THE CONVENIENT SETTING FOR DENJOY–CARLEMAN DIFFERENTIABLE MAPPINGS OF BEURLING AND ROUMIEU TYPE

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Abstract. We prove in a uniform way that all Denjoy–Carleman differentiable function classes of Beurling type $C^{(M)}$ and of Roumieu type $C^{\{M\}}$, admit a convenient setting if the weight sequence $M = (M_k)$ is log-convex and of moderate growth: For $\mathcal{C}$ denoting either $C^{(M)}$ or $C^{\{M\}}$, the category of $\mathcal{C}$-mappings is cartesian closed in the sense that $\mathcal{C}(E, \mathcal{C}(F, G)) \cong \mathcal{C}(E \times F, G)$ for convenient vector spaces. Applications to manifolds of mappings are given: The group of $\mathcal{C}$-diffeomorphisms is a regular $\mathcal{C}$-Lie group if $\mathcal{C} \supseteq C^\omega$, but not better.

1. Introduction

Denjoy–Carleman differentiable functions form classes of smooth functions that are described by growth conditions on the Taylor expansion. The growth is prescribed in terms of a sequence $M = (M_k)$ of positive real numbers which serves as a weight for the iterated derivatives: for compact K the sets

$$\Big\{ \frac{f^{(k)}(x)}{\rho^k\, k!\, M_k} : x \in K, k \in \mathbb{N} \Big\}$$

are required to be bounded. The positive real number ρ is subject to either a universal or an existential quantifier, thereby dividing the Denjoy–Carleman classes into those of Beurling type, denoted by $C^{(M)}$, and those of Roumieu type, denoted by $C^{\{M\}}$, respectively. For the constant sequence $M = (M_k) = (1)$, as Beurling type we recover the real and imaginary parts of all entire functions on the one hand, and as Roumieu type the real analytic functions on the other hand, where $1/\rho$ plays the role of a radius of convergence. Moreover, Denjoy–Carleman classes are divided into quasianalytic and non-quasianalytic classes, depending on whether the mapping to infinite Taylor expansions is injective on the class or not.

That a class of mappings $\mathcal{C}$ admits a convenient setting means essentially that we can extend the class to mappings between admissible infinite dimensional spaces $E, F, \dots$ so that $\mathcal{C}(E, F)$ is again admissible and we have $\mathcal{C}(E \times F, G)$ canonically $\mathcal{C}$-diffeomorphic to $\mathcal{C}(E, \mathcal{C}(F, G))$. This property is called the *exponential law*; it includes the basic assumption of variational calculus. Usually the exponential law

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comes hand in hand with (partially nonlinear) uniform boundedness theorems which are easy $\mathcal{C}$-detection principles.

The class C^∞ of smooth mappings admits a convenient setting. This is due originally to [9], [10], and [20], [21]. For the C^∞ convenient setting one can test smoothness along smooth curves. Also real analytic (C^ω) mappings admit a convenient setting, due to [22]: A mapping is C^ω if and only if it is C^∞ and in addition is weakly C^ω along weakly C^ω-curves (i.e., curves whose compositions with any bounded linear functional are C^ω); indeed, it suffices to test along affine lines instead of weakly C^ω-curves. See the book [23] for a comprehensive treatment, or the three appendices in [25] for a short overview of the C^∞ and C^ω cases. We shall use convenient calculus of C^∞-mappings in this paper, and we shall reprove that C^ω admits a convenient calculus.

We now describe what was known about convenient settings for Denjoy-Carleman classes before: In [25] we developed the convenient setting for non-quasianalytic log-convex Denjoy–Carleman classes of Roumieu type $C^{\{M\}}$ having moderate growth, and we showed that moderate growth and a condition that guarantees stability under composition (like log-convexity) are necessary. There a mapping is $C^{\{M\}}$ if and only if it is weakly $C^{\{M\}}$ along all weakly $C^{\{M\}}$-curves. The method of proof relies on the existence of $C^{\{M\}}$ partitions of unity.

We succeeded in [26] to prove that *some* quasianalytic log-convex Denjoy–Carleman classes of Roumieu type $C^{\{M\}}$ having moderate growth admit a convenient setting. The method consisted of representing $C^{\{M\}}$ as the intersection of all larger non-quasianalytic log-convex classes $C^{\{L\}}$. A mapping is $C^{\{M\}}$ if and only if it is weakly $C^{\{L\}}$ along each weakly $C^{\{L\}}$-curve for each non-quasianalytic log-convex $L \geq M$. We constructed countably many classes which satisfy all these requirements, but many reasonable quasianalytic classes $C^{\{M\}}$, like the real analytic class, are not covered by this approach.

In this paper we prove that *all* log-convex Denjoy–Carleman classes of moderate growth admit a convenient setting. This is achieved through a change of philosophy: instead of *testing along curves* as in our previous approaches [25] and [26] we *test along Banach plots*, i.e., mappings of the respective weak class defined in open subsets of Banach spaces. By 'weak' we mean: the mapping is in the class after composing it with any bounded linear functional. In this way we are able to treat all Denjoy–Carleman classes uniformly, no matter if quasianalytic, non-quasianalytic, of Beurling, or of Roumieu type, including C^ω and real and imaginary parts of entire functions. Furthermore, it makes the proofs shorter and more transparent.

Smooth mappings between Banach spaces are $C^{(M)}$ or $C^{\{M\}}$ if their derivatives satisfy the boundedness conditions alluded to above. A smooth mapping between admissible locally convex vector spaces is $C^{(M)}$ or $C^{\{M\}}$ if and only if it maps Banach plots of the respective class to Banach plots of the same class. This implies stability under composition, see Theorem 4.9.

We equip the spaces of $C^{(M)}$ or $C^{\{M\}}$ mappings between Banach spaces with natural locally convex topologies which are just the usual ones if the involved Banach spaces are finite dimensional, see Section 4.1. In order to show completeness we need to work with Whitney jets on compact subsets of Banach spaces satisfying growth conditions of Denjoy–Carleman type, see Proposition 4.1. Having found nothing in the literature we introduce Whitney jets on Banach spaces in Section 3.

In Theorem 7.1 we show that the Roumieu type classes of Denjoy–Carleman differentiable mappings studied in the present paper coincide bornologically with the classes considered previously in [25] and [26] and, most notably, with the structure C^ω of real analytic mappings introduced in [22] (see also [23]). We want to stress that thereby we provide a considerably simpler proof for the real analytic convenient setting. But for the results that testing along curves suffices one still has to rely on [22], [25], and [26].

For a class of mappings $\mathcal{C}$ that admits a convenient setting one can hope that the space $\mathcal{C}(A,B)$ of all $\mathcal{C}$-mappings between finite dimensional $\mathcal{C}$-manifolds (with A compact for simplicity) is again a $\mathcal{C}$-manifold, that composition is $\mathcal{C}$, and that the group $\operatorname{Diff}^{\mathcal{C}}(A)$ of all $\mathcal{C}$-diffeomorphisms of A is a regular infinite dimensional $\mathcal{C}$-Lie group. In Section 9 this is proved for all log-convex Denjoy–Carleman classes of moderate growth $C^{\{M\}}$ and for the classes $C^{(M)}$ containing C^ω.

A further area of application is the perturbation theory for linear unbounded operators; see [27] and [30].

This paper is organized as follows. In Section 2 we recall basic facts about Denjoy–Carleman classes $C^{[M]}$ (which stands for $C^{\{M\}}$ or $C^{(M)}$) in finite dimensions and discuss corresponding sequence spaces. In Section 3 we introduce Whitney jets on Banach spaces. In Section 4 we define $C^{[M]}$-mappings in infinite dimensions, first between Banach spaces with the aid of jets and then between convenient vector spaces, and we show that they form a category, if $M=(M_k)$ is log-convex. In Section 5 we prove that this category is cartesian closed, if $M=(M_k)$ has moderate growth. In Section 6 we show the $C^{[M]}$ uniform boundedness principle. In Section 7 we prove that the structures studied in this paper coincide bornologically with the structures considered in our previous work [25], [26], [22], and [23]. In Section 8 we further study the spaces of $C^{[M]}$-mappings. In Section 9 we apply this theory to prove that the space of $C^{[M]}$-mappings between finite dimensional (compact) manifolds is naturally an infinite dimensional $C^{[M]}$-manifold, and that the group of $C^{[M]}$-diffeomorphisms of a compact manifold is a $C^{[M]}$-regular Lie group.

Notation. We use $\mathbb{N}=\mathbb{N}_{>0}\cup\{0\}$. For each multi-index $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$, we write $\alpha!=\alpha_1!\cdots\alpha_n!$, $|\alpha|=\alpha_1+\cdots+\alpha_n$, and $\partial^\alpha=\partial^{|\alpha|}/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$.

A sequence $r=(r_k)$ of reals is called increasing if $r_k\le r_{k+1}$ for all k.

We write $f^{(k)}(x)=d^kf(x)$ for the k-th order Fréchet derivative of f at x; by d_v^k we mean k times iterated directional derivatives in direction v.

For a convenient vector space E and a closed absolutely convex bounded subset $B\subseteq E$, we denote by E_B the linear span of B equipped with the Minkowski functional $\|x\|_B=\inf\{\lambda>0 : x\in\lambda B\}$. Then E_B is a Banach space. If $U\subseteq E$ then $U_B:=i_B^{-1}(U)$, where $i_B:E_B\to E$ is the inclusion of E_B in E.

We denote by E^* (resp. E') the dual space of continuous (resp. bounded) linear functionals. $L(E_1,\dots,E_k;F)$ is the space of k-linear bounded mappings $E_1\times\cdots\times E_k\to F$; if $E_i=E$ for all i, we also write $L^k(E,F)$. If E and F are Banach spaces, then $\|\ \|_{L^k(E,F)}$ denotes the operator norm on $L^k(E,F)$. By $L^k_{\text{sym}}(E,F)$ we denote the subspace of symmetric k-linear bounded mappings. We write oE for the open unit ball in a Banach space E.

The notation $C^{[M]}$ stands *locally constantly* for either $C^{(M)}$ or $C^{\{M\}}$; this means: Statements that involve more than one $C^{[M]}$ symbol must not be interpreted by mixing $C^{(M)}$ and $C^{\{M\}}$.

From Section 2.3 on, if not specified otherwise, a positive sequence $M = (M_k)$ is assumed to satisfy $M_0 = 1 \le M_1$. In Section 9 we also assume that $M = (M_k)$ is log-convex and has moderate growth, and in the Beurling case $C^{[M]} = C^{(M)}$ we additionally require $C^\omega \subseteq C^{(M)}$.

2. Denjoy–Carleman differentiable functions in finite dimensions

2.1. **Denjoy–Carleman differentiable functions of Beurling and Roumieu type in finite dimensions.** Let $M = (M_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers. Let $U \subseteq \mathbb{R}^n$ be open, $K \subseteq U$ compact, and $\rho > 0$. Consider the set

$$\Big\{\frac{\partial^\alpha f(x)}{\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|}} : x \in K, \alpha \in \mathbb{N}^n\Big\}. \tag{1}$$

We define the *Denjoy–Carleman classes*

$$C^{(M)}(U) := \{f \in C^\infty(U) : \forall \text{ compact } K \subseteq U\ \forall \rho > 0 : (1) \text{ is bounded}\},$$
$$C^{\{M\}}(U) := \{f \in C^\infty(U) : \forall \text{ compact } K \subseteq U\ \exists \rho > 0 : (1) \text{ is bounded}\}.$$

The elements of $C^{(M)}(U)$ are said to be of *Beurling type*; those of $C^{\{M\}}(U)$ of *Roumieu type*. If $M_k = 1$, for all k, then $C^{(M)}(U)$ consists of the restrictions to U of the real and imaginary parts of all entire functions, while $C^{\{M\}}(U)$ coincides with the ring $C^\omega(U)$ of real analytic functions on U. We shall also write $C^{[M]}$ and thereby mean that $C^{[M]}$ stands for either $C^{(M)}$ or $C^{\{M\}}$.

A sequence $M = (M_k)$ is *log-convex* if $k \mapsto \log(M_k)$ is convex, i.e.,

$$M_k^2 \le M_{k-1}\,M_{k+1} \quad \text{for all } k. \tag{2}$$

If $M = (M_k)$ is log-convex, then $k \mapsto (M_k/M_0)^{1/k}$ is increasing and

$$M_l\,M_k \le M_0\,M_{l+k} \quad \text{for all } l,k \in \mathbb{N}. \tag{3}$$

Let us assume $M_0 = 1$ from now on. Furthermore, we have that $k \mapsto k!M_k$ is log-convex (since Euler's Γ-function is so); if $M = (M_k)$ satisfies this weaker condition we say that it is *weakly log-convex*. If $M = (M_k)$ is weakly log-convex, then $C^{[M]}(U)$ is a ring, for all open subsets $U \subseteq \mathbb{R}^n$.

If $M = (M_k)$ is log-convex, then (see the proof of [25, 2.9]) we have

$$M_1^j\,M_k \ge M_j\,M_{\alpha_1}\cdots M_{\alpha_j} \quad \text{for all } \alpha_i \in \mathbb{N}_{>0} \text{ with } \alpha_1 + \cdots + \alpha_j = k. \tag{4}$$

Condition (4) implies that the class of $C^{[M]}$-mappings is stable under composition. This is due to [34] in the Roumieu case, see also [7] or [1, 4.7]; the same proof works in the Beurling case. We reproof it in Theorem 4.9; compare also with Lemma 2.3. For a partial converse, see [31].

If $M = (M_k)$ is log-convex, then the inverse function theorem for $C^{\{M\}}$ holds ([17]; see also [1, 4.10]), and $C^{\{M\}}$ is closed under solving ODEs (due to [18]). If additionally we have $M_{k+1}/M_k \to \infty$, then also $C^{(M)}$ is closed under taking the inverse and solving ODEs (again by [17] and [18]). See [39], [40], [32], and Section 9.2 for Banach space versions of these results.

Suppose that $M = (M_k)$ and $N = (N_k)$ are such that $\sup_k (M_k/N_k)^{1/k} < \infty$, i.e.

$$\exists C, \rho > 0\ \forall k \in \mathbb{N} :\ M_k \le C\rho^k N_k. \tag{5}$$

Then $C^{(M)}(U) \subseteq C^{(N)}(U)$ and $C^{\{M\}}(U) \subseteq C^{\{N\}}(U)$. The converse is true if $M = (M_k)$ is weakly log-convex: In the Roumieu case the inclusion $C^{\{M\}}(U) \subseteq$

$C^{\{N\}}(U)$ implies (5) thanks to the existence of a function $f \in C^{\{M\}}(\mathbb{R})$ such that $|f^{(k)}(0)| \geq k!\, M_k$ for all k (see [38, Thm. 1]; and also Section 2.2). In the Beurling case the equivalence of $C^{(M)}(U) \subseteq C^{(N)}(U)$ and (5) follows from the closed graph theorem; see Bruna [2]. As a consequence we see that the following three conditions are equivalent: $C^\omega(U)$ is contained in $C^{\{M\}}(U)$, the restrictions of entire functions are contained in $C^{(M)}(U)$, and $\underline{\lim}\, M_k^{1/k} > 0$.

$C^{[M]}$ is *stable under derivations* (alias *derivation closed*) if

$$\sup_{k \in \mathbb{N}_{>0}} \Big(\frac{M_{k+1}}{M_k}\Big)^{\frac{1}{k}} < \infty. \tag{6}$$

The converse is true if $M = (M_k)$ is weakly log-convex: $C^{\{M\}}$ is stable under derivations if and only if (6) holds.

A sequence $M = (M_k)$ is said to have *moderate growth* if

$$\sup_{j,k \in \mathbb{N}_{>0}} \Big(\frac{M_{j+k}}{M_j\, M_k}\Big)^{\frac{1}{j+k}} < \infty. \tag{7}$$

Moderate growth implies (6) and thus stability under derivations. If $M = (M_k)$ is weakly log-convex and has moderate growth, then $C^{[M]}(U)$ is *stable under ultra-differential operators*, see [15, 2.11 and 2.12].

For sequences $M = (M_k)$ and $N = (N_k)$ of positive real numbers we define

$$\begin{aligned} M \lhd N \quad &:\Leftrightarrow \quad \forall \rho > 0\ \exists C > 0 : M_k \leq C \rho^k N_k\ \forall k \in \mathbb{N} \\ &\Leftrightarrow \quad \lim_{k\to\infty} \Big(\frac{M_k}{N_k}\Big)^{\frac{1}{k}} = 0. \end{aligned}$$

If $M \lhd N$, then we have $C^{\{M\}}(U) \subseteq C^{(N)}(U)$. If $M = (M_k)$ is weakly log-convex, also the converse is true: $C^{\{M\}}(U) \subseteq C^{(N)}(U)$ implies $M \lhd N$. This follows from the existence of a function $f \in C^{\{M\}}(\mathbb{R})$ with $|f^{(k)}(0)| \geq k!\, M_k$ for all k (see [38, Thm. 1]). As a consequence $C^\omega(U)$ is contained in $C^{(M)}(U)$ if and only if $M_k^{1/k} \to \infty$.

Theorem 2.1 (Denjoy–Carleman [6], [3])**.** *For a sequence $M = (M_k)$ of positive real numbers the following statements are equivalent:*

1. $C^{[M]}$ *is* quasianalytic, *i.e., for open connected $U \subseteq \mathbb{R}^n$ and each $x \in U$, the Taylor series homomorphism centered at x from $C^{[M]}(U,\mathbb{R})$ into the space of formal power series is injective.*
2. $\sum_{k=1}^\infty \frac{1}{m_k^{\flat(i)}} = \infty$ *where* $m_k^{\flat(i)} := \inf\{(j!\, M_j)^{1/j} : j \geq k\}$ *is the increasing minorant of* $(k!\, M_k)^{1/k}$.
3. $\sum_{k=1}^\infty (\frac{1}{M_k^{\flat(lc)}})^{1/k} = \infty$ *where* $M_k^{\flat(lc)}$ *is the log-convex minorant of* $k!\, M_k$, *given by* $M_k^{\flat(lc)} := \inf\{(j!\, M_j)^{\frac{l-k}{l-j}} (l!\, M_l)^{\frac{k-j}{l-j}} : j \leq k \leq l, j < l\}$.
4. $\sum_{k=0}^\infty \frac{M_k^{\flat(lc)}}{M_{k+1}^{\flat(lc)}} = \infty$.

For contemporary proofs of the equivalence of (2), (3), (4) and quasianalyticity of $C^{\{M\}}$, see for instance [14, 1.3.8] or [35, 19.11]. For the equivalence of these conditions to the quasianalyticity of $C^{(M)}$, see [15, 4.2].

2.2. **Sequence spaces.** Let $M = (M_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers, and $\rho > 0$. We consider (where $\mathcal{F}$ stands for 'formal power series')

$$\mathcal{F}^M_\rho := \Big\{(f_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \exists C > 0\, \forall k \in \mathbb{N} : |f_k| \le C\rho^k k! M_k\Big\},$$
$$\mathcal{F}^{(M)} := \bigcap_{\rho>0} \mathcal{F}^M_\rho, \quad \text{and} \quad \mathcal{F}^{\{M\}} := \bigcup_{\rho>0} \mathcal{F}^M_\rho.$$

Lemma. *Consider the following conditions for two positive sequences $M^i = (M^i_k)$, $i = 1, 2$, and $0 < \sigma < \infty$:*

1. $\sup_k (M^1_k/M^2_k)^{1/k} = \sigma$.
2. *For all $\rho > 0$ we have $\mathcal{F}^{M^1}_\rho \subseteq \mathcal{F}^{M^2}_{\rho\sigma}$.*
3. $\mathcal{F}^{\{M^1\}} \subseteq \mathcal{F}^{\{M^2\}}$.
4. $\mathcal{F}^{(M^1)} \subseteq \mathcal{F}^{(M^2)}$.
5. $M^1 \lhd M^2$.
6. $\mathcal{F}^{\{M^1\}} \subseteq \mathcal{F}^{(M^2)}$.

Then we have (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) *and* (5) $\Leftrightarrow$ (6).

Proof. (1) $\Rightarrow$ (2) Let $f = (f_k) \in \mathcal{F}^{M^1}_\rho$, i.e., there is a $C > 0$ such that $|f_k| \le C\rho^k k! M^1_k \le C(\rho\sigma)^k k! M^2_k$, for all k. So $f \in \mathcal{F}^{M^2}_{\rho\sigma}$.

(2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) follow by definition.

(3) $\Rightarrow$ (1) Let $f_k := k! M^1_k$. Then $f = (f_k) \in \mathcal{F}^{\{M^1\}} \subseteq \mathcal{F}^{\{M^2\}}$, so there exists $\rho > 0$ such that $k! M^1_k \le \rho^{k+1} k! M^2_k$ for all k.

