An Explicit Shadow Price for the Growth-Optimal Portfolio with Transaction Costs

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Joint work with Stefan Gerhold and Walter Schachermayer

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Outline

Introduction

Shadow Prices

Growth-Optimal Portfolio under Transaction Costs

Outlook

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Frictionless markets:

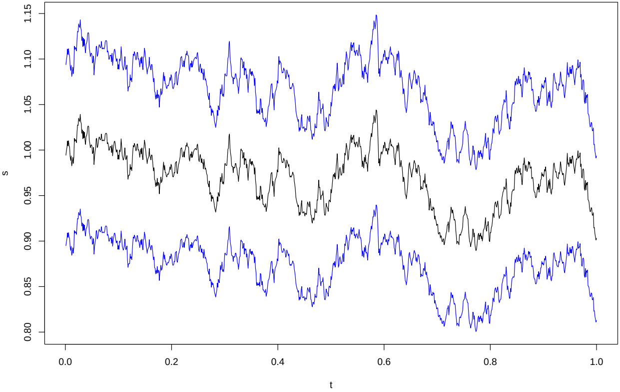
- Securities can be bought and sold for the same price
- Most optimizers of financial problems involve continuous trading, not possible in reality
- Applies to investment strategies, hedges, etc.

Proportional transaction costs:

- Pay higher ask price when buying securities, only receive lower bid price when selling
- Much more realistic optimal trading strategies
- Connection to frictionless markets?

Shadow Prices

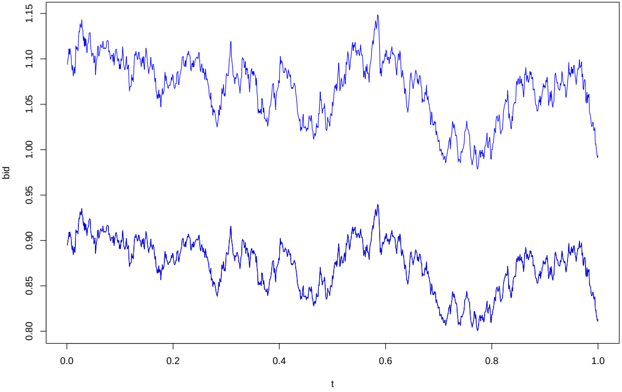
A general principle



Optimal portfolio **with transaction costs**?

Shadow Prices

A general principle



Optimal portfolio **with transaction costs**?

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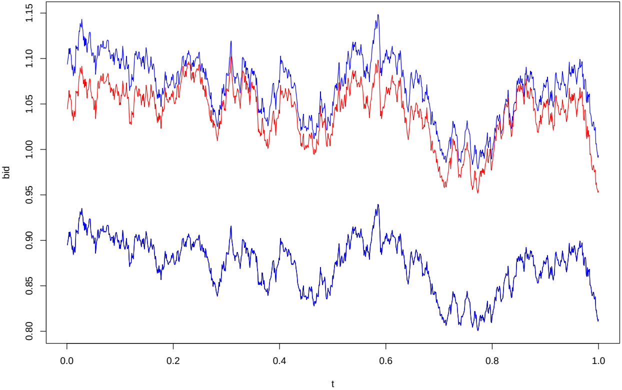
Shadow Prices

A general principle



Optimal portfolio **without transaction costs** for shadow price

$\Longleftrightarrow$

Optimal portfolio **with transaction costs**

Shadow Prices

Literature

Structural results:

- Jouini & Kallal (1995), various more recent articles: No-arbitrage
- Cvitanić, Pham & Touzi (1999): Superhedging
- Lamberton, Pham & Schweizer (1998): Local risk minimization
- Cvitanić & Karatzas (1996), Loewenstein (2000): Portfolio optimization

Computations in the Black-Scholes model:

- Kallsen & M-K (2010): Infinite-horizon optimal consumption
- Gerhold, M-K & Schachermayer: Growth-optimal portfolio $\Rightarrow$ Focus of this talk!

