Risk measures for multivariate random variables in markets with transaction costs

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Happy Birthday, Walter!

Outline

1. Risk measures under transaction costs

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- d assets (may include different currencies), discrete time Θ, $(\Omega, (\mathcal{F}_t)_{t\in\Theta}, P)$
- portfolio vector in physical units (numéraire-free): V_t (# of units in d assets at time t)
- proportional transaction costs at time t: closed convex cone $\mathbb{R}^d_+ \subseteq K_t(\omega) \subseteq \mathbb{R}^d$ (solvency cone), positions transferrable into nonnegative positions
- lets focus on time $t = 0$ and $t = T$
- claim $X \in L^0_d(\Omega, \mathcal{F}_T, P; \mathbb{R}^d)$: payoff (in physical units) at time T
- a portfolio vector $u \in M$ ($M \subseteq \mathbb{R}^d$ linear subspace of eligible assets) compensates the risk of X if

$$X + u\mathbb{1} \in A$$

for some set $A \subseteq L^0_d$ of acceptable positions.

1. Risk measures under transaction costs

Primal Representation

Risk measures and acceptance sets are one-to-one via

$$R_A(X) = \{u \in M : X + u\mathbb{1} \in A\}$$

and

$$A_R = \{X \in L^p_d : 0 \in R(X)\}.$$

A function $R : L_d^p \to \mathcal{P}(M)$ is called M-translative iff

$$\forall X \in L_d^p,\ \forall u \in M :\ R(X + u1\!\!1) = R(X) - u.$$

One-to-one properties for M-translative function R and $A \subseteq L_d^p$:

	R	A
finite at zero	$R(0) \neq \emptyset$ $R(0) \neq M$	$M1\!\!1 \cap A \neq \emptyset$ $M1\!\!1 \cap (L_d^p \backslash A) \neq \emptyset$
market compatible	$L_d^p(K_T)$- monotone $R(X) = R(X) + K_0^M$	$A + L_d^p(K_T) \subseteq A$ $A + K_0^M 1\!\!1 \subseteq A$
	convex positively homogeneous subadditive closed images lsc	convex cone $A + A \subseteq A$ directionally closed closed

Let $\mathbb{G}_M = \{D \subseteq M : D = \operatorname{cl}\operatorname{co}(D + K_0^M)\}$.

Dual Representation, $1 \leq p \leq \infty$

A function $R\colon L_d^p \to \mathbb{G}_M$ is a market compatible lsc convex risk measure, finite at zero, if and only if there is a penalty function $-\alpha_R\colon \mathcal{W}^q \to \mathbb{G}_M$ such that for all $X \in L_d^p$

$$R(X) = \bigcap_{(Q,w)\in\mathcal{W}^q} \left[\left(E^Q[-X] + G(w)\right) \cap M - \alpha_R(Q,w)\right].$$

where

$$\mathcal{W}^q = \{(Q,w) \in \mathcal{M}_{1,d}^P \times K_0^+ \backslash M^\perp + M^\perp : \operatorname{diag}(w)\frac{dQ}{dP} \in L_d^q(K_T^+)\}.$$

- $\mathcal{M}_{1,d}^P$ vector probability measures with components Q_i (i=1,...,d), $\frac{dQ_i}{dP} \in L^q$ and $E^Q[X] = (E^{Q_1}[X_1], ..., E^{Q_d}[X_d])^T$.
- $G(w) = \{u \in \mathbb{R}^d : w^T u \geq 0\}$.

Dual Representation, $1 \leq p \leq \infty$

The function R is a market compatible lsc coherent risk measures, finite at zero, if and only if there is a nonempty set $\mathcal{W}^q_R \subseteq \mathcal{W}^q$ such that

$$R(X) = \bigcap_{(Q,w)\in\mathcal{W}^q_R} \left[\left(E^Q[-X] + G(w)\right) \cap M \right]$$

Proof: Set-valued convex analysis.

▷ Hamel, Heyde, Rudloff (2010): Set–valued risk measures with random transaction costs. Submitted for publication.

▷ Hamel, Heyde (2010): Duality for set–valued measures of risk. SIAM Journal on Financial Mathematics 1.

▷ Jouini, Touzi, Meddeb (2004): Vector-valued measure of risk, Fin.&Stoch. 8.

2. Examples:

- AV@R
- Worst case risk measure
- superhedging price
- risk measures considered in ▷ Artzner, Delbaen, Koch-Medina (09): *Risk Measures and Efficient Use of Capital.* ASTIN Bulletin 39.

2.1 The link between risk measures and superreplication.

recall the scalar case:

$$\rho(-X) = \sup_{Q \in \mathcal{M}^e(S)} E^Q[X]$$

is a coherent risk measure.

