Modelling of Dependent Credit Rating Transitions

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Introduction

Motivation:

- Volcano on Iceland erupted and caused that most of the flights in Europe had to be cancelled for a few days.
- That caused simultaneous losses of the airlines.

$\Rightarrow$ Credit quality of the airlines is simultaneously affected.

Previous literature:

- Dependence introduced by interacting intensities
- No simultaneous credit rating transitions possible!

Main modeling assumption:
Firms may simultaneously change their credit rating in continuous time.

Outline

1. Model
   - General framework
   - General model
   - Examples
2. Simulation
3. Maximum Likelihood Estimation
   - MLE for the extended strongly coupled random walk
   - Asymptotic properties of the estimator

The Marked Point Process

Definition (Marked point process)

- $(\tau_i)_{i\in\mathbb{N}}$: random time with values in $(0,\infty]$, and $\tau_i < \tau_{i+1}$ on $\{\tau_i < \infty\}$ and $\tau_i = \tau_{i+1} = \infty$ on $\{\tau_i = \infty\}$
- $(\rho_i)_{i\in\mathbb{N}}$: random mark with $\rho_i \in E$ on $\{\tau_i < \infty\}$ and $\rho_i := \rho_\infty$ on $\{\tau_i = \infty\}$, where ρ_∞ external point of E.

We call $\big((\tau_i,\rho_i)\big)_{i\in\mathbb{N}}$ a **marked point process**.

Mark space E:

$$E = \Big\{ r : S \times I \to S \Big| \ r \text{ is } \big(\mathcal{P}(S)\otimes\mathcal{I}\big)\text{-}\mathcal{P}(S) \text{ measurable} \Big\}$$

- $S = \{1,\ldots,K\}$: credit rating classes, where K means firm is in default and 1 is best rating class
- Measurable space $(I,\mathcal{I})$: state space of idiosyncratic component

The General Framework

- $F = \{1, \dots, n\}$: set of firms, $n \in \mathbb{N}$ is number of firms
- $X = \big((X_t(1), \dots, X_t(n))\big)_{t \geq 0}$: credit rating process
- $\big((\tau_i, \rho_i)\big)_{i \in \mathbb{N}}$: marked point process
- $U_i(j)$: I-valued random variable for $i \in \mathbb{N}$ and $j \in F$.

Definition (General framework)

We say that the process $X = (X_t)_{t \geq 0}$ with state space S^n **follows the general framework**, if

1. $X_t = X_0$ for $t \in [0, \tau_1)$, and
2. for each $i \in \mathbb{N}$ and firm $j \in F$

$$X_t(j) = \rho_i\big(X_{\tau_i -}(j), U_i(j)\big) \quad \text{for } t \in [\tau_i, \tau_{i+1}).$$

Remark: Process is in general not Markovian.

Markov Process in the General Framework

Additional assumptions to obtain a Markov process:

1. Random times $(\tau_i)_{i\in\mathbb{N}}$:
   jump times of a Poisson process with intensity $\lambda > 0$
2. Random marks $(\rho_i)_{i\in\mathbb{N}}$: i. i. d. sequence
3. Idiosyncratic components $\{U_i(j) : i \in \mathbb{N}, j \in F\}$:
   i. i. d. collection
4. $(\rho_i)_{i\in\mathbb{N}}$, $\{U_i(j) : i \in \mathbb{N}, j \in F\}$, X_0 and the Poisson process are pairwise independent.

In the following:
We assume that these additional assumptions are satisfied.

Dynamics of the General Model

Assumption for the general model:

All firms with the same rating may simultaneously change only to the same rating class or remain in their rating class.

Dynamics of the general model:

- Possible rating transitions are given by a map $s : S \to S$:
  - Each firm with rating 1 either remains in this class or changes its rating to $s(1)$,
  - each firm with rating 2 remains in 2 or changes to $s(2)$,
  - and so on ...
- The probability that a firm actually changes is given by p_x, where $x \in S$ is the current rating of the firm.

Definition of the General Model

Definition (General model)

We say that the Markov jump process $X = (X_t)_{t \geq 0}$ **follows the general model with parameters** (λ, P, p), if it follows the general framework with the additional assumptions:

- P probability distribution on S^S and $p \in [0,1]^S$
- Each ρ_i takes a. s. only values in $\{r_s : s \in S^S\} \subset E$ where

$$r_s(x, u) = \begin{cases} s(x), & \text{if } u \in [0, p_x], \\ x, & \text{if } u \in [p_x, 1]. \end{cases}$$

- $\mathbf{P}[\rho_i = r_s] = P(s)$ for each $s \in S$
- $U_i(j)$: uniformly distributed on $I = [0,1]$ for $i \in \mathbb{N}$, $j \in F$

Example 1: The Strongly Coupled Random Walk

Dynamics:

Only firms in one rating class may simultaneously change to the same rating class or remain in their rating class.