Growth-Optimal Portfolio under Transaction Costs

Frictionless case

- Bond normalized to $S^0 = 1$
- Stock modelled as geometric Brownian motion

$$dS_t/S_t = \mu dt + \sigma dW_t$$

- Goal: Maximize **asymptotic logarithmic growth-rate**

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}[\log(\varphi^0_T + \varphi_T S_T)]$$

  over all admissible strategies (φ^0, φ).

- Optimal to keep fraction wealth in stocks equal to Merton proportion $\theta = \mu/\sigma^2$
- Also holds for general Itô processes by inserting drift resp. diffusion coefficient $\mu_t(\omega)$ resp. $\sigma_t(\omega)$

Growth-Optimal Portfolio under Transaction Costs

With transaction costs

- Bond $S^0 = 1$
- Can buy stocks only at higher **ask price** S_t, where

$$dS_t/S_t = \mu dt + \sigma dW_t$$

- Can sell them only at lower **bid price** $(1-\lambda)S_t$, $\lambda > 0$
- Goal: Maximize

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}[\log(\varphi^0_T + \varphi^+_T(1-\lambda)S_T - \varphi^-_T S_T)]$$

- Trading strategies of infinite variation are ruled out
- What about the growth-optimal portfolio?
- Studied by Taksar, Klass & Assaff (1988) using stochastic control theory

Without transaction costs (Merton (1971))

- Fixed fraction $\theta = \mu/\sigma^2$ of wealth in stock (e.g. 31%)

With transaction costs (Taksar, Klass & Assaff (1988)):

- Minimal trading to keep fraction of wealth in stock in fixed corridor around $\theta = \mu/\sigma^2$ (e.g. 20-40%)
- Do nothing in the interior of this **no-trade region**
- Corridor determined by the zero of a deterministic function

Goal here:

- Compute shadow price
- Find asymptotics of the corridor for small transaction costs

Growth-Optimal Portfolio under Transaction Costs

Ansatz for the shadow price

Suppose we start at time t_0 with...

- Ask price $S_{t_0} = 1$, $\varphi^0_{t_0} \geq 0$ bonds, $\varphi_{t_0} \geq 0$ stocks such that

$$\pi_{t_0} = \frac{\varphi_{t_0} S_{t_0}}{\varphi^0_{t_0} + \varphi_{t_0} S_{t_0}} = \frac{1}{c+1}, \quad c = \varphi^0_{t_0}/\varphi_{t_0},$$

  lies on the **buying boundary** of the no-trade region

Then:

- If S_t increases, so does π_t, until S_t reaches $\bar{s} > 1$ at time t_1 such that

$$\pi_{t_1} = \frac{\varphi_{t_1} S_{t_1}}{\varphi^0_{t_1} + \varphi_{t_1} S_{t_1}} = \frac{1}{c/\bar{s} + 1}$$

  lies on the **selling boundary**, c constant
- Shadow price $\tilde{S} = g(S)$ during this excursion!

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Growth-Optimal Portfolio under Transaction Costs

Ansatz for the shadow price

Ansatz $\tilde{S}_t = g(S_t)$ for $g : [1, \bar{s}] \to [1, (1-\lambda)\bar{s}]$

- $g(1) = 1$ such that $\tilde{S} = S$ at buying boundary
- $g(\bar{s}) = (1-\lambda)\bar{s}$ such that $\tilde{S} = (1-\lambda)S$ at selling boundary
- Itô's formula: $dg(S_t)/g(S_t) = \tilde{\mu}_t dt + \tilde{\sigma}_t dW_t$
- Frictionless log-optimizer for $\tilde{S}$ given by

$$\frac{\varphi^1_{t_0}\tilde{S}_t}{\varphi^0_{t_0} + \varphi_{t_0}\tilde{S}_t} = \frac{g(S_t)}{c + g(S_t)} = \frac{\tilde{\mu}_t}{\tilde{\sigma}^2_t}$$

- Yields ODE for g:

$$g''(s) = \frac{2g'(s)^2}{c + g(s)} - \frac{2\mu g'(s)}{\sigma^2 s}$$

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Growth-Optimal Portfolio under Transaction Costs

Ansatz for the shadow price

Ansatz $\tilde{S}_t = g(S_t)$ for $g : [1, \bar{s}] \to [1, (1-\lambda)\bar{s}]$ such that

$$g''(s) = \frac{2g'(s)^2}{c + g(s)} - \frac{2\theta g'(s)}{s}$$

- Merton proportion $\theta = \mu/\sigma^2$
- $g(1) = 1$, $g(\bar{s}) = (1-\lambda)\bar{s}$
- $\bar{s}$, c still unknown, two more boundary conditions?

