Habilitationsschrift

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Perturbation theory for polynomials and linear operators
&
The convenient setting for Denjoy–Carleman differentiable mappings

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Preface

Introduction

This *Habilitationsschrift* consists of my recent research papers [**LR07**], [**Rai09d**], [**Rai09f**], [**Rai09e**], [**KMR09a**], [**KMR09b**], [**KMR09d**], and [**KMR09c**]. The first five articles are published, resp. accepted for publication, in refereed journals. The last three very recent papers are submitted. They represent a continuation and (in some sense) a completion of the work initiated and performed in the first five articles, and should therefore be included in this collection.

The work presented here centers on two main themes which are intertwined: first the perturbation theory for polynomials and linear operators, and secondly the convenient setting for Denjoy–Carleman differentiable mappings. Indeed, in order to extend (in appropriate form) the perturbation results, obtained first for polynomials and matrices, also to unbounded linear operators on an infinitely dimensional Hilbert space, it was necessary to develop calculus for ultradifferentiable (Denjoy–Carleman) mappings beyond Banach spaces.

My research papers [**KLMR05**], [**KLMR06**], [**HR06**], [**KLMR08b**], [**Rai09b**], [**Rai09c**], [**Rai09a**], [**KLMR08a**], and [**LMR09**] are not included in the *Habilitationsschrift*, but they are related and at least some are implicitly incorporated, especially in [**LR07**] and [**Rai09d**].

The treatise is structured into three parts. Part 1 comprises the papers [**LR07**], [**Rai09d**], [**Rai09f**], and [**Rai09e**] which contribute to the **perturbation theory for polynomials and matrices**. Part 2 consists of [**KMR09a**] and [**KMR09b**] in which we develop **the convenient setting for Denjoy–Carleman classes**. In Part 3, i.e. [**KMR09d**] and [**KMR09c**], we are thus able to prove the counterparts in the **perturbation theory for unbounded operators** of the results found in the first part. Evidently, this division is not to be understood in the most stringent sense; for instance, [**Rai09d**] might as well belong to the third part.

The following sections in this preface are related to the three parts and provide an overview of the contained articles. The bibliographies are independently attached for each paper and also for the preface. If the cited paper can be found in this collection, the citation is supplemented with a page reference. All other references are confined to the preface.

1. Perturbation theory for polynomials and matrices

By *perturbation theory for polynomials* we understand the study of the regularity of the roots of a polynomial depending on parameters. This is an old topic with important applications, foremost in the perturbation theory for linear operators and in PDEs. In the last decade several new contributions to this subject appeared. Some of them are based on a recent deeper understanding of resolution of singularities, which opens new ways to study perturbation of polynomials.

In full generality the problem reads as follows: Consider a family of univariate monic polynomials

$$P(x)(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j(x) z^{n-j}, \tag{1.1}$$

where the coefficients $a_j : U \to \mathbb{C}$ are complex valued functions defined in an open subset $U \subseteq \mathbb{R}^q$. Given that the coefficients a_j are regular (of some kind), is it possible to find n regular functions $\lambda_j : U \to \mathbb{C}$ such that $\lambda_1(x), \ldots, \lambda_n(x)$ represent the roots of $P(x)(z) = 0$ for all $x \in U$?[1]

For a long time the problem was only studied under the additional assumption of hyperbolicity: P is called *hyperbolic* if all roots λ_i are real. [**LR07**], p. 3, and [**Rai09f**], p. 39, contribute to the hyperbolic perturbation problem. Until recently only sparse results in very special situations were known in the general case. A systematic study of the perturbation theory for *complex* polynomials (and its applications to the perturbation theory for *normal* operators) was initiated in [**Rai09d**], p. 23, and continued in [**Rai09e**], p. 57 (with some contributions in [**Rai09f**], p. 39).

For the greater part the contributions by the author to this topic are guided by geometrical ideas.

1.1. Hyperbolic polynomials. The notion of hyperbolic polynomials originates from the theory of partial differential equations. It probably appeared for the first time in the fundamental paper [**Går51**]. This (more general than our) notion of hyperbolicity reflects an algebraic condition necessary for the well-posedness of a Cauchy problem[2] (see also [**Går59**], [**Hör63, Hör83b**], and [**ABG70, ABG73**]). Hyperbolic polynomials have also recently found important applications in convex optimization and semi-definite programming.

Let us assume that P in (1.1) is hyperbolic (i.e. has all roots real). The study of the smoothness of the roots of hyperbolic polynomials depending on a parameter started with Rellich's seminal contributions to the perturbation theory for linear operators (see [**Rel37a, Rel37b, Rel39, Rel40, Rel42, Rel69**]). In [**Rel37a**] he showed that real analytic curves of hyperbolic polynomials admit real analytic roots. However, if the coefficients of P are just C^∞, then we cannot hope for C^∞ roots: Glaeser [**Gla63**] gave an example of a non-negative C^∞ function defined in $\mathbb{R}$ not admitting a C^2 square root. Actually, the roots can in general not be parameterized by $C^{1,\alpha}$ functions for any $\alpha > 0$, see [**BBCP06**]. In [**AKLM98**] it was shown that, if the coefficients are C^∞ and no two of the increasingly ordered (thus continuous) roots meet of infinite order of flatness, then C^∞ roots exist.

In general (without nonflatness conditions) we have the following: If the coefficients a_j are C^n (resp. C^{2n}), then the roots admit parameterizations by C^1 (resp. twice differentiable) functions, and this statement is best possible in both assumption and conclusion. This result comprises the contributions of [**Bro79**], [**Man85**], [**KLM03**], and [**COP08**]. The main portion is Bronshtein's theorem proved in [**Bro79**]: The roots of a C^n curve of hyperbolic polynomials P can be chosen differentiable with locally bounded derivative, thus locally Lipschitz. This theorem is

[1] Given that, for some $x_0 \in U$, all roots of $P(x_0)$ are distinct, locally near x_0 the roots have the regularity of the coefficients, by the implicit function theorem. However, at points where different roots come together the problem is highly nontrivial.

[2] A polynomial P of degree m in n variables $\xi = (\xi_1, \ldots, \xi_n)$ and with principal part P_m is said to be *hyperbolic with respect to a real vector* N, if $P_m(N) \neq 0$ and there is τ_0 such that $P(\xi + \tau N) \neq 0$, if $\xi \in \mathbb{R}^n$ and $\operatorname{Im}(\tau) < \tau_0$. The Cauchy problem for the differential operator $P(D)$ (where $D_j = -i\partial_j$) with data on a non-characteristic hyperplane $\langle x \mid N \rangle = 0$ (i.e., $P_m(N) \neq 0$) cannot be solved in general, unless P is hyperbolic with respect to N.

very delicate and only poorly understood in the literature so far, although it has important consequences in PDE theory[3]. See [**Rai**] for a detailed presentation. Since local Lipschitzness can be tested along C^∞ curves (cf. [**Bom67**] and [**KM97**]), we immediately get a multiparameter version: The increasingly ordered roots of a C^n (multiparameter) family of hyperbolic polynomials P are locally Lipschitz. In [**Wak86**] a completely different proof of this result is given[4]. In the multiparameter case we cannot expect the roots to be C^1, even when the coefficients are real analytic[5]. However, due to [**KP08**], if P is real analytic, there exists a modification $\Phi : W \to U$, namely a locally finite composition of blow-ups with smooth centers, such that the roots of $P \circ \Phi$ can be parameterized locally by real analytic functions. Moreover, [**KP08**] contains a new proof for the multiparameter version of Bronshtein's theorem for real analytic coefficients using resolution of singularities.

Further results on the perturbation problem for hyperbolic polynomials appeared in [**Die70**], [**CC04**], [**BCP06**], [**ST06a, ST06b**], [**Tar06**].

Contributions by the author. The paper [**LR07**], p. 3, investigates the smoothness of the roots of curves of hyperbolic polynomials having certain symmetries. By this we mean that the roots $\lambda_1, \ldots, \lambda_n$ of P fulfill some linear relations, i.e., there is a (proper) linear subspace U of $\mathbb{R}^n$ such that $(\lambda_1(t), \ldots, \lambda_n(t)) \in U$, for all $t \in \mathbb{R}$. Then the curve P lies in the semialgebraic subset $E(U)$ of $E(\mathbb{R}^n) \subseteq \mathbb{R}^n$, where $E = (E_1, \ldots, E_n) : \mathbb{R}^n \to \mathbb{R}^n$ and E_i denotes the i-th elementary symmetric function.[6] The symmetries of the roots of P are represented by the action of the group W on U which is inherited from the action of the symmetric group S_n on $\mathbb{R}^n$ by permuting the coordinates: $W = W(U) := N(U)/Z(U)$, where $N(U) := \{\tau \in \mathrm{S}_n : \tau.U = U\}$ and $Z(U) := \{\tau \in \mathrm{S}_n : \tau.x = x \text{ for all } x \in U\}$.