($\mathcal{M}^e(S)$ is the set of equivalent martingale measures)

- $(V_t)_{t=0}^T$ self-financing portfolio process if

$$V_t - V_{t-1} \in -K_t \quad P-a.s. \quad \forall t \in \Theta \quad (V_{-1} \equiv 0)$$

- attainable claims at zero cost

$$A_t := \{V_t : V \text{ is self-fin. portf. process}\}, \quad t \in \Theta$$

Superhedging Theorem (Kabanov 99, Schachermayer 04, Pennanen, Penner 08,...)

Under robust NA the following conditions are equivalent

- The claim $X \in L^0_d(\Omega, \mathcal{F}_T, P; \mathbb{R}^d)$ can be superhedged with initial endowment $v \in \mathbb{R}^d$, i.e. $X \in v + A_T$
- For every consistent pricing process $(Z_t)_{t=0}^T$ with $E\left[(X^T Z_T)^-\right] < \infty$, we have

$$\forall\, Z \in \mathrm{CPP}: \quad E[X^T Z_T] \leq v^T Z_0.$$

- $v \in R_{-A_T}(-X)$ with

$$R_{-A_T}(-X) := \left\{ u \in \mathbb{R}^d \colon -X + u\mathbb{1} \in -A_T \right\}.$$

$R_{-A_T} : L^0_d \to \mathbb{G}$ is a closed coherent market compatible risk measure and has the following representation

$$R_{-A_T}(-X) = \bigcap_{\left\{(Q,w)\in\mathcal{W}^1_{\{0,\dots,T\}}:\ E^Q[X^-]<\infty\right\}} \left(E^Q[X] + G(w)\right).$$

$$\begin{aligned}\mathcal{W}^1_{\{0,\dots,T\}} = &\Big\{ (Q,w) \in \mathcal{M}^P_{1,d} \times \mathbb{R}^d \backslash \{0\} :\ \text{for all } t = 0,\dots,T \\ & E\left[\operatorname{diag}(w)\frac{dQ}{dP}\Big|\mathcal{F}_t\right] \in L^1_d\left(\Omega,\mathcal{F}_t,P;K_t^+\right)\Big\} \subseteq \mathcal{W}^1.\end{aligned}$$

One-to-one relation between CPP Z_t and $(Q,w) \in \mathcal{W}^1_{\{0,\dots,T\}}$ via: $\frac{dQ_i}{dP} = \frac{1}{w_i}(Z_T)_i$ and $w = E[Z_T] = Z_0$.

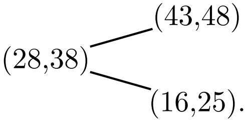
Consistent pricing process: adapted $\mathbb{R}^d_+$-valued process $Z = (Z_t)_{t=0}^T$ such that

- Z is a martingale under P
- $Z_t \in K_t^+ \quad P-a.s., t \in \Theta \quad (K_t^+ :$ pos. dual cone of $K_t)$

2.1 Risk measures and superreplication

Example:

- riskless asset, risky stock with bid-ask prices at $t = 0, T$:



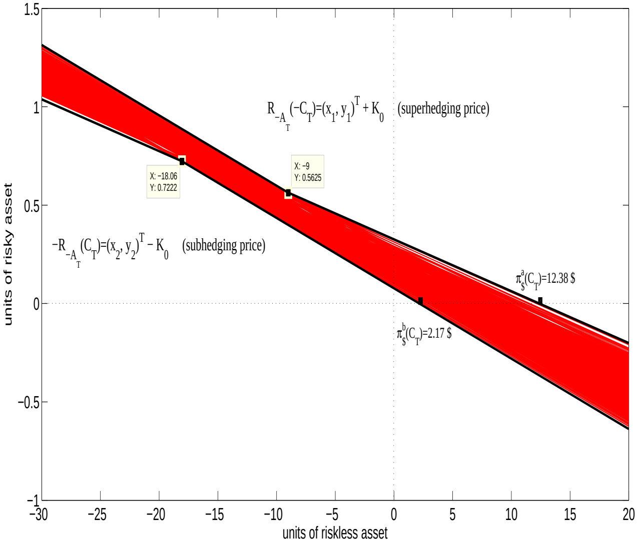
- European call, physical delivery, $K = 30$, i.e.
  $C_T(\omega_1) = (-30, 1)$ and $C_T(\omega_2) = (0, 0)$

Recall: superhedging price

$$R_{-A_T}(-X) = \bigcap_{(Q,w)\in\mathcal{W}^1_{\{1,\dots,T\}}} \left(E^Q\left[X\right] + G\left(w\right)\right)$$

and the subhedging price

$$-R_{-A_T}(X) = \bigcap_{(Q,w)\in\mathcal{W}^1_{\{1,\dots,T\}}} \left(E^Q\left[X\right] - G\left(w\right)\right)$$



R_{-A_T}(−C_T)=(x_1, y_1)^T + K_0 (superhedging price)
−R_{-A_T}(C_T)=(x_2, y_2)^T − K_0 (subhedging price)
X: −18.06
Y: 0.7222
X: −9
Y: 0.5625
π^a_$(C_T)=12.38 $
π^b_$(C_T)=2.17 $
units of risky asset
units of riskless asset
1.5
1
0.5
0
−0.5
−1
−30
−25
−20
−15
−10
−5
0
5
10
15
20

Scalarization: real-valued function $\varphi_{R,v}\colon L_d^p \to \mathbb{R} \cup \{\pm\infty\}$ given by

$$\varphi_{R,v}(X) = \inf_{u \in R(X)} v^T u$$

for $v \in K_0^+$.