$\tilde{S} = g(S)$ should remain in $[(1-\lambda)S, S]$

- Diffusion coefficient of $\tilde{S}/S$ should vanish as $S_t \to 1$ or $S_t \to \bar{s}$
- Leads to $g'(1) = 1$ and $g'(\bar{s}) = 1 - \lambda$
- Free boundary value problem?

Growth-Optimal Portfolio under Transaction Costs

Computing the candidate

- General solution to ODE with $g(1) = g'(1) = 1$:

$$g(s) = \frac{-cs + (2\theta - 1 + 2c\theta)s^{2\theta}}{s - (2 - 2\theta + c(2\theta - 1))s^{2\theta}}.$$

- Plugging this into $g(\bar{s}) = (1 - \lambda)\bar{s}$, $g'(\bar{s}) = 1 - \lambda$ yields

$$\bar{s} = \bar{s}(c) = \left(\tfrac{c}{(2\theta-1+2c\theta)(2-2\theta-c(2\theta-1))}\right)^{1/(2\theta-1)}.$$

and

$$\left(\tfrac{c}{(2\theta-1+2c\theta)(2-2\theta-c(2\theta-1)}\right)^{\frac{1-\theta}{\theta-1/2}} - \frac{1}{1-\lambda}(2\theta - 1 + 2c\theta)^2 = 0$$

Growth-Optimal Portfolio under Transaction Costs

Computing the candidate

- Elementary analysis: Exists unique solution c to

$$\left(\frac{c}{(2\theta-1+2c\theta)(2-2\theta-c(2\theta-1)}\right)^{\frac{1-\theta}{\theta-1/2}} - \frac{1}{1-\lambda}(2\theta-1+2c\theta)^2 = 0$$

- Define

$$\bar{s} = \bar{s}(c) = \left(\frac{c}{(2\theta-1+2c\theta)(2-2\theta-c(2\theta-1))}\right)^{1/(2\theta-1)} .$$

- Compute boundaries of the no trade region:

$$1/(1+c) \leq 1/(1+c/\bar{s})$$

- But: No explicit solution for c.
- However: Fractional Taylor expansions!

Growth-Optimal Portfolio under Transaction Costs

Fractional Taylor expansions for small λ

Theorem (Gerhold, M-K, Schachermayer (2010))

*For p_k and q_k that can be **algorithmically computed**:*

$$c = \frac{1-\theta}{\theta} + \sum_{k=1}^{\infty} q_k(\theta) \left(\frac{6}{\theta(1-\theta)}\right)^{k/3} \lambda^{k/3}$$

This yields expansions of arbitrary order for no-trade region:

$$\frac{1}{1+c} = \theta - \left(\tfrac{3}{4}\theta^2(1-\theta)^2\right)^{1/3} \lambda^{1/3} + \tfrac{3}{20}(2\theta^2 - 2\theta + 1)\lambda + O(\lambda^{4/3})$$