Parameters:

- Independent Poisson processes with intensity $\lambda_x > 0$ for each rating class $x \in S$
- Stochastic transition function $P^{\mathrm{c}} : S \times S \to [0, 1]$: probability for transitions from x to y given Poisson process of x jumps
- $p_x \in [0, 1]$: probability that a firm with rating x actually changes the rating

Embedding in the General Model

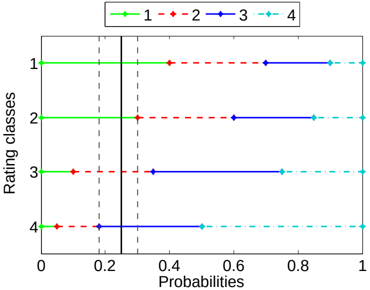
Define $\lambda = \sum_{x \in S} \lambda_x$ and the distribution P on S^S by

$$P(s) = \begin{cases} \frac{\lambda_x}{\lambda} P^{\mathrm{c}}(x,y), & \text{if there exist } x, y \in S \text{ with } x \neq y, \\ & s(x) = y,\ s(u) = u \text{ for all } u \in S \setminus \{x\} \\ \sum\limits_{x \in S} \frac{\lambda_x}{\lambda} P^{\mathrm{c}}(x,x), & \text{if } s(x) = x \text{ for all } x \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Definition (Strongly coupled random walk)

We say that the Markov jump process X is a **strongly coupled random walk process** with parameters $\big((\lambda_x)_{x \in S}, P^{\mathrm{c}}, p\big)$, if X follows the general model with parameters (λ, P, p).

Example 2: The Scheme Model



For each $x \in S$ the interval $[0,1]$ is divided into K subintervals with length p_{xy} for the y-th subinterval.
The subinterval containing V represents the rating class $\tilde{s}(x)$.

- $(p_{xy})_{x,y \in S} \in [0,1]^{S \times S}$: stochastic transition function
- V: random variable, uniformly distributed on $[0,1]$
- S^S-valued random function $\tilde{s}$:

$$\tilde{s}(x) = \max\left\{ y \in S : \sum_{k=1}^{y-1} p_{xk} \leq V \right\}, \quad \text{for } x \in S.$$

Definition of the Scheme Model

The distribution of $\tilde{s}$ is given by

$$P^{\mathrm{s}}(s) = \max\left\{\min_{x\in S}\sum_{k=1}^{s(x)} p_{xk} - \max_{x\in S}\sum_{k=1}^{s(x)-1} p_{xk}, 0\right\}.$$

Definition (Scheme model)

- $(p_{xy})_{x,y\in S} \in [0,1]^{S\times S}$: stochastic transition function
- P^{s}: probability distribution of $\tilde{s}$
- $\lambda > 0$ and $p = (p_x)_{x\in S}$ is a vector in $[0,1]^S$

We say that the Markov jump process X follows the **scheme model** with parameters $\big(\lambda, (p_{xy})_{x,y\in S}, p\big)$, if X follows the general model with parameters $(\lambda, P^{\mathrm{s}}, p)$.

Embedding of a Model with Fewer Firms

Theorem (Embedding property)

- *X rating process in general framework with n firms*
- *Y rating process in general framework with $m < n$ firms*

$\Rightarrow$ *Distribution of rating transitions of first m firms of X*
$=$ *Distribution of rating transitions of Y*

- Q-matrix $\mu \in \mathbb{R}^{K \times K}$ of the transitions of the individual firms is the same for all firms.
- Correspondence of parameters:

$$(\mu, p) \Rightarrow (\lambda_x, P^{\mathrm{c}}, p) \quad \text{or} \quad (\lambda, P^{\mathrm{s}}, p)$$

- **Extended strongly coupled random walk:**
  $p_x = 0$: independent rating transitions of firms in class x

Loss of a Credit Portfolio

Credit portfolio:

- $n = 100$ credits with amount $C = 1$ and maturity $T = 15$.
- Obligors change credit rating according to process X.
- $K = 8$ rating classes, default K is an absorbing state.
- Recovery rate: $\delta = 0.4$
- Default-free interest rate is zero.