(5) $\Rightarrow$ (6) Let $f = (f_k) \in \mathcal{F}^{M^1}_\rho$. As $M^1 \lhd M^2$, for each $\sigma > 0$ there exists $C > 0$ such that $|f_k| \le C(\frac{\sigma}{\rho})^k k! M_k$ for all k. So $f \in \mathcal{F}^{M^2}_{\frac{\sigma}{\rho}}$ for all σ.

(6) $\Rightarrow$ (5) Since $(k! M_k) \in \mathcal{F}^{\{M^1\}} \subseteq \mathcal{F}^{(M^2)}$, for each $\rho > 0$ there is $C > 0$ such that $k! M^1_k \le C\rho^k k! M^2_k$ for all k, i.e., $M^1 \lhd M^2$. □

Theorem 2.2. *Let $M = (M_k)$ be a (weakly) log-convex sequence of positive real numbers. Then we have*

$$\mathcal{F}^{\{M\}} = \bigcap_L \mathcal{F}^{(L)} = \bigcap_L \mathcal{F}^{\{L\}}, \tag{8}$$

where the intersections are taken over all (weakly) log-convex $L = (L_k)$ with $M \lhd L$.

Proof. The inclusions $\mathcal{F}^{\{M\}} \subseteq \bigcap_L \mathcal{F}^{(L)} \subseteq \bigcap_L \mathcal{F}^{\{L\}}$ follow from Lemma 2.2. So it remains to prove that $\mathcal{F}^{\{M\}} \supseteq \bigcap_L \mathcal{F}^{\{L\}}$. Let $f = (f_k) \notin \mathcal{F}^{\{M\}}$, i.e.,

$$\overline{\lim} \Big(\frac{|f_k|}{k! M_k}\Big)^{\frac{1}{k}} = \infty. \tag{9}$$

We must show that there exists a (weakly) log-convex $L = (L_k)$ with $M \lhd L$ such that $f \notin \mathcal{F}^{\{L\}}$.

Choose $a_j, b_j > 0$ with $a_j \nearrow \infty$, $b_j \searrow 0$, and $a_j b_j \to \infty$. Now (9) implies that there exists a strictly increasing sequence $k_j \in \mathbb{N}$ such that

$$\Big(\frac{|f_{k_j}|}{(k_j)! M_{k_j}}\Big)^{\frac{1}{k_j}} \ge a_j.$$

Passing to a subsequence we may assume that $k_0 > 0$ and that

$$1 < \beta_j := b_j \Big(\frac{|f_{k_j}|}{(k_j)! M_{k_j}} \Big)^{\frac{1}{k_j}} \nearrow \infty.$$

Passing to a subsequence again we may also get

$$\beta_{j+1} \ge (\beta_j)^{k_j}. \tag{10}$$

We define a piecewise affine function ϕ by setting

$$\phi(k) := \begin{cases} 0 & \text{if } k = 0, \\ k_j \log \beta_j & \text{if } k = k_j, \\ c_j + d_j k & \text{for the minimal } j \text{ with } k \le k_j, \end{cases}$$

where c_j and d_j are chosen such that ϕ is well defined and $\phi(k_{j-1}) = c_j + d_j k_{j-1}$, i.e., for $j \ge 1$,

$$\begin{aligned} c_j + d_j k_j &= k_j \log \beta_j, \\ c_j + d_j k_{j-1} &= k_{j-1} \log \beta_{j-1}, \qquad \text{and} \\ c_0 &= 0, \\ d_0 &= \log \beta_0. \end{aligned}$$

This implies first that $c_j \le 0$ and then

$$\begin{aligned} \log \beta_j \le d_j = \frac{k_j \log \beta_j - k_{j-1} \log \beta_{j-1}}{k_j - k_{j-1}} &\le \frac{k_j}{k_j - k_{j-1}} \log \beta_j \\ &\overset{(10)}{\le} \frac{\log \beta_{j+1}}{k_j - k_{j-1}} \le \log \beta_{j+1}. \end{aligned}$$

Thus $j \mapsto d_j$ is increasing and so ϕ is convex. The fact that all $c_j \le 0$ implies that $\phi(k)/k$ is increasing.

Now let

$$L_k := e^{\phi(k)} \cdot M_k.$$

Then $L = (L_k)$ is (weakly) log-convex, since so is $M = (M_k)$. As $\phi(k)/k$ is increasing and $e^{\phi(k_j)/k_j} = \beta_j \to \infty$, we find $M \lhd L$. Finally, $f \notin \mathcal{F}^{\{L\}}$, since we have

$$\Big(\frac{|f_{k_j}|}{(k_j)! L_{k_j}} \Big)^{\frac{1}{k_j}} = \Big(\frac{|f_{k_j}|}{(k_j)! M_{k_j}} \Big)^{\frac{1}{k_j}} \cdot e^{-\phi(k_j)/k_j} = \Big(\frac{|f_{k_j}|}{(k_j)! M_{k_j}} \Big)^{\frac{1}{k_j}} \cdot \beta_j^{-1} = b_j^{-1} \to \infty.$$

The proof is complete. □

Remark. (1) If $M_0 = 1 \le M_1$ we also have $L_0 = 1 \le L_1$.

(2) The proof also shows that, if $M = (M_k)$ is just any positive sequence, then (8) still holds if the intersections are taken over all positive sequences $L = (L_k)$ with $M \lhd L$.

Lemma 2.3. *Let $M = (M_k)$ and $L = (L_k)$ be sequences of positive real numbers. Then for the composition of formal power series we have*

$$\mathcal{F}^{[M]} \circ \mathcal{F}^{[L]}_{>0} \subseteq \mathcal{F}^{[M \circ L]}, \tag{11}$$

where $(M \circ L)_k := \max\{M_j L_{\alpha_1} \dots L_{\alpha_j} : \alpha_i \in \mathbb{N}_{>0}, \alpha_1 + \dots + \alpha_j = k\}$.

Here $\mathcal{F}^{[L]}_{>0}$ is the space of formal power series in $\mathcal{F}^{[L]}$ with vanishing constant term.

Proof. Let $f \in \mathcal{F}^{(M)}$ and $g \in \mathcal{F}^{(L)}$ (resp. $f \in \mathcal{F}^{\{M\}}$ and $g \in \mathcal{F}^{\{L\}}$). For $k > 0$ we have (inspired by [8])

$$\frac{(f \circ g)_k}{k!} := \sum_{j=1}^k \frac{f_j}{j!} \sum_{\substack{\alpha \in \mathbb{N}^j_{>0} \\ \alpha_1+\dots+\alpha_j=k}} \frac{g_{\alpha_1}}{\alpha_1!} \cdots \frac{g_{\alpha_j}}{\alpha_j!},$$

$$\begin{aligned}
\frac{|(f \circ g)_k|}{k!(M \circ L)_k} &\le \sum_{j=1}^k \frac{|f_j|}{j! M_j} \sum_{\substack{\alpha \in \mathbb{N}^j_{>0} \\ \alpha_1+\dots+\alpha_j=k}} \frac{|g_{\alpha_1}|}{\alpha_1! L_{\alpha_1}} \cdots \frac{|g_{\alpha_j}|}{\alpha_j! L_{\alpha_j}} \\
&\le \sum_{j=1}^k \rho_f^j C_f \sum_{\substack{\alpha \in \mathbb{N}^j_{>0} \\ \alpha_1+\dots+\alpha_j=k}} \rho_g^k C_g^j \le \sum_{j=1}^k \rho_f^j C_f \binom{k-1}{j-1} \rho_g^k C_g^j \\
&= \rho_g^k \rho_f C_f C_g \sum_{j=1}^k (\rho_f C_g)^{j-1} \binom{k-1}{j-1} = \rho_g^k \rho_f C_f C_g (1 + \rho_f C_g)^{k-1} \\
&= (\rho_g(1+\rho_f C_g))^k \frac{\rho_f C_f C_g}{1+\rho_f C_g}.
\end{aligned}$$

This implies (11) in the Roumieu case. For the Beurling case, let $\tau > 0$ be arbitrary, and choose $\sigma > 0$ such that $\tau = \sqrt{\sigma} + \sigma$. If we set $\rho_g = \sqrt{\sigma}$ and $\rho_f = \sqrt{\sigma}/C_g$, then $f \circ g \in \mathcal{F}^{M \circ L}_\tau$. $\square$

2.3. **Convention.** For a positive sequence $M = (M_k) \in (\mathbb{R}_{>0})^{\mathbb{N}}$ consider the following properties:

(0) $M_0 = 1 \le M_1$.
(1) $M = (M_k)$ is weakly log-convex, i.e., $k \mapsto \log(k!\, M_k)$ is convex.
(2) $M = (M_k)$ is log-convex, i.e., $k \mapsto \log(M_k)$ is convex.
(3) $M = (M_k)$ is derivation closed, i.e., $k \mapsto (\frac{M_{k+1}}{M_k})^{\frac{1}{k}}$ is bounded.
(4) $M = (M_k)$ has moderate growth, i.e., $(j,k) \mapsto (\frac{M_{j+k}}{M_j\, M_k})^{\frac{1}{j+k}}$ is bounded.
(5) $\frac{M_{k+1}}{M_k} \to \infty$.
(6) $M_k^{1/k} \to \infty$, or equivalently, $C^\omega \subseteq C^{(M)}$.

Henceforth, if not specified otherwise, we assume that $M = (M_k)$, $N = (N_k)$, $L = (L_k)$, etc., satisfy condition (0). It will be explicitly stated when some of the other properties (1)–(6) are assumed.

Remarks. Let $M = (M_k)$ be a positive sequence. We may replace $(M_k)_k$ by $(C\rho^k M_k)_k$ with $C, \rho > 0$ without changing $\mathcal{F}^{[M]}$ (see Section 2.2). In particular, it is no loss of generality to assume that $M_1 > 1$ (put $C\rho > 1/M_1$) and $M_0 = 1$ (put $C := 1/M_0$). Each one of the properties (1)–(6) is preserved by this modification. Furthermore $M = (M_k)$ is quasianalytic if and only if the modified sequence is so, since $(M_k^{\flat(lc)})_k$ (see Theorem 2.1) is modified in the same way.

Conditions (0) and (1) together imply that $k \mapsto k!\, M_k$ is monotone increasing, while (0) and (2) together imply that $k \mapsto M_k$ is monotone increasing.

3. Whitney jets on Banach spaces

3.1. **Whitney jets.** Let E and F be Banach spaces. For open $U \subseteq E$ consider the space $C^\infty(U,F)$ of arbitrarily often Fréchet differentiable mappings $f : U \to F$. For such f we have the derivatives $f^{(k)} : U \to L^k_{\mathrm{sym}}(E,F)$, where $L^k_{\mathrm{sym}}(E,F)$ denotes the space of symmetric k-linear bounded mappings $E \times \cdots \times E \to F$. We also have the iterated uni-directional derivatives $d^k_v f(x) \in F$ defined by

$$d^k_v f(x) := \Big(\frac{d}{dt}\Big)^k f(x + t\,v)|_{t=0}.$$

Let $j^\infty : C^\infty(U,F) \to J^\infty(U,F) := \prod_{k\in\mathbb{N}} C(U, L^k_{\mathrm{sym}}(E,F))$ be the jet mapping $f \mapsto (f^{(k)})_{k\in\mathbb{N}}$. On $L^k_{\mathrm{sym}}(E,F)$ we consider the operator norm

$$\|\ell\|_{L^k_{\mathrm{sym}}(E,F)} := \sup\Big\{\|\ell(v_1,\dots,v_k)\|_F : \|v_j\|_E \le 1 \text{ for all } j \in \{1,\dots,k\}\Big\}.$$

Note that by the polarization equality (see [23, 7.13.1])

$$\begin{aligned}\sup\{\|\ell(v,\dots,v)\|_F : \|v\|_E \le 1\} &\le \|\ell\|_{L^k_{\mathrm{sym}}(E,F)} \\ &\le (2e)^k \sup\{\|\ell(v,\dots,v)\|_F : \|v\|_E \le 1\}\end{aligned}$$

For an infinite jet $f = (f^k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} L^k_{\mathrm{sym}}(E,F)^X$ on a subset $X \subseteq E$ let the Taylor polynomial $(T^n_y f)^k : X \to L^k_{\mathrm{sym}}(E,F)$ of order n at y be

$$(T^n_y f)^k(x)(v_1,\dots,v_k) := \sum_{j=0}^n \frac{1}{j!}\, f^{j+k}(y)(x-y,\dots,x-y,v_1,\dots v_k)$$

and the remainder

$$(R^n_y f)^k(x) := f^k(x) - (T^n_y f)^k(x) = (T^n_x f)^k(x) - (T^n_y f)^k(x) \in L^k_{\mathrm{sym}}(E,F).$$

Let

$$\|f\|_k := \sup\{\|f^k(x)\|_{L^k_{\mathrm{sym}}(E,F)} : x \in X\} \in [0,+\infty] \quad \text{and}$$

$$|||f|||_{n,k} := \sup\Big\{(n+1)!\frac{\|(R^n_y f)^k(x)\|_{L^k_{\mathrm{sym}}(E,F)}}{\|x-y\|^{n+1}} : x,y \in X, x \ne y\Big\} \in [0,+\infty].$$

By Taylor's theorem, for $f \in C^\infty(U,F)$ and $[x,y] \subseteq U$ we have

$$\begin{aligned}(R^n_y f)^k(x) &= f^{(k)}(x) - \sum_{j\le n} \frac{f^{(k+j)}(y)(x-y)^j}{j!} \\ &= \int_0^1 \frac{(1-t)^n}{n!}\, f^{(k+n+1)}(y + t(x-y))(x-y)^{n+1}\,dt\end{aligned}$$

and hence for convex $X \subseteq U$:

$$\begin{aligned}&|||j^\infty f|_X|||_{n,k} := \\ &= \sup\Big\{(n+1)!\frac{\|(R^n_y f)^k(x)(v_1,\dots,v_k)\|_F}{\|x-y\|^{n+1}} : \|v_j\|_E \le 1, x,y \in X, x \ne y\Big\} \\ &\le \sup\Big\{\frac{\|f^{(k+n+1)}(x)(v_1,\dots,v_k,x-y,\dots,x-y)\|_F}{\|x-y\|^{n+1}} : \|v_j\|_E \le 1, x \ne y\Big\} \\ &\le \|j^\infty f|_X\|_{n+k+1}. \qquad (12)\end{aligned}$$

We supply $C^\infty(U,F)$ with the semi-norms

$$f \mapsto \|j^\infty f|_K\|_n \qquad \text{for all compact } K \subseteq U \text{ and all } n \in \mathbb{N}.$$

For compact convex $K \subseteq E$ the space $C^\infty(E \supseteq K, F)$ of Whitney jets on K is defined by

$$\begin{aligned} &C^\infty(E \supseteq K, F) := \\ &\quad = \Big\{ f = (f^k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} C(K, L^k_{\mathrm{sym}}(E,F)) : |||f|||_{n,k} < \infty \text{ for all } n,k \in \mathbb{N} \Big\} \end{aligned}$$

and is supplied with the seminorms $\|\ \|_n$ for $n \in \mathbb{N}$ together with $|||\ |||_{n,k}$ for $n,k \in \mathbb{N}$.

Lemma 3.1. *For Banach spaces E and F and compact convex $K \subseteq E$ the space $C^\infty(E \supseteq K, F)$ is a Fréchet space.*

Proof. The injection of $C^\infty(E \supseteq K, F)$ into $\prod_{k\in\mathbb{N}} C(K, L^k_{\mathrm{sym}}(E,F))$ is continuous by definition and $C(K, L^k_{\mathrm{sym}}(E,F))$ is a Banach space, so a Cauchy sequence $(f_p)_p$ in $C^\infty(E \supseteq K, F)$ has an infinite jet $f_\infty = (f^k_\infty)_k$ as component-wise limit in $\prod_{k\in\mathbb{N}} C(K, L^k_{\mathrm{sym}}(E,F))$ with respect to the seminorms $\|\ \|_n$. This is the limit also with respect to the finer structure of $C^\infty(E \supseteq K, F)$ with the additional seminorms $|||\ |||_{n,k}$ as follows: For given $n,k \in \mathbb{N}$ and $\epsilon > 0$ there exists by the Cauchy condition a p_0 such that $|||f_p - f_q|||_{n,k} < \epsilon/3$ for all $p,q \ge p_0$. By the convergence $f_q \to f_\infty$ in $\prod_{k\in\mathbb{N}} C(K, L^k_{\mathrm{sym}}(E,F))$ there exists for given $x,y \in K$ with $x \ne y$ a $q \ge p_0$ such that for all $m \le k+n$

$$\|f_q - f_\infty\|_m \le \frac{\|x-y\|^{n+1}}{(n+1)!}\,\frac{\epsilon}{3}\,\min\{1, e^{-\|x-y\|}\}$$

and hence

$$\begin{aligned} &\|(T^n_y f_q)^k(x) - (T^n_y f_\infty)^k(x)\|_{L^k_{\mathrm{sym}}(E,F)} \le \\ &\qquad \le \sum_{j=0}^n \|f_q^{k+j}(y) - f_\infty^{k+j}(y)\|_{L^{k+j}_{\mathrm{sym}}(E,F)}\,\frac{\|x-y\|^j}{j!} \\ &\qquad \le \sum_{j=0}^n \|f_q - f_\infty\|_{k+j}\,\frac{\|x-y\|^j}{j!} \\ &\qquad \le \frac{\|x-y\|^{n+1}}{(n+1)!}\,\frac{\epsilon}{3}\,e^{-\|x-y\|}\,\sum_{j=0}^n \frac{\|x-y\|^j}{j!} \le \frac{\|x-y\|^{n+1}}{(n+1)!}\,\frac{\epsilon}{3}. \end{aligned}$$

So

$$\begin{aligned} &(n+1)!\,\frac{\|(R^n_y f_p)^k(x) - (R^n_y f_\infty)^k(x)\|_{L^k_{\mathrm{sym}}(E,F)}}{\|x-y\|^{n+1}} \le \\ &\qquad \le |||f_p - f_q|||_{n,k} + (n+1)!\,\frac{\|(R^n_y f_q)^k(x) - (R^n_y f_\infty)^k(x)\|_{L^k_{\mathrm{sym}}(E,F)}}{\|x-y\|^{n+1}} \\ &\qquad \le \frac{\epsilon}{3} + (n+1)!\,\frac{\|f^k_q(x) - f^k_\infty(x)\|_{L^k_{\mathrm{sym}}(E,F)}}{\|x-y\|^{n+1}} \\ &\qquad\qquad + (n+1)!\,\frac{\|(T^n_y f_q)^k(x) - (T^n_y f_\infty)^k(x)\|_{L^k_{\mathrm{sym}}(E,F)}}{\|x-y\|^{n+1}} \end{aligned}$$

$$\leq 3\,\frac{\epsilon}{3} = \epsilon$$

and finally

$$|||f_p - f_\infty|||_{n,k} \leq \epsilon \text{ for all } p \geq p_0.$$

Consequently,

$$|||f_\infty|||_{n,k} \leq |||f_\infty - f_p|||_{n,k} + |||f_p|||_{n,k} < \infty,$$

i.e., $f_\infty \in C^\infty(E \supseteq K, F)$. □

4. The category of Denjoy–Carleman differentiable mappings

4.1. Spaces of Denjoy–Carleman jets and mappings between Banach spaces. Let E and F be Banach spaces, $K \subseteq E$ compact, and $\rho > 0$. Let

$$C^M_\rho(E \supseteq K, F) := \Big\{(f^m)_m \in \prod_{m\in\mathbb{N}} C(K, L^m_{\mathrm{sym}}(E,F)) : \|f\|_\rho < \infty\Big\},$$

where

$$\|f\|_\rho := \max\bigg\{\sup\Big\{\frac{\|f\|_m}{m!\rho^m M_m} : m \in \mathbb{N}\Big\}, \\ \sup\Big\{\frac{|||f|||_{n,k}}{(n+k+1)!\,\rho^{n+k+1}\,M_{n+k+1}} : k,n \in \mathbb{N}\Big\}\bigg\},$$

cf. [4, 11], [5, 11] and [37, 3], and, for an open neighborhood U of K in E, let

$$C^M_{K,\rho}(U,F) := \Big\{f \in C^\infty(U,F) : j^\infty f|_K \in C^M_\rho(E \supseteq K, F)\Big\}$$

supplied with the semi-norm $f \mapsto \|j^\infty f|_K\|_\rho$. This space is not Hausdorff and for infinite dimensional E it(s Hausdorff quotient) will not always be complete. This is the reason for considering the jet spaces $C^M_\rho(E \supseteq K, F)$ instead. Note that for convex K we have $|||j^\infty f|_K|||_{n,k} \leq \|j^\infty f|_K\|_{n+k+1}$ by (12) and hence the seminorm $f \mapsto \|j^\infty f|_K\|_\rho$ on $C^M_{K,\rho}(U,F)$ coincides with

$$f \mapsto \sup\Big\{\frac{\|f^{(n)}(x)\|_{L^n_{\mathrm{sym}}(E,F)}}{n!\rho^n M_n} : x \in K, n \in \mathbb{N}\Big\} =: \|f\|_{K,\rho}.$$

Thus

$$C^M_{K,\rho}(U,F) = \Big\{f \in C^\infty(U,F) : (\|j^\infty f|_K\|_m)_m \in \mathcal{F}^M_\rho\Big\}$$

and the bounded subsets $\mathcal{B} \subseteq C^M_{K,\rho}(U,F)$ are exactly those $\mathcal{B} \subseteq C^\infty(U,F)$ for which $(b_m)_m \in \mathcal{F}^M_\rho$, where $b_m := \sup\{\|j^\infty f|_K\|_m : f \in \mathcal{B}\}$.

