

# On refined volatility smile expansion in the Heston model

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# Heston Model

- Dynamics

$$\begin{aligned}dS_t &= S_t \sqrt{V_t} dW_t, & S_0 &= 1, \\dV_t &= (a + bV_t) dt + c \sqrt{V_t} dZ_t, & V_0 &= v_0 > 0,\end{aligned}$$

- Correlated Brownian motions

$$d\langle W, Z \rangle_t = \rho dt, \quad \rho \in [-1, 1]$$

- Parameters

$$a \geq 0, b \leq 0, c > 0$$

## Density and smile asymptotics

- Consider a fixed maturity  $T > 0$ .
- $D_T :=$  density of  $S_T$ .
- How heavy are the tails?

$$D_T(x) \sim ? \quad (x \rightarrow 0, \infty)$$

- Implied Black-Scholes volatility ( $k = \log K$  is the log-strike)

$$\sigma_{BS}^2(k, T) \sim ? \quad (k \rightarrow \pm\infty)$$

## Known results

- Leading term of smile asymptotics: Lee's moment formula. Andersen, Piterbarg (2007); Benaim, Friz (2008)
- Drăgulescu, Yakovenko (2002): Stationary variance regime. Leading growth order of distribution function of  $S_T$ , by (non-rigorous) saddle-point argument
- Gulisashvili-Stein (2009): Precise density asymptotics for uncorrelated Heston model

## Main results (right tail), SG et al. 2010

- Density asymptotics for  $x \rightarrow \infty$

$$D_T(x) = A_1 x^{-A_3} e^{A_2 \sqrt{\log x}} (\log x)^{-3/4+a/c^2} (1 + O((\log x)^{-1/2}))$$

- Implied volatility for  $k = \log K \rightarrow \infty$

$$\sigma_{BS}(k, T) \sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O\left(\frac{\varphi(k)}{k^{1/2}}\right)$$

( $\varphi$  arbitrary function tending to  $\infty$ )

## Interpretation of smile expansion

- Implied volatility for  $k = \log K \rightarrow \infty$

$$\sigma_{BS}(k, T) \sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O\left(\frac{\varphi(k)}{k^{1/2}}\right)$$

- $\beta_1$  does not depend on  $\sqrt{v_0}$
- $\beta_2$  depends linearly on  $\sqrt{v_0}$
- Changes of  $\sqrt{v_0}$  have second-order effects
- Increase  $\sqrt{v_0}$ : parallel shift, slope not affected
- Changes in mean-reversion level  $\bar{v} = -a/b$  seen only in  $\beta_3$

## General remarks

- Constants depend on: critical moment, critical slope, critical curvature
- Critical moment etc. defined in a model-free manner
- Closed form of Fourier (Mellin) transform not needed
- Work only with affine principles (Riccati equations)

## Lee's moment formula (2004)

- Model-free result
- Relates critical moment to implied volatility

$$s^* := \sup\{s : E[S_T^s] < \infty\}$$

$$s^* =: \frac{1}{2\beta_1^2} + \frac{\beta_1^2}{8} + \frac{1}{2}$$

$$\limsup_{k \rightarrow \infty} \frac{\sigma_{BS}(k, T)\sqrt{T}}{\sqrt{k}} = \beta_1$$

- Refinements by Benaim, Friz (2008), Gulisashvili (2009)



## Heston Model: Mgf of log-spot $X_t$

- Moment generating function

$$E[e^{sX_t}] = \exp(\phi(s, t) + v_0\psi(s, t))$$

- Riccati equations

$$\partial_t \phi = F(s, \psi), \quad \phi(0) = 0,$$

$$\partial_t \psi = R(s, \psi), \quad \psi(0) = 0$$

$$F(s, v) = av,$$

$$R(s, v) = \frac{1}{2}(s^2 - s) + \frac{1}{2}c^2v^2 + bv + s\rho cv$$

- Explicit solution possible, but cumbersome expression

# Moment explosion

- Critical moment for time  $T$

$$s^* := \sup \{s \geq 1 : E[S_T^s] < \infty\}$$

- Explosion time for moment of order  $s$

$$T^*(s) = \sup \{t \geq 0 : E[S_t^s] < \infty\}$$

- Critical slope, critical curvature:

$$\sigma := -\partial_s T^*|_{s^*} \geq 0 \quad \text{and} \quad \kappa := \partial_s^2 T^*|_{s^*}$$

