
- S satisfies the **structure condition (SC)**, if $\eta = 0$, i.e.

$$A = \int d\langle M \rangle \lambda,$$

and the **mean-variance tradeoff process (MVT)**

$$K_t := \int_0^t \lambda_u^\top d\langle M \rangle_u \lambda_u = \int_0^t \lambda_u dA_u < +\infty.$$

- Expected future gains** $Z(\vartheta)$ and **GKW decomposition** of $Y(\vartheta)$

$$\begin{aligned} Z_t(\vartheta) &:= E \left[\int_t^T \vartheta_u dS_u \middle| \mathcal{F}_t \right] =: Y_t(\vartheta) - \int_0^t \vartheta_u dA_u \\ &= Y_0(\vartheta) + \int_0^t \xi_u(\vartheta) dM_u + L_t(\vartheta) - \int_0^t \vartheta_u dA_u \end{aligned}$$

Local mean-variance efficiency in continuous time

- Idea: Combine **recursive optimisation** with a **limiting argument**.

Definition

A strategy $\hat{\vartheta} \in \Theta$ is **locally mean-variance efficient** (in continuous time) if

$$\lim_{n \rightarrow \infty} u^{\Pi_n}[\hat{\vartheta}, \delta] := \lim_{n \rightarrow \infty} \sum_{t_i, t_{i+1} \in \Pi_n} \frac{U_{t_i}(\hat{\vartheta}) - U_{t_i}(\hat{\vartheta} + \delta \mathbb{1}_{(t_i, t_{i+1}]})}{E[B_{t_{i+1}} - B_{t_i} | \mathcal{F}_{t_i}]} \mathbb{1}_{(t_i, t_{i+1}]} \geq 0 \quad P \otimes B\text{-a.e.}$$

for any increasing sequence (Π_n) of partitions such that $|\Pi_n| \rightarrow 0$ and any $\delta \in \Theta$.

- Inspired by the concept of local risk-minimisation (LRM); Schweizer (88, 08).

$$\lim_{n \rightarrow \infty} u^{\Pi_n}[\hat{\vartheta}, \delta] = \left(\gamma(\xi(\hat{\vartheta}) + \hat{\vartheta}) - \lambda + \frac{\gamma}{2} \delta \right)^\top \tilde{c}^M \delta - \delta^\top \eta \quad P \otimes B\text{-a.e.}$$

- Remarks:** 1) Convergence without any additional assumptions, i.e. boundedness assumptions on δ and continuity of A .
2) Generalises also results for LRM.

The LMVE strategy $\hat{\vartheta}$ in continuous time

Theorem

1) The LMVE strategy $\hat{\vartheta} \in \Theta$ exists if and only if

- i) S satisfies (SC) with $\lambda \in L^2(M)$, i.e. $K_T \in L^1(P)$.
- ii) $\hat{\vartheta} = \frac{1}{\gamma}\lambda - \xi(\hat{\vartheta})$, i.e. $\hat{J}(\hat{\vartheta}) = \hat{\vartheta}$, where $\hat{J}(\psi) := \frac{1}{\gamma}\lambda - \xi(\psi)$ for $\psi \in \Theta$ and $\xi(\psi)$ is the integrand in the GKW decomposition of $Y_T(\psi) = \int_0^T \psi_u dA_u$.

2) If K is bounded and continuous, $\hat{J}(\cdot)$ is a contraction on $(\Theta, \|\cdot\|_{\beta, \infty})$ where

$$\|\vartheta\|_{\beta, \infty} := \left\| \left(\int_0^T \frac{1}{\mathcal{E}(-\beta K)_u} \vartheta_u^\top d\langle M \rangle_u \vartheta_u \right)^{\frac{1}{2}} \right\|_{L^2(P)} \sim \|\vartheta\|_{L^2(M)} + \|\vartheta\|_{L^2(A)}.$$

In particular, the LMVE strategy $\hat{\vartheta}$ is given as the limit $\hat{\vartheta} = \lim_{n \rightarrow \infty} \vartheta^n$ in $(\Theta, \|\cdot\|_{\beta, \infty})$, where $\vartheta^{n+1} = \hat{J}(\vartheta^n)$ for $n \geq 1$, for any $\vartheta^0 = \vartheta \in \Theta$.

- **Remark:** Using the “salami technique” of Monat and Stricker (1996) one can drop the assumption that K is continuous in 2).

Global description of $\xi(\hat{\vartheta})$ via FS decomposition

Theorem

The LMVE strategy $\hat{\vartheta} \in \Theta$ exists if and only if S satisfies (SC) and the MVT process $K_T \in L^1(P)$ and can be written as

$$K_T = \hat{K}_0 + \int_0^T \hat{\xi} dS + \hat{L}_T \quad (2)$$

with $\hat{K}_0 \in L^2(\mathcal{F}_0)$, $\hat{\xi} \in L^2(M)$ such that $\hat{\xi} - \lambda \in L^2(A)$ and $\hat{L} \in \mathcal{M}_0^2(P)$ strongly orthogonal to M . In that case, $\hat{\vartheta} = \frac{1}{\gamma}(\lambda - \hat{\xi})$, $\xi(\hat{\vartheta}) = \frac{1}{\gamma}\hat{\xi}$ and $U(\hat{\vartheta}) = \dots$ (2).