If the restrictions $E_i|_U$, $1 \le i \le n$, generate the algebra $\mathbb{R}[U]^W$ of W-invariant polynomials on U, we gave in [**LR07**], p. 3, refined conditions for the existence of a C^∞ parameterization of the roots of P. These conditions were formulated in terms of the two natural stratifications carried by U and $E(U) = U/W$: the orbit type stratification with respect to W and the restriction of the orbit type stratification with respect to S_n (also called *ambient* stratification). We proved that in general the orbit type stratification is coarser. Now we could apply previous work: By a result in [**AKLM00**], a C^∞ curve P in $U/W = E|_U(U) \subseteq \mathbb{R}^n$ admits a C^∞ lift λ to U (i.e. $P = E \circ \lambda$) if P is *normally nonflat* with respect to the orbit type stratification, i.e., (roughly speaking) P does not meet lower dimensional orbit type strata with infinite order of flatness. Evidently, the lift λ provides a parameterization of the roots of P. The condition that no two roots of P meet of infinite order is equivalent to normal nonflatness with respect to the ambient stratification. The orbit type stratification being coarser than the ambient stratification, we obtained weaker conditions guaranteeing the existence of C^∞ roots. It might happen that roots of P meet of infinite order, while P is not normally nonflat with respect to the orbit

[3] It enabled Bronshtein [**Bro80**] to prove well-posedness of the hyperbolic Cauchy problem $P(x, D)u(x) = f(x)$ with non-constant coefficients in Gevrey space G^s with $s = r/(r-1)$, where the multiplicity of the characteristic roots of $P(x, D)$ does not exceed r. It is in general not possible to go beyond the limits of Gevrey spaces of order s.

[4] Actually in [**Wak86**] a more general version is shown: If all a_i are in $C^{k,\alpha}$, where $0 < \alpha \le 1$, then on any open relatively compact set the increasingly ordered roots of P satisfy a Hölder condition with exponent $\min\{1, (k+\alpha)/n\}$.

[5] For example $P(x_1, x_2)(z) = z^2 - (x_1^2 + x_2^2)$, $x_1, x_2 \in \mathbb{R}$.

[6] The space of monic hyperbolic polynomials of degree n can be identified with the orbit space $\mathbb{R}^n/\mathrm{S}_n$ of the standard representation of the symmetric group S_n in $\mathbb{R}^n$ by permuting the coordinates (the roots). By Vieta's formulas, for the coefficients a_j and the roots λ_j of P we have $a_j = E_j(\lambda_1, \ldots, \lambda_n)$. Thus $\mathbb{R}^n/\mathrm{S}_n$ may in turn be identified with the semialgebraic subset $E(\mathbb{R}^n) \subseteq \mathbb{R}^n$.

type stratification. Along the same lines we obtained improved conditions also for C^1 (resp. twice differentiable) roots: instead of P being C^n (resp. C^{2n}) required by the general theory[7], we need less derivatives of the coefficients, if P has certain symmetries. Here we used results previously proven in our papers [**KLMR05**], [**KLMR06**], and [**KLMR08a**].[8]

The paper [**Rai09f**], p. 39, answers questions posed by K. Kurdyka and E. Bierstone: Does a non-negative C^∞ function definable in some o-minimal expansion of the real field[9] admit C^∞ admissible[10] square roots? What are sufficient conditions for the existence of C^p (for $p \in \mathbb{N}$) arrangements of the roots?

The first question is motivated by the observation that all counter-examples (e.g. in [**Gla63**], [**AKLM98**], [**BBCP06**]), which show that C^∞ coefficients do in general not imply the existence of C^∞ roots, are *oscillating* in nature. This means that some iterated derivative switches sign infinitely often near some point, where the multiplicity of the roots changes. Definability excludes oscillation, but an infinitely flat function may be definable in some o-minimal expansion of the reals. Indeed, we proved that definability of the coefficients guarantees C^∞ solvability of C^∞ curves of hyperbolic polynomials. Thus oscillatory behaviour, rather than flat contact alone, is responsible for the loss of smoothness. An essential building block of the proof was the following lemma: If $\mathbb{R} \ni t \mapsto f(t) \in \mathbb{R}$ is definable and continuous, then $t \mapsto t^p f(t)$ belongs to C^p near 0 (for all $p \in \mathbb{N}$).

As for the second question, we provided sufficient conditions for the existence of C^p parameterizations of the roots, in terms of the differentiability of the coefficients and the maximal order of contact of the roots, in both the definable and the non-definable case. In particular, we gave a simple proof of Bronshtein's theorem in the special case of definable coefficients: C^n curves P admit C^1 roots. These conditions are sharp in the definable case and under certain circumstances also in the non-definable case.

The proofs are quite technical, but the principles behind are simple: If not all roots of $P(t_0)$ coincide, then, near t_0, P factors into polynomials of the initial regularity each of which has the property that its roots coincide at t_0.[11] So (by treating each factor separately) we may assume that all roots of $P(t_0)$ coincide. By a change of variables we can assume that $a_1 = 0$ identically.[12] Then all coefficients a_j $(2 \le j \le n)$ must vanish at t_0, and hyperbolicity forces a_j to vanish at least of

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[7] At the time we wrote [**LR07**] it was only known that C^{2n} (resp. C^{3n}) coefficients imply the existence of C^1 (resp. twice differentiable) roots. The sharp conditions were recently established by [**COP08**].

[8] [**AKLM00**] and [**KLMR05, KLMR06, KLMR08a**] study a lifting problem which generalizes the perturbation problem for (curves of) hyperbolic polynomials: Can a smooth curve in the orbit space V/G of an orthogonal finite dimensional representation of a compact Lie group G be lifted smoothly to the representation space V? Here V/G is identified with the semialgebraic subset $\sigma(V) \subseteq \mathbb{R}^n$, where $\sigma = (\sigma_1, \ldots, \sigma_n) : V \to \mathbb{R}^n$ and $\sigma_1, \ldots, \sigma_n$ constitute a system of homogeneous generators of the algebra $\mathbb{R}[V]^G$ of G-invariant polynomials on V.

[9] See [**vdDM96**] for a concise exposition of *o-minimality*, and also [**vdD98**]. Let $\mathcal{M} = \bigcup_{n \in \mathbb{N}_{>0}} \mathcal{M}_n$, where each $\mathcal{M}_n$ is a family of subsets of $\mathbb{R}^n$. Then $\mathcal{M}$ is said to be an *o-minimal structure* on (or *expansion* of) $(\mathbb{R}, +, \cdot)$ if the following conditions are satisfied:

(1) Each $\mathcal{M}_n$ is closed under finite set-theoretical operations.
(2) If $A \in \mathcal{M}_n$ and $B \in \mathcal{M}_m$, then $A \times B \in \mathcal{M}_{n+m}$.
(3) If $A \in \mathcal{M}_{n+m}$ and $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is the natural projection, then $\pi(A) \in \mathcal{M}_m$.
(4) If $f, g_1, \ldots, g_l \in \mathbb{R}[X_1, \ldots, X_n]$, then $\{x : f(x) = 0, g_1(x) > 0, \ldots, g_l(x) > 0\} \in \mathcal{M}_n$.
(5) $\mathcal{M}_1$ consists of all finite unions of open intervals and points.

For a fixed o-minimal structure $\mathcal{M}$, A is $\mathcal{M}$-*definable* if $A \in \mathcal{M}_n$ for some n. A mapping $f : \mathbb{R}^n \supseteq A \to \mathbb{R}^m$ is $\mathcal{M}$-*definable* if its graph is $\mathcal{M}$-definable.

[10] g is an admissible square root of f if $f = g^2$.

[11] This follows from the inverse function theorem and is a kind of Hensel's lemma.

[12] Replace x by $x - a_1/n$. This is sometimes called a Tschirnhausen transformation.

order jr where r is a positive integer. So we can consider the polynomial $P_{(r)}$ with coefficients $a_j(t)/(t-t_0)^{jr}$, thereby loosing up to nr derivatives. If λ_j parameterize the roots of $P_{(r)}$, then $t \mapsto (t-t_0)^r\lambda_j(t)$ parameterize the root of P.[13] Thus we have reduced the problem to $P_{(r)}$. If not all roots of $P_{(r)}(t_0)$ coincide, we can factor $P_{(r)}$ (which lowers the degree) and proceed by induction. Otherwise it turns out that the roots of $P_{(r)}$ must vanish of infinite order at t_0. Then, by the observation above, the roots are as differentiable as we like near t_0 in the definable case. In general we have to exclude those points.

1.2. Complex polynomials. If the hyperbolicity assumption is dropped, much less regularity of the roots of P (given in (1.1)) can be expected. They can in general not satisfy a local Lipschitz condition (even if the coefficients are real analytic)[14], but they may have weaker regularity properties.

The roots of a continuous family P are continuous as a whole[15] and satisfy a Hölder condition of order $1/n$, due to [**Ost40**]. But the single roots do in general not allow continuous parameterizations, if P is non-hyperbolic and depends on more than just one parameter.[16] Continuous curves of polynomials P still admit continuous parameterizations of its roots (e.g. [**Kat76**, II 5.2]).

[**Len75**] studied α-th roots, $\alpha \in \mathbb{R}_{\geq 1}$, of a non-negative C^p function in $\mathbb{R}^q$ which is p-flat at all its zeros and satisfies a weak Lojasiewicz type inequality. Also some results for functions in one real variable are included. But in many examples the roots are actually of much higher differentiability than predicted by those results.[17] This phenomenon was described by [**Mac78**], who determined the actual class of differentiability of $f^{1/r}$ if f is an exactly i-flat C^{m+i} function of one real variable, for special values of r, m, i. In [**Rei80**] the results were shown (in a much shorter way) for all possible values of r, m, i.