## Explicit Explosion time for the Heston model

- Explosion time for moment of order  $s$

$$T^*(s) = \frac{2}{\sqrt{-\Delta(s)}} \left( \arctan \frac{\sqrt{-\Delta(s)}}{s\rho c + b} + \pi \right),$$

$$\Delta(s) := (s\rho c + b)^2 - c^2 (s^2 - s)$$

- Critical moment  $s^*$ : Find numerically from

$$T^*(s^*) = T.$$

## Mellin (Fourier) inversion

- Mellin transform of spot:  $M(u) = E[e^{(u-1)X_T}]$
- Analytic in a complex strip
- Density of  $S_T$  by Mellin inversion:

$$D_T(x) = \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} x^{-u} M(u) du.$$

- Valid for contour in analyticity strip of the Mellin transform
- Justification: exponential decay of  $M(u)$  at  $\pm i\infty$ .

## Analyticity and growth

- Mellin transform analytic in a strip

$$u_- < \Re(u) < u^* = s^* + 1$$

- Leading order of density for  $x \rightarrow \infty$

$$x^{-u^* - \varepsilon} \ll D_T(x) \ll x^{-u^* + \varepsilon},$$

depends on *location* of singularity

- Refinement: lower order factors depend on *type* of singularity

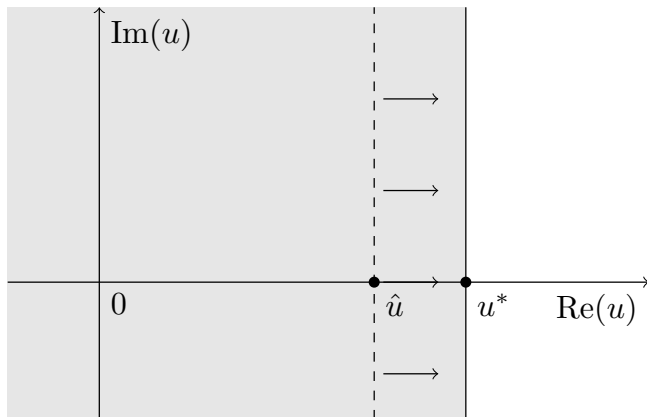
## Saddle point method

- Recall:

$$D_T(x) = \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} x^{-u} M(u) du$$

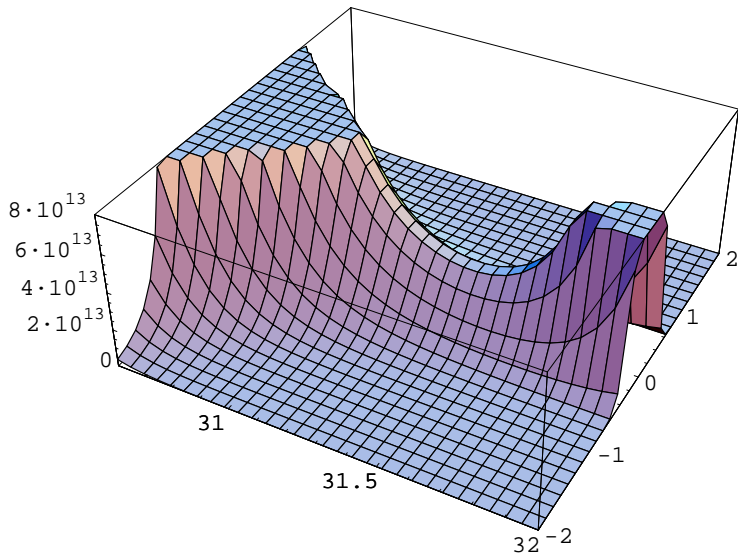
- Shift contour to the right, close to the singularity.
- Let it pass through a saddle point of the integrand.
- For large  $x$ , the integral is concentrated around the saddle.
- Local expansion of integrand yields expansion of whole integral.
- (Laplace, Riemann, Debye...)