For open convex $U \subseteq E$ and compact convex $K \subseteq U$ let

$$C^{(M)}(E \supseteq K, F) := \bigcap_{\rho>0} C^M_\rho(E \supseteq K, F),$$

$$C^{\{M\}}(E \supseteq K, F) := \bigcup_{\rho>0} C^M_\rho(E \supseteq K, F), \text{ and}$$

$$C^{[M]}(U,F) := \Big\{f \in C^\infty(U,F) : \forall K : (f^{(k)}|_K) \in C^{[M]}(E \supseteq K, F)\Big\}.$$

That means, we consider the projective limit

$$C^{(M)}(E \supseteq K, F) := \varprojlim_{\rho>0} C^M_\rho(E \supseteq K, F),$$

the inductive limit

$$C^{\{M\}}(E \supseteq K, F) := \varinjlim_{\rho>0} C^M_\rho(E \supseteq K, F),$$

and the projective limits

$$C^{[M]}(U, F) := \varprojlim_{K \subseteq U} C^{[M]}(E \supseteq K, F),$$

where K runs through all compact convex subsets of U.

Furthermore, we consider the projective limit

$$C^{(M)}_K(U, F) := \varprojlim_{\rho>0} C^M_{K,\rho}(U, F),$$

and the inductive limit

$$C^{\{M\}}_K(U, F) := \varinjlim_{\rho>0} C^M_{K,\rho}(U, F).$$

Thus

$$C^{[M]}_K(U, F) = \Big\{ f \in C^\infty(U, F) : (\|j^\infty f|_K\|_m)_m \in \mathcal{F}^{[M]} \Big\}.$$

Furthermore, the bounded subsets $\mathcal{B} \subseteq C^{[M]}_K(U, F)$ are exactly those $\mathcal{B} \subseteq C^\infty(U, F)$ for which $(b_m)_m \in \mathcal{F}^{[M]}$, where $b_m := \sup\{\|j^\infty f|_K\|_m : f \in \mathcal{B}\}$.

Finally, the projective limits

$$\varprojlim_{K \subseteq U} C^{[M]}_K(U, F) = \Big\{ f \in C^\infty(U, F) : \forall K : (\|j^\infty f|_K\|_m)_m \in \mathcal{F}^{[M]} \Big\},$$

where K runs through all compact convex subsets of U, are for $E = \mathbb{R}^n$ and $F = \mathbb{R}$ exactly the vector spaces of Section 2.1 and the topology is the usual one.

For the inductive limits with respect to $\rho > 0$ it suffices to take $\rho \in \mathbb{N}$ only.

Proposition 4.1. *We have the following completeness properties:*

(1) *The spaces $C^M_\rho(E \supseteq K, F)$ are Banach spaces.*
(2) *The spaces $C^{(M)}(E \supseteq K, F)$ are Fréchet spaces.*
(3) *The spaces $C^{\{M\}}(E \supseteq K, F)$ are compactly regular (i.e., compact subsets are contained and compact in some step) (LB)-spaces hence (c^∞-)complete, webbed and (ultra-)bornological.*
(4) *The spaces $C^{[M]}(U, F)$ are complete spaces.*
(5) *As locally convex spaces*

$$C^{[M]}(U, F) := \varprojlim_{K \subseteq U} C^{[M]}(E \supseteq K, F) = \varprojlim_{K \subseteq U} C^{[M]}_K(U, F).$$

Proof. (1) The injection $C^M_\rho(E \supseteq K, F) \to \prod_{k\in\mathbb{N}} C(K, L^k_{\mathrm{sym}}(E,F))$ is by definition continuous and $C(K, L^k_{\mathrm{sym}}(E,F))$ is a Banach space, so a Cauchy sequence $(f_p)_p$ in $C^M_\rho(E \supseteq K, F)$ has an infinite jet $f_\infty = (f^k_\infty)_k$ as component-wise limit in $\prod_{k\in\mathbb{N}} C(K, L^k_{\mathrm{sym}}(E,F))$. This is the limit also with respect to the finer structure of $C^M_\rho(E \supseteq K, F)$ as follows: For fixed n, k and $x \ne y$ we have that $(R^n_y f_p)^k(x)$ converges to $(R^n_y f_\infty)^k(x)$. So we choose for $\epsilon > 0$ a $p_0 \in \mathbb{N}$ such that $\|f_p - f_q\|_\rho < \epsilon/2$ for all $p, q \ge p_0$ and given x, y, n, and k we can choose $q > p_0$ such that

$$(n+1)!\, \frac{\|(R^n_y f_q)^k(x) - (R^n_y f_\infty)^k(x)\|_{L^k_{\mathrm{sym}}(E,F)}}{(n+k+1)!\rho^{n+k+1} M_{n+k+1} \|x-y\|^{n+1}} < \frac{\epsilon}{2}$$

and

$$\frac{\|f^n_q(x) - f^n_\infty(x)\|_{L^n_{\mathrm{sym}}(E,F)}}{n!\,\rho^n\,M_n} < \frac{\epsilon}{2}.$$

Thus

$$(n+1)!\, \frac{\|(R^n_y f_p)^k(x) - (R^n_y f_\infty)^k(x)\|_{L^k_{\mathrm{sym}}(E,F)}}{(n+k+1)!\rho^{n+k+1} M_{n+k+1} \|x-y\|^{n+1}} < \\ \|f_p - f_q\|_\rho + (n+1)!\, \frac{\|(R^n_y f_q)^k(x) - (R^n_y f_\infty)^k(x)\|_{L^k_{\mathrm{sym}}(E,F)}}{(n+k+1)!\rho^{n+k+1} M_{n+k+1} \|x-y\|^{n+1}} < \epsilon$$

and hence

$$\frac{|||f_p - f_\infty|||_{n,k}}{(n+k+1)!\rho^{n+k+1} M_{n+k+1}} \le \epsilon$$

and similarly for $\frac{\|f_p - f_\infty\|_n}{n!\,\rho^n\,M_n}$. Thus $\|f_p - f_\infty\|_\rho \le \epsilon$ for all $p \ge p_0$.

(2) This is obvious; they are countable projective limits of Banach spaces.

(3) For finite dimensional E and F it is shown in [37] that the connecting mappings are nuclear. For infinite dimensional E the connecting mappings in $C^{\{M\}}(E \supseteq K, F) = \varinjlim_{\rho>0} C^M_\rho(E \supseteq K, F)$ cannot be compact, since the set $\{\ell \in E' : \|\ell\| \le 1\}$ is bounded in $C^M_\rho(E \supseteq K, \mathbb{R})$ for each $\rho \ge 1$. In fact, $\|\ell\|_0 = \sup\{|\ell(x)| : x \in K\} \le \sup\{\|x\| : x \in K\}$, $\|\ell\|_1 = \|\ell\| \le 1$ and $\|\ell\|_m = 0$ for $m \ge 2$, moreover, $(R^n_y \ell)^k = 0$ for $n + k \ge 1$ and $(R^0_y \ell)^0(x) = \ell(x-y)$. It is not relatively compact in any of the spaces $C^M_\rho(E \supseteq K, \mathbb{R})$, $\rho \ge 1$, since it is not even pointwise relatively compact in $C(K, L(E, \mathbb{R}))$.

In order to show that the (LB)-space in (3) is compactly regular it suffices by [29, Satz 1] to verify condition (M) of [33]: There exists a sequence of increasing 0-neighborhoods $U_n \subseteq C^M_n(E \supseteq K, F)$, such that for each n there exists an $m \ge n$ for which the topologies of $C^M_k(E \supseteq K, F)$ and of $C^M_m(E \supseteq K, F)$ coincide on U_n for all $k \ge m$.

For $\rho' \ge \rho$ we have $\|f\|_{\rho'} \le \|f\|_\rho$. So consider the ϵ-balls $U^\rho_\epsilon(f) := \{g : \|g - f\|_\rho \le \epsilon\}$ in $C^M_\rho(E \supseteq K, F)$. It suffices to show that for $\rho > 0$, $\rho_1 := 2\rho$, $\rho_2 > \rho_1$, $\epsilon > 0$, and $f \in U^\rho_1 := U^\rho_1(0)$ there exists a $\delta > 0$ such that $U^{\rho_2}_\delta(f) \cap U^\rho_1 \subseteq U^{\rho_1}_\epsilon(f)$. Since $f \in U^\rho_1$ we have

$$\|f\|_n \le n!\rho^n M_n \text{ and } |||f|||_{n,k} \le (n+k+1)!\rho^{n+k+1} M_{n+k+1} \text{ for all } n, k.$$

Let $\frac{1}{2^N} < \frac{\epsilon}{2}$ and $\delta := \epsilon \left(\frac{\rho_1}{\rho_2}\right)^{N-1}$. Let $g \in U^{\rho_2}_\delta(f) \cap U^\rho_1$, i.e.,

$$\|g\|_n \le n!\rho^n M_n \text{ for all } n,$$

$$\|g-f\|_n \le \delta\, n!\rho_2^n M_n \text{ for all } n,$$
$$|||g|||_{n,k} \le (n+k+1)!\rho^{n+k+1} M_{n+k+1} \text{ for all } n,k,$$
$$|||g-f|||_{n,k} \le \delta\,(n+k+1)!\rho_2^{n+k+1} M_{n+k+1} \text{ for all } n,k.$$

Then

$$\begin{aligned}\|g-f\|_n &\le \|g\|_n + \|f\|_n \le 2\, n!\rho^n M_n = 2\, n!\rho_1^n M_n \,\frac{1}{2^n}\\ &< \epsilon\, n!\rho_1^n M_n \quad \text{ for } n \ge N\end{aligned}$$

and

$$\|g-f\|_n \le \delta\, n!\rho_2^n M_n \le \epsilon\, n!\rho_1^n M_n \quad \text{ for } n < N.$$

Moreover,

$$\begin{aligned}|||g-f|||_{n,k} &\le |||g|||_{n,k} + |||f|||_{n,k} \le 2\,(n+k+1)!\rho^{n+k+1} M_{n+k+1}\\ &= 2\,(n+k+1)!\rho_1^{n+k+1} M_{n+k+1}\,\frac{1}{2^{n+k+1}}\\ &< \epsilon\,(n+k+1)!\rho_1^{n+k+1} M_{n+k+1} \quad \text{ for } n+k+1 \ge N\end{aligned}$$

and

$$\begin{aligned}|||g-f|||_{n,k} &\le \delta\,(n+k+1)!\rho_2^{n+k+1} M_{n+k+1}\\ &\le \epsilon\,(n+k+1)!\rho_1^{n+k+1} M_{n+k+1} \quad \text{ for } n+k+1 < N.\end{aligned}$$

(4) This is obvious; they are projective limits of complete spaces.

(5) Since $j^\infty|_K : C^M_{K,\rho}(U,F) \to C^M_\rho(E \supseteq K,F)$ is by definition a well-defined continuous linear mapping, it induces such mappings $C^{[M]}_K(U,F) \to C^{[M]}(E \supseteq K,F)$ and $\varprojlim_K C^{[M]}_K(U,F) \to \varprojlim_K C^{[M]}(E \supseteq K,F)$. The last mapping is obviously injective (use $K := \{x\}$ for the points $x \in U$).

Conversely, let $f^k_K \in C(K, L^k_{\mathrm{sym}}(E,F))$ be given, such that for each K there exists $\rho > 0$ (resp. each $\rho > 0$) we have $(f^k_K)_{k\in\mathbb{N}} \in C^M_\rho(E \supseteq K,F)$ and such that $f^k_K|_{K'} = f^k_{K'}$. They define an infinite jet $(f^k)_{k\in\mathbb{N}} \in J^\infty(U,F)$ by setting $f^k(x) := f^k_{\{x\}}(x)$ which satisfies $f^k|_K = f^k_K$ for all $k \in \mathbb{N}$ and all K.

We claim that $f^0 \in C^\infty(U,F)$ and $(f^0)^{(k)} = f^k$ for all k, i.e., $j^\infty f^0|_K = (f^k_K)_k$ for all $k \in \mathbb{N}$ and all K.
By [23, 5.20] it is enough to show by induction that $d^k_v f^0(x) = f^k(x)(v,\dots,v)$. For $k = 0$ this is obvious, so let $k > 0$. Then

$$\begin{aligned}d^k_v f^0(x) :&= \lim_{t\to 0} \frac{d^{k-1}_v f^0(x+t\,v) - d^{k-1}_v f^0(x)}{t}\\ &= \lim_{t\to 0} \frac{f^{k-1}(x+t\,v)(v^{k-1}) - f^{k-1}(x)(v^{k-1})}{t}\\ &= \lim_{t\to 0} \frac{(R^1_x f)^{k-1}(x+t\,v)(v^{k-1})}{t} + f^k(x)(v^k) = f^k(x)(v^k).\end{aligned}$$

Finally, f^0 defines an element in $\varprojlim_K C^{[M]}_K(U,F)$, since $\forall K$ we have $f^0 \in C^M_{K,\rho}(U,F) = \{g \in C^\infty(U,F) : j^\infty g|_K \in C^M_\rho(E \supseteq K,F)\}$ for some (resp. all) $\rho > 0$.

That this bijection is an isomorphism follows, since the seminorm $\| \ \|_{K,\rho}$ on $C^M_{K,\rho}(U,F)$ is the pull-back of the seminorm $\| \ \|_\rho$ on $C^M_\rho(E \supseteq K, F)$. $\square$

4.2. **Spaces of Denjoy–Carleman differentiable mappings between convenient vector spaces.** For convenient vector spaces E and F, and c^∞-open $U \subseteq E$, we define:

$$\begin{aligned}
C^{(M)}_b(U,F) &:= \Big\{ f \in C^\infty(U,F) : \forall B\ \forall \text{ compact } K \subseteq U_B\ \forall \rho > 0 : \\
&\qquad \Big\{ \frac{f^{(k)}(x)(v_1,\dots,v_k)}{k!\,\rho^k\,M_k} : k \in \mathbb{N}, x \in K, \|v_i\|_B \le 1 \Big\} \text{ is bounded in } F \Big\} \\
&= \Big\{ f \in C^\infty(U,F) : \forall B\ \forall \text{ compact } K \subseteq U_B\ \forall \rho > 0 : \\
&\qquad \Big\{ \frac{d^k_v f(x)}{k!\,\rho^k\,M_k} : k \in \mathbb{N}, x \in K, \|v\|_B \le 1 \Big\} \text{ is bounded in } F \Big\}, \quad \text{and} \\
C^{\{M\}}_b(U,F) &:= \Big\{ f \in C^\infty(U,F) : \forall B\ \forall \text{ compact } K \subseteq U_B\ \exists \rho > 0 : \\
&\qquad \Big\{ \frac{f^{(k)}(x)(v_1,\dots,v_k)}{k!\,\rho^k\,M_k} : k \in \mathbb{N}, x \in K, \|v_i\|_B \le 1 \Big\} \text{ is bounded in } F \Big\} \\
&= \Big\{ f \in C^\infty(U,F) : \forall B\ \forall \text{ compact } K \subseteq U_B\ \exists \rho > 0 : \\
&\qquad \Big\{ \frac{d^k_v f(x)}{k!\,\rho^k\,M_k} : k \in \mathbb{N}, x \in K, \|v\|_B \le 1 \Big\} \text{ is bounded in } F \Big\}.
\end{aligned}$$

Here B runs through all closed absolutely convex bounded subsets in E, E_B is the vector space generated by B with the Minkowski functional $\|v\|_B = \inf\{\lambda \ge 0 : v \in \lambda B\}$ as complete norm, and $U_B = U \cap E_B$. For Banach spaces E and F obviously

$$C^{[M]}_b(U,F) = C^{[M]}(U,F).$$

Now we define the spaces of main interest in this paper:

$$\boxed{C^{[M]}(U,F) := \Big\{ f \in C^\infty(U,F) : \forall \ell \in F^*\ \forall B : \ell \circ f \circ i_B \in C^{[M]}(U_B,\mathbb{R}) \Big\},}$$

where B again runs through all closed absolutely convex bounded subsets in E, the mapping $i_B : E_B \to E$ denotes the inclusion of E_B in E, and $U_B = i_B^{-1}(U) = U \cap E_B$. It will follow from Lemmas 4.2 and 4.3 that for Banach spaces E and F this definition coincides with the one given earlier in Section 4.1.

We equip $C^{[M]}(U,F)$ with the initial locally convex structure induced by all linear mappings

$$C^{[M]}(U,F) \xrightarrow{C^{[M]}(i_B,\ell)} C^{[M]}(U_B,\mathbb{R}), \quad f \mapsto \ell \circ f \circ i_B.$$

Then $C^{[M]}(U,F)$ is a convenient vector space as c^∞-closed subspace in the product $\prod_{\ell,B} C^{[M]}(U_B,\mathbb{R})$, since smoothness can be tested by composing with the inclusions $E_B \to E$ and with the $\ell \in F^*$, see [23, 2.14.4 and 1.8]. This shows at the same time, that

$$C^{[M]}(U,F) = \Big\{ f \in F^U : \forall \ell \in F^*\ \forall B : \ell \circ f \circ i_B \in C^{[M]}(U_B,\mathbb{R}) \Big\}.$$

Lemma 4.2 ($C^{(M)} = C_b^{(M)}$). *Let E, F be convenient vector spaces, and let $U \subseteq E$ be c^∞-open. Then a mapping $f : U \to F$ is $C^{(M)}$ (i.e., is in $C^{(M)}(U,F)$) if and only if f is $C_b^{(M)}$.*

Proof. Let $f : U \to F$ be C^∞. We have the following equivalences, where B runs through all closed absolutely convex bounded subsets in E:

$$\begin{aligned}
&f \in C^{(M)}(U,F)\\
&\iff \forall \ell \in F^*\ \forall B\ \forall K \subseteq U_B \text{ compact } \ \forall \rho > 0 :\\
&\qquad \Big\{\frac{(\ell \circ f)^{(k)}(x)(v_1,\dots,v_k)}{\rho^k\, k!\, M_k} : x \in K, k \in \mathbb{N}, \|v_i\|_B \le 1\Big\} \text{ is bounded in } \mathbb{R}\\
&\iff \forall B\ \forall K \subseteq U_B \text{ compact } \ \forall \rho > 0\ \forall \ell \in F^* :\\
&\qquad \ell\Big(\Big\{\frac{f^{(k)}(x)(v_1,\dots,v_k)}{\rho^k\, k!\, M_k} : x \in K, k \in \mathbb{N}, \|v_i\|_B \le 1\Big\}\Big) \text{ is bounded in } \mathbb{R}\\
&\iff \forall B\ \forall K \subseteq U_B \text{ compact } \ \forall \rho > 0 :\\
&\qquad \Big\{\frac{f^{(k)}(x)(v_1,\dots,v_k)}{\rho^k\, k!\, M_k} : x \in K, k \in \mathbb{N}, \|v_i\|_B \le 1\Big\} \text{ is bounded in } F\\
&\iff f \in C_b^{(M)}(U,F) \quad \square
\end{aligned}$$

In the Roumieu case $C^{\{M\}}$ the corresponding equality holds only under additional assumptions:

Lemma 4.3 ($C^{\{M\}} = C_b^{\{M\}}$). *Let E, F be convenient vector spaces, and let $U \subseteq E$ be c^∞-open. Assume that there exists a Baire vector space topology on the dual F^* for which the point evaluations ev_x are continuous for all $x \in F$. Then a mapping $f : U \to F$ is $C^{\{M\}}$ if and only if f is $C_b^{\{M\}}$.*

Proof. ($\Rightarrow$) Let B be a closed absolutely convex bounded subset of E. Let K be compact in U_B. We consider the sets

$$A_{\rho,C} := \Big\{\ell \in F^* : \frac{|(\ell \circ f)^{(k)}(x)(v_1,\dots,v_k)|}{\rho^k\, k!\, M_k} \le C \text{ for all } x \in K, k \in \mathbb{N}, \|v_i\|_B \le 1\Big\}$$

which are closed subsets in F^* for the given Baire topology. We have $\bigcup_{\rho,C} A_{\rho,C} = F^*$. By the Baire property there exist ρ and C such that the interior $\mathrm{int}(A_{\rho,C})$ of $A_{\rho,C}$ is non-empty. If $\ell_0 \in \mathrm{int}(A_{\rho,C})$, then for each $\ell \in F^*$ there is a $\delta > 0$ such that $\delta\ell \in \mathrm{int}(A_{\rho,C}) - \ell_0$, and, hence, for all $k \in \mathbb{N}$, $x \in K$, and $\|v_i\|_B \le 1$, we have

$$\begin{aligned}
|(\ell \circ f)^{(k)}(x)(v_1,\dots)| &\le \tfrac{1}{\delta}\Big(|((\delta\,\ell + \ell_0) \circ f)^{(k)}(x)(v_1,\dots)| + |(\ell_0 \circ f)^{(k)}(x)(v_1,\dots)|\Big)\\
&\le \tfrac{2C}{\delta}\, \rho^k\, k!\, M_k.
\end{aligned}$$

So the set

$$\Big\{\frac{f^{(k)}(x)(v_1,\dots,v_k)}{\rho^k\, k!\, M_k} : x \in K, k \in \mathbb{N}, \|v_i\|_B \le 1\Big\}$$

is weakly bounded in F and hence bounded. Since B and K were arbitrary, we obtain $f \in C_b^{\{M\}}(U,F)$.