[**Spa99**] proved that the roots of C^∞ curves of polynomials P either with degree $n = 2, 3$ (the case $n = 4$ is announced) or of the form $P(t)(z) = z^n - f(t)$ can be parameterized by locally absolutely continuous functions.[18] Absolute continuity is optimal in some sense (see below). Here essential use of the explicit solution formulas available in those cases was made.

In several variables the following was known: Due to [**CJS83**], for each non-negative C^k function $f : U \to \mathbb{R}$, $U \subseteq \mathbb{R}^q$ open, $k \geq 2$, the gradient $\nabla(f^{1/k})$ belongs to L^1_{loc}. It was shown in [**CL03**] that $\nabla(f^{1/k})$ even belongs to $L_w^{k/k-2}$.[19] This result is optimal among L^p spaces and it generalizes Glaeser's classical theorem on the square root of a non-negative function [**Gla63**].[20]

[13] Hereby we gain back r derivatives in the definable case and in very few situations also in the non-definable case.

[14] For example $P(x)(z) = z^2 - x$, $x \in \mathbb{R}$.

[15] The roots λ_j of P form an *unordered* n-tuple of complex numbers $\lambda = [\lambda_1, \dots, \lambda_n]$. They are continuous with respect to the *distance* $d(\lambda, \mu) = \min_{\sigma \in \mathrm{S}_n} \max_{1 \leq j \leq n} |\lambda_j - \mu_{\sigma(j)}|$.

[16] For example $P(x_1, x_2)(z) = z^2 - (x_1 + ix_2)$, $x_1, x_2 \in \mathbb{R}$, $i = \sqrt{-1}$.

[17] In [**Len75**] $(x^4)^{1/2}$ is only C^3.

[18] Actually, only C^5, C^{25}, and C^{2n+1} is needed in the three cases, respectively.

[19] For $1 \leq p < \infty$ and $K \subseteq U$ compact, $f \in L^p_w(K)$ means that

$$\sup_{M>0} M^p \lambda(\{x \in K : |f(x)| > M\}) < \infty,$$

where λ is the Lebesgue measure. For $1 \leq r < p < \infty$ one has $L^p(K) \subseteq L^p_w(K) \subseteq L^r(K)$.

[20] A C^1 function $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ with $f'' \in L^\infty(\mathbb{R})$ satisfies $f'(t)^2 \leq 2f(t)\|f''\|_{L^\infty(\mathbb{R})}$ (Glaeser's inequality). This inequality implies immediately that, if $f \in C^2(U, \mathbb{R}_{\geq 0})$, $U \subseteq \mathbb{R}^q$ open, then $\nabla(f^{1/2}) \in L^\infty_{\text{loc}}(U)$.

Contributions by the author. In [**Rai09d**], p. 23, we proved that any continuous arrangement λ_j of the roots of a C^∞ curve P (which always exists) is actually locally absolutely continuous, if no two λ_j have infinite order of contact. The proof consists of first showing that the roots of P admit a desingularization by means of *local power substitutions*, and secondly ascertaining that absolute continuity is preserved by pullback with the *inverses* of local power substitutions. More precisely: If hyperbolicity is lacking, the coefficients a_j may well all vanish at some point t_0, but not of the required order jr, for some integer r.[21] The idea was to modify P such that its coefficients do vanish of the required order. We proved that, for each t_0 there exists an $N \in \mathbb{N}_{>0}$ such that $t \mapsto P(t_0 \pm (t-t_0)^N)$ admits C^∞ parameterizations of its roots near t_0.[22] Both signs are necessary if N is even; otherwise a loss of information occurs. Now in order to establish local absolute continuity for the roots of P itself, we only had to show that absolute continuity is preserved by pullback with the inverses of local power substitutions.[23]

This conclusion is optimal in the sense that the roots cannot be chosen with first order derivatives in L^p_{loc} for any $1 < p \le \infty$.[24] On the other hand, finding the optimal assumptions on P for admitting locally absolutely continuous roots is an open problem. In particular, it is unclear whether we may drop the condition on the roots not meeting of infinite order.[25] We settled this question in a special case: In [**Rai09f**], p. 39, we found the weakest possible assumptions for locally absolutely continuous roots, if the coefficients of P are definable in an o-minimal expansion of the real field. Remarkably, it suffices that the coefficients be just continuous.[26] Moreover, the roots of definable C^∞ curves P can be desingularized by means of local power substitutions (even when roots meet of infinite order).

Another topic addressed in [**Rai09d**], p. 23, is finding the conditions for the existence of differentiable parameterizations of the roots of P. Evidently, a necessary condition is that there exists a continuous choice of the roots such that whenever two of them meet they meet of order ≥ 1.[27] We showed that this condition is also sufficient, provided that the coefficients a_j of P belong to C^n.

Furthermore, we discussed a reformulation of the problem of finding smooth roots of P in terms of a lifting problem which had been treated in [**AKLM00**] and [**KLMR05, KLMR06, KLMR08a**] (see [8]). Based on the results for the lifting problem we could formulate implicit sufficient conditions on a curve of polynomials P for allowing C^∞, C^1, or twice differentiable parameterizations of its roots, respectively.

In [**Rai09e**], p. 57, we investigated the general case when P depends on several real parameters. Then the single roots will not admit continuous parameterizations (see [16]) and power substitutions alone will not suffice to desingularize the roots of P (also see by [16]). A further construction is required, namely *blow-ups* with smooth centers[28], familiar from *resolution of singularities*. Our goal was to pursue perturbation theory for polynomials P with coefficients a_j as general as possible.

[21] In that case the roots cannot be smooth at t_0.

[22] This is a Puiseux type result.

[23] If $f \in AC([0,r])$ (resp. $f \in AC([-r,0])$), then $t \mapsto f(\sqrt[N]{t}) \in AC([0,r^N])$ (resp. $t \mapsto f(-\sqrt[N]{|t|}) \in AC([-r^N,0])$).

[24] For example $P(x)(z) = z^n - x$, $x \in \mathbb{R}$, if $n \ge \frac{p}{p-1}$, for $1 < p < \infty$, and if $n \ge 2$, for $p = \infty$.

[25] Spagnolo's results [**Spa99**] indicate that it might be true. This problem, which is in some sense the analogue of Bronshtein's theorem for complex polynomials, is of particular interest for PDE theory. It requires new methods.

[26] Actually, any definable continuous function $f : \mathbb{R} \to \mathbb{C}$ is even locally absolutely continuous.

[27] This condition is automatically satisfied if P is hyperbolic.

[28] Note that blow-ups are *invisible* in dimension one; they reduce to the identity mapping.

So we worked within the framework of the *largest* function classes admitting resolution of singularities. Due to [**BM97**] (see also [**BM04**] and [**RSW03**]) it had to be a subring $\mathcal{C}$ of C^∞ that includes polynomial but excludes flat functions (in other words, is *quasianalytic*), and is closed under composition, differentiation, division by a coordinate, and taking the inverse. For instance, $\mathcal{C}$ can be any quasianalytic Denjoy–Carleman class C^M, where the weight sequence M satisfies some mild conditions (see 2.2), in particular, $\mathcal{C}$ can be the class of real analytic functions C^ω.

The first main theorem in [**Rai09e**], p. 57, is the following: If the coefficients of P are $\mathcal{C}$ functions on a $\mathcal{C}$-manifold M, then for each compact subset $K \subseteq M$ there exist a neighborhood W of K and a finite covering $\{\pi_k : U_k \to W\}$ of W by $\mathcal{C}$ mappings, where each π_k is a composite of finitely many mappings each of which is either a *local blow-up*[29] with smooth center or a *local power substitution*[30], such that, for all k, the family of polynomials $P \circ \pi_k$ admits a $\mathcal{C}$ parameterization of its roots.[31] If P is hyperbolic, local blow-ups suffice.[32]

At the core of the proof lies the following line of arguments: The assertion is local. So we may assume that the parameters vary in an open neighborhood of $0 \in \mathbb{R}^q$. We can reduce to the case that all roots of $P(0)$ coincide and equal 0.[33] Set $A_j(x) = a_j(x)^{\frac{n!}{j}}$. Using resolution of singularities in $\mathcal{C}$, we find a finite covering $\{\pi_k : U_k \to U\}$ of a neighborhood U of 0 by finite composites of local blow-ups, such that, for each k, the non-zero $A_j \circ \pi_k$ and its pairwise non-zero differences $A_i \circ \pi_k - A_j \circ \pi_k$ have simultaneously only *normal crossings*[34]. Let $x_0 \in U_k$. Then in suitable local coordinates $x_0 = 0$ and either $A_j \circ \pi_k = 0$ or $(A_j \circ \pi_k)(x) = x^{\delta_j} A_j^k(x)$, where $A_j^k(0) \neq 0$. It turns out that the multi-indices δ_j are totally ordered[35]. Set $\alpha := \min \delta_j$. If $\alpha = 0$, not all roots of $(P \circ \pi_k)(x_0)$ coincide, and we may use induction. If $\alpha \neq 0$, we have $(A_j \circ \pi_k)(x) = x^\alpha \tilde{A}_j^k(x)$ for all j, where some $\tilde{A}_j^k(0) \neq 0$. For all i, write $\alpha_i/n! = \beta_i/\gamma_i$ where $\beta_i, \gamma_i \in \mathbb{N}$ are relatively prime (and $\gamma_i > 0$). Then after a local power substitution ψ_γ with exponent $\gamma = (\gamma_1, \ldots, \gamma_q)$, each $a_j \circ \pi_k \circ \psi_\gamma$ is divisible by $x^{j\beta}$ (where $\beta = (\beta_1, \ldots, \beta_q)$). Now consider the polynomial P^k with coefficients $x \mapsto (a_j \circ \pi_k \circ \psi_\gamma)(x)/x^{j\beta}$. If P^k admits a $\mathcal{C}$ parameterization λ_j^k of its roots, then the functions $x \mapsto x^\beta \lambda_j^k(x)$ form a choice of $\mathcal{C}$ roots of the family $x \mapsto (P \circ \pi_k \circ \psi_\gamma)(x)$. By construction not all roots of $P^k(0)$ coincide, and we may proceed by induction.