- If the **minimal martingale measure** exists, i.e. $\frac{d\hat{P}}{dP} := \mathcal{E}(-\lambda \cdot M)_T \in L^2(P)$ and strictly positive, and $K_T \in L^2(P)$, then

$$Z_t(\hat{\vartheta}) = \frac{1}{\gamma} \left(\hat{K}_0 + \int_0^t \hat{\xi} dS + \hat{L}_t - K_t \right) = \frac{1}{\gamma} \hat{E}[K_T - K_t | \mathcal{F}_t],$$

and $\hat{\xi}$ is related to the GKW of K_T under \hat{P} ; see Choulli et al. (2010).

- Application in concrete models: 1) λ , 2) K , 3) $\mathcal{E}(-\lambda \cdot M)$ and 4) $\hat{\xi} \dots$

Discretisation of the financial market

- Let $(\Pi_n)_{n \in \mathbb{N}}$ be increasing such that $|\Pi_n| \rightarrow 0$ and $S = S_0 + M + A$.

Discretisation of processes

- $S_t^n := S_{t_i}$, $M_t^n := M_{t_i}$ and $A_t^n := A_{t_i}$ for $t \in [t_i, t_{i+1})$ and all $t_i \in \Pi_n$.

Discretisation of filtration

- $\mathcal{F}_t^n := \mathcal{F}_{t_i}$ for $t \in [t_i, t_{i+1})$ and all $t_i \in \Pi_n$ and $\mathbb{F}^n := (\mathcal{F}_t^n)_{0 \leq t \leq T}$.

Canonical decomposition of $S^n = S_0 + \bar{M}^n + \bar{A}^n \in \mathcal{S}^2(P, \mathbb{F}^n)$

- $\bar{A}_t^n := \sum_{k=1}^i E[\Delta A_{t_k}^n | \mathcal{F}_{t_{k-1}}] = A_t^n - \mathbf{M}_t^{\mathbf{A},n}$
- $\bar{M}_t^n := M_t^n + \mathbf{M}_t^{\mathbf{A},n}$ for $t \in [t_i, t_{i+1})$

where the “**discretisation error**” is given by the \mathbb{F}^n -martingale

$$\mathbf{M}_t^{\mathbf{A},n} := \sum_{k=1}^i (\Delta A_{t_k}^n - E[\Delta A_{t_k}^n | \mathcal{F}_{t_{k-1}}]) \quad \text{for } t \in [t_i, t_{i+1}).$$

Convergence of solutions $\widehat{\vartheta}^n$

- Due to time inconsistency usual abstract arguments don't work.
- Work with **global description** directly to show

$$\widehat{\vartheta}^n = \frac{1}{\gamma}(\lambda^n - \widehat{\xi}^n) \xrightarrow{L^2(M)} \widehat{\vartheta} = \frac{1}{\gamma}(\lambda - \widehat{\xi}), \quad \text{as } |\Pi^n| \rightarrow 0.$$

- **Discrete-** and **continuous-time FS decomposition**

$$K_T^n = \widehat{K}_0^n + \sum_{t_i \in \Pi_n} \widehat{\xi}_{t_i}^n \Delta S_{t_i}^n + \widehat{L}_T^n \quad \text{and} \quad K_T = \widehat{K}_0 + \int_0^T \widehat{\xi}_u dS_u + \widehat{L}_T.$$

- For this we establish

$$1) \quad \lambda^n = \sum_{t_i, t_{i+1} \in \Pi_n} \frac{\Delta \bar{A}_{t_{i+1}}^n}{E[(\Delta M_{t_{i+1}}^n)^2 | \mathcal{F}_{t_i}]} \mathbb{1}_{(t_i, t_{i+1}]} \xrightarrow{L^2(M)} \lambda$$

$$2) \quad K_T^n = \sum_{t_i, t_{i+1} \in \Pi_n} \lambda_{t_{i+1}}^n \Delta \bar{A}_{t_{i+1}}^n \xrightarrow{L^2(P)} K_T = \int_0^T \lambda_u dA_u$$

$$3) \quad 2), |\Pi_n| \rightarrow 0 \text{ implies } \widehat{\xi}^n \xrightarrow{L^2(M)} \widehat{\xi}.$$

- **Problem** to control the “discretisation error” $M^{A,n}$.
- **Simple sufficient condition:** $K = \int k dt$ and k uniformly bounded.

Thank you for your attention!

<http://www.math.ethz.ch/~czichowc>

Some references

-  Basak and Chabakauri. *Dynamic Mean-Variance Asset Allocation*. (2007). Forthcoming in *Review of Financial Studies*.
-  Björk and Murgoci. *A General Theory of Markovian Time Inconsistent Stochastic Control Problems*. Working paper, Stockholm School of Economics, (2008).
-  Ekeland and Lazrak. *Being serious about non-commitment: subgame perfect equilibrium in continuous time*, (2006). Preprint, Univ. of British Columbia.
-  Choulli, Vandaele and Vanmaele. *The Föllmer-Schweizer decomposition: Comparison and description*. *Stoch. Pro. and Appl.*, (2010), 853-872.
-  Schweizer. *Hedging of Options in a General Semimartingale Model*. Diss. ETHZ no. 8615, ETH Zürich (1988).
-  Schweizer. *A Guided Tour through Quadratic Hedging Approaches*. In *Option Pricing, Interest Rates and Risk Management*, Cambridge Univ. Press (2001).
-  Schweizer. *Local risk-minimization for multidimensional assets and payment streams*. In *Advances in Mathematics of Finance*, Banach Center Publ. (2008).
-  Strotz. *Myopia and inconsistency in dynamic utility maximization*. *Review of Financial Studies* (1956), 165-180.