($\Leftarrow$) is obvious. $\square$

The following example shows that the additional assumption in Lemma 4.3 cannot be dropped.

Example 4.4. By [38, Thm. 1], for each weakly log-convex sequence $M = (M_k)$ there exists $f \in C^{\{M\}}(\mathbb{R},\mathbb{R})$ such that $|f^{(k)}(0)| \geq k!\, M_k$ for all $k \in \mathbb{N}$. *Then $g : \mathbb{R}^2 \to \mathbb{R}$ given by $g(s,t) = f(st)$ is $C^{\{M\}}$, whereas there is no reasonable topology on $C^{\{M\}}(\mathbb{R},\mathbb{R})$ such that the associated mapping $g^\vee : \mathbb{R} \to C^{\{M\}}(\mathbb{R},\mathbb{R})$ is $C_b^{\{M\}}$.* For a topology on $C^{\{M\}}(\mathbb{R},\mathbb{R})$ to be reasonable we require only that all evaluations $\mathrm{ev}_t : C^{\{M\}}(\mathbb{R},\mathbb{R}) \to \mathbb{R}$ are bounded linear functionals.

Proof. The mapping g is obviously $C^{\{M\}}$. If $g^\vee$ were $C_b^{\{M\}}$, for $s = 0$ there existed ρ such that

$$\Big\{\frac{(g^\vee)^{(k)}(0)}{k!\,\rho^k\, M_k} : k \in \mathbb{N}\Big\}$$

was bounded in $C^{\{M\}}(\mathbb{R},\mathbb{R})$. We apply the bounded linear functional ev_t for $t = 2\rho$ and then get

$$\frac{|(g^\vee)^{(k)}(0)(2\rho)|}{k!\,\rho^k\, M_k} = \frac{(2\rho)^k |f^{(k)}(0)|}{k!\,\rho^k\, M_k} \geq 2^k,$$

a contradiction. □

This example shows that for $C_b^{\{M\}}$ one cannot expect cartesian closedness. Using cartesian closedness, i.e., Theorem 5.2, and Lemma 5.1 this also shows (for $F = C^{\{M\}}(\mathbb{R},\mathbb{R})$ and $U = \mathbb{R} = E$) that

$$C^{\{M\}}(U,F) \supsetneq \bigcap_{B,V} C_b^{\{M\}}(U_B, F_V),$$

where F_V is the completion of $F/p_V^{-1}(0)$ with respect to the seminorm p_V induced by the absolutely convex closed 0-neighborhood V.

If we compose $g^\vee$ with the restriction mapping $(\mathrm{incl}_{\mathbb{N}})^* : C^{\{M\}}(\mathbb{R},\mathbb{R}) \to \mathbb{R}^{\mathbb{N}} := \prod_{t\in\mathbb{N}} \mathbb{R}$, then we get a $C^{\{M\}}$-curve, since the continuous linear functionals on $\mathbb{R}^{\mathbb{N}}$ are linear combinations of coordinate projections ev_t with $t \in \mathbb{N}$. However, this curve cannot be $C_b^{\{M\}}$ as the argument above for $t > \rho$ shows.

In the following Lemmas 4.5 and 4.6 we find projective descriptions for $C^{(M)}(U,F)$ and $C^{\{M\}}(U,F)$, if E, F are Banach spaces, and $U \subseteq E$ is open. This is of vital importance for the development of the convenient setting of $C^{\{M\}}(U,F)$. The spaces $C^{(M)}(U,F)$, however, already are projective by definition, and thus Lemma 4.5 just gives a further projective description; see also Theorem 8.5. We include and use Lemma 4.5 in order to treat the Beurling and Roumieu case in a uniform and efficient way.

Lemma 4.5. *Let E, F be Banach spaces, $U \subseteq E$ open, and $f : U \to F$ a C^∞-mapping. The following are equivalent:*

(1) *f is $C^{(M)} = C_b^{(M)}$.*
(2) *For each sequence (r_k) with $r_k\, \rho^k \to 0$ for some $\rho > 0$ and each compact $K \subseteq U$, the set*

$$\Big\{\frac{f^{(k)}(a)(v_1,\dots,v_k)}{k! M_k}\, r_k : a \in K, k \in \mathbb{N}, \|v_i\| \leq 1\Big\}$$

is bounded in F.

(3) *For each sequence (r_k) satisfying $r_k > 0$, $r_k r_\ell \ge r_{k+\ell}$, and $r_k\,\rho^k \to 0$ for some $\rho > 0$, each compact $K \subseteq U$, and each $\delta > 0$, the set*
$$\Big\{\frac{f^{(k)}(a)(v_1,\dots,v_k)}{k! M_k}\, r_k\,\delta^k : a \in K, k \in \mathbb{N}, \|v_i\| \le 1\Big\}$$
is bounded in F.

Proof. (1) $\Rightarrow$ (2) For (r_k) and K, and $\rho > 0$ such that $r_k\,\rho^k \to 0$,
$$\Big\|\frac{f^{(k)}(a)}{k!\,M_k} r_k\Big\|_{L^k(E,F)} = \Big\|\frac{f^{(k)}(a)}{k!\,\rho^k\,M_k}\Big\|_{L^k(E,F)} \cdot |r_k \rho^k|$$
is bounded uniformly in $k \in \mathbb{N}$ and $a \in K$ (by Lemma 4.2).

(2) $\Rightarrow$ (3) Apply (2) to the sequence $(r_k\delta^k)$.

(3) $\Rightarrow$ (1) Let $a_k := \sup_{a\in K} \|\frac{f^{(k)}(a)}{k!\,M_k}\|_{L^k(E,F)}$. By the following lemma, the a_k are the coefficients of a power series with infinite radius of convergence. Thus a_k/ρ^k is bounded for every $\rho > 0$. □

Lemma. *For a formal power series $\sum_{k\ge 0} a_k t^k$ with real coefficients the following are equivalent:*

(4) *The radius of convergence is infinite.*

(5) *For each sequence (r_k) satisfying $r_k > 0$, $r_k r_\ell \ge r_{k+\ell}$, and $r_k\,\rho^k \to 0$ for some $\rho > 0$, and each $\delta > 0$, the sequence $(a_k r_k \delta^k)$ is bounded.*

Proof. (4) $\Rightarrow$ (5) The series $\sum a_k r_k \delta^k = \sum (a_k (\frac{\delta}{\rho})^k) r_k \rho^k$ converges absolutely for each δ. Hence $(a_k r_k \delta^k)$ is bounded.

(5) $\Rightarrow$ (4) Suppose that the radius of convergence ρ is finite. So $\sum_k |a_k| n^k = \infty$ for $n > \rho$. Set $r_k = 1/n^k$. Then, by (5),
$$a_k n^k 2^k = a_k r_k n^{2k} 2^k = a_k r_k (2n^2)^k < C,$$
for some $C > 0$ and all k. Consequently, $\sum_k |a_k| n^k \le C \sum_k \frac{1}{2^k}$, a contradiction. □

Lemma 4.6. *Let E, F be Banach spaces, $U \subseteq E$ open, and $f : U \to F$ a C^∞-mapping. The following are equivalent:*

(1) *f is $C^{\{M\}} = C_b^{\{M\}}$.*

(2) *For each sequence (r_k) with $r_k\,\rho^k \to 0$ for all $\rho > 0$, and each compact $K \subseteq U$, the set*
$$\Big\{\frac{f^{(k)}(a)(v_1,\dots,v_k)}{k! M_k}\, r_k : a \in K, k \in \mathbb{N}, \|v_i\| \le 1\Big\}$$
is bounded in F.

(3) *For each sequence (r_k) satisfying $r_k > 0$, $r_k r_\ell \ge r_{k+\ell}$, and $r_k\,\rho^k \to 0$ for all $\rho > 0$, and each compact $K \subseteq U$, there exists $\delta > 0$ such that*
$$\Big\{\frac{f^{(k)}(a)(v_1,\dots,v_k)}{k! M_k}\, r_k\,\delta^k : a \in K, k \in \mathbb{N}, \|v_i\| \le 1\Big\}$$
is bounded in F.

Proof. (1) $\Rightarrow$ (2) For K, there exists $\rho > 0$ such that
$$\Big\|\frac{f^{(k)}(a)}{k!\,M_k} r_k\Big\|_{L^k(E,F)} = \Big\|\frac{f^{(k)}(a)}{k!\,\rho^k\,M_k}\Big\|_{L^k(E,F)} \cdot |r_k \rho^k|$$

is bounded uniformly in $k \in \mathbb{N}$ and $a \in K$ (by Lemma 4.3).

(2) $\Rightarrow$ (3) Use $\delta = 1$.

(3) $\Rightarrow$ (1) Let $a_k := \sup_{a\in K} \|\frac{f^{(k)}(a)}{k!\,M_k}\|_{L^k(E,F)}$. Using [23, 9.2(4⇒1)] these are the coefficients of a power series with positive radius of convergence. Thus a_k/ρ^k is bounded for some $\rho > 0$. □

Definition 4.7 (Banach plots)**.** Let E be a convenient vector space. A $C^{[M]}$ *(Banach) plot in* E is a mapping $c : D \to E$ of class $C^{[M]}$, where D is an open set in some Banach space F. It suffices to only consider the open unit ball $D = oF$.

Theorem 4.8. *Let $M = (M_k)$ be log-convex. Let $U \subseteq E$ be c^∞-open in a convenient vector space E, let F be a Banach space, and let $f : U \to F$ be a mapping. Then:*

$$f \in C^{[M]}(U, F) \implies f \circ c \in C^{[M]}, \text{ for all } C^{[M]}\text{-plots } c.$$

Note that the converse ($\Leftarrow$) holds by Section 4.2.

Proof. We treat first the Beurling case $C^{(M)}$: We have to show that $f \circ c$ is $C^{(M)}$ for each $C^{(M)}$-plot $c : G \supseteq D \to E$. By Lemma 4.5(3), it suffices to show that, for each sequence (r_k) satisfying $r_k > 0$, $r_k r_\ell \ge r_{k+\ell}$, and $r_k\, t^k \to 0$ for some $t > 0$, each compact $K \subseteq D$, and each $\delta > 0$, the set

$$\Big\{\frac{(f\circ c)^{(k)}(a)(v_1,\dots,v_k)}{k!M_k}\, r_k\, \delta^k : a \in K, k \in \mathbb{N}, \|v_i\|_G \le 1\Big\} \tag{13}$$

is bounded in F.

So let δ, the sequence (r_k), and a compact (and without loss of generality convex) subset $K \subseteq D$ be fixed. For each $\ell \in E^*$ the set

$$\Big\{\frac{(\ell\circ c)^{(k)}(a)(v_1,\dots,v_k)}{k!M_k}\, r_k\, (2\delta)^k : a \in K, k \in \mathbb{N}, \|v_i\|_G \le 1\Big\} \tag{14}$$

is bounded in $\mathbb{R}$, by Lemma 4.5(2) applied to the sequence $(r_k(2\delta)^k)$. Thus, the set

$$\Big\{\frac{c^{(k)}(a)(v_1,\dots,v_k)}{k!M_k}\, r_k\, (2\delta)^k : a \in K, k \in \mathbb{N}, \|v_i\|_G \le 1\Big\} \tag{15}$$

is contained in some closed absolutely convex bounded subset B of E and hence

$$\frac{\|c^{(k)}(a)\|_{L^k(G,E_B)}\, r_k\, \delta^k}{k!M_k} \le \frac{1}{2^k}.$$

Furthermore $c : K \to E_B$ is Lipschitzian, since

$$c(x) - c(y) = \int_0^1 c'(y + t(x-y))\,(x-y)\,dt \in \frac{M_1\,\|x-y\|_G}{2\,r_1\,\delta}\,B,$$

and hence $c(K)$ is compact in E_B. By Faà di Bruno's formula for Banach spaces (see [8] for the 1-dimensional version), for $k \ge 1$,

$$\frac{(f\circ c)^{(k)}(a)}{k!} = \operatorname{sym}\Big(\sum_{j\ge 1}\ \sum_{\substack{\alpha\in\mathbb{N}_{>0}^j\\ \alpha_1+\dots+\alpha_j=k}} \frac{1}{j!} f^{(j)}(c(a))\Big(\frac{c^{(\alpha_1)}(a)}{\alpha_1!},\dots,\frac{c^{(\alpha_j)}(a)}{\alpha_j!}\Big)\Big),$$

where sym denotes symmetrization. Using (4) and $f \in C^{(M)}(U,F)$, we find that for each $\rho > 0$ there is $C > 0$ so that, for all $a \in K$ and $k \in \mathbb{N}_{>0}$,

$$\begin{aligned}
&\Big\| \frac{(f\circ c)^{(k)}(a)}{k! M_k}\, r_k\, \delta^k \Big\|_{L^k(G,F)} \\
&\qquad \le \sum_{j\ge 1} M_1^j \sum_{\substack{\alpha\in\mathbb{N}_{>0}^j\\ \alpha_1+\dots+\alpha_j=k}} \underbrace{\frac{\|f^{(j)}(c(a))\|_{L^j(E_B,F)}}{j! M_j}}_{\le C\,\rho^j} \prod_{i=1}^j \underbrace{\frac{\|c^{(\alpha_i)}(a)\|_{L^{\alpha_i}(G,E_B)}\, r_{\alpha_i}\, \delta^{\alpha_i}}{\alpha_i! M_{\alpha_i}}}_{\le 1/2^{\alpha_i}} \\
&\qquad \le \sum_{j\ge 1} M_1^j \binom{k-1}{j-1} C\,\rho^j\, \frac{1}{2^k} = M_1\rho(1+M_1\,\rho)^{k-1} C\, \frac{1}{2^k} \le \frac{C}{2},
\end{aligned} \tag{16}$$

as required, where in the last inequality we set $\rho := 1/M_1$.

Let us now consider the Roumieu case $C^{\{M\}}$: Let now $c : G \supseteq D \to E$ be a $C^{\{M\}}$-plot. We have to show that $f \circ c$ is $C^{\{M\}}$. By Lemma 4.6(3), it suffices to show that for each sequence (r_k) satisfying $r_k > 0$, $r_k r_\ell \ge r_{k+\ell}$, and $r_k\, t^k \to 0$ for all $t > 0$, and each compact $K \subseteq D$, there exists $\delta > 0$ such that the set (13) is bounded in F.

By Lemma 4.6(2) (applied to $(r_k 2^k)$ instead of (r_k)), for each $\ell \in E^*$, each sequence (r_k) with $r_k\, t^k \to 0$ for all $t > 0$, and each compact $K \subseteq D$, the set (14) with $\delta = 1$ is bounded in $\mathbb{R}$, and, thus, the set (15) with $\delta = 1$ is contained in some closed absolutely convex bounded subset B of E. Using that $f \in C^{\{M\}}(U,F)$ and computing as above we find that, for some $\rho > 0$ and $C > 0$ and $\delta := \frac{2}{1+M_1\,\rho}$, the left-hand side of (16) is bounded by $\frac{C\, M_1\, \rho}{1+M_1\,\rho}$. □

Theorem 4.9 ($C^{[M]}$ is a category)**.** *Let $M = (M_k)$ be log-convex. Let E, F, G be convenient vector spaces, $U \subseteq E$, $V \subseteq F$ be c^∞-open, and $f : U \to F$, $g : V \to G$, and $f(U) \subseteq V$. Then:*

$$f, g \in C^{[M]} \implies g \circ f \in C^{[M]}.$$

Proof. By Section 4.2, we must show that for all closed absolutely convex bounded $B \subseteq E$ and for all $\ell \in G^*$ the composite $\ell \circ g \circ f \circ i_B : U_B \to \mathbb{R}$ belongs to $C^{[M]}$.

$$\begin{tikzcd}
U \arrow[r, "f"] & V \arrow[r, "g"] \arrow[dr, "\ell\circ g"'] & G \arrow[d, "\ell"] \\
U_B \arrow[u, "i_B"] \arrow[ur, "f\circ i_B"'] & & \mathbb{R}
\end{tikzcd}$$

By assumption, $f \circ i_B$ and $\ell \circ g$ are $C^{[M]}$. So the assertion follows from Theorem 4.8. □

5. The exponential law

Lemma 5.1. *Let E be a Banach space, and $U \subseteq E$ be open. Let F be a convenient vector space, and let $\mathcal{S}$ be a family of bounded linear functionals on F which together detect bounded sets (i.e., $B \subseteq F$ is bounded if and only if $\ell(B)$ is bounded for all $\ell \in \mathcal{S}$). Then:*

$$f \in C^{[M]}(U,F) \iff \ell \circ f \in C^{[M]}(U,\mathbb{R}), \text{ for all } \ell \in \mathcal{S}.$$

Proof. For C^∞-curves this follows from [23, 2.1 and 2.11], and, by composing with such, it follows for C^∞-mappings $f : U \to F$.

In the Beurling case $C^{(M)}$: By Lemma 4.5, for $\ell \in F^*$, the function $\ell \circ f$ is $C^{(M)}$ if and only if the set

$$\Big\{\frac{(\ell\circ f)^{(k)}(a)(v_1,\dots,v_k)}{k!M_k}\, r_k : a \in K, k \in \mathbb{N}, \|v_i\| \le 1\Big\} \tag{17}$$

is bounded, for each sequence (r_k) with $r_k\,\rho^k \to 0$ for some $\rho > 0$ and each compact $K \subseteq U$. So the smooth mapping $f : U \to F$ is $C^{(M)}$ if and only if the set

$$\Big\{\frac{f^{(k)}(a)(v_1,\dots,v_k)}{k!M_k}\, r_k : a \in K, k \in \mathbb{N}, \|v_i\| \le 1\Big\} \tag{18}$$

is bounded in F, for each such (r_k) and K. This is in turn equivalent to $\ell \circ f \in C^{(M)}$ for all $\ell \in \mathcal{S}$, since $\mathcal{S}$ detects bounded sets.

The same proof works in the Roumieu case $C^{\{M\}}$ if we use Lemma 4.6 and demand that $r_k\,\rho^k \to 0$ for all $\rho > 0$. □

Theorem 5.2 (Cartesian closedness)**.** *We have:*

1. *Let $M = (M_k)$ be weakly log-convex and have moderate growth. Then, for convenient vector spaces E_1, E_2, and F and c^∞-open sets $U_1 \subseteq E_1$ and $U_2 \subseteq E_2$, we have the exponential law:*
$$f \in C^{[M]}(U_1 \times U_2, F) \iff f^\vee \in C^{[M]}(U_1, C^{[M]}(U_2, F)).$$
The direction $(\Leftarrow)$ holds without the assumption that $M = (M_k)$ has moderate growth. The direction $(\Rightarrow)$ holds without the assumption that $M = (M_k)$ is weakly log-convex.
2. *Let $M = (M_k)$ be log-convex and have moderate growth. Then the category of $C^{[M]}$-mappings between convenient real vector spaces is cartesian closed, i.e., satisfies the exponential law.*

Note that $C^{[M]}$ is not necessarily a category if $M = (M_k)$ is just weakly log-convex.

Proof. (2) is a direct consequence of (1) and Theorem 4.9. Let us prove (1). We have $C^\infty(U_1 \times U_2, F) \cong C^\infty(U_1, C^\infty(U_2, F))$, by [23, 3.12]; thus, in the following all mappings are assumed to be smooth. We have the following equivalences, where $B \subseteq E_1 \times E_2$ and $B_i \subseteq E_i$ run through all closed absolutely convex bounded subsets, respectively:

$$\begin{aligned} &f \in C^{[M]}(U_1 \times U_2, F)\\ &\iff \forall \ell \in F^*\ \forall B : \ell \circ f \circ i_B \in C^{[M]}((U_1 \times U_2)_B, \mathbb{R})\\ &\iff \forall \ell \in F^*\ \forall B_1, B_2 : \ell \circ f \circ (i_{B_1} \times i_{B_2}) \in C^{[M]}((U_1)_{B_1} \times (U_2)_{B_2}, \mathbb{R}) \end{aligned}$$

For the second equivalence we use that every bounded $B \subseteq E_1 \times E_2$ is contained in $B_1 \times B_2$ for some bounded $B_i \subseteq E_i$, and, thus, the inclusion $(E_1 \times E_2)_B \to (E_1)_{B_1} \times (E_1)_{B_2}$ is bounded.