In the second part of [**Rai09e**], p. 57, we used the aforementioned desingularization result in order to investigate the regularity of the roots of the *original* family

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[29] A local blow-up Φ over an open subset U of M means the composition $\Phi = \iota \circ \varphi$ of a blow-up $\varphi : U' \to U$ with smooth center and of the inclusion $\iota : U \to M$.

[30] A local power substitution is a mapping of $\mathcal{C}$-manifolds $\Psi : V \to M$ of the form $\Psi = \iota \circ \psi$, where $\iota : W \to M$ is the inclusion of a coordinate chart W of M and $\psi : V \to W$ is given by

$$(y_1, \ldots, y_q) = \psi_{\gamma,\epsilon}(x_1, \ldots, x_q) := ((-1)^{\epsilon_1} x_1^{\gamma_1}, \ldots, (-1)^{\epsilon_q} x_q^{\gamma_q}),$$

for some $\gamma = (\gamma_1, \ldots, \gamma_q) \in (\mathbb{N}_{>0})^q$ and all $\epsilon = (\epsilon_1, \ldots, \epsilon_q) \in \{0,1\}^q$, where $y_1, \ldots, y_q$ denote the coordinates of W (and $q = \dim M$). Since the involved manifolds are real, we have to consider all possible sign combinations.

[31] If $\mathcal{C} = C^\omega$ it is enough to substitute powers at the last step after all local blow-ups (see [**Rai09e**], p. 57) which follows from the Abhyankar–Jung theorem [**Abh55**], [**Jun08**]. It seems that one can produce a proof of a $\mathcal{C}$ version of the Abhyankar–Jung theorem along the lines of Luengo's approach [**Lue83**]. However, the proof in [**Lue83**] contains a gap as pointed out by Kiyek and Vicente [**KV04**].

[32] For hyperbolic P with real analytic coefficients this was proved in [**KP08**].

[33] By means of the inverse function theorem in $\mathcal{C}$ and Tschirnhausen's transformation (see [11] and [12]).

[34] A function has only *normal crossings* if locally it is just a monomial times a unit.

[35] Here we need that also the pairwise non-zero differences $A_i \circ \pi_k - A_j \circ \pi_k$ have only normal crossings.

of polynomials P. The idea was to apply the same strategy as in [**Rai09d**], p. 23: Find regularity properties shared by $\mathcal{C}$ functions and preserved by pulling back with the *inverses* (where they exist) of local power substitutions and local blow-ups. In one dimension the absolutely continuous functions coincide with the Sobolev space $W^{1,1}$ (possibly after modifying the function on a set of measure zero). So the first guess might be that the roots admit parameterizations in $W^{1,1}_{\text{loc}}$. However, in contrast to the 1-parameter case, multiparameter families P do not allow roots in $W^{1,1}_{\text{loc}}$ (and VMO).[36] Instead we proved that the roots of P admit a parameterization by "piecewise Sobolev $W^{1,1}_{\text{loc}}$" functions. More precisely: Let us denote by $\mathcal{W}^{\mathcal{C}}$ the class of all functions f defined, bounded, and of class $\mathcal{C}$ on the complement of a closed nullset with finite 1-codimensional Hausdorff measure such that its classical gradient belongs to L^1.[37] A $\mathcal{W}^{\mathcal{C}}$ function on an open bounded subset U of $\mathbb{R}^q$ is also a *special function of bounded variation* (SBV).[38] As it turned out, $\mathcal{W}^{\mathcal{C}}$ was suitable for the aforementioned strategy. So we showed that the roots of a $\mathcal{C}$ family of polynomials P admit a parameterization by $\mathcal{W}^{\mathcal{C}}_{\text{loc}}$ functions, and hence by SBV_{loc} functions. As a corollary we obtained that the mapping $\sigma : \mathbb{C}^n \to \mathbb{C}^n$ from roots to coefficients[39] has local $\mathcal{W}^{\mathcal{C}}$ (resp. SBV) sections.

The conclusion is best possible in the following sense: We cannot expect that the roots admit arrangements having gradients in L^p_{loc} for any $1 < p \le \infty$.[40] Then again it is an open question, whether the roots can be parameterized by SBV functions, if the coefficients just belong to a wider function class. But this requires a new approach, since quasianalyticity is a crucial ingredient for resolution of singularities.

We also obtained some new results for *subanalytic* functions.[41] Since any continuous subanalytic function admits a rectilinearization (see [**BM90**] and [**Par94**]), the method developed in [**Rai09e**], p. 57, yielded that: Any continuous subanalytic function belongs to $\mathcal{W}^{C^\omega}_{\text{loc}}$ (resp. SBV_{loc}). Moreover, the roots of continuous subanalytic families P admit arrangements in $\mathcal{W}^{C^\omega}_{\text{loc}}$ (resp. SBV_{loc}).

1.3. Normal matrices. Perturbation theory for linear operators is a classical topic with numerous applications in the natural sciences. At the heart of this theory stands the problem of choosing the eigenvalues and the eigenvectors of a family of operators as smoothly as possible. Obviously, regularity properties possessed by the roots of polynomials immediately translate to the same properties for the eigenvalues of matrices. It is remarkable that in many cases the eigenvectors reflect strong regularity properties as well.

The systematic study of the problem started in the 1930s with Rellich's work [**Rel37a, Rel37b, Rel39, Rel40, Rel42, Rel69**] and it culminated with Kato's

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[36] For example $P(x)(z) = z^2 - x$, $x \in \mathbb{C}$. The roots $\pm\sqrt{x}$ must have a jump along some ray. Hence the distributional derivative of $\sqrt{x}$ with respect to angle contains a delta distribution which is not in L^1_{loc}. Moreover, one can show that $\sqrt{x}$ has not *vanishing mean oscillation* (VMO).

[37] For example, the Heaviside function belongs to $\mathcal{W}^{\mathcal{C}}((-1,1))$, but the function $f(x) := \sin 1/|x|$ does not.

[38] An L^1 function has *bounded variation* if its distributional derivative is a finite Radon measure. It is called *special* if the Cantor part of its derivative vanishes. This notion was introduced in [**DGA88**], see also [**AFP00**].

[39] $\sigma = (\sigma_1, \ldots, \sigma_n)$ where $\sigma_i(z) = \sum_{j_1 < \cdots < j_i} z_{j_1} \cdots z_{j_i}$.

[40] For example $P(x)(z) = z^n - x_1 \cdots x_q$, $x = (x_1, \ldots, x_q) \in \mathbb{R}^q$, if $n \ge \frac{p}{p-1}$, for $1 < p < \infty$, and if $n \ge 2$, for $p = \infty$.

[41] Let M be a real analytic manifold. A subset $X \subseteq M$ is called *subanalytic* if each point of M admits a neighborhood V such that $X \cap V$ is a projection of a relatively compact semianalytic set. Let U be an open subanalytic subset of $\mathbb{R}^q$. A function $f : U \to \mathbb{R}$ is called *subanalytic* if the closure in $\mathbb{R}^q \times \mathbb{RP}^1$ of the graph of f is a subanalytic subset of $\mathbb{R}^q \times \mathbb{RP}^1$.

celebrated monograph [**Kat76**]. See also [**Bau72**] for an account of finite dimensional analytic perturbation theory.

Given that the literature on perturbation theory is huge, we will mention here only a small number of known result which are directly related to the contributions by the author.

Let $A = (A_{ij})_{1 \le i,j \le n}$ be a family of complex matrices. If $\mathbb{C} \ni z \mapsto A(z)$ is holomorphic, then all eigenvalues, all eigenprojections, and all eigennilpotents are holomorphic with at most algebraic singularities at discrete points (cf. [**Kat76**, II.1.8]). If $\mathbb{R} \ni t \mapsto A(t)$ is a real analytic curve of Hermitian complex matrices, then, due to [**Rel37a**], the eigenvalues and the eigenvectors of A can be chosen real analytically in t. This no longer true if A is not Hermitian. Due to [**AKLM98**], the eigenvalues and the eigenvectors of a C^∞ curve of Hermitian complex matrices admit C^∞ parameterizations, if no two unequal continuous[42] eigenvalues meet of infinite order. The nonflatness condition in this statement is essential.[43] Sometimes the eigenvalues show a greater regularity than predicted by the corresponding perturbation problem for polynomials: Due to [**Rel69**], the eigenvalues of a C^1 curve of symmetric matrices A can be chosen C^1.[44] In general there do not exist $C^{1,\alpha}$ eigenvalues for $\alpha > 0$, even if the curve of symmetric matrices is C^∞ (see [**AKLM98**, 7.4] and [**KM03**]).

