Dynamic Modelling of CDOs

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joint work with J. Zabczyk

Introduction

Essentials of securitization

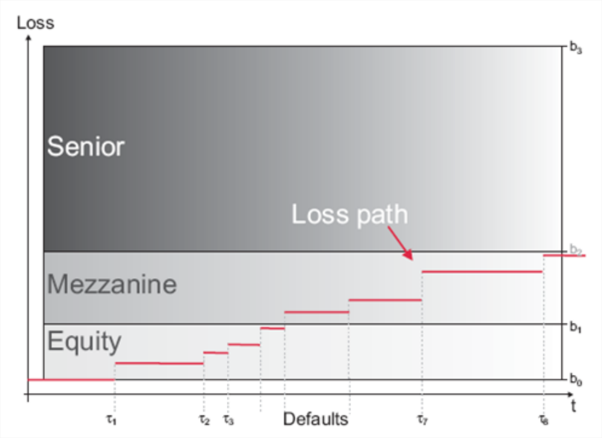
- Consider a CDO as a pool of m defaultable entities.
- Default i occurs at τ_i with associated loss q_i
- Cumulative loss

$$A_t = \sum_{i=1}^{m} q_i 1_{\{\tau_i \leq t\}}.$$

- Normalize the total nominal to 1, set $\mathcal{I} := [0, 1]$.
- Loss is split into tranches: a tranche refers to an interval $(x_i, x_{i-1}] \subset \mathcal{I}$,

$$0 = x_0 < x_1 < \cdots < x_k = 1$$

Partition of losses into tranches



Examples: Traded indicies (iTraxx, CDX)

Single tranche CDOs

A STCDO is specified by

- a number of future dates $T_0 < T_1 < \cdots < T_m$,
- a *tranche* with lower and upper detachment points $x_1 < x_2$,
- a fixed spread S.

We write

$$H(x) := (x_2 - x)^+ - (x_1 - x)^+ = \int_{(x_1,x_2]} 1_{\{x \le y\}} dy.$$

An investor in this STCDO

- receives $SH(A_{T_k})$ at T_k, $k = 1, \dots, m-1$ (payment leg),
- pays $H(A_t) - H(A_{t-})$ at any time when $\Delta A_t \neq 0$. (default leg)

Filipović, Overbeck and Schmidt (2009)

- A security which pays $\mathbf{1}_{\{A_T \le x\}}$ at T is called (T,x)-*bond*. Its price at time $t \le T$ is denoted by $P(t,T,x)$.

Proposition

The value of the STCDO at time $t \le T_1$ is

$$\Gamma(t,S) = \int_{(x_1,x_2]} \left(S \sum_{i=1}^{n} P(t,T_i,y) + P(t,T_n,y) - P(t,T_0,y) + \gamma(t,y) \right) dy$$

with

$$\gamma(t,y) = \int_{T_0}^{T_n} \mathbb{E}\left[r_u e^{-\int_t^u r_s\, ds} 1_{\{A_u \le y\}} \mid \mathcal{F}_t \right] du.$$

Solving $\Gamma = 0$ for S gives the fair spread.

Drift condition

(A1) $A_t = \sum_{s \le t} \Delta A_s$ is an increasing marked point process with compensator $\nu^A(t, dx)\, dt$ and values in $[0, 1]$.

- Consider $\lambda(t, x)$, such that

$$M_t^x = 1_{\{A_t \le x\}} + \int_0^t 1_{\{A_s \le x\}} \lambda(s, x)\, ds$$

is a martingale.

- Consider a d-dimensional Lévy process Z such that $\mathbb{E}(\, e^{-\langle u, Z_t \rangle}) = e^{tJ(u)}$ $u \in \mathbb{R}^d$ with

$$J(u) = \langle m, u \rangle + \frac{1}{2} \langle \Sigma u, u \rangle + \int_{\mathbb{R}^d} \left(e^{-\langle u, z \rangle} - 1 + 1_{\{|z| \le 1\}}(z) \langle u, z \rangle \right) \tilde{\nu}(dz). \tag{1}$$

Forward-rate approach:

We consider

$$P(t, T, x) = 1_{\{A_t \le x\}} \exp\Big(- \int_t^T f(t, u, x)\, du \Big)$$

where

$$\begin{aligned} f(t, T, x) = f(0, T, x) + \int_0^t a(s, T, x) ds + \int_0^t \langle b(s, T, x), dZ_s \rangle \\ + \int_0^t \int_{\mathcal{I}} c(s, T, x; y)\, \mu^A(ds, dy) \end{aligned} \tag{2}$$