On the other hand, we have:

$$\begin{aligned} &f^\vee \in C^{[M]}(U_1, C^{[M]}(U_2, F))\\ &\iff \forall B_1 : f^\vee \circ i_{B_1} \in C^{[M]}((U_1)_{B_1}, C^{[M]}(U_2, F))\\ &\iff \forall \ell \in F^*\ \forall B_1, B_2 : C^{[M]}(i_{B_2}, \ell) \circ f^\vee \circ i_{B_1} \in C^{[M]}((U_1)_{B_1}, C^{[M]}((U_2)_{B_2}, \mathbb{R})) \end{aligned}$$

For the second equivalence we use Lemma 5.1 and the fact that the linear mappings $C^{[M]}(i_{B_2},\ell)$ generate the bornology.

These considerations imply that in order to prove cartesian closedness we may restrict to the case that $U_i \subseteq E_i$ are open in Banach spaces E_i and $F=\mathbb{R}$.

(**Direction** $\Rightarrow$) We assume that $M=(M_k)$ has moderate growth. Let $f \in C^{[M]}(U_1\times U_2,\mathbb{R})$. It is clear that $f^\vee$ takes values in $C^{[M]}(U_2,\mathbb{R})$.

Claim. *$f^\vee : U_1 \to C^{[M]}(U_2,\mathbb{R})$ is C^∞ with $d^j f^\vee = (\partial_1^j f)^\vee$.*
Since $C^{[M]}(U_2,\mathbb{R})$ is a convenient vector space, by [23, 5.20] it is enough to show that the iterated unidirectional derivatives $d_v^j f^\vee(x)$ exist, equal $\partial_1^j f(x,\quad)(v^j)$, and are separately bounded for x, resp. v, in compact subsets. For $j=1$ and fixed x, v, and y consider the smooth curve $c: t\mapsto f(x+tv,y)$. By the fundamental theorem

$$\begin{aligned}\frac{f^\vee(x+tv)-f^\vee(x)}{t}(y) - (\partial_1 f)^\vee(x)(y)(v) &= \frac{c(t)-c(0)}{t} - c'(0)\\ &= t\int_0^1 s\int_0^1 c''(tsr)\,dr\,ds\\ &= t\int_0^1 s\int_0^1 \partial_1^2 f(x+tsrv,y)(v,v)\,dr\,ds.\end{aligned}$$

Since $(\partial_1^2 f)^\vee(K_1)(oE_1^2)$ is obviously bounded in $C^{[M]}(U_2,\mathbb{R})$ for each compact subset $K_1\subseteq U_1$, this expression is Mackey convergent to 0 in $C^{[M]}(U_2,\mathbb{R})$ as $t\to 0$. Thus $d_v f^\vee(x)$ exists and equals $\partial_1 f(x,\quad)(v)$.

Now we proceed by induction, applying the same arguments as before to $(d_v^j f^\vee)^\wedge : (x,y)\mapsto \partial_1^j f(x,y)(v^j)$ instead of f. Again $(\partial_1^2(d_v^j f^\vee)^\wedge)^\vee(K_1)(oE_1^2) = (\partial_1^{j+2} f)^\vee(K_1)(oE_1,oE_1,v,\dots,v)$ is bounded, and also the separated boundedness of $d_v^j f^\vee(x)$ follows. So the claim is proved.

We have to show that $f^\vee : U_1 \to C^{[M]}(U_2,\mathbb{R})$ is $C^{[M]}$.
In the Beurling case $C^{(M)}$:

$$\begin{tikzcd}
U_1 \arrow[r, "f^\vee"] & C^{(M)}(U_2,\mathbb{R}) \arrow[r, equal] & \varprojlim_{K_2}\varprojlim_{\rho_2} C^M_{\rho_2}(E_2\supseteq K_2,\mathbb{R}) \arrow[r, "\ell"] \arrow[d] & \mathbb{R} \\
 & & \varprojlim_{\rho_2} C^M_{\rho_2}(E_2\supseteq K_2,\mathbb{R}) \arrow[d] & \\
K_1 \arrow[uu, hook] \arrow[r, dotted] & C^M_{K_2,\rho_2}(U_2,\mathbb{R}) \arrow[r] & C^M_{\rho_2}(E_2\supseteq K_2,\mathbb{R}) \arrow[uur, dotted, bend right] &
\end{tikzcd} \tag{19}$$

By Lemma 5.1, it suffices to show that $f^\vee : U_1 \to C^M_{\rho_2}(E_2\supseteq K_2,\mathbb{R})$ is $C_b^{(M)} = C^{(M)}$ (see Lemma 4.2) for each compact $K_2\subseteq U_2$ and each $\rho_2>0$, since every $\ell\in C^{(M)}(U_2,\mathbb{R})^*$ factors over some $C^M_{\rho_2}(E_2\supseteq K_2,\mathbb{R})$. Thus it suffices to prove that, for all compact $K_1\subseteq U_1$, $K_2\subseteq U_2$ and all $\rho_1,\rho_2>0$, the set

$$\Big\{\frac{d^{k_1} f^\vee(x_1)(v_1^1,\dots,v_{k_1}^1)}{k_1!\,\rho_1^{k_1}\,M_{k_1}} : x_1\in K_1, k_1\in\mathbb{N}, \|v_j^1\|_{E_1}\le 1\Big\} \tag{20}$$

is bounded in $C^M_{K_2,\rho_2}(U_2,\mathbb{R})$, or, equivalently, for all K_1,K_2,ρ_1,ρ_2 the set

$$\Big\{\frac{\partial_2^{k_2}\partial_1^{k_1} f(x_1,x_2)(v_1^1,\dots,v_{k_1}^1;v_1^2,\dots,v_{k_2}^2)}{k_2!\,k_1!\,\rho_2^{k_2}\,\rho_1^{k_1}\,M_{k_2}\,M_{k_1}} : x_i\in K_i, k_i\in\mathbb{N}, \|v_j^i\|_{E_i}\le 1\Big\} \tag{21}$$

is bounded in $\mathbb{R}$.

Since $M=(M_k)$ has moderate growth, i.e., $M_{k_1+k_2}\le \sigma^{k_1+k_2}M_{k_1}M_{k_2}$ for some $\sigma>0$, we obtain, for $x_1\in K_1$, $k_1\in\mathbb{N}$, and $\|v_j^1\|_{E_1}\le 1$,

$$\begin{aligned}
&\Big\|\frac{d^{k_1}f^\vee(x_1)(v_1^1,\dots,v_{k_1}^1)}{k_1!\,\rho_1^{k_1}\,M_{k_1}}\Big\|_{K_2,\rho_2}\\
&= \sup\Big\{\frac{|\partial_2^{k_2}\partial_1^{k_1} f(x_1,x_2)(v_1^1,\dots,v_{k_1}^1;v_1^2,\dots,v_{k_2}^2)|}{k_2!\,k_1!\,\rho_2^{k_2}\,\rho_1^{k_1}\,M_{k_2}\,M_{k_1}} : x_2\in K_2, k_2\in\mathbb{N}, \|v_j^2\|_{E_2}\le 1\Big\}\\
&\le \sup\Big\{(2\sigma)^{k_1+k_2}\frac{|\partial_2^{k_2}\partial_1^{k_1} f(x_1,x_2)(v_1^1,\dots;v_1^2,\dots)|}{(k_1+k_2)!\,\rho_1^{k_1}\,\rho_2^{k_2}\,M_{k_1+k_2}} : x_2\in K_2, k_2\in\mathbb{N}, \|v_j^2\|_{E_2}\le 1\Big\}.
\end{aligned} \tag{22}$$

If for given $\rho_1,\rho_2>0$ we set $\rho:=\frac{1}{2\sigma}\min\{\rho_1,\rho_2\}$, then (22) is bounded by

$$\sup\Big\{\frac{|\partial_2^{k_2}\partial_1^{k_1} f(x_1,x_2)(v_1^1,\dots;v_1^2,\dots)|}{(k_1+k_2)!\,\rho^{k_1+k_2}\,M_{k_1+k_2}} : x_i\in K_i, k_i\in\mathbb{N}, \|v_j^i\|_{E_i}\le 1\Big\} \tag{23}$$

which is finite, since f is $C^{(M)}$. Thus, $f^\vee$ is $C^{(M)}$.

In the Roumieu case $C^{\{M\}}$:

$$\begin{tikzcd}
U_1 \arrow[r, "f^\vee"] & C^{\{M\}}(U_2,\mathbb{R}) \arrow[r, equal] & \varprojlim_{K_2}\varinjlim_{\rho_2} C^M_{\rho_2}(E_2\supseteq K_2,\mathbb{R}) \arrow[r, "\ell"] \arrow[d] & \mathbb{R} \\
 & & \varinjlim_{\rho_2} C^M_{\rho_2}(E_2\supseteq K_2,\mathbb{R}) \arrow[ur, dotted] & \\
K_1 \arrow[uu, hook] \arrow[r, dotted] & C^M_{K_2,\rho_2}(U_2,\mathbb{R}) \arrow[r] & C^M_{\rho_2}(E_2\supseteq K_2,\mathbb{R}) \arrow[u] &
\end{tikzcd} \tag{24}$$

By Lemma 5.1, it suffices to show that $f^\vee : U_1\to \varinjlim_{\rho_2} C^M_{\rho_2}(E_2\supseteq K_2,\mathbb{R})$ is $C_b^{\{M\}}\subseteq C^{\{M\}}$ for each compact $K_2\subseteq U_2$, since every $\ell\in C^{\{M\}}(U_2,\mathbb{R})^*$ factors over some $\varinjlim_{\rho_2} C^M_{\rho_2}(E_2\supseteq K_2,\mathbb{R})$. Thus it suffices to prove that, for all compact $K_1\subseteq U_1$ and $K_2\subseteq U_2$ there exists $\rho_1>0$, such that the set (20) is bounded in $\varinjlim_{\rho_2} C^M_{\rho_2}(E_2\supseteq K_2,\mathbb{R})$. For that it suffices to show that for all K_1,K_2 there are ρ_1,ρ_2 so that the set (21) is bounded in $\mathbb{R}$.

Since f is $C^{\{M\}}$, there exists $\rho>0$ so that (23) is finite, by Proposition 4.1(3). If we set $\rho_i:=2\sigma\rho$, then (22) is bounded by (23). It follows that $f^\vee$ is $C^{\{M\}}$.

(**Direction** $\Leftarrow$) Let $f^\vee : U_1\to C^{[M]}(U_2,\mathbb{R})$ be $C^{[M]}$. Clearly, $f^\vee : U_1\to C^{[M]}(U_2,\mathbb{R})\to C^\infty(U_2,\mathbb{R})$ is C^∞, see Proposition 8.1, and so it remains to show that $f\in C^{[M]}(U_1\times U_2,\mathbb{R})$.

In the Beurling case $C^{(M)}$: Consider diagram (19). For each compact $K_2\subseteq U_2$ and each $\rho_2>0$, the mapping $f^\vee : U_1\to C^M_{\rho_2}(E_2\supseteq K_2,\mathbb{R})$ is $C^{(M)}=C_b^{(M)}$. That means that, for all compact $K_1 \subseteq U_1$, $K_2 \subseteq U_2$ and all $\rho_1, \rho_2 > 0$, the set (20) is bounded in $C^M_{\rho_2}(E_2 \supseteq K_2, \mathbb{R})$. Since it is contained in $C^M_{K_2,\rho_2}(U_2, \mathbb{R}) = \{g \in C^\infty(U_2, \mathbb{R}) : j^\infty g|_{K_2} \in C^M_{\rho_2}(E_2 \supseteq K_2, \mathbb{R})\}$ and $\|g\|_{K_2,\rho_2} = \|j^\infty g|_{K_2}\|_{\rho_2}$, it is also bounded in this space, and hence the set (21) is bounded.

Since $M = (M_k)$ is weakly log-convex, thus, $k_1!\, k_2!\, M_{k_1} M_{k_2} \le (k_1+k_2)!\, M_{k_1+k_2}$, we have, for $x_1 \in K_1$, $k_1 \in \mathbb{N}$, and $\|v^1_j\|_{E_1} \le 1$,

(25)

$$\begin{aligned}
&\Big\|\frac{d^{k_1} f^\vee(x_1)(v^1_1, \dots, v^1_{k_1})}{k_1!\, \rho_1^{k_1}\, M_{k_1}}\Big\|_{K_2,\rho_2} \\
&= \sup\Big\{\frac{|\partial_2^{k_2}\partial_1^{k_1} f(x_1,x_2)(v^1_1,\dots,v^1_{k_1}; v^2_1,\dots,v^2_{k_2})|}{k_2!\, k_1!\, \rho_2^{k_2}\, \rho_1^{k_1}\, M_{k_2}\, M_{k_1}} : x_2 \in K_2, k_2 \in \mathbb{N}, \|v^2_j\|_{E_2} \le 1\Big\} \\
&\ge \sup\Big\{\frac{|\partial_2^{k_2}\partial_1^{k_1} f(x_1,x_2)(v^1_1,\dots; v^2_1,\dots)|}{(k_1+k_2)!\, \rho_1^{k_1}\, \rho_2^{k_2}\, M_{k_1+k_2}} : x_2 \in K_2, k_2 \in \mathbb{N}, \|v^2_j\|_{E_2} \le 1\Big\}.
\end{aligned}$$

This implies that f is $C^{(M)}$.

In the Roumieu case $C^{\{M\}}$: Consider diagram (24). For each compact $K_2 \subseteq U_2$, the mapping $f^\vee : U_1 \to \varinjlim_{\rho_2} C^M_{\rho_2}(E_2 \supseteq K_2, \mathbb{R})$ is $C^{\{M\}}$. The inductive limit is regular, by Proposition 4.1(3). So the dual space $(\varinjlim_{\rho_2} C^M_{\rho_2}(E_2 \supseteq K_2, \mathbb{R}))^*$ can be equipped with the Baire topology of the countable limit $\varprojlim_{\rho_2} C^M_{\rho_2}(E_2 \supseteq K_2, \mathbb{R})^*$ of Banach spaces. Thus, the mapping $f^\vee : U_1 \to \varinjlim_{\rho_2} C^M_{\rho_2}(E_2 \supseteq K_2, \mathbb{R})$ is $C^{\{M\}}_b$, by Lemma 4.3. By regularity, for each compact $K_1 \subseteq U_1$ there exists $\rho_1 > 0$ so that the set (20) is contained and bounded in $C^M_{\rho_2}(E_2 \supseteq K_2, \mathbb{R})$ for some $\rho_2 > 0$. Since this set is contained in $C^M_{K_2,\rho_2}(U_2, \mathbb{R}) = \{g \in C^\infty(U_2, \mathbb{R}) : j^\infty g|_{K_2} \in C^M_{\rho_2}(E_2 \supseteq K_2, \mathbb{R})\}$ and $\|g\|_{K_2,\rho_2} = \|j^\infty g|_{K_2}\|_{\rho_2}$, it is also bounded in this space, and hence the set (21) is bounded. Then (25) implies that f is $C^{\{M\}}$. The proof is complete. $\square$

Remarks 5.3. Theorem 8.2 below states that, if $M = (M_k)$ is (weakly) log-convex, E, F are convenient vector spaces, and $U \subseteq E$ is c^∞-open, then

$$C^{\{M\}}(U, F) = \varprojlim_{L} C^{(L)}(U, F) \tag{26}$$

as vector spaces with bornology, where the projective limits are taken over all (weakly) log-convex $L = (L_k)$ with $M \lhd L$. Using this equality we can give an alternative proof of the direction

$$f^\vee \in C^{\{M\}}(U_1, C^{\{M\}}(U_2, F)) \Rightarrow f \in C^{\{M\}}(U_1 \times U_2, F)$$

in Theorem 5.2 as follows: If $f^\vee \in C^{\{M\}}(U_1, C^{\{M\}}(U_2, F))$ then $f^\vee \in C^{(L)}(U_1, C^{(L)}(U_2, F))$ for all $L = (L_k)$ with $M \lhd L$, by (26). By cartesian closedness of $C^{(L)}$ (i.e., Theorem 5.2, the implication which holds without moderate growth), we have $f \in C^{(L)}(U_1 \times U_2, F)$ for all L, and, by (26) again, $f \in C^{\{M\}}(U_1 \times U_2, F)$.

The proof of (26) in Theorem 8.2 uses the $C^{\{M\}}$ uniform boundedness principle, i.e., Theorem 6.1, and the proof of the latter uses completeness of the inductive limit $\varinjlim_\rho C^M_\rho(E \supseteq K, F)$, where E, F are Banach spaces and $K \subseteq E$ is compact, see Proposition 4.1. Here is a direct proof of (26), where we only assume that $M = (M_k)$ is positive:

The spaces coincide as vector spaces by Sections 4.1, 4.2, and by Theorem 2.2.

For K compact in a Banach space E and $\rho > 0$, the inclusion $C^M_\rho(E \supseteq K, \mathbb{R}) \to C^L_\sigma(E \supseteq K, \mathbb{R})$ is continuous for all $\sigma > 0$ if $M \lhd L$. It follows that the inclusion $\varinjlim_\rho C^M_\rho(E \supseteq K, \mathbb{R}) \to \varprojlim_\sigma C^L_\sigma(E \supseteq K, \mathbb{R})$ is continuous. This implies that the inclusion $C^{\{M\}}(U, F) \to C^{(L)}(U, F)$ is continuous (by definition of the structure in Section 4.2).

Conversely, let $\mathcal{B}$ be a bounded set in $\varprojlim_L C^{(L)}(U, F)$, i.e., bounded in each $C^{(L)}(U, F)$. We claim that $\mathcal{B}$ is bounded in $C^{\{M\}}(U, F)$. We may assume without loss of generality that E is a Banach space and $F = \mathbb{R}$ (by composing with $C^{\{M\}}(i_B, \ell)$). Let $K \subseteq U$ be compact and $b_k := \sup\{\|j^\infty f|_K\|_k : f \in \mathcal{B}\}$. For all $L = (L_k)$ with $M \lhd L$ the set $\mathcal{B}$ is bounded in $C^{(L)}(U, F)$ by assumption, i.e., $(b_k)_k \in \bigcap_L \mathcal{F}^{(L)} = \mathcal{F}^{\{M\}}$ by Theorem 2.2. From this follows that $\mathcal{B}$ is bounded in $C^{\{M\}}_K(U, F)$ and by Proposition 4.1(5) also in $C^{\{M\}}(U, F)$.

Note that this independently proves that $C^{\{M\}}(U, F)$ is c^∞-complete since so is $\varprojlim_L C^{(L)}(U, F)$. Moreover, it provides an independent proof of the regularity of the inductive limit involved in the definition of $C^{\{M\}}(U, F)$ if E and F are Banach spaces (cf. Proposition 4.1 and the remark after Theorem 8.6).