If A depends on several variables, we cannot hope for differentiable eigenvalues.[45] However, [**KP08**] proved that, given a real analytic family $\mathbb{R}^q \supseteq U \ni x \mapsto A(x)$ of symmetric matrices, there exists a modification $\pi : W \to U$, namely a locally finite composite of blow-ups with smooth center, such that $A \circ \pi$ admits a real analytic diagonalization, locally.[46]

Contributions by the author. In [**Rai09d**], p. 23, and [**Rai09e**], p. 57, we used our results on the regularity of the roots of complex polynomials in order to study the perturbation problem for *normal* matrices.

We considered the 1-parameter case in [**Rai09d**], p. 23. Let $\mathbb{R} \ni t \mapsto A(t)$ be a C^∞ (resp. C^ω) curve of normal complex $n \times n$ matrices such that no two unequal continuously chosen eigenvalues meet of infinite order. We showed that for each t_0 there exists a $N \in \mathbb{N}_{>0}$ such that $t \mapsto A(t_0 \pm (t-t_0)^N)$ admits a C^∞ (resp. C^ω) parameterization of its eigenvalues and its eigenvectors. Consequently, the eigenvalues and the eigenvectors of A itself can be parameterized locally absolutely continuously.

In [**Rai09e**], p. 57, the multiparameter case was investigated. Recall that $\mathcal{C}$ stands for a quasianalytic subring of C^∞ that includes polynomial functions and is closed under composition, differentiation, division by a coordinate, and taking the inverse. We proved that the eigenvalues and the eigenvectors of a $\mathcal{C}$ family $M \ni x \mapsto A(x)$, M a $\mathcal{C}$-manifold, of normal complex $n \times n$ matrices A allow a desingularization by means of local blow-ups and local power substitutions: For each compact subset $K \subseteq M$ there exist a neighborhood W of K and a finite covering $\{\pi_k : U_k \to W\}$ of W by $\mathcal{C}$ mappings, where each π_k is a composite of finitely many mappings each of which is either a local blow-up with smooth center or a local power substitution, such that, for all k, the family $A \circ \pi_k$ admits a $\mathcal{C}$

[42] For instance, by ordering them increasingly.

[43] The eigenvectors of $A(t) = e^{-\frac{1}{t^2}} \begin{pmatrix} \cos\frac{2}{t} & \sin\frac{2}{t} \\ \sin\frac{2}{t} & -\cos\frac{2}{t} \end{pmatrix}$, $t \in \mathbb{R} \setminus \{0\}$, $A(0) = 0$, cannot be chosen continuously near 0. The eigenvalues however are C^∞.

[44] For (hyperbolic) polynomials, one needs in general at least C^n coefficients in order to have C^1 roots.

[45] For example $A(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{pmatrix}$, $x_1, x_2 \in \mathbb{R}$.

[46] [**KP08**] contains also a corresponding result for antisymmetric matrices.

parameterization of its eigenvalues and its eigenvectors. For Hermitian A, local blow-ups suffice (if $\mathcal{C} = C^\omega$ this is due to [**KP08**]). We could conclude that the eigenvalues and the eigenvectors of a $\mathcal{C}$ family of normal complex matrices A admit parameterizations by $\mathcal{W}^{\mathcal{C}}_{\text{loc}}$ functions, and, thus, by SBV_{loc} functions.

Note that these desingularization results for the eigenvectors are no longer true, if we do not demand normality[47] of A, or if we do not insist on a condition preventing flat contact of the eigenvalues[43] (such as quasianalyticity).

More contributions to the perturbation theory for matrices are presented in 3.1 below, where they are stated in greater generality.

2. The convenient setting for Denjoy–Carleman classes

One motivation for developing the convenient setting for Denjoy–Carleman differentiable mappings was the intension to extend (in appropriate form) our quasianalytic perturbation results for matrices to unbounded linear operators with compact resolvents and common domain of definition. A thorough execution of that project requires a differential calculus (a convenient setting, see 2.1) for quasianalytic classes of mappings beyond Banach spaces.

We decided to work within the framework of Denjoy–Carleman classes which are described by growth conditions on the iterated derivatives. A different approach to ultradifferentiable functions based on decay properties of the Fourier transform was proposed by [**Beu61**] and modified by [**BMT90**]; here we shall not expand on that.

As it turned out, in order to be able to treat quasianalytic Denjoy–Carleman classes we had first to understand the non-quasianalytic classes. For the latter we developed the convenient setting in [**KMR09a**], p. 91. Utilizing that we succeeded to establish the convenient setting for some quasianalytic Denjoy–Carleman classes in [**KMR09b**], p. 121. Apart from perturbation theory (see 3.1) we gave applications to manifolds of ultradifferentiable mappings.

2.1. Convenient setting. Let $\mathcal{S}$ be a class of mappings (like C^∞, real analytic C^ω, holomorphic $\mathcal{H}$,...). That $\mathcal{S}$ admits a *convenient setting* means essentially that we can extend the class $\mathcal{S}$ to mappings between admissible locally convex vector spaces $E, F, \dots$ so that $\mathcal{S}(E, F)$ is again admissible and we have $\mathcal{S}(E \times F, G)$ canonically $\mathcal{S}$-diffeomorphic to $\mathcal{S}(E, \mathcal{S}(F, G))$ (the *exponential law*). Note that this is the starting point of the classical calculus of variations, where a smooth curve in a space of functions was assumed to be just a smooth function in one variable more. It is also the source of the name convenient calculus. The exponential law and some other obvious properties already determine the convenient calculus. Usually it comes hand in hand with (partly nonlinear) uniform boundedness theorems which are easy $\mathcal{S}$-detection principles.

In the following let $\mathcal{S}$ stand for C^∞, $\mathcal{H}$, or C^ω. The convenient setting for these function classes was established by [**Frö80, Frö81, Kri82, Kri83**], [**KN85**], and [**KM90**], respectively. For the classes $\mathcal{L}\text{ip}^k$ (i.e. all derivatives up to order k exist and are locally Lipschitz) and $C^{k,\alpha}$ (i.e. C^k and the highest derivative is locally Hölder) it was developed by [**FGK83**] and [**Fau91**], but only in a weaker sense. For a comprehensive exposition see [**KM97**] (and also [**FK88**]), for a concise overview without proofs the appendix in [**KMR09a**], p. 91.

[47] Any choice of eigenvectors for $A(x) = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}$, $x \in \mathbb{R}$, has a pole at 0. The two parameter family $A(x_1, x_2) = \begin{pmatrix} 0 & x_1^2 \\ x_2^2 & 0 \end{pmatrix}$, $x_1, x_2 \in \mathbb{R}$, has the eigenvalues $\pm x_1 x_2$. But its eigenvectors cannot be chosen continuously near 0, even after applying blow-ups or power substitutions.

Let E be a locally convex vector space. We consider the final topology with respect to the set $C^\infty(\mathbb{R}, E)$[48] (a curve in E is called C^∞ if all derivatives exist and are continuous — this is a concept without problems). This topology is called the *c^∞-topology* on E. It coincides with the usual Mackey closure topology[49]. In general[50] it is finer than the given locally convex topology, and it is not a vector space topology, since addition is no longer jointly continuous. On Fréchet spaces it coincides with the given locally convex topology.

The class of locally convex vector spaces admissible to convenient $\mathcal{S}$ calculus is the class of *convenient* vector spaces, which satisfy some mild completeness conditions. A locally convex vector space E is said to be *convenient* if it is Mackey complete[51], equivalently, if a curve $c : \mathbb{R} \to E$ is C^∞ if and only if it is scalarwise C^∞.[52] A complex locally convex vector space is called convenient if the underlying real space is convenient.

The main properties of the convenient calculus for $\mathcal{S}$ mappings (where $\mathcal{S} \in \{C^\infty, \mathcal{H}, C^\omega\}$) are the following:

(1) For c^∞-open subsets $U \subseteq E$, $V \subseteq F$, ... in convenient vector spaces we can define $\mathcal{S}$ *mappings*, and the space $\mathcal{S}(U, F)$ is again convenient in a suitable locally convex structure. Any $\mathcal{S}$ mapping is continuous for the c^∞-topologies.[53] If E, F are Banach spaces, $\mathcal{S}(U, F)$ coincides with the classically defined space.[54]

(2) A mapping $f : U \to F$ is $\mathcal{S}$ if and only if $\ell \circ f$ is $\mathcal{S}$ for all ℓ in a subset of E' (the dual consisting of all bounded linear functionals) which describes the bornology. Multilinear $\mathcal{S}$ mappings are exactly the bounded ones. The inclusion $L(E, F) \subseteq \mathcal{S}(E, F)$ gives a bornological embedding (where the first space carries the topology of uniform convergence on bounded sets).

(3) The category of $\mathcal{S}$ mappings is *cartesian closed*, i.e., the exponential law holds: We have a linear $\mathcal{S}$ diffeomorphism $\mathcal{S}(U \times V, G) = \mathcal{S}(U, \mathcal{S}(V, G))$.

(4) Uniform boundedness principles: (i) A mapping $f : U \to L(F, G)$ is $\mathcal{S}$ if and only if $\mathrm{ev}_x \circ f : U \to G$ is $\mathcal{S}$ for all $x \in F$. (ii) A linear mapping $f : E \to \mathcal{S}(V, G)$ is $\mathcal{S}$ (equivalently, bounded) if and only if $\mathrm{ev}_x \circ f : E \to G$ is $\mathcal{S}$ for all $x \in F$.