## New integration contour



- Contour runs through saddle point  $\hat{u} = \hat{u}(x)$
- Moves to the right as  $x \rightarrow \infty$

# The surface $|x^{-u}M(u)|$





## Asymptotics of $\psi$ and $\phi$ near critical moment

- Recall  $M(u) = \exp(\phi(u-1, t) + v_0\psi(u-1, t))$
- For  $u \rightarrow u^*$  we have (with  $\beta := \sqrt{2v_0}/c\sqrt{\sigma}$ )

$$\psi(u-1, T) = \frac{\beta^2}{u^* - u} + \text{const} + O(u^* - u),$$

$$\phi(u-1, T) = \frac{2a}{c^2} \log \frac{1}{u^* - u} + \text{const} + O(u^* - u)$$

- Found from Riccati equations

## Saddle point method

- Finding the saddle point:  $0 =$  derivative of integrand
- Use only first order expansion:

$$0 = \frac{\partial}{\partial u} x^{-u} \exp\left(\frac{\beta^2}{u^* - u}\right)$$

- Approximate saddle point at

$$\hat{u}(x) = u^* - \beta/\sqrt{\log x}$$

## New integration contour

- Contour depends on  $x$ :

$$u = \hat{u}(x) + iy, \quad -\infty < y < \infty$$

- Divide contour into three parts:

$$|y| < (\log x)^{-\alpha} \quad (\text{central part}),$$

upper tail, lower tail (symmetric)

- Uniform local expansion at saddle point  $\Rightarrow$  need large  $\alpha$
- Tails negligible  $\Rightarrow$  need small  $\alpha$
- Can take  $\frac{2}{3} < \alpha < \frac{3}{4}$

## Local expansion

- Recall Mellin transform

$$M(u) = \exp(\phi(u-1, t) + v_0\psi(u-1, t))$$

- Determine singular expansions of  $\phi$  and  $\psi$  from Riccati equations
- Abbreviation  $L := \log x$
- Local expansion of the integrand:

$$x^{-u}M(u) = Cx^{-u^*} \exp\left(2\beta L^{1/2} + \frac{a}{c^2} \log L - \beta^{-1}L^{3/2}y^2 + o(1)\right)$$

- Gaussian integral

$$\begin{aligned} & \int_{-L^{-\alpha}}^{L^{-\alpha}} \exp(-\beta^{-1}L^{3/2}y^2)dy \\ &= \beta^{1/2}L^{-3/4} \int_{-\beta^{-1/2}L^{3/4-\alpha}}^{\beta^{-1/2}L^{3/4-\alpha}} \exp(-w^2)dw \\ &\sim \beta^{1/2}L^{-3/4} \int_{-\infty}^{\infty} \exp(-w^2)dw = \sqrt{\pi}\beta^{1/2}L^{-3/4} \end{aligned}$$

## Tail estimate

- Finding saddle point + local expansion fairly routine
- Problem: Verify concentration
- Needs some insight into behaviour of function away from saddle point
- Show exponential decay by ODE comparison

## Result of saddle point method

- Density asymptotics for  $x \rightarrow \infty$

$$D_T(x) = A_1 x^{-A_3} e^{A_2 \sqrt{\log x}} (\log x)^{-3/4+a/c^2} (1 + O((\log x)^{-1/2}))$$