Example 5.4 (Cartesian closedness fails without moderate growth)**.** *Let us assume that $M = (M_k)$ is weakly log-convex and has non-moderate growth (for instance, $M_k = q^{k^2}$, $q > 1$, see* [38, 2.1.3]*). Then:*

1. *There exists $f \in C^{\{M\}}(\mathbb{R}^2, \mathbb{C})$ such that $f^\vee : \mathbb{R} \to C^{\{M\}}(\mathbb{R}, \mathbb{C})$ is not $C^{\{M\}}$.*
2. *There exists a weakly log-convex $N = (N_k)$ with $M \lhd N$ and an $f \in C^{(N)}(\mathbb{R}^2, \mathbb{C})$ such that $f^\vee : \mathbb{R} \to C^{(N)}(\mathbb{R}, \mathbb{C})$ is not $C^{(N)}$.*

Proof. (1) There is a function $g \in C^{\{M\}}(\mathbb{R}, \mathbb{C})$ such that $g^{(k)}(0) = i^k h_k$ and $h_k \geq k!\, M_k$ for all k; see [38, Thm. 1]. Defining $f(s, t) := g(s + t)$, we obtain a function $f \in C^{\{M\}}(\mathbb{R}^2, \mathbb{C})$ with

$$\partial^\alpha f(0, 0) = i^{|\alpha|} h_{|\alpha|}, \quad h_{|\alpha|} \geq |\alpha|!\, M_{|\alpha|} \quad \text{for all } \alpha \in \mathbb{N}^2.$$

Since $M = (M_k)$ has non-moderate growth, there exist $j_n \nearrow \infty$ and $k_n > 0$ such that

$$\Big(\frac{M_{k_n+j_n}}{M_{k_n} M_{j_n}}\Big)^{\frac{1}{k_n+j_n}} \geq n.$$

Consider the linear functional $\ell : C^{\{M\}}(\mathbb{R}, \mathbb{C}) \to \mathbb{C}$ given by

$$\ell(g) = \sum_n \frac{i^{3j_n} g^{(j_n)}(0)}{j_n!\, M_{j_n}\, n^{j_n}}.$$

This functional is continuous, since

$$\Big|\sum_n \frac{i^{3j_n} g^{(j_n)}(0)}{j_n!\, M_{j_n}\, n^{j_n}}\Big| \leq \sum_n \frac{|g^{(j_n)}(0)|}{j_n!\, \rho^{j_n}\, M_{j_n}} \frac{\rho^{j_n}}{n^{j_n}} \leq C(\rho)\, \|g\|_{[-1,1],\rho} < \infty,$$

for suitable ρ, where

$$C(\rho) := \sum_n \Big(\frac{\rho}{n}\Big)^{j_n} < \infty,$$

for all ρ. But $\ell \circ f^\vee$ is not $C^{\{M\}}$, since

$$\begin{aligned}
\|\ell \circ f^\vee\|_{[-1,1],\rho_1} &= \sup_{k\in\mathbb{N}, t\in[-1,1]} \frac{|(\ell\circ f^\vee)^{(k)}(t)|}{\rho_1^k\, k!\, M_k}\\
&\ge \sup_k \frac{1}{\rho_1^k\, k!\, M_k}\Big|\sum_n \frac{i^{3j_n} f^{(j_n,k)}(0,0)}{j_n!\, M_{j_n} n^{j_n}}\Big|\\
&= \sup_k \frac{1}{\rho_1^k\, k!\, M_k}\Big|\sum_n \frac{i^{4j_n+k} h_{(j_n,k)}}{j_n!\, M_{j_n} n^{j_n}}\Big|\\
&= \sup_k \frac{1}{\rho_1^k\, k!\, M_k}\sum_n \frac{h_{(j_n,k)}}{j_n!\, M_{j_n} n^{j_n}}\\
&\ge \sup_n \frac{1}{\rho_1^{k_n}\, k_n!\, M_{k_n}}\frac{h_{(j_n,k_n)}}{j_n!\, M_{j_n}\, n^{j_n}}\\
&\ge \sup_n \frac{(j_n+k_n)!\, M_{j_n+k_n}}{\rho_1^{k_n}\, k_n!\, j_n!\, M_{k_n}\, M_{j_n}\, n^{j_n}} \ge \sup_n \frac{n^{j_n+k_n}}{\rho_1^{k_n}\, n^{j_n}} = \infty,
\end{aligned}$$

for all $\rho_1 > 0$.

(2) By Theorem 8.2, we have, for convenient vector spaces E, F and c^∞-open subsets $U \subseteq E$,

$$C^{\{M\}}(U,F) = \bigcap_N C^{(N)}(U,F),$$

where the intersection is taken over all weakly log-convex $N = (N_k)$ with $M \lhd N$. Let f be the function in (1). Then there exist weakly log-convex sequences $N^i = (N^i_k)$, $i = 1,2$, with $M \lhd N^i$ such that $f^\vee : \mathbb{R} \to C^{(N^2)}(\mathbb{R},\mathbb{C})$ is not $C^{(N^1)}$. By the lemma below there exists a weakly log-convex sequence $N = (N_k)$ such that $M \lhd N \le N^i$ for $i = 1,2$. Since $f \in C^{\{M\}}(\mathbb{R}^2,\mathbb{C}) \subseteq C^{(N)}(\mathbb{R}^2,\mathbb{C})$, the mapping $f^\vee$ has values in $C^{(N)}(\mathbb{R},\mathbb{C})$ and thus factors over the inclusion $C^{(N)}(\mathbb{R},\mathbb{C}) \to C^{(N^2)}(\mathbb{R},\mathbb{C})$ which is obviously continuous. It follows that $f^\vee : \mathbb{R} \to C^{(N)}(\mathbb{R},\mathbb{C})$ is not $C^{(N^1)}$ and consequently not $C^{(N)}$.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "f^\vee \notin C^{(N^1)}"] \arrow[dr, "f^\vee \notin C^{(N^1)} \supseteq C^{(N)}"'] & C^{(N^2)}(\mathbb{R},\mathbb{C}) \\
& C^{(N)}(\mathbb{R},\mathbb{C}) \arrow[u, hook]
\end{tikzcd}$$

By Theorem 5.2, $N = (N_k)$ has non-moderate growth. $\square$

Lemma. *Let $M = (M_k)$, $N^i = (N^i_k)$, $i = 1,2$, be weakly log-convex with $M \lhd N^i$ for $i = 1,2$. Then there exists a weakly log-convex sequence $N = (N_k)$ such that $M \lhd N \le N^i$ for $i = 1,2$.*

Proof. Set $\bar N = (\bar N_k) := (\min\{N^1_k, N^2_k\})$ and $N = (N_k)$, where $(k!N_k)$ is the log-convex minorant of $(k!\bar N_k)$. Note that $N_0 = 1 \le N_1$. We have $N \le \bar N \le N^i$, and $M \lhd N^i$ implies $M \lhd \bar N$. It remains to show that $M \lhd N$.

We claim that $C^{(N)}_{\text{global}}(\mathbb{R},\mathbb{R}) = C^{(\bar N)}_{\text{global}}(\mathbb{R},\mathbb{R})$, where for a sequence $L = (L_k) \in (\mathbb{R}_{>0})^{\mathbb{N}}$ we set

$$C^{[L]}_{\text{global}}(\mathbb{R},\mathbb{R}) := \Big\{ f \in C^\infty(\mathbb{R},\mathbb{R}) : (\sup_{x\in\mathbb{R}} |f^{(k)}(x)|)_k \in \mathcal{F}^{[L]} \Big\}.$$

In the Roumieu case this a theorem due to Cartan and Gorny, see [19, IV E]; the same proof with obvious modifications yields the Beurling version, i.e., the claim.

Now $M \lhd \bar{N}$ implies $C^{\{M\}}_{\text{global}}(\mathbb{R},\mathbb{R}) \subseteq C^{(\bar{N})}_{\text{global}}(\mathbb{R},\mathbb{R}) = C^{(N)}_{\text{global}}(\mathbb{R},\mathbb{R})$. The function $\tilde{g} := \operatorname{Re} g + \operatorname{Im} g$, where g is the function from the proof of (1), is actually an element of $C^{\{M\}}_{\text{global}}(\mathbb{R},\mathbb{R})$ and satisfies $|\tilde{g}^{(k)}(0)| \ge k!\, M_k$ for all k; see [38, Thm. 1]. Thus $\tilde{g} \in C^{(N)}_{\text{global}}(\mathbb{R},\mathbb{R})$ and therefore $M \lhd N$. □

Corollary 5.5 (Canonical mappings)**.** *Let $M = (M_k)$ be log-convex and have moderate growth. Let E, F, etc., be convenient vector spaces and let U and V be c^∞-open subsets of such. Then we have:*
(1) *The* exponential law *holds:*

$$C^{[M]}(U, C^{[M]}(V,G)) \cong C^{[M]}(U \times V, G)$$

is a linear $C^{[M]}$-diffeomorphism of convenient vector spaces.
The following canonical mappings are $C^{[M]}$.

(2) $\operatorname{ev} : C^{[M]}(U,F) \times U \to F, \quad \operatorname{ev}(f,x) = f(x)$

(3) $\operatorname{ins} : E \to C^{[M]}(F, E \times F), \quad \operatorname{ins}(x)(y) = (x,y)$

(4) $(\quad)^\wedge : C^{[M]}(U, C^{[M]}(V,G)) \to C^{[M]}(U \times V, G)$

(5) $(\quad)^\vee : C^{[M]}(U \times V, G) \to C^{[M]}(U, C^{[M]}(V,G))$

(6) $\operatorname{comp} : C^{[M]}(F,G) \times C^{[M]}(U,F) \to C^{[M]}(U,G)$

(7) $C^{[M]}(\quad,\quad) : C^{[M]}(F,F_1) \times C^{[M]}(E_1,E) \to C^{[M]}\Big(C^{[M]}(E,F), C^{[M]}(E_1,F_1)\Big)$

$\qquad (f,g) \mapsto (h \mapsto f \circ h \circ g)$

(8) $\prod : \prod C^{[M]}(E_i,F_i) \to C^{[M]}\Big(\prod E_i, \prod F_i\Big)$

Proof. This is a direct consequence of cartesian closedness, i.e., Theorem 5.2. See [25, 5.5] or [23, 3.13] for the detailed arguments. □

6. Uniform boundedness principles

Theorem 6.1 ($C^{[M]}$ uniform boundedness principle)**.** *Let E, F, G be convenient vector spaces and let $U \subseteq F$ be c^∞-open. A linear mapping $T : E \to C^{[M]}(U,G)$ is bounded if and only if $\operatorname{ev}_x \circ T : E \to G$ is bounded for every $x \in U$.*

Proof. ($\Rightarrow$) For $x \in U$ and $\ell \in G^*$, the linear mapping $\ell \circ \operatorname{ev}_x = C^{[M]}(x,\ell) : C^{[M]}(U,G) \to \mathbb{R}$ is continuous, thus ev_x is bounded. Therefore, if T is bounded then so is $\operatorname{ev}_x \circ T$.

($\Leftarrow$) Suppose that $\operatorname{ev}_x \circ T$ is bounded for all $x \in U$. By the definition of $C^{[M]}(U,G)$ in Section 4.2 it is enough to show that T is bounded in the case that E and F are Banach spaces and $G = \mathbb{R}$. By Section 4.1, $C^{[M]}(U,\mathbb{R}) = \varprojlim_K C^{[M]}(F \supseteq K, \mathbb{R})$, by Proposition 4.1(2), $C^{(M)}(F \supseteq K, \mathbb{R})$ is a Fréchet space, and by Proposition 4.1(3), $C^{\{M\}}(F \supseteq K,\mathbb{R})$ is an (LB)-space, so $C^{[M]}(F \supseteq K,\mathbb{R})$ is webbed and hence the closed graph theorem [23, 52.10] gives the desired result.

$$
\begin{tikzcd}
E \arrow[r, "T"] \arrow[dr] & C^{[M]}(U,\mathbb{R}) \arrow[r, "\mathrm{ev}_x"] \arrow[d, equal] & \mathbb{R} \\
& \varprojlim_K C^{[M]}(F \supseteq K,\mathbb{R}) \arrow[r] & C^{[M]}(F \supseteq K,\mathbb{R}) \arrow[u, "(\mathrm{ev}_x)^0"']
\end{tikzcd} \tag{27}
$$

□

Remark 6.2. Alternatively, the $C^{\{M\}}$ uniform boundedness principle follows from the $C^{(M)}$ uniform boundedness principle and from the remark after Theorem 8.2, since the structure of $C^{\{M\}}(U,F) = \varprojlim_L C^{(L)}(U,F)$ is initial with respect to the inclusions $\varprojlim_L C^{(L)}(U,F) \to C^{(L)}(U,F)$ for all L. This is no circular argument, since the first identity in Theorem 8.2 was proved in Remark 5.3 without using the $C^{\{M\}}$ uniform boundedness principle, i.e., Theorem 6.1.

7. Relation to previously considered structures

In [25] and [26] we have developed the convenient setting for all *reasonable* non-quasianalytic and some quasianalytic (namely, $\mathcal{L}$-intersectable, see Section 7.1) Denjoy–Carleman classes of Roumieu type. We have worked with a definition which is based on *testing along curves*. The resulting structures were denoted by C^M in [25] and [26] and will be denoted by $C^{\{M\}}_{\text{curve}}$ in this section; this notation does not appear elsewhere in this paper. We shall now show that they coincide bornologically with the structure $C^{\{M\}}$ studied in the present paper. Furthermore, we prove that the bornologies induced by $C^{\{1\}}$ and the structure C^ω of real analytic mappings introduced in [22] are isomorphic; here 1 denotes the constant sequence $(1)_k$. Note that $C^{\{1\}}$ is not $\mathcal{L}$-intersectable (see [26, 1.8]).

7.1. **Testing along curves.** Let $M = (M_k)$ be log-convex, E and F convenient vector spaces, and U a c^∞-open subset in E. If $M = (M_k)$ is non-quasianalytic we set

$$C^{\{M\}}_{\text{curve}}(U,F) := \Big\{ f \in F^U : \forall \ell \in F^* \; \forall c \in C^{\{M\}}(\mathbb{R},U) : \ell \circ f \circ c \in C^{\{M\}}(\mathbb{R},\mathbb{R}) \Big\}.$$

If $M = (M_k)$ is quasianalytic and $\mathcal{L}$*-intersectable*, i.e., $\mathcal{F}^{\{M\}} = \bigcap_{L \in \mathcal{L}(M)} \mathcal{F}^{\{L\}}$, where

$$\mathcal{L}(M) := \Big\{ L = (L_k) : L \geq M, L \text{ is non-quasianalytic log-convex} \Big\},$$

we define

$$C^{\{M\}}_{\text{curve}}(U,F) := \bigcap_{L \in \mathcal{L}(M)} C^{\{L\}}_{\text{curve}}(U,F).$$

Note that non-quasianalytic log-convex sequences are trivially $\mathcal{L}$-intersectable. For non-quasianalytic $M = (M_k)$ we supply $C^{\{M\}}_{\text{curve}}(U,F)$ with the initial locally convex structure induced by all linear mappings:

$$C^{\{M\}}_{\text{curve}}(U,F) \xrightarrow{C^{\{M\}}_{\text{curve}}(c,\ell)} C^{\{M\}}(\mathbb{R},\mathbb{R}), \quad f \mapsto \ell \circ f \circ c, \qquad \ell \in F^*, c \in C^{\{M\}}(\mathbb{R},U),$$

and for quasianalytic and $\mathcal{L}$*-intersectable* $M = (M_k)$ by all inclusions

$$C^{\{M\}}_{\text{curve}}(U,F) \longrightarrow C^{\{L\}}_{\text{curve}}(U,F), \qquad L \in \mathcal{L}(M).$$

In both cases $C^{\{M\}}_{\text{curve}}(U,F)$ is a convenient vector space.

Let $C^\omega(\mathbb{R},\mathbb{R})$ denote the real analytic functions $f:\mathbb{R}\to\mathbb{R}$ and set

$$C^\omega(U,F) := \Big\{ f \in C^\infty(U,F) : \forall \ell \in F^* \ \forall c \in C^\omega(\mathbb{R},U) : \ell\circ f\circ c \in C^\omega(\mathbb{R},\mathbb{R}) \Big\},$$

where $C^\omega(\mathbb{R},U)$ is the space of all weakly C^ω-curves in U. We equip $C^\omega(U,\mathbb{R})$ with the initial locally convex structure induced by the family of mappings

$$\begin{aligned} C^\omega(U,\mathbb{R}) &\xrightarrow{c^*} C^\omega(\mathbb{R},\mathbb{R}), \quad f\mapsto f\circ c, \qquad c\in C^\omega(\mathbb{R},U),\\ C^\omega(U,\mathbb{R}) &\xrightarrow{c^*} C^\infty(\mathbb{R},\mathbb{R}), \quad f\mapsto f\circ c, \qquad c\in C^\infty(\mathbb{R},U), \end{aligned}$$

where $C^\infty(\mathbb{R},\mathbb{R})$ carries the topology of compact convergence in each derivative separately, and where $C^\omega(\mathbb{R},\mathbb{R})$ is equipped with the final locally convex topology with respect to the embeddings (restriction mappings) of all spaces of holomorphic mappings from a neighborhood V of $\mathbb{R}$ in $\mathbb{C}$ mapping $\mathbb{R}$ to $\mathbb{R}$, and each of these spaces carries the topology of compact convergence. The space $C^\omega(U,F)$ is equipped with the initial locally convex structure induced by all mappings

$$C^\omega(U,F) \xrightarrow{\ell_*} C^\omega(U,\mathbb{R}), \quad f\mapsto \ell\circ f, \qquad \ell\in F^*.$$

This is again a convenient vector space.

Theorem 7.1. *Let $M=(M_k)$ be log-convex, E and F convenient vector spaces, and U a c^∞-open subset in E. We have:*

(1) *If $M=(M_k)$ is $\mathcal{L}$-intersectable, then*

$$C^{\{M\}}(U,F) = C^{\{M\}}_{\text{curve}}(U,F)$$

as vector spaces with bornology.

(2) *If 1 denotes the constant sequence, then*

$$C^{\{1\}}(U,F) = C^\omega(U,F)$$

as vector spaces with bornology.

Proof. (1) If $M=(M_k)$ is non-quasianalytic, then $C^{\{M\}}(U,F)$ and $C^{\{M\}}_{\text{curve}}(U,F)$ coincide as vector spaces, by [26, 2.8]. If $M=(M_k)$ is quasianalytic and $\mathcal{L}$-intersectable, then the non-quasianalytic case implies that

$$C^{\{M\}}_{\text{curve}}(U,F) = \bigcap_{L\in\mathcal{L}(M)} C^{\{L\}}_{\text{curve}}(U,F) = \bigcap_{L\in\mathcal{L}(M)} C^{\{L\}}(U,F) = C^{\{M\}}(U,F)$$

as vector spaces, where the last equality is a consequence of the definition of $C^{\{M\}}(U,F)$ (see Section 4.2) and of [26, 1.6] (applied to $C^{\{M\}}(U_B,\mathbb{R})$). The fact that both spaces $C^{\{M\}}(U,F)$ and $C^{\{M\}}_{\text{curve}}(U,F)$ are convenient and satisfy the uniform boundedness principle with respect to the set of point evaluations, see Theorem 6.1 and [26, 2.9], implies that the identity is a bornological isomorphism.

(2) We show first that $C^{\{1\}}(U,F)=C^\omega(U,F)$ as vector spaces. By Section 4.2 and [23, 10.6], it suffices to consider the case that U is open in a Banach space E and $F=\mathbb{R}$.

Let $f\in C^\omega(U,\mathbb{R})$. By [22, 2.4 and 2.7] or [23, 10.1 and 10.4], this is equivalent to f being smooth and being locally given by its convergent Taylor series. Let $K\subseteq U$

be compact. Since the Taylor series of f converges locally, there exist constants $C, \rho > 0$ such that

$$\frac{\|f^{(k)}(a)\|_{L^k(E,\mathbb{R})}}{k!} \le C\rho^k, \qquad \text{for all } a \in K, k \in \mathbb{N},$$

that is, $f \in C^{\{1\}}(U, \mathbb{R})$.
Conversely, the above estimate for compact subsets K of affine lines in E implies that the restriction of f to each affine line is real analytic and hence $f \in C^\omega(U, \mathbb{R})$ by [23, 10.1].

The bornologies coincide, since both spaces are convenient and satisfy the uniform boundedness principle with respect to the set of point evaluations, see Theorem 6.1 and [22, 5.6] or [23, 11.12]. □

8. More on function spaces

Proposition 8.1 (Inclusions)**.** *Let $M = (M_k)$, $N = (N_k)$ be positive sequences, E, F convenient vector spaces, and $U \subseteq E$ a c^∞-open subset. We have:*

1. $C^{(M)}(U, F) \subseteq C^{\{M\}}(U, F) \subseteq C^\infty(U, F)$.
2. *If there exist $C, \rho > 0$ so that $M_k \le C\rho^k N_k$ for all k, then*
$$C^{(M)}(U, F) \subseteq C^{(N)}(U, F) \quad \text{and} \quad C^{\{M\}}(U, F) \subseteq C^{\{N\}}(U, F).$$
3. *If for each $\rho > 0$ there exists $C > 0$ so that $M_k \le C\rho^k N_k$ for all k, i.e., $M \lhd N$, then*
$$C^{\{M\}}(U, F) \subseteq C^{(N)}(U, F).$$
4. *For $U \ne \emptyset$ and $F \ne \{0\}$ we have:*
$$C^\omega(U, F) \subseteq C^{(M)}(U, F) \iff M_k^{1/k} \to \infty, \quad \text{and}$$
$$C^\omega(U, F) \subseteq C^{\{M\}}(U, F) \iff \varliminf M_k^{1/k} > 0.$$

All these inclusions are bounded.

Proof. The inclusions in (1), (2), and (3) follow immediately from the definitions in Sections 4.1 and 4.2 and Lemma 2.2. Here we use that $C^{\{1\}}(U, F) = C^\omega(U, F)$ as vector spaces with bornology, see Theorem 7.1.

The directions ($\Leftarrow$) in (4) are direct consequences of (2) and (3). The directions ($\Rightarrow$) follow, since they have been shown in Section 2.1 for $E = F = \mathbb{R}$.

All inclusions are bounded, since all spaces are convenient and satisfy the uniform boundedness principle, cf. Theorem 6.1 and [23, 5.26] for C^∞. □

Theorem 8.2. *Let $M = (M_k)$ be (weakly) log-convex, E and F convenient vector spaces, and U a c^∞-open subset in E. We have*

$$C^{\{M\}}(U, F) = \varprojlim_L C^{(L)}(U, F) = \varprojlim_L C^{\{L\}}(U, F)$$

as vector spaces with bornology, where the projective limits are taken over all (weakly) log-convex sequences $L = (L_k)$ with $M \lhd L$.