(5) If $f : U \to F$ is $\mathcal{S}$ then the derivative $df : U \times E \to F$ is $\mathcal{S}$, and also $df : U \to L(E, F)$ is $\mathcal{S}$. The chain rule holds.

(6) The following canonical mappings are $\mathcal{S}$.

$$\begin{aligned}
&\mathrm{ev} : \mathcal{S}(E, F) \times E \to F, \quad (f, x) \mapsto f(x)\\
&\mathrm{ins} : E \to \mathcal{S}(F, E \times F), \quad x \mapsto (y \mapsto (x, y))\\
&(\quad)^\wedge : \mathcal{S}(E, \mathcal{S}(F, G)) \to \mathcal{S}(E \times F, G), \quad f^\wedge(x, y) = f(x)(y)\\
&(\quad)^\vee : \mathcal{S}(E \times F, G) \to \mathcal{S}(E, \mathcal{S}(F, G)), \quad f^\vee(x)(y) = f(x, y)\\
&\mathrm{comp} : \mathcal{S}(F, G) \times \mathcal{S}(E, F) \to \mathcal{S}(E, G), \quad (f, g) \mapsto f \circ g
\end{aligned}$$

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[48] The set $C^\infty(\mathbb{R}, E)$ does not depend on the locally convex topology of E, only on its associated bornology (the system of bounded sets).

[49] The final topology w.r.t. all Mackey convergent sequences $x_n \to x$ (i.e., there exists a sequence $\lambda_n \to \infty$ in $\mathbb{R}$ with $\lambda_n(x_n - x)$ bounded).

[50] On the space of test functions for example.

[51] Mackey Cauchy sequences (i.e., $\lambda_{nm}(x_n - x_m)$ is bounded for some $\lambda_{nm} \to \infty$ in $\mathbb{R}$) converge in E.

[52] $\ell \circ c$ is C^∞ for all continuous (equivalently, bounded) linear functionals ℓ on E.

[53] There are $\mathcal{S}$ mappings which are not continuous w.r.t. the given locally convex topologies. This is unavoidable. For example the evaluation $E \times E^* \to \mathbb{R}$ is jointly continuous if and only if E is normable, but it is always of class $\mathcal{S}$.

[54] Actually, the notion of C^∞ coincides with all other reasonable classical definitions on Fréchet spaces.

$$\mathcal{S}(\ \ ,\ \ ) : \mathcal{S}(F, F_1) \times \mathcal{S}(E_1, E) \to \mathcal{S}(\mathcal{S}(E, F), \mathcal{S}(E_1, F_1)),$$
$$(f, g) \mapsto (h \mapsto f \circ h \circ g)$$
$$\prod : \prod \mathcal{S}(E_i, F_i) \to \mathcal{S}(\prod E_i, \prod F_i), \quad \prod((f_i)_i)((x_i)_i) = (f_i(x_i))_i$$

Usually the hardest part is to prove that the notion of $\mathcal{S}$ mapping used on convenient vector spaces coincides with the classical definition on Banach spaces. For $\mathcal{S} = C^\infty$ a mapping $f : U \to F$ is C^∞ if and only if $f \circ c$ is C^∞ for each $c \in C^\infty(\mathbb{R}, U)$. In finite dimensions ($U \subseteq \mathbb{R}^n$ and $F = \mathbb{R}$) this is due to [**Bom67**]; it is turned into a definition in infinite dimensions. For $\mathcal{S} = \mathcal{H}$ let $\mathbb{D} \subseteq \mathbb{C}$ be the open unit disk and let $\mathcal{H}(\mathbb{D}, E)$ be the space of all mappings $c : \mathbb{D} \to E$ such that $\ell \circ c : \mathbb{D} \to \mathbb{C}$ is holomorphic for each continuous (equivalently, bounded) complex-linear functional ℓ on E. A mapping $f : E \to F$ between complex convenient vector spaces (or c^∞-open sets therein) is called $\mathcal{H}$ if $f \circ c$ is in $\mathcal{H}(\mathbb{D}, F)$ for each $c \in \mathcal{H}(\mathbb{D}, E)$ (cf. [**Fan30, Fan33**]). One can show that a mapping is $\mathcal{H}$ if and only if it is separately $\mathcal{H}$ (generalized Hartog's theorem), so by the classical Hartog's theorem we have recovered the usual definition in finite dimensions. For $\mathcal{S} = C^\omega$ a curve $c : \mathbb{R} \to E$ is called C^ω if $\ell \circ c$ is C^ω for every continuous (equivalently, bounded) linear functional ℓ on E.[55] A mapping $f : U \to F$ is called C^ω if it is C^∞ (i.e., maps C^∞ curves to C^∞ curves) and maps C^ω curves to C^ω curves. Actually, it suffices that a C^∞ mapping f be C^ω along all affine lines in E. Thus, by [**Boc70, Sic70, BS71**], we have recovered the classical definition on Banach spaces. For $S \in \{\mathcal{L}\mathrm{ip}^k, C^{k,\alpha}\}$, a mapping $f : U \to F$ is $\mathcal{S}$ if and only if it is $\mathcal{S}$ along C^∞ curves; but the exponential law, for instance, does not hold in these cases.

2.2. Denjoy–Carleman classes. Denjoy–Carleman differentiable functions form spaces of functions intermediate between real analytic and C^∞. They are described by growth conditions on the Taylor expansions. Under appropriate conditions the fundamental results of calculus still hold: stability under differentiation, composition, solving ODEs, taking the inverse. See the survey [**Thi08**] (also [**KMR09a**], p. 91) and references therein. Denjoy–Carleman classes, more generally ultradifferentiable function classes (and ultradistributions), play an important role in harmonic analysis and PDEs.

Let $M = (M_k)$ be an increasing sequence of positive real numbers with $M_0 = 1$. Let $U \subseteq \mathbb{R}^n$ be open. A Denjoy–Carleman class $C^M(U)$ is the set of functions $f \in C^\infty(U)$ such that, for all compact $K \subseteq U$, there exist positive constants C and ρ such that[56]

$$|\partial^\alpha f(x)| \leq C\rho^{|\alpha|}|\alpha|!M_{|\alpha|} \quad \text{for all } \alpha \in \mathbb{N}^n, x \in K. \tag{2.1}$$

For the constant sequence $M_k = 1$ we get the real analytic functions. The following table relates properties of the weight sequence M with properties of C^M. Note that, for the sake of brevity, the conditions for M therein are not always minimal; e.g., for $C^M(U)$ to be a ring it is enough that M is *weakly log-convex* (i.e., $(k!M_k)_k$ is log-convex). (The mapping $T_a : C^M(U) \to \mathcal{F}_n^M$ is the Taylor series homomorphism at $a \in U$, where $\mathcal{F}_n^M$ denotes the ring of formal power series $F = \sum_\alpha F_\alpha x^\alpha$ in n variables such that, for some $C, \rho > 0$, $|F_\alpha| \leq C\rho^{|\alpha|}M_{|\alpha|}$ for all α.)

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[55] Surprisingly enough one has to deviate from the most obvious notion of real analytic curves (i.e., locally given by power series which converge in the topology of E) in order to get a meaningful theory.

[56] This definition yields Denjoy–Carleman differentiable functions of *Roumieu type*. If we require that for every compact $K \subseteq U$ and *every* $\rho > 0$ there exists $C > 0$ such that (2.1) holds, then we obtain Denjoy–Carleman differentiable functions of *Beurling type*.

Properties of M		**Properties of C^M**
M increasing, $M_0 = 1$, (always assumed below this line)	$\Rightarrow$	$C^\omega(U) \subseteq C^M(U) \subseteq C^\infty(U)$
M is log-convex (always assumed below this line), i.e., $M_k^2 \le M_{k-1}\, M_{k+1}$ for all k. Then: $(M_k)^{1/k}$ is increasing, $M_l\, M_k \le M_{l+k}$ for all l, k, and $M_1^k\, M_k \ge M_j\, M_{\alpha_1} \cdots M_{\alpha_j}$ for $\alpha_i \in \mathbb{N}_{>0}, \alpha_1 + \cdots + \alpha_j = k$.	$\Rightarrow$	$C^M(U)$ is a ring. C^M is closed under composition. C^M is closed under applying the inverse function theorem. C^M is closed under solving ODEs.
$\sup_{k\in\mathbb{N}_{>0}}(M_k/N_k)^{1/k} < \infty$	$\Leftrightarrow$	$C^M(U) \subseteq C^N(U)$
$\sup_{k\in\mathbb{N}_{>0}}(M_k)^{1/k} < \infty$	$\Leftrightarrow$	$C^\omega(U) = C^M(U)$
$\lim_{k\to\infty}(M_k)^{1/k} = \infty$	$\Leftrightarrow$	$C^\omega(U) \subsetneq C^M(U)$
$\sup_{k\in\mathbb{N}_{>0}}(M_{k+1}/M_k)^{1/k} < \infty$ (always assumed below this line)	$\Leftrightarrow$	C^M is closed under derivation.
$\sum_{k=0}^\infty \frac{M_k}{(k+1)M_{k+1}} = \infty$ or, equivalently, $\sum_{k=1}^\infty (\frac{1}{k!\,M_k})^{1/k} = \infty$	$\Leftrightarrow$[57]	C^M is quasianalytic, i.e., $T_a : C^M(U) \to \mathcal{F}_n^M$ is injective (not surjective if $C^\omega(U) \subsetneq C^M(U)$).
$\sum_{k=0}^\infty \frac{M_k}{(k+1)M_{k+1}} < \infty$	$\Leftrightarrow$	C^M is non-quasianalytic. Then C^M partitions of unity exist.
$\lim_{k\to\infty}(M_k)^{1/k} = \infty$ and $\sum_{k=j}^\infty \frac{M_k}{(k+1)M_{k+1}} \le C \frac{M_j}{M_{j+1}}$ for all $j \in \mathbb{N}$ and some C	$\Leftrightarrow$	$C^\omega(U) \subsetneq C^M(U)$ and $T_a : C^M(U) \to \mathcal{F}_n^M$ is surjective, i.e., C^M is strongly non-quasianalytic.
M has moderate growth, i.e., $\sup_{j,k\in\mathbb{N}_{>0}}(\frac{M_{j+k}}{M_j\, M_k})^{1/(j+k)} < \infty$	$\Rightarrow$	C^M is cartesian closed (see below)
M is strongly regular, i.e., it is strongly non-quasianalytic and has moderate growth.	$\Rightarrow$	Whitney's extension theorem holds in C^M.
$\delta > 0$ and $M_k = (k!)^\delta$ for $k \in \mathbb{N}$. Then M is strongly regular.	$\Leftrightarrow$	C^M is the Gevrey class $G^{1+\delta}$.