Superhedging price in USD:

- $\pi_\$^a(X)$ is a scalarization of the set–valued risk measure $R^M_{-A_T}(-X)$ with $v = e^\$$ and $M = \{te^\$, t \in \mathbb{R}\}$.
- can be calculated by solving a (dyn.) linear program.

References: Bensaid, Lesne, Pages, Scheinkman (92), Rutkowski (98), Stettner (97), Roux (08), Jouini, Kallal (95).

▷ Hamel, Rudloff (2010): On price bounds in markets with transaction costs. Working Paper.

2.2 Efficient use of capital

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based on ▷ Artzner, Delbaen, Koch-Medina (09): *Risk Measures and Efficient Use of Capital.* ASTIN Bulletin 39.

- multiple eligible assets, one leading (domestic) currency
- random exchange rates in the future
- no transaction costs (scalar risk measure)
  $\rho : L^0 \to \mathbb{R} \cup \{+\infty\}$

$$\rho_{\mathcal{A}}(X) = \inf\{m : X + Z \in \mathcal{A},\ Z \in \mathcal{M} \text{ with } \pi(Z) = m\},$$

$\mathcal{M}$: future values of portfolios of eligible assets
$\pi(Z)$: today's price of $Z \in \mathcal{M}$ in domestic currency
$\mathcal{A} \subseteq L^0$ acceptance set

It holds: ρ is a scalarization of a set-valued risk measure.

Toy-Example:

- 2 currencies, current exchange rate: 1
- future exchange rate: $(\frac{1}{2}, 1, 2)$
- random payoff in domestic currency $X_1 = (-16, 1, -7)$
- capital requirement for worst case risk measure ($\mathcal{A} = L^0_+$):
  - if only domestic currency is eligible: need 16 units
  - if both currencies are eligible: (4,6) is minimal capital requirement (4 units of domestic currency and 6 of foreign), costs: 10

Connection to set–valued risk measure:

- capital requirement for $A = L^1_2(K_T)$ (worst case risk measure)? $X = (X_1, 0)$

$$R_A(X) = \{u \in \mathbb{R}^2 : X + u \in A\}$$
$$= \{u \in \mathbb{R}^2 : u_1 + 2u_2 \geq 16, 2u_1 + u_2 \geq 14\}.$$

the unique minimal point w.r.t. K_0 is $(4, 6)$

...Connection to set–valued risk measure:

- market compatible risk measure $R_{\widetilde{A}}$ with $\widetilde{A} = \mathrm{cl}\,(A + K_0)$ (takes trading at $t = 0$ into account): set of minimal points are all initial positions that can be exchanged into $(4, 6)$.
- $\rho_{\mathcal{A}}$ is a scalarization of R_A, but also of $R_{\widetilde{A}}$

$$\begin{aligned}\rho(X_1) &= \inf\left\{\sum_{i=1}^{2} \pi_i u_i : u \in R_A(X_1, 0)\right\}\\ &= \inf\left\{\sum_{i=1}^{2} \pi_i u_i : u \in R_{\widetilde{A}}(X_1, 0)\right\}.\end{aligned}$$

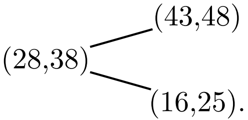
Advantage of set–valued approach:

- theory embedded in set–valued approach via scalarization
- yields more solutions
- transaction costs can be included
- future payoffs also in foreign currencies possible
- illiquidities can be modelled
- dual representation of risk measures possible

▷ Hamel, Heyde, Rudloff (2010): Set–valued risk measures with random transaction costs. Submitted for Publication.

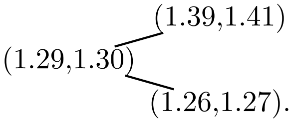
back to the binomial example:

- riskless asset, risky stock with bid-ask prices at $t = 0, T$:



- European call, $C_T(\omega_1) = (-30, 1)$, $C_T(\omega_2) = (0, 0)$

Recall: price bounds in \$: $(2.17, 12.38)$.
add another asset: e.g. Euro



new price bounds in \$: $(2.17, 5.54)$.

▷ Hamel, Rudloff (2010): On price bounds in markets with transaction costs. Working Paper.

Thank you!