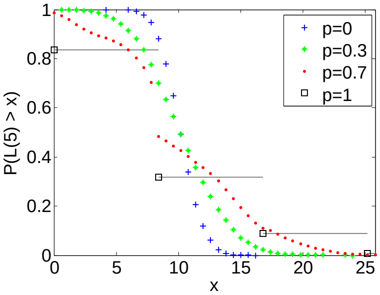
Loss of the credit portfolio:

$$L(t) = \sum_{i=1}^{n} C\,(1-\delta)\mathbb{1}_{\{X_{t\wedge T}(i)=K\}}, \quad \text{for } t \geq 0.$$

Empirical Excess Loss Distribution

- X_0: 16 firms in rating class 1, 14 firms each in 2 to 7
- $p_x = p$ for all $x \in S$
- Intensity μ of individual credit rating transitions is based on data of Standard & Poor's.

Empirical excess loss distribution (5 000 simulations):



strongly coupled random walk — scheme model

Histogram of Simulated Losses

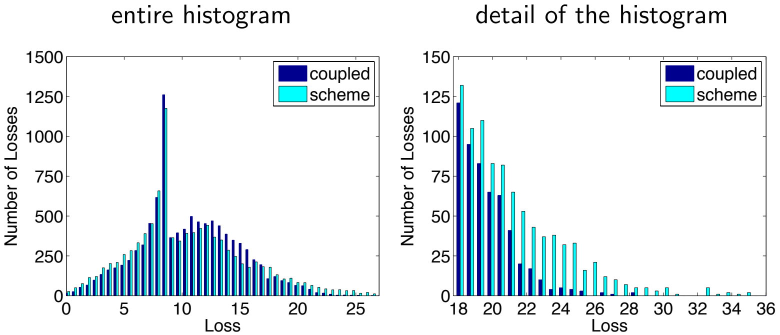


Figure: Histogram of the simulated losses for the strongly coupled random walk (coupled) and the scheme model (scheme) where $p = 0.5$ and $t = 5$, based on 10 000 simulations.

Estimation for Strongly Coupled Random Walk

Notation:

- Set of parameters: $\Theta = \big([0,\infty)^{K-1} \times [0,1]\big)^K$
- $\theta \in \Theta$: Set $\theta = (\theta_x)_{x \in S}$ with $\theta_x = (\mu_{x,1}, \ldots, \mu_{x,x-1}, \mu_{x,x+1}, \ldots, \mu_{x,K}, p_x)$
- X follows extended strongly coupled random walk process with true parameter θ_0 and n firms.

Parameter estimation:
Given observations of sample paths of X, which parameter $\hat{\theta}$ is likeliest to be the true parameter θ_0?

Likelihood Function

Likelihood function:

$$\mathcal{L}(\theta) = \left(\prod_{\substack{x,y \in S \\ x \neq y}} \prod_{\substack{a,b=1 \\ a \geq b}}^{n} \left(\mu_{xy} p_x^{b-1} (1-p_x)^{a-b} \right)^{N_{x,y,a,b}} \right)$$

$$\times \left(\prod_{j=1}^{k} \mathbf{P}\left[X_0 = z_j \right] \right) \exp \left\{ - \sum_{x \in S} \mu_x \sum_{a=1}^{n} T_{x,a} \sum_{j=0}^{a-1} (1-p_x)^j \right\}$$

- $N_{x,y,a,b} \in \mathbb{N}_0$: total number of simultaneous rating changes of b firms from x to $y \neq x$, a firms originally with rating x in the observed k paths
- $T_{x,a} \in [0, T]$: total time that exactly a firms have rating x
- $z_j \in S^n$: initial rating in the j-th observed path

Maximum Likelihood Estimator

Theorem (Maximum Likelihood Estimator)

The parameters in $\hat{\Theta} \subset \Theta$ are exactly the MLE, where for $\hat{\theta} \in \hat{\Theta}$ holds:

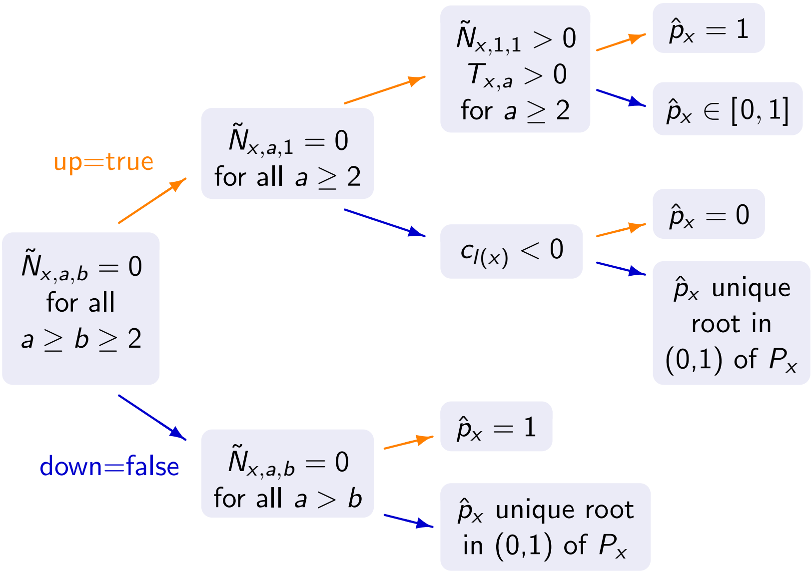
1. *$\hat{p}_x$ for $x \in S$ is either 0, 1, the unique root in $(0, 1)$ of polynomial P_x, or arbitrary depending on $\tilde{N}_{x,a,b}$ and $T_{x,a}$.*
2. *For $x \in S$ with $T_{x,a} > 0$ for $a \geq 1$:*

$$\hat{\mu}_{xy} = \frac{\sum_{a=1}^{n} \sum_{b=1}^{a} N_{x,y,a,b}}{\sum_{a=1}^{n} T_{x,a} \sum_{j=0}^{a-1} (1 - \hat{p}_x)^j}, \quad \text{for } y \in S \text{ with } x \neq y$$

3. *$\hat{\theta}_x$ arbitrary for $x \in S$ with $T_{x,a} = 0$ for all $a \in \mathbb{N}$.*

- $\tilde{N}_{x,a,b} \in \mathbb{N}_0$: total number of rating changes of b firms with rating x, where a firms originally in class x

Maximum Likelihood Estimator for p



Definition of Polynomial P_x

Definition of polynomial P_x:

- $P_x(p) = c_0 + c_1 p + c_2 p^2 + \ldots + c_n p^n$ has coefficients

$$c_0 = \sum_{\substack{a,b=1 \\ a \geq b}}^{n} (b-1)\widetilde{N}_{x,a,b} \sum_{i=1}^{n} i \; T_{x,i}$$

$$c_j = (-1)^j \sum_{\substack{a,b=1 \\ a \geq b}}^{n} \widetilde{N}_{x,a,b} \sum_{i=j}^{n} \binom{i}{j} \left(\frac{i-j}{j+1} \, b + a - i \right) T_{x,i}$$

for $j \in \{1, \ldots, n\}$

Definition of $c_{l(x)}$:

- $l(x) \in \{0, \ldots, n\}$ is the maximal index such that $c_j = 0$ for all $j \in \{0, \ldots, l(x) - 1\}$.

Consistency of the Maximum Likelihood Estimator

Theorem (Consistency)

- $\Theta \ni \theta_0 = (\mu_{x,1}, \ldots, \mu_{x,x-1}, \mu_{x,x+1}, \ldots, \mu_{x,K}, p_x)_{x \in S}$
- $\hat{\theta}^k$ *for* $k \in \mathbb{N}$: *MLE for the observed first* k *paths*

Assume $\mu_x > 0$ *for all* $x \in S$ *and the expected time is positive, that more than one firm has rating* x *in the path of* X.

Then the maximum likelihood estimator of θ_0 *is* **strongly asymptotically consistent**, *i. e.*

$$\hat{\theta}^k \to \theta_0, \quad a.\,s.\text{ for } k \to \infty.$$

Asymptotic Normality of the MLE

Theorem (Asymptotic normality)

- $\Theta \ni \theta_0 = (\mu_{x,1}, \dots, \mu_{x,x-1}, \mu_{x,x+1}, \dots, \mu_{x,K}, p_x)_{x \in S}$
- $\hat{\theta}^k$ *for* $k \in \mathbb{N}$*: MLE for the observed first* k *paths*

Assume $\mu_{xy} \in (0, \infty)$ *and* $p_x \in (0,1)$ *for* $x, y \in S$ *with* $x \neq y$*.*

Then the maximum likelihood estimator is ***asymptotically normal****, i.e.* $\sqrt{k}(\hat{\theta}^k - \theta_0)$ *converges to a normal distribution for* $k \to \infty$ *with mean zero and the inverse Fisher matrix as covariance matrix.*