Note that, if M is log-convex, closed under derivation, and quasianalytic, then $\mathcal{C} = C^M$ admits resolution of singularities.

Let M be log-convex. For any $\rho > 0$ and $K \subseteq U$ compact with smooth boundary,

$$C^M_\rho(K) := \{f \in C^\infty(K) : \|f\|_{\rho,K} < \infty\}$$

with

$$\|f\|_{\rho,K} := \sup\left\{\frac{|\partial^\alpha f(x)|}{\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|}} : \alpha \in \mathbb{N}^n, x \in K\right\}$$

57 This is a version of the famous Denjoy–Carleman theorem [**Den21, Car26**]. For contemporary proofs see for instance [**Hör83a**] or [**Rud87**].

is a Banach space. The space $C^M(U)$ carries the projective limit topology over compact $K \subseteq U$ of the inductive limit over $\rho \in \mathbb{N}_{>0}$:

$$C^M(U) = \varprojlim_{K \subseteq U} \Big(\varinjlim_{\rho \in \mathbb{N}_{>0}} C^M_\rho(K) \Big), \tag{2.2}$$

where $\varinjlim_\rho C^M_\rho(K)$ is a Silva space[58].

Contributions by the author. In [**KMR09a**], p. 91, we proved that *non-quasianalytic* Denjoy–Carleman differentiable functions admit a convenient setting with the properties (1)–(6) as explained in 2.1. More precisely: Let L be non-quasianalytic and log-convex. We soon noticed that the most naive notion of C^L mappings in infinite dimensions (by simply requiring the growth conditions on the derivatives in the obvious way) did not work out: the exponential law and scalar-wise testing would fail. This led us to the following definitions. A curve $c : \mathbb{R} \to E$ in a convenient vector space E is called C^L if $\ell \circ c$ is C^L for all continuous linear functionals $\ell \in E^*$ (equivalently E'). It turned out to be actually sufficient to test with bounded linear functionals which together detect bounded sets.[59] A mapping $f : U \to F$ between convenient vector spaces, U c^∞-open in E, is called C^L if f is C^∞ and it maps C^L curves to C^L curves. Then C^L is obviously stable under composition. We proved that this notion coincides on Banach spaces with the classical definition of C^L given by growth conditions on the derivatives. Moreover, we showed that f is C^L if and only if it is C^L along all C^L curves (so in the definition 'C^∞' is superfluous). The reason for this is the fact[60] that, thanks to C^L partitions of unity for non-quasianalytic L, for any sufficiently fast converging sequences of points x_n and directions v_n we can construct a C^L curve through the points x_n having the v_n as its tangent vectors.

We equipped the space $C^L(U,F)$ with the initial locally convex structure with respect to the family of mappings

$$C^L(U,F) \xrightarrow{C^L(c,\ell)} C^L(\mathbb{R},\mathbb{R}), \quad f \mapsto \ell \circ f \circ c, \quad \ell \in E^*, c \in C^L(\mathbb{R}, U), \tag{2.3}$$

where $C^L(\mathbb{R},\mathbb{R})$ carries the locally convex structure described in (2.2). This structure is weaker than the structure in (2.2), but the bornology is the same. The space $C^L(U,F)$ is convenient.

The uniform boundedness principles for C^L were derived using the closed graph theorem for the webbed space $\varinjlim_{\rho \in \mathbb{N}_{>0}} C^L_\rho(K)$ (where $K \subseteq \mathbb{R}$ compact). As corollaries we obtained (among others) a multitude of different description of the bornology of $C^L(U,F)$.

Not surprisingly, the derivative of a C^L mapping $f : U \to F$ is again C^L, provided that L is closed under derivation.

For cartesian closedness, i.e., $C^L(U \times V, G) \cong C^L(U, C^L(V,G))$, we have to impose an additional condition which is required for the direction 'from left to right' (without that condition the implication is wrong): L must be of *moderate growth*, i.e., $\sup_{j,k \in \mathbb{N}_{>0}} (\frac{L_{j+k}}{L_j\, L_k})^{1/(j+k)} < \infty$.[61] Thus we proved that, if L is non-quasianalytic, log-convex, and of moderate growth, then there is a linear C^L diffeomorphism $C^L(U \times V, G) = C^L(U, C^L(V,G))$. The proof is first carried out for $U = V = G = \mathbb{R}$ and then obtained for the general situation via the structure (2.3).

[58] An inductive limit of Banach spaces such that the canonical mappings are compact.

[59] $\mathcal{T} \subseteq E'$ such that $B \subseteq E$ is bounded if and only if $\ell(B)$ is bounded for all $\ell \in \mathcal{T}$.

[60] It already appeared in [**Bom67**].

[61] Moderate growth essentially means that $\frac{|\partial_1^j \partial_2^k f(x,y)|}{j!k! M_j M_k \rho_1^j \rho_2^k}$ is bounded if and only if $\frac{\|f^{(n)}(z)\|}{n! M_n \rho^n}$ is bounded. Note that moderate growth implies closedness under derivation.

Finally, we gave applications to the theory of manifolds of mappings. Let L be non-quasianalytic, log-convex, and of moderate growth. A C^L-manifold is a C^∞ manifold modelled on a c^∞-open subset of a convenient vector space such that all chart changings are C^L mappings; likewise for C^L-bundles and C^L Lie groups.[62] We proved the following: If A and B are finite dimensional C^L-manifolds with A compact, then the space $C^L(A,B)$ of all C^L mappings $A \to B$ is a C^L-manifold modelled on convenient vector spaces $C^L(A \leftarrow f^*TB)$ of C^L sections of pullback bundles along $f : A \to B$. Moreover, a curve $c : \mathbb{R} \to C^L(A,B)$ is C^L if and only if $c^\wedge : \mathbb{R} \times A \to B$ is C^L. As a corollary, composition $C^L(A_2,B) \times C^L(A_1,A_2) \to C^L(A_1,B)$, $(f,g) \mapsto f \circ g$, is C^L but not better[63] (here A_i, B are finite dimensional C^L-manifolds with A_i compact). For a compact C^L-manifold A, the group $\operatorname{Diff}^L(A)$ of all C^L diffeomorphisms of A is an open subset of the C^L-manifold $C^L(A,A)$. Moreover, it is a C^L-regular[64] C^L Lie group (not better): Inversion and composition are C^L. Its Lie algebra consists of all C^L-vector fields on A, with the negative of the usual bracket as Lie bracket. The exponential mapping is C^L. It is not surjective onto any neighborhood of Id_A.

In [**KMR09b**], p. 121, we developed the convenient setting (with the properties (1)–(6) as explained in 2.1) for some *quasianalytic* Denjoy–Carleman classes. Let Q be a quasianalytic log-convex weight sequence. The lack of C^Q partitions of unity prevented that we just used the approach that had worked in the non-quasianalytic case. Indeed, a mapping which sends C^Q curves to C^Q curves need not be C^Q (even in $\mathbb{R}^2$ and for $C^Q = C^\omega$).[65] In the real analytic case we get a characterization of C^ω if we additionally require that C^∞ curves are sent to C^∞ curves. This is what the C^ω convenient setting is based on. There is also a subtlety in the proof of the C^ω exponential law which was resolved by using that on Banach spaces real analytic mappings extend locally as holomorphic mappings on the complexification. So, in order to follow the strategy for C^ω, one first would have to show that a mapping is C^Q if and only if it is C^∞ and maps C^Q curves to C^Q curves. That is a difficult open problem. And even if that were accomplished there is still the mentioned subtlety, and now the Taylor series will not converge. Consequently, we had to come up with a completely different method.