- Constants in terms of critical moment and critical slope:

$$A_3 = u^* = s^* + 1 \quad \text{and} \quad A_2 = 2 \frac{\sqrt{2v_0}}{c\sqrt{\sigma}}$$

- Easily extended to full asymptotic expansion

## Explicit expression for constant factor

- From closed form of  $\phi$  and  $\psi$ :

$$\begin{aligned} A_1 &= \frac{1}{2\sqrt{\pi}} (2v_0)^{1/4 - a/c^2} c^{2a/c^2 - 1/2} \sigma^{-a/c^2 - 1/4} \\ &\times \exp\left(-v_0 \left(\frac{b + s^* \rho c}{c^2} + \frac{\kappa}{c^2 \sigma^2}\right) - \frac{aT}{c^2} (b + c \rho s^*)\right) \\ &\times \left(\frac{2\sqrt{b^2 + 2bc\rho s^* + c^2 s^*(1 - (1 - \rho^2)s^*)}}{c^2 s^*(s^* - 1) \sinh \frac{1}{2} \sqrt{b^2 + 2bc\rho s^* + c^2 s^*(1 - (1 - \rho^2)s^*)}}\right)^{2a/c^2} \end{aligned}$$



## Call prices and Smile asymptotics

- Gulisashvili (2009): Assumes that density of spot varies regularly at infinity

$$D_T(x) = x^{-\gamma} h(x),$$

$h$  varies slowly at infinity,  $\gamma > 2$

- Expansions of call prices and implied volatility
- Similarly for left tail

## Smile asymptotics

- Implied volatility for log-strike  $k \rightarrow \infty$

$$\sigma_{BS}(k, T)\sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O\left(\frac{\varphi(k)}{k^{1/2}}\right)$$

- Constants

$$\beta_1 = \sqrt{2} \left( \sqrt{A_3 - 1} - \sqrt{A_3 - 2} \right),$$

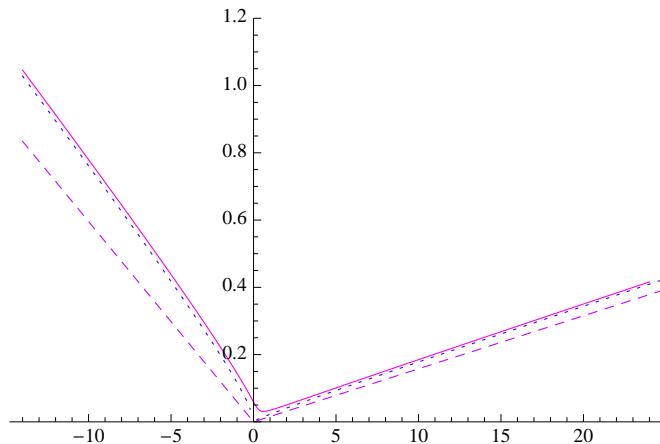
$$\beta_2 = \frac{A_2}{\sqrt{2}} \left( \frac{1}{\sqrt{A_3 - 2}} - \frac{1}{\sqrt{A_3 - 1}} \right),$$

$$\beta_3 = \frac{1}{\sqrt{2}} \left( \frac{1}{4} - \frac{a}{c^2} \right) \left( \frac{1}{\sqrt{A_3 - 1}} - \frac{1}{\sqrt{A_3 - 2}} \right)$$

- Call price for strike  $K \rightarrow \infty$

$$C(K) = \frac{A_1}{(-A_3 + 1)(-A_3 + 2)} K^{-A_3+2} e^{A_2 \sqrt{\log K}} (\log K)^{-\frac{3}{4} + \frac{a}{c^2}} \\ \times \left( 1 + O\left((\log K)^{-\frac{1}{4}}\right) \right)$$

# Smile asymptotics



**Figure:** Implied variance  $\sigma(k, 1)^2$  in terms of log-strikes compared to the first order (dashed) and third order (dotted) approximations.

## References

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