No-arbitrage condition

$$e^{-\int_0^t r_s ds} P(t, T, x) \text{ are local martingales for all } (T, x). \tag{3}$$

Under some technical assumptions, we have that

Theorem

(3) *is equivalent to*

$$\int_t^s a(t,u,x)du = J\Big(\int_t^s b(t,u,x)du\Big) + \int_{\mathcal{I}} \Big(e^{-\int_t^s c(t,u,x;y)\,du} - 1\Big) 1_{\{L_t + y \le x\}} \nu^A(t,dy) \tag{4}$$

$$f(t,t,x) = r_t + \lambda(t,x), \tag{5}$$

where (4) *and* (5) *hold on* $\{A_t \le x\}$, $\mathbb{Q} \otimes dt$*-a.s.*

1. In Filipović, Overbeck, Schmidt (2009) also tractable affine models are developed.
2. Variance-Minimizing Hedging Strategies are developed in Filipović, Schmidt (2010) which lead to explicit strategies in a affine one-factor model

Market models

Forward rate modelling

Denote $\mathcal{T} := \{T_0, \dots, T_n\}$, $\delta_k := T_{k+1} - T_k$ and let

$$P(t, T, x) = p(t, T, x) 1_{\{A_t \le x\}}, \tag{6}$$

$(p(t, T, x))_{0 \le t \le T}$ a strictly positive special semimartingale with $p(T, T, x) = 1$.

Definition

The (T_k, T_{k+1}, x)-*spread* is given by

$$F(t, T_k, T_{k+1}, x) := \frac{P(t, T_k, x)}{P(t, T_{k+1}, x)}. \tag{7}$$

Proposition.

Forward spreads given on $\{A_t \leq x_i\}$ by

$$\frac{dF(t, T_k, T_{k+1}, x_i)}{F(t-, T_k, T_{k+1}, x_i)} = \alpha_{ki}(t)dt + \langle \beta_{ki}(t), dW(t)\rangle$$
$$+ \int_{\mathbb{R}^d} \left(e^{\langle \beta_{ki}(t), z\rangle} - 1 \right) \mu(dt, dz) + \int_{\mathcal{I}} \left(e^{\gamma_{ki}(t, A_{t-}; y)} - 1 \right) 1_{\{A_{t-}+y \leq x_i\}} \mu^A(dt, dy),$$

$(T_k, x_i) \in \mathcal{S}$, $k < n$ and zero on $\{A_t > x_i\}$ constitute an arbitrage-free market if

$$\alpha_{ki}(t) = -\lambda(t, x_i) + \sum_{j=\eta(t)}^{k} \langle \beta_{ji}(t), \Sigma \beta_{ki}(t)\rangle$$
$$+ \int_{\mathbb{R}^d} \left(e^{\langle \beta_{ki}(t), z\rangle} + \left(e^{-\langle \beta_{ki}(t), z\rangle} - 1 \right) \prod_{j=\eta(t)}^{k-1} e^{-\langle \beta_{ji}(t), z\rangle} - 1 \right) \nu(dz)$$
$$+ \int_{\mathcal{I}} \left(e^{\gamma_{ki}(t, A_{t-}; y)} + \left(e^{-\gamma_{ki}(t, A_{t-}; y)} - 1 \right) \prod_{j=\eta(t)}^{k-1} e^{-\gamma_{ji}(t, A_{t-}; y)} \right) 1_{\{A_t + y \leq x_i\}} \nu^A(t, dy)$$

for all $(T_k, x_i), (T_{k+1}, x_i) \in \mathcal{S}$.

- Similar techniques as in interest rate markets can be applied to value CDOs and options on CDOs.
- Grbac, Eberlein, Schmidt (2010) study directly discrete rates.
- Statistical results show that in shorter time periods affine models are appropriate.

Further issues

- Model risk
- Measuring the risk of credit and market risk simultaneously.

[1] A Brace, D. Gatarek, M. Musiela: The market model of interest rate dynamics (1995).

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[3] D Filipović, L Overbeck, T Schmidt: Dynamic CDO term structure modelling (2009).

[4] D Filipović, T Schmidt: "Pricing and Hedging of CDOs: A Top-Down Approach", (2010). Contemporary Quantitative Finance,Chiarella, C. and Novikov, A. (Eds.)

[5] T Schmidt, J Zabczyk: CDO term structure modelling with Lévy processes and the relation to market models (2010). arXiv:1007.1706