Proof. The three spaces coincide as vector spaces: By Section 4.2 it suffices to assume that E and F are Banach spaces, and by Section 4.1 and Proposition 4.1(5) it suffices to apply Theorem 2.2 to the sequence $(\|j^\infty f|_K\|_m)$.

Each space is convenient (see Section 4.2; projective limits preserve c^∞-completeness) and each space satisfies the uniform boundedness principle with respect to the set of point evaluations (see Theorem 6.1; the structure of $\varprojlim_L C^{[L]}(U,F)$ is initial with respect to the inclusions $\varprojlim_L C^{[L]}(U,F) \to C^{[L]}(U,F)$ for all L). Thus the identity between any two of the three spaces is a bornological isomorphism. $\square$

Remark. By the remark after Theorem 2.2 the statement of the theorem still holds, if $M = (M_k)$ is just any positive sequence, where the projective limits are now taken over all positive sequences $L = (L_k)$ with $M \lhd L$.

Proposition 8.3 (Derivatives)**.** *Let $M = (M_k)$ be a positive sequence and set $M_{+1} = (M_{k+1})$. Let E and F be convenient vector spaces, and $U \subseteq E$ a c^∞-open subset. Then we have:*

(1) *Multilinear mappings between convenient vector spaces are $C^{[M]}$ if and only if they are bounded.*
(2) *If $f : E \supseteq U \to F$ is $C^{[M]}$, then the derivative $df : U \to L(E,F)$ is $C^{[M_{+1}]}$, where the space $L(E,F)$ of all bounded linear mappings is considered with the topology of uniform convergence on bounded sets. If $M_{+1} = (M_{k+1})$ is weakly log-convex (which is the case if $M = (M_k)$ is weakly log-convex), also $(df)^\wedge : U \times E \to F$ is $C^{[M_{+1}]}$,*
(3) *The chain rule holds.*

Proof. (1) If f is $C^{[M]}$ then it is smooth and hence bounded by [23, 5.5]. Conversely, if f is multilinear and bounded then it is smooth, again by [23, 5.5]. Furthermore, $f \circ i_B$ is multilinear and continuous and all derivatives of high order vanish. Thus f is $C^{[M]}$, by Section 4.2.

(2) Since f is smooth, by [23, 3.18] the mapping $df : U \to L(E,F)$ exists and is smooth. We have to show that $(df) \circ i_B : U_B \to L(E,F)$ is $C^{[M_{+1}]}$, for all closed absolutely convex bounded subsets $B \subseteq E$. By the uniform boundedness principle [23, 5.18] and by Lemma 5.1 it suffices to show that the mapping $U_B \ni x \mapsto \ell(df(i_B(x))(v)) \in \mathbb{R}$ is $C^{[M_{+1}]}$ for each $\ell \in F^*$ and $v \in E$.

Since $\ell \circ f$ is $C^{(M)}$ (resp. $C^{\{M\}}$), for each closed absolutely convex bounded $B \subseteq E$, each compact $K \subseteq U_B$, and each $\rho > 0$ (resp. some $\rho > 0$) the set

$$\Big\{ \frac{\|d^k(\ell \circ f \circ i_B)(a)\|_{L^k(E_B,\mathbb{R})}}{k!\, \rho^k\, M_k} : a \in K, k \in \mathbb{N} \Big\}$$

is bounded, say by $C > 0$. The assertion follows in both cases from the following computation. For $v \in E$ and those B containing v we then have:

$$\begin{aligned}
\|d^k(L(\ell,v) \circ df) \circ i_B)(a)\|_{L^k(E_B,\mathbb{R})} &= \|d^k(d(\ell \circ f)(\quad)(v)) \circ i_B)(a)\|_{L^k(E_B,\mathbb{R})} \\
&= \|d^{k+1}(\ell \circ f \circ i_B)(a)(v, \dots)\|_{L^k(E_B,\mathbb{R})} \\
&\le \|d^{k+1}(\ell \circ f \circ i_B)(a)\|_{L^{k+1}(E_B,\mathbb{R})} \|v\|_B \le C\,(k+1)!\,\rho^{k+1}\, M_{k+1} \\
&= C\rho\,((k+1)^{1/k}\rho)^k\, k!\, M_{k+1} \le C\rho\,(2\rho)^k\, k!\,(M_{+1})_k.
\end{aligned}$$

By Proposition 8.4 below also $(df)^\wedge$ is $C^{[M_{+1}]}$, if $M = (M_k)$ is weakly log-convex.

(3) This is valid even for all smooth f by [23, 3.18]. $\square$

Proposition 8.4. *We have:*

(1) *For convenient vector spaces E and F, the following topologies have the same bounded subsets in $L(E,F)$:*
   - *The topology of uniform convergence on bounded subsets of E.*
   - *The topology of pointwise convergence.*
   - *The trace topology of $C^\infty(E,F)$.*
   - *The trace topology of $C^{[M]}(E,F)$.*

(2) *Let $M=(M_k)$ be weakly log-convex, E, F, and G convenient vector spaces, and $U\subseteq E$ a c^∞-open subset. A mapping $f:U\times F\to G$ which is linear in the second variable is $C^{[M]}$ if and only if $f^\vee:U\to L(F,G)$ is well defined and $C^{[M]}$.*

Analogous results hold for spaces of multilinear mappings.

Proof. (1) That the first three topologies on $L(E,F)$ have the same bounded sets has been shown in [23, 5.3 and 5.18]. The inclusion $C^{[M]}(E,F)\to C^\infty(E,F)$ is bounded by Proposition 8.1. Conversely, the inclusion $L(E,F)\to C^{[M]}(E,F)$ is bounded by the uniform boundedness principle, i.e., Theorem 6.1.

(2) The assertion for C^∞ is true by [23, 3.12] since $L(E,F)$ is closed in $C^\infty(E,F)$.

Suppose that f is $C^{[M]}$. We have to show that $f^\vee\circ i_B$ is $C^{[M]}$ into $L(F,G)$, for all closed absolutely convex bounded subsets $B\subseteq E$. By the uniform boundedness principle [23, 5.18] and by Lemma 5.1 it suffices to show that the mapping $U_B\ni x\mapsto \ell\big(f^\vee(i_B(x))(v)\big)=\ell\big(f(i_B(x),v)\big)\in\mathbb{R}$ is $C^{[M]}$ for each $\ell\in G^*$ and $v\in F$; this is obviously true.

Conversely, let $f^\vee:U\to L(F,G)$ be $C^{[M]}$. By (1) the inclusion $L(F,G)\to C^{[M]}(F,G)$ is bounded linear, and so $f^\vee:U\to C^{[M]}(F,G)$ is $C^{[M]}$. By cartesian closedness, i.e., Theorem 5.2 (the direction which holds without moderate growth), $f:U\times F\to G$ is $C^{[M]}$ and linearity in the second variable is obvious. □

Remark. We may prove $f^\vee\in C^{[M]}(U,L(F,G))\Rightarrow f\in C^{[M]}(U\times F,G)$ without using cartesian closedness: By composing with $\ell\in G^*$ we may assume that $G=\mathbb{R}$. By induction we have:

$$d^kf(x,w_0)\big((v_k,w_k),\dots,(v_1,w_1)\big)=d^k(f^\vee)(x)(v_k,\dots,v_1)(w_0)+\\ +\sum_{i=1}^k d^{k-1}(f^\vee)(x)(v_k,\dots,\widehat{v_i},\dots,v_1)(w_i)$$

Thus for B, B' closed absolutely convex bounded subsets of E, F, respectively, $K\subseteq U_B$ compact, and $x\in K$ we have:

$$\begin{aligned}&\|d^kf(x,w_0)\|_{L^k(E_B\times F_{B'},\mathbb{R})}\le\\ &\le\|d^k(f^\vee)(x)(\dots)(w_0)\|_{L^k(E_B,\mathbb{R})}+\sum_{i=1}^k\|d^{k-1}(f^\vee)(x)\|_{L^{k-1}(E_B,L(F_{B'},\mathbb{R}))}\\ &\le\|d^k(f^\vee)(x)\|_{L^k(E_B,L(F_{B'},\mathbb{R}))}\|w_0\|_{B'}+k\,\|d^{k-1}(f^\vee)(x)\|_{L^{k-1}(E_B,L(F_{B'},\mathbb{R}))}\\ &\le C\,\rho^k\,k!\,M_k\|w_0\|_{B'}+k\,C\,\rho^{k-1}\,(k-1)!\,M_{k-1}=C\,\rho^k\,k!\,M_k\Big(\|w_0\|_{B'}+\tfrac{M_{k-1}}{\rho\,M_k}\Big),\end{aligned}$$

for all $\rho>0$ and some $C=C(\rho)$ (resp. for some $C,\rho>0$), since the mapping $L(i_{B'},\mathbb{R})\circ f^\vee\circ i_B:U_B\to L(F_{B'},\mathbb{R})$ is $C^{[M]}$. Since $k\mapsto k!\,M_k$ is increasing (see the remarks in Section 2.3), we have $\frac{M_{k-1}}{M_k}\le k\le 2^k$, and we may conclude that f is $C^{[M]}$.

Let $r = (r_k)$ be a positive sequence, E and F Banach spaces, and $K \subseteq E$ compact convex. Consider

$$C^M_{(r_k)}(E \supseteq K, F) := \Big\{(f^m)_m \in \prod_{m\in\mathbb{N}} C(K, L^m_{\text{sym}}(E,F)) : \|f\|_{(r_k)} < \infty\Big\},$$

where

$$\|f\|_{(r_k)} := \max\Big\{\sup\Big\{\frac{\|f\|_m}{m!\, r_m\, M_m} : m \in \mathbb{N}\Big\}, \\ \sup\Big\{\frac{|||f|||_{n,k}}{(n+k+1)!\, r_{n+k+1}\, M_{n+k+1}} : k,n \in \mathbb{N}\Big\}\Big\}.$$

If $(r_k) = (\rho^k)$ for some $\rho > 0$ we just write ρ instead of (r_k) as indices and recover the spaces introduced in Section 4.1. Similarly as in Proposition 4.1(1) one shows that the spaces $C^M_{(r_k)}(E \supseteq K, F)$ are Banach spaces.

Theorem 8.5. *Let E and F be Banach spaces and let $U \subseteq E$ be open and convex. Then we have*

$$C^{(M)}(U,F) = \varprojlim_{K,(r_k)} C^M_{(r_k)}(E \supseteq K, F)$$

as vector spaces with bornology. Here K runs through all compact convex subsets of U ordered by inclusion and (r_k) runs through all sequences of positive real numbers for which $\rho^k/r_k \to 0$ for some $\rho > 0$.

Proof. Note first that the elements of the space $\varprojlim_{K,(r_k)} C^M_{(r_k)}(E \supseteq K, F)$ are smooth functions $f : U \to F$ which can be seen as in the proof of Proposition 4.1(5). By Lemma 4.5 it coincides with $C^{(M)}(U,F)$ as vector space.

Obviously the identity is continuous from left to right. The space on the right-hand side is as a projective limit of Banach spaces convenient and $C^{(M)}(U,F)$ satisfies the uniform boundedness principle, i.e., Theorem 6.1, with respect to the set of point evaluations. Thus the identity from right to left is bounded. □

Theorem 8.6. *Let E and F be Banach spaces and let $U \subseteq E$ be open and convex. Then we have*

$$C^{\{M\}}(U,F) = \varprojlim_{K,(r_k)} C^M_{(r_k)}(E \supseteq K, F)$$

as vector spaces with bornology. Here K runs through all compact convex subsets of U ordered by inclusion and (r_k) runs through all sequences of positive real numbers for which $\rho^k/r_k \to 0$ for all $\rho > 0$.

Proof. The proof is literally identical with the proof of Theorem 8.5, where we replace $C^{(M)}$ with $C^{\{M\}}$ and use Lemma 4.6 instead of Lemma 4.5. □

Remark. Let us prove that the identity $\varprojlim_{K,(r_k)} C^M_{(r_k)}(E \supseteq K, F) \to C^{\{M\}}(U,F)$ is bounded without using the $C^{\{M\}}$ uniform boundedness principle, i.e., Theorem 6.1: Let $\mathcal{B}$ be a bounded set in $\varprojlim_{K,(r_k)} C^M_{(r_k)}(E \supseteq K, F)$, i.e., for each compact K and each (r_k) with $\rho^k/r_k \to 0$ for all $\rho > 0$ the set $\mathcal{B}$ is bounded in $C^M_{(r_k)}(E \supseteq K, F)$, i.e.,

$$\sup\{\|f|_K\|_{(r_k)} : f \in \mathcal{B}\} < \infty.$$

Since the elements of $\varprojlim_{K,(r_k)} C^M_{(r_k)}(E \supseteq K, F)$ are the infinite jets of smooth functions, we may estimate $|||f|_K|||_{n,k}$ by $\|f|_K\|_{n+k+1}$ by (12), and so the sequence

$$a_k := \sup\Big\{ \frac{\|f|_K\|_k}{k!\, M_k} : f \in \mathcal{B} \Big\} < \infty$$

satisfies $\sup_k a_k/r_k < \infty$ for each (r_k) as above. By [23, 9.2], these are the coefficients of a power series with positive radius of convergence. Thus a_k/ρ^k is bounded for some $\rho > 0$. That means that $\mathcal{B}$ is contained and bounded in $C^M_\rho(E \supseteq K, F)$.

This also provides an independent proof of the completeness of $C^{\{M\}}(U, F)$ and of the regularity of the involved inductive limit (cf. Proposition 4.1 and Remark 5.3).

Lemma 8.7. *For convenient vector spaces E, F, G, and c^∞-open $V \subseteq F$ the flip of variables induces an isomorphism $L(E, C^{[M]}(V, G)) \cong C^{[M]}(V, L(E, G))$ as vector spaces.*

Proof. For $f \in C^{[M]}(V, L(E, G))$ consider $\tilde f(x) := \mathrm{ev}_x \circ f \in C^{[M]}(V, G)$ for $x \in E$. By the uniform boundedness principle, i.e., Theorem 6.1, the linear mapping $\tilde f$ is bounded, since $\mathrm{ev}_y \circ \tilde f = f(y) \in L(E, G)$ for $y \in V$.

If conversely $\ell \in L(E, C^{[M]}(V, G))$, we consider $\tilde\ell(y) = \mathrm{ev}_y \circ \ell \in L(E, G)$ for $y \in V$. Since the bornology of $L(E, G)$ (see Proposition 8.4) is generated by $\mathcal{S} := \{\mathrm{ev}_x : x \in E\}$ and since $\mathrm{ev}_x \circ \tilde\ell = \ell(x) \in C^{[M]}(V, G)$, it follows that $\tilde\ell : V \to L(E, G)$ is $C^{[M]}$, by Lemma 5.1 (and by composing with all $i_B : V_B \to V$). □

Lemma 8.8. *Let E be a convenient vector space and let $U \subseteq E$ be c^∞-open. By $\lambda^{[M]}(U)$ we denote the c^∞-closure of the linear subspace generated by $\{\mathrm{ev}_x : x \in U\}$ in $C^{[M]}(U, \mathbb{R})'$ and let $\delta : U \to \lambda^{[M]}(U)$ be given by $x \mapsto \mathrm{ev}_x$. Then $\lambda^{[M]}(U)$ is the free convenient vector space over $C^{[M]}$, i.e., for every convenient vector space G the $C^{[M]}$-mapping δ induces a bornological isomorphism*

$$L(\lambda^{[M]}(U), G) \cong C^{[M]}(U, G).$$

Proof. The proof goes along the same lines as in [11, 5.1.1] and [23, 23.6]. Note first that $\lambda^{[M]}(U)$ is a convenient vector space, since it is c^∞-closed in the convenient vector space $C^{[M]}(U, \mathbb{R})'$. Moreover, δ is $C^{[M]}$, by Lemma 5.1 (and by composing with all $i_B : U_B \to U$), since $\mathrm{ev}_h \circ \delta = h$ for all $h \in C^{[M]}(U, \mathbb{R})$. So $\delta^* : L(\lambda^{[M]}(U), G) \to C^{[M]}(U, G)$ is a well-defined linear mapping. This mapping is injective, since each bounded linear mapping $\lambda^{[M]}(U) \to G$ is uniquely determined on $\delta(U) = \{\mathrm{ev}_x : x \in U\}$. Let now $f \in C^{[M]}(U, G)$. Then $\ell \circ f \in C^{[M]}(U, \mathbb{R})$ for every $\ell \in G^*$, and hence $\tilde f : C^{[M]}(U, \mathbb{R})' \to \prod_{G^*} \mathbb{R}$ given by $\tilde f(\phi) = (\phi(\ell \circ f))_{\ell \in G^*}$ is a well-defined bounded linear mapping. Since it maps ev_x to $\tilde f(\mathrm{ev}_x) = \delta(f(x))$, where $\delta : G \to \prod_{G^*} \mathbb{R}$ denotes the bornological embedding given by $y \mapsto (\ell(y))_{\ell \in G^*}$, it induces a bounded linear mapping $\tilde f : \lambda^{[M]}(U) \to G$ satisfying $\tilde f \circ \delta = f$. Thus δ^* is a linear bijection. That it is a bornological isomorphism follows from the uniform boundedness principle, i.e., Theorem 6.1, and from Proposition 8.4. □

Theorem 8.9 (Canonical isomorphisms)**.** *Let $M = (M_k)$ and $N = (N_k)$ be positive sequences. Let E, F be convenient vector spaces and let W_i be c^∞-open subsets in such. We have the following natural bornological isomorphisms:*

(1) $C^{(M)}(W_1, C^{(N)}(W_2, F)) \cong C^{(N)}(W_2, C^{(M)}(W_1, F))$,

(2) $C^{\{M\}}(W_1, C^{\{N\}}(W_2, F)) \cong C^{\{N\}}(W_2, C^{\{M\}}(W_1, F))$,
(3) $C^{(M)}(W_1, C^{\{N\}}(W_2, F)) \cong C^{\{N\}}(W_2, C^{(M)}(W_1, F))$,
(4) $C^{[M]}(W_1, C^{\infty}(W_2, F)) \cong C^{\infty}(W_2, C^{[M]}(W_1, F))$.
(5) $C^{[M]}(W_1, C^{\omega}(W_2, F)) \cong C^{\omega}(W_2, C^{[M]}(W_1, F))$.
(6) $C^{[M]}(W_1, L(E, F)) \cong L(E, C^{[M]}(W_1, F))$.
(7) $C^{[M]}(W_1, \ell^{\infty}(X, F)) \cong \ell^{\infty}(X, C^{[M]}(W_1, F))$.
(8) $C^{[M]}(W_1, \mathcal{L}\mathrm{ip}^k(X, F)) \cong \mathcal{L}\mathrm{ip}^k(X, C^{[M]}(W_1, F))$.

In (7) *the space X is an ℓ^{∞}-space, i.e., a set together with a bornology induced by a family of real valued functions on X, cf.* [11, 1.2.4]. *In* (8) *the space X is a $\mathcal{L}\mathrm{ip}^k$-space, cf.* [11, 1.4.1]. *The spaces $\ell^{\infty}(X, F)$ and $\mathcal{L}\mathrm{ip}^k(X, F)$ are defined in* [11, 3.6.1 and 4.4.1].

Proof. Let $\mathcal{C}^1$ and $\mathcal{C}^2$ denote any of the functions spaces mentioned above and X_1 and X_2 the corresponding domains. In order to show that the flip of coordinates $f \mapsto \tilde{f}$, $\mathcal{C}^1(X_1, \mathcal{C}^2(X_2, F)) \to \mathcal{C}^2(X_2, \mathcal{C}^1(X_1, F))$ is a well-defined bounded linear mapping we have to show:

- $\tilde{f}(x_2) \in \mathcal{C}^1(X_1, F)$, which is obvious, since $\tilde{f}(x_2) = \mathrm{ev}_{x_2} \circ f : X_1 \to \mathcal{C}^2(X_2, F) \to F$.
- $\tilde{f} \in \mathcal{C}^2(X_2, \mathcal{C}^1(X_1, F))$, which we will show below.
- $f \mapsto \tilde{f}$ is bounded and linear, which follows by applying the appropriate uniform boundedness theorems for $\mathcal{C}^2$ and $\mathcal{C}^1$, since $f \mapsto \mathrm{ev}_{x_1} \circ \mathrm{ev}_{x_2} \circ \tilde{f} = \mathrm{ev}_{x_2} \circ \mathrm{ev}_{x_1} \circ f$ is bounded and linear.