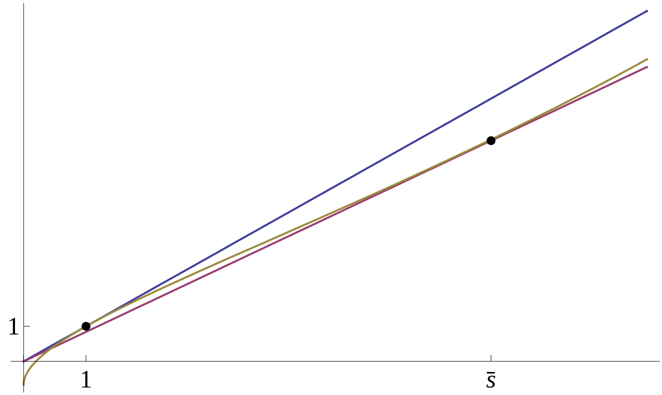
$$\frac{1}{1+c/\overline{s}} = \theta + \left(\tfrac{3}{4}\theta^2(1-\theta)^2\right)^{1/3} \lambda^{1/3} - \tfrac{1}{20}(26\theta^2 - 26\theta + 3)\lambda + O(\lambda^{4/3})$$

- Compare Janeček & Shreve (2004) for first terms with consumption

Growth-Optimal Portfolio under Transaction Costs

Verification

Up to now: Heuristic derivation of candidate $\tilde{S}_t = g(S_t)$



- Only during **one excursion** of S_t from 1 to $\bar{s}$
- What happens when S hits the boundaries?

Growth-Optimal Portfolio under Transaction Costs
Verification

- Start at the buying boundary at time t_0 with $S_{t_0} = 1$
- $\tilde{S}_{t_0} = g(S_{t_0}) = g(1)$
- If S moves down, we should still have $\tilde{S} = S$
- If $S_{t_0} \neq 1$, everything should scale with S_{t_0}
- Hence until S_t reaches $\bar{s} > 1$, let

$$m_t = \min_{t_0 \leq t} S_t, \quad \tilde{S}_t = m_t g(\tfrac{S_t}{m_t})$$

After S_t hits $\bar{s}$ at time σ_1:

$$m_t = \max_{\sigma_1 \leq t} S_t/\bar{s}, \quad \tilde{S}_t = m_t g(\tfrac{S_t}{m_t})$$

until $S_t/m_t \leq 1$. Continue in an obvious way.

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Growth-Optimal Portfolio under Transaction Costs

Verification

- Have defined continuous process $\tilde{S} = mg(S/m)$
- Moves between $[(1-\lambda)S, S]$
- But why should this be a nice process?

Theorem (Gerhold, M-K, Schachermayer (2010))

$\tilde{S} = mg(S/m)$ is an Itô process with bounded coefficients, which satisfies

$$d\tilde{S}_t = g'\left(\frac{S_t}{m_t}\right) dS_t + \frac{1}{2m_t} g''\left(\frac{S_t}{m_t}\right) d\langle S, S\rangle_t$$

- Frictionless log-optimal portfolio is well-known
- Number of stocks only increases resp. decreases when $\tilde{S} = S$ resp. $\tilde{S} = (1-\lambda)S$ by construction
- Hence, $\tilde{S}$ is a **shadow price**!

Growth-Optimal Portfolio under Transaction Costs

Construction and results

Construction of the shadow price:

- Itô process with bounded coefficients
- Function of ask price S and its running minima resp. maxima during Brownian excursions
- Determined up to solution of one dimensional equation

Asymptotic expansions in terms of $\lambda^{1/3}$ (for $0 < \lambda < \lambda_0$):

- Expansions of arbitrary order for no-trade region
- Can also determine asymptotic growth rate

$$\delta = \frac{\mu^2}{2\sigma^2} - \left(\frac{3\sigma^3}{\sqrt{128}}\theta^2(1-\theta)^2\right)^{2/3}\lambda^{2/3} + O(\lambda^{4/3})$$

- Compare Janeček & Shreve (2004), Shreve & Soner (1994), Rogers (2004) for first term with consumption

Outlook

Beyond Black-Scholes

Work in progress: Shadow price and asymptotics for...

- Log-utility from consumption
- Asymptotic power growth rate

Future topics:

- General existence
- Asymptotics formulas beyond Black-Scholes
- Extensions to utility-based pricing and hedging

For more details:

- Gerhold, S., Muhle-Karbe, J., and Schachermayer, W. (2010). *The dual optimizer for the growth-optimal portfolio under transaction costs*. Preprint. Available at `www.mat.univie.ac.at/~muhlekarbe`.