All occurring function spaces are convenient and satisfy the uniform $\mathcal{S}$-boundedness theorem, where $\mathcal{S}$ is the set of point evaluations:

$C^{[M]}$ by Section 4.2 and Theorem 6.1.
C^{∞} by [23, 2.14.3 and 5.26]
C^{ω} by [23, 11.11 and 11.12] or by Theorems 6.1 and 7.1,
L by [23, 2.14.3 and 5.18]
ℓ^{∞} by [23, 2.15, 5.24, and 5.25] or [11, 3.6.1 and 3.6.6]
$\mathcal{L}\mathrm{ip}^k$ by [11, 4.4.2 and 4.4.7]

It remains to check that $\tilde{f}$ is of the appropriate class:

(1)–(4) For $\alpha \in \{(M), \{M\}\}$ and $\beta \in \{(N), \{N\}, \infty\}$ we have

$$\begin{aligned} C^{\alpha}(W_1, C^{\beta}(W_2, F)) &\cong L(\lambda^{\alpha}(W_1), C^{\beta}(W_2, F)) && \text{by Lemma 8.8} \\ &\cong C^{\beta}(W_2, L(\lambda^{\alpha}(W_1), F)) && \text{by Lemma 8.7, [23, 3.13.4 and 5.3]} \\ &\cong C^{\beta}(W_2, C^{\alpha}(W_1, F)) && \text{by Lemma 8.8.} \end{aligned}$$

(5) follows from (2), (3), and Theorem 7.1.
(6) is exactly Lemma 8.7.
(7) follows from (6), using the free convenient vector spaces $\ell^1(X)$ over the ℓ^{∞}-space X, see [11, 5.1.24 or 5.2.3], satisfying $\ell^{\infty}(X, F) \cong L(\ell^1(X), F)$.
(8) follows from (6), using the free convenient vector spaces $\lambda^k(X)$ over the $\mathcal{L}\mathrm{ip}^k$-space X, see [11, 5.1.24 or 5.2.3], satisfying $\mathcal{L}\mathrm{ip}^k(X, F) \cong L(\lambda^k(X), F)$. □

9. Manifolds of $C^{[M]}$-mappings

9.1. **Hypothesis.** In this section we assume that $M = (M_k)$ is log-convex and has moderate growth. In the Beurling case $C^{[M]} = C^{(M)}$ we also require that $C^\omega \subseteq C^{(M)}$, equivalently, $M_k^{1/k} \to \infty$ or $M_{k+1}/M_k \to \infty$.

For the equivalence of $C^\omega \subseteq C^{(M)}$ and $M_k^{1/k} \to \infty$, see Proposition 8.1(4). Moreover, $M_k^{1/k} \to \infty$ implies $M_{k+1}/M_k \to \infty$, since $M_k^{1/k}$ is increasing, by log-convexity (see Section 2.1), and thus $M_{k+1}/M_k \geq M_k^{1/k}$. Conversely, if $M_{k+1}/M_k \to \infty$ then for each $n \in \mathbb{N}$ there is k_n so that $M_k/M_{k-1} \geq n$ for all $k \geq k_n$. It follows that $M_k/M_{k_n-1} \geq n^{k-k_n+1}$ and thus $M_k^{1/k} \to \infty$. This is needed for the $C^{(M)}$ inverse function theorem (see Sections 2.1 and 9.2).

9.2. **Tools for $C^{[M]}$-analysis.** We collect here results which are needed below (see also Section 2.1):

(1) On open sets in $\mathbb{R}^n$, $C^{[M]}$-vector fields have $C^{[M]}$-flows, see [18] and [40].
(2) Between Banach spaces, the $C^{[M]}$ implicit function theorem holds. This is essentially due to [39], but in [39] only the Roumieu case is treated and the $C^{\{M\}}$-conditions are global. So we shall indicate briefly how to obtain the result we need (cf. [32]):

Theorem. *Let $M = (M_k)$ be log-convex. In the Beurling case $C^{[M]} = C^{(M)}$ we also assume $M_{k+1}/M_k \to \infty$. Let E, F be Banach spaces, $U \subseteq E$, $V \subseteq F$ open, and $f : U \to V$ a C^∞-diffeomorphism. We have:*

(3) *Let $K \subseteq U$ be compact. If $f \in C^{[M]}_K(U, F)$ then $f^{-1} \in C^{[M]}_{f(K)}(V, E)$.*
(4) *If $f \in C^{[M]}(U, F)$ then $f^{-1} \in C^{[M]}(V, E)$.*

Proof. By Proposition 4.1(5), (3) implies (4). The proof of [39, Thm. 2] with small obvious modifications provides a proof of (3) in the Roumieu case (see also [36, 3.4.5]).

For the Beurling case let $f \in C^{(M)}_K(U, F)$ and

$$L_k := \frac{1}{k!} \sup_{x \in K} \|f^{(k)}(x)\|_{L^k(E,F)}.$$

Then $L \lhd M$ and since $M_{k+1}/M_k \to \infty$ there exists a log-convex sequence $N = (N_k)$ satisfying $N_{k+1}/N_k \to \infty$ and such that $L \leq N \lhd M$, by [16, Lemma 6]. Thus, $f \in C^{\{N\}}_K(U, F)$ and, by the Roumieu case, $f^{-1} \in C^{\{N\}}_{f(K)}(V, E)$. Since $N \lhd M$, we have $f^{-1} \in C^{(M)}_{f(K)}(V, E)$, by Proposition 8.1. □

The $C^{[M]}$ implicit function theorem follows in the standard way.

9.3. **$C^{[M]}$-manifolds.** A $C^{[M]}$-manifold is a smooth manifold such that all chart changings are $C^{[M]}$-mappings. They will be considered with the topology induced by the c^∞-topology on the charts. Likewise for $C^{[M]}$-bundles and $C^{[M]}$ Lie groups.

A mapping between $C^{[M]}$-manifolds is $C^{[M]}$ if and only if it maps $C^{[M]}$-plots (i.e., $C^{[M]}$-mappings from open sets (or unit balls) of Banach spaces into the domain manifold) to such.

Note that any finite dimensional (always assumed paracompact) C^∞-manifold admits a C^∞-diffeomorphic real analytic structure thus also a $C^{[M]}$-structure.

Maybe, any finite dimensional $C^{[M]}$-manifold admits a $C^{[M]}$-diffeomorphic real analytic structure. This would follow from:

Conjecture. *Let X be a finite dimensional real analytic manifold. Consider the space $C^{[M]}(X,\mathbb{R})$ of all $C^{[M]}$-functions on X, equipped with the (obvious) Whitney $C^{[M]}$-topology. Then $C^\omega(X,\mathbb{R})$ is dense in $C^{[M]}(X,\mathbb{R})$.*

This conjecture is the analogue of [13, Proposition 8]. It was proved in the non-quasianalytic Beurling case $C^{(M)}$ for X open in $\mathbb{R}^n$ by [28].

The proofs of the following results are similar to the proofs given in [26, Section 5], using other analytical tools. For the convenience of the reader, we give full proofs here, sometimes with more details.

9.4. **Spaces of $C^{[M]}$-sections.** Let $p : E \to B$ be a $C^{[M]}$ vector bundle (possibly infinite dimensional). The space $C^{[M]}(B \leftarrow E)$ of all $C^{[M]}$-sections is a convenient vector space with the structure induced by

$$C^{[M]}(B \leftarrow E) \to \prod_\alpha C^{[M]}(u_\alpha(U_\alpha), V)$$
$$s \mapsto \mathrm{pr}_2 \circ \psi_\alpha \circ s \circ u_\alpha^{-1}$$

where $B \supseteq U_\alpha \xrightarrow{u_\alpha} u_\alpha(U_\alpha) \subseteq W$ is a $C^{[M]}$-atlas for B which we assume to be modeled on a convenient vector space W, and where $\psi_\alpha : E|_{U_\alpha} \to U_\alpha \times V$ form a vector bundle atlas over charts U_α of B.

Lemma. *Assume Hypothesis 9.1. Let D be the open unit ball in a Banach space. A mapping $c : D \to C^{[M]}(B \leftarrow E)$ is a $C^{[M]}$-plot if and only if $c^\wedge : D \times B \to E$ is $C^{[M]}$.*

Proof. By the description of the structure on $C^{[M]}(B \leftarrow E)$ we may assume by Lemma 5.1 that B is c^∞-open in a convenient vector space W and that $E = B \times V$. Then we have $C^{[M]}(B \leftarrow B \times V) \cong C^{[M]}(B, V)$. Thus the statement follows from the exponential law, i.e., Theorem 5.2. $\square$

Let $U \subseteq E$ be an open neighborhood of $s(B)$ for a section s and let $q : F \to B$ be another vector bundle. The set $C^{[M]}(B \leftarrow U)$ of all $C^{[M]}$-sections $s' : B \to E$ with $s'(B) \subseteq U$ is c^∞-open in the convenient vector space $C^{[M]}(B \leftarrow E)$ if B is compact and thus finite dimensional, since then it is open in the coarser compact-open topology. An immediate consequence of the lemma is the following: If $U \subseteq E$ is an open neighborhood of $s(B)$ for a section s and if $f : U \to F$ is a fiber respecting $C^{[M]}$-mapping where $F \to B$ is another vector bundle, then $f_* : C^{[M]}(B \leftarrow U) \to C^{[M]}(B \leftarrow F)$ is $C^{[M]}$ on the open neighborhood $C^{[M]}(B \leftarrow U)$ of s in $C^{[M]}(B \leftarrow E)$. We have $(d(f_*)(s)v)_x = d(f|_{U\cap E_x})(s(x))(v(x))$.

Theorem 9.1. *Assume Hypothesis 9.1. Let A and B be finite dimensional $C^{[M]}$-manifolds with A compact and B equipped with a $C^{[M]}$ Riemann metric. Then the space $C^{[M]}(A,B)$ of all $C^{[M]}$-mappings $A \to B$ is a $C^{[M]}$-manifold modeled on convenient vector spaces $C^{[M]}(A \leftarrow f^*TB)$ of $C^{[M]}$-sections of pullback bundles along $f : A \to B$. Moreover, a mapping $c : D \to C^{[M]}(A,B)$ is a $C^{[M]}$-plot if and only if $c^\wedge : D \times A \to B$ is $C^{[M]}$.*

If the $C^{[M]}$-structure on B is induced by a real analytic structure, then there exists a real analytic Riemann metric which in turn is $C^{[M]}$.

Proof. $C^{[M]}$-vector fields have $C^{[M]}$-flows by Section 9.2; applying this to the geodesic spray we get the $C^{[M]}$ exponential mapping $\exp : TB \supseteq U \to B$ of the Riemann metric, defined on a suitable open neighborhood of the zero section. We may assume that U is chosen in such a way that $(\pi_B, \exp) : U \to B \times B$ is a $C^{[M]}$-diffeomorphism onto an open neighborhood V of the diagonal, by the $C^{[M]}$ inverse function theorem, see Section 9.2.

For $f \in C^{[M]}(A, B)$ we consider the pullback vector bundle

$$\begin{tikzcd} A \times TB & A \times_B TB \arrow[l, hook'] \arrow[r, equal] & f^*TB \arrow[r, "\pi_B^* f"] \arrow[d, "f^*\pi_B"'] & TB \arrow[d, "\pi_B"] \\ & & A \arrow[r, "f"] & B \end{tikzcd}$$

Then the convenient space of sections $C^{[M]}(A \leftarrow f^*TB)$ is canonically isomorphic to the space $C^{[M]}(A, TB)_f := \{h \in C^{[M]}(A, TB) : \pi_B \circ h = f\}$ via $s \mapsto (\pi_B^* f) \circ s$ and $(\mathrm{Id}_A, h) \hookleftarrow h$. Now let

$$\begin{aligned} U_f &:= \{g \in C^{[M]}(A, B) : (f(x),\ g(x)) \in V \text{ for all } x \in A\}, \\ u_f &: U_f \to C^{[M]}(A \leftarrow f^*TB), \\ u_f(g)(x) &= (x, \exp_{f(x)}^{-1}(g(x))) = (x, ((\pi_B, \exp)^{-1} \circ (f, g))(x)). \end{aligned}$$

Then $u_f : U_f \to \{s \in C^{[M]}(A \leftarrow f^*TB) : s(A) \subseteq f^*U = (\pi_B^* f)^{-1}(U)\}$ is a bijection with inverse $u_f^{-1}(s) = \exp \circ (\pi_B^* f) \circ s$, where we view $U \to B$ as a fiber bundle. The set $u_f(U_f)$ is c^∞-open in $C^{[M]}(A \leftarrow f^*TB)$ for the topology described above in Section 9.4, since A is compact and the push forward u_f is $C^{[M]}$, since it respects $C^{[M]}$-plots, by the lemma in Section 9.4.

Now we consider the atlas $(U_f, u_f)_{f \in C^{[M]}(A,B)}$ for $C^{[M]}(A, B)$. Its chart change mappings are given for $s \in u_g(U_f \cap U_g) \subseteq C^{[M]}(A \leftarrow g^*TB)$ by

$$\begin{aligned} (u_f \circ u_g^{-1})(s) &= (\mathrm{Id}_A, (\pi_B, \exp)^{-1} \circ (f, \exp \circ (\pi_B^* g) \circ s)) \\ &= (\tau_f^{-1} \circ \tau_g)_*(s), \end{aligned}$$

where $\tau_g(x, Y_{g(x)}) := (x, \exp_{g(x)}(Y_{g(x)}))$ is a $C^{[M]}$-diffeomorphism $\tau_g : g^*TB \supseteq g^*U \to (g \times \mathrm{Id}_B)^{-1}(V) \subseteq A \times B$ which is fiber respecting over A. The chart change $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ is defined on an open subset and it is also $C^{[M]}$, since it respects $C^{[M]}$-plots, by the lemma in Section 9.4.

Finally for the topology on $C^{[M]}(A, B)$ we take the identification topology from this atlas (with the c^∞-topologies on the modeling spaces $C^{[M]}(A \leftarrow f^*TB)$), which is obviously finer than the compact-open topology and thus Hausdorff.

The equation $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ shows that the $C^{[M]}$-structure does not depend on the choice of the $C^{[M]}$ Riemannian metric on B.

The statement on $C^{[M]}$-plots follows from the lemma in Section 9.4. □

Corollary 9.2. *Assume Hypothesis 9.1. Let A_1, A_2 and B be finite dimensional $C^{[M]}$-manifolds with A_1 and A_2 compact. Then composition*

$$C^{[M]}(A_2, B) \times C^{[M]}(A_1, A_2) \to C^{[M]}(A_1, B), \quad (f, g) \mapsto f \circ g$$

is $C^{[M]}$.

Proof. Composition maps $C^{[M]}$-plots to $C^{[M]}$-plots, so it is $C^{[M]}$. □

Example 9.3. The result in Corollary 9.2 is best possible in the following sense: If $N = (N_k)$ is another weakly log-convex sequence such that $C^{[N]} \subsetneq C^{[M]}$ (or equivalently, $\inf(N_k/M_k)^{1/k} = 0$ and $\sup(N_k/M_k)^{1/k} < \infty$), then composition

$$C^{[M]}(S^1,\mathbb{R}) \times C^{[M]}(S^1,S^1) \to C^{[M]}(S^1,\mathbb{R}), \quad (f,g) \mapsto f \circ g$$

is **not** $C^{[N]}$ with respect to the canonical real analytic manifold structures.

Namely, there exists $f \in C^{[M]}(S^1,\mathbb{R}) \setminus C^{[N]}(S^1,\mathbb{R})$. We consider f as a periodic function $\mathbb{R} \to \mathbb{R}$. The universal covering space of $C^{[M]}(S^1,S^1)$ consists of all $2\pi\mathbb{Z}$-equivariant mappings in $C^{[M]}(\mathbb{R},\mathbb{R})$, namely the space of all $g+\mathrm{Id}_{\mathbb{R}}$ for 2π-periodic $g \in C^{[M]}$. Thus $C^{[M]}(S^1,S^1)$ is a real analytic manifold and $t \mapsto (x \mapsto x+t)$ induces a real analytic curve c in $C^{[M]}(S^1,S^1)$. But $f_* \circ c$ is not $C^{(N)}$ (resp. $C^{\{N\}}$) since:

$$\frac{(\partial_t^k|_{t=0}(f_* \circ c)(t))(x)}{k!\rho^k N_k} = \frac{\partial_t^k|_{t=0} f(x+t)}{k!\rho^k N_k} = \frac{f^{(k)}(x)}{k!\rho^k N_k}$$

which is unbounded in k for x in a suitable compact set and for some (resp. all) $\rho > 0$, since $f \notin C^{(N)}$ (resp. $f \notin C^{\{N\}}$).

Theorem 9.4. *Assume Hypothesis 9.1. Let A be a compact (thus finite dimensional) $C^{[M]}$-manifold. Then the group $\mathrm{Diff}^{[M]}(A)$ of all $C^{[M]}$-diffeomorphisms of A is an open subset of the $C^{[M]}$-manifold $C^{[M]}(A,A)$. Moreover, it is a $C^{[M]}$-regular $C^{[M]}$ Lie group: Inversion and composition are $C^{[M]}$. Its Lie algebra consists of all $C^{[M]}$-vector fields on A, with the negative of the usual bracket as Lie bracket. The exponential mapping is $C^{[M]}$. It is not surjective onto any neighborhood of Id_A.*

Following [24], see also [23, 38.4], a $C^{[M]}$-Lie group G with Lie algebra $\mathfrak{g} = T_eG$ is called $C^{[M]}$-regular if the following holds:

- For each $C^{[M]}$-curve $X \in C^{[M]}(\mathbb{R},\mathfrak{g})$ there exists a $C^{[M]}$-curve $g \in C^{[M]}(\mathbb{R},G)$ whose right logarithmic derivative is X, i.e.,

$$\begin{cases} g(0) & = e \\ \partial_t g(t) & = T_e(\mu^{g(t)})X(t) = X(t).g(t) \end{cases}$$

  The curve g is uniquely determined by its initial value $g(0)$, if it exists.
- Put $\mathrm{evol}^r_G(X) = g(1)$, where g is the unique solution required above. Then $\mathrm{evol}^r_G : C^{[M]}(\mathbb{R},\mathfrak{g}) \to G$ is required to be $C^{[M]}$ also.

Proof. The group $\mathrm{Diff}^{[M]}(A)$ is c^∞-open in $C^{[M]}(A,A)$, since the C^∞-diffeomorphism group $\mathrm{Diff}(A)$ is c^∞-open in $C^\infty(A,A)$, by [23, 43.1], and since $\mathrm{Diff}^{[M]}(A) = \mathrm{Diff}(A) \cap C^{[M]}(A,A)$, by Section 9.2. So $\mathrm{Diff}^{[M]}(A)$ is a $C^{[M]}$-manifold and composition is $C^{[M]}$, by Theorem 9.1 and Corollary 9.2. To show that inversion is $C^{[M]}$ let c be a $C^{[M]}$-plot in $\mathrm{Diff}^{[M]}(A)$. By Theorem 9.1, the mapping $c^\wedge : D \times A \to A$ is $C^{[M]}$ and $(\mathrm{inv} \circ c)^\wedge : D \times A \to A$ satisfies the Banach manifold implicit equation $c^\wedge(t, (\mathrm{inv} \circ c)^\wedge(t,x)) = x$ for $x \in A$. By the Banach $C^{[M]}$ implicit function theorem, see Section 9.2, the mapping $(\mathrm{inv} \circ c)^\wedge$ is locally $C^{[M]}$ and thus $C^{[M]}$. By Theorem 9.1 again, $\mathrm{inv} \circ c$ is a $C^{[M]}$-plot in $\mathrm{Diff}^{[M]}(A)$. So $\mathrm{inv} : \mathrm{Diff}^{[M]}(A) \to \mathrm{Diff}^{[M]}(A)$ is $C^{[M]}$. The Lie algebra of $\mathrm{Diff}^{[M]}(A)$ is the convenient vector space of all $C^{[M]}$-vector fields on A, with the negative of the usual Lie bracket (compare with the proof of [23, 43.1]).

To show that $\operatorname{Diff}^{[M]}(A)$ is a $C^{[M]}$-regular Lie group, we choose a $C^{[M]}$-plot in the space of $C^{[M]}$-curves in the Lie algebra of all $C^{[M]}$ vector fields on A, that is $c : D \to C^{[M]}(\mathbb{R}, C^{[M]}(A \leftarrow TA))$. By the lemma in Section 9.4, the plot c corresponds to a $(D\times\mathbb{R})$-time-dependent $C^{[M]}$ vector field $c^{\wedge\wedge} : D\times\mathbb{R}\times A \to TA$. Since $C^{[M]}$-vector fields have $C^{[M]}$-flows and since A is compact, $\operatorname{evol}^r(c^\wedge(s))(t) = \operatorname{Fl}_t^{c^\wedge(s)}$ is $C^{[M]}$ in all variables, by Section 9.2. Thus $\operatorname{Diff}^{[M]}(A)$ is a $C^{[M]}$-regular $C^{[M]}$ Lie group.

The exponential mapping is evol^r applied to constant curves in the Lie algebra, i.e., it consists of flows of autonomous $C^{[M]}$ vector fields. That the exponential mapping is not surjective onto any $C^{[M]}$-neighborhood of the identity follows from [23, 43.5] for $A = S^1$. This example can be embedded into any compact manifold, see [12]. □

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