The idea was to describe quasianalytic classes C^Q as intersections of non-quasianalytic classes. In fact, due to [**Bom65**], for each quasianalytic log-convex weight sequence Q, we have $C^Q = \bigcap_{L \in \mathcal{L}_w(Q)} C^L$, where $\mathcal{L}_w(Q)$ denotes the set of all non-quasianalytic, *weakly log-convex* $L \geq Q$ (weakly log-convex means that $(k!L_k)_k$ is log-convex). But we had to improve on this result: convenient calculus is based on composition, and stability of C^L under composition necessitates log-convexity (instead of just weak log-convexity). Indeed we proved that, under a technical condition on Q, we have $C^Q = \bigcap_{L \in \mathcal{L}(Q)} C^L$, where $\mathcal{L}(Q)$ denotes the set of all non-quasianalytic, *log-convex* $L \geq Q$. If C^Q is representable in this manner, we say that Q is *$\mathcal{L}$-intersectable*. However we did not get all quasianalytic Denjoy–Carleman classes that way, in particular, the real analytic class: The smallest $\mathcal{L}$-intersection,

[62] Note that any finite dimensional (always assumed paracompact) C^∞-manifold admits a C^∞-diffeomorphic real analytic structure, thus also a C^L-structure. We do not know whether any finite dimensional C^L-manifold admits a C^L-diffeomorphic real analytic structure.

[63] If N is another non-quasianalytic log-convex weight sequence of moderate growth with $(N_k/L_k)^{1/k} \searrow 0$ then composition is not C^N.

[64] A C^L Lie group G with Lie algebra $\mathfrak{g} = T_eG$ is called C^L-regular if the following holds: For each C^L-curve $X \in C^L(\mathbb{R},\mathfrak{g})$ there exists a C^L-curve $g \in C^L(\mathbb{R},G)$ (uniquely determined by its initial value $g(0)$) whose right logarithmic derivative is X, and the mapping $\operatorname{evol}^r_G : C^M(\mathbb{R},\mathfrak{g}) \to G$, $\operatorname{evol}^r_G(X) = g(1)$, is C^L.

[65] The mapping $(x,y) \mapsto \frac{xy^2}{x^2+y^2}$ is not differentiable but *arc-analytic*, i.e., analytic along analytic arcs. Arc-analytic functions need not even be continuous, cf. [**BMP91**].

obtained by taking the intersection of all C^L with non-quasianalytic log-convex L, turns out to be the Denjoy–Carleman class C^Q with $Q_k = (k\log(k+e))^k/k!$, and is strictly larger than C^ω (cf. [**Rud62**]).[66] We constructed countably many $\mathcal{L}$-intersectable quasianalytic log-convex weight sequences Q which also satisfy all other conditions necessary for convenient calculus (like moderate growth).

So we were led to the following definition: For Q quasianalytic log-convex $\mathcal{L}$-intersectable, a mapping $f : E \supseteq U \to F$ between convenient vector spaces (U c^∞-open in E) is called C^Q if it is C^L (i.e., it maps C^L curves to C^L curves) for all $L \in \mathcal{L}(Q)$. The space $C^Q(U,F)$ is equipped with the initial convenient structure induced by the family of mappings

$$C^Q(U,F) \longrightarrow C^L(U,F), \quad L \in \mathcal{L}(Q),$$

where $C^L(U,F)$ carries the structure described in (2.3).

With this definition we succeeded to develop the convenient setting for C^Q mappings with the properties (1)–(6) as described in 2.1. In order to do so we had to prove stronger versions of many results of [**KMR09a**], p. 91, for non-quasianalytic L which are *not* derivation closed, and sometimes even *not* log-convex. For the C^Q exponential law $C^Q(U\times V,G) \cong C^Q(U,C^Q(V,G))$, for instance, we could not just use the C^L exponential law, since for an $\mathcal{L}$-intersectable Q of moderate growth, there is no guarantee that each $L \in \mathcal{L}(Q)$ has moderate growth (needed for the exponential law). Instead we used that, (i) for each $L^1, L^2 \in \mathcal{L}_w(Q)$ there is a $L \in \mathcal{L}_w(Q)$ such that $L \le L^1, L^2$, and (ii) for each $L \in \mathcal{L}_w(Q)$ there exists a $L' \in \mathcal{L}_w(Q)$ such that $L'_{j+k} \le C^{j+k} L_j L_k$ for some $C > 0$ and all j,k. We were not able to show (i) for $\mathcal{L}(Q)$ instead of $\mathcal{L}_w(Q)$, and therefore we could not reduce the exponential law to the case $U = V = G = \mathbb{R}$ (as in [**KMR09a**], p. 91). Nevertheless, we successfully reduced to the Banach space situation, where the C^Q structure can equivalently be described by boundedness conditions (in the spirit of (2.2)).

Among other applications we have the following canonical bornological isomorphisms (induced by a flip of variables): Let M be non-quasianalytic log-convex or quasianalytic log-convex $\mathcal{L}$-intersectable; likewise M'. Let E, F be convenient vector spaces and let W_i be c^∞-open subsets in such. Then:[67]

$$\begin{aligned}
&C^M(W_1, C^{M'}(W_2,F)) \cong C^{M'}(W_2, C^M(W_1,F))\\
&C^M(W_1, C^{\infty}(W_2,F)) \cong C^{\infty}(W_2, C^M(W_1,F))\\
&C^M(W_1, C^{\omega}(W_2,F)) \cong C^{\omega}(W_2, C^M(W_1,F))\\
&C^M(W_1, L(E,F)) \cong L(E, C^M(W_1,F))\\
&C^M(W_1, \ell^\infty(X,F)) \cong \ell^\infty(X, C^M(W_1,F))\\
&C^M(W_1, \mathcal{L}\mathrm{ip}^k(X,F)) \cong \mathcal{L}\mathrm{ip}^k(X, C^M(W_1,F))
\end{aligned}$$

Again we gave applications to manifolds of mappings: Let Q be quasianalytic log-convex $\mathcal{L}$-intersectable of moderate growth. The space $C^Q(A,B)$ of all C^Q mappings between finite dimensional C^Q-manifolds (with A compact for simplicity) is again a C^Q-manifold, composition is C^Q, and the group $\mathrm{Diff}^Q(A)$ of all C^Q diffeomorphisms of A is a C^Q-regular C^Q Lie group (not better). In the proofs we used the fact that a mapping between C^Q-manifolds is C^Q if and only if it maps

[66] [**Ban46**] showed that C^ω is the intersection of all C^L, where L runs through all non-quasianalytic sequences with $(k!L_k)^{1/k}$ increasing. Weakly log-convex L fulfill the latter condition.

[67] For a definition of the spaces $\ell^\infty(X,F)$ and $\mathcal{L}\mathrm{ip}^k(W,F)$ see [**FK88**, 3.6.1 and 4.4.1].

C^Q Banach plots to C^Q Banach plots.[68] A C^Q *Banach plot* in a C^Q-manifold X is a C^Q mapping $E \supseteq D \to X$ from an open unit ball D in a Banach space E.

3. Perturbation theory for unbounded operators

Let us resume the discussion of the contributions to the perturbation theory for linear operators started in 1.3. The theory developed in [**KMR09a**], p. 91, and in [**KMR09b**], p. 121, enabled us to generalize our results for matrices to infinite dimensional unbounded operators and to prove several new results.

The analytic perturbation problem for unbounded self-adjoint operators is treated extensively in [**Kat76**]. However, it involves a lengthy struggle with several different notions of analyticity in infinite dimension, which is easily resolved by the convenient approach discussed in 2.1.

3.1. Unbounded normal operators. Let $t \mapsto A(t)$ for $t \in T$ be a parameterized family of unbounded self-adjoint (or normal) operators in a Hilbert space H with common domain of definition and with compact resolvent.

Let $L = (L_k)$ and $Q = (Q_k)$ be increasing sequences of positive real numbers with $L_0 = Q_0 = 1$. Let us assume that L is non-quasianalytic log-convex and that Q is quasianalytic log-convex $\mathcal{L}$-intersectable.

That $A(t)$ is a C^ω, C^L, C^Q, C^∞, or $C^{k,\alpha}$ family of unbounded operators means the following: There is a dense subspace V of the Hilbert space H such that V is the domain of definition of each $A(t)$, and such that $A(t)^* = A(t)$ in the self-adjoint case, or $A(t)$ has closed graph and $A(t)A(t)^* = A(t)^*A(t)$ wherever defined in the normal case. Moreover, we require that $t \mapsto \langle A(t)u, v\rangle$ is of the respective differentiability class for each $u \in V$ and $v \in H$.

If $t \in T = \mathbb{R}$ and all $A(t)$ are self-adjoint then the following holds:

- (A) If $A(t)$ is real analytic in $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ may be parameterized real analytically in t.
- (B) If $A(t)$ is C^∞ in $t \in \mathbb{R}$ and if no two unequal continuously parameterized eigenvalues meet of infinite order at any $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ can be parameterized C^∞ in t.
- (C) If $A(t)$ is C^∞ in $t \in \mathbb{R}$, then the eigenvalues of $A(t)$ may be parameterized twice differentiably in t (not better[69]).
- (D) If $A(t)$ is $C^{1,\alpha}$ in $t \in \mathbb{R}$ for some $\alpha > 0$, then the eigenvalues of $A(t)$ may be parameterized in a C^1 way in t.

Part (A) is due to [**Rel42**] (see also [**Bau72**] and [**Kat76**, VII.3.9]). Part (B) was proved in [**AKLM98**]; the nonflatness condition is essential (see [43]). (C) and (D) were proved in [**KM03**].

Contributions by the author. If $t \in T = \mathbb{R}$ and all $A(t)$ are self-adjoint we have furthermore:

- (E) If $A(t)$ is C^Q in $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ may be parameterized C^Q in t.
- (F) If $A(t)$ is C^L in $t \in \mathbb{R}$ and if no two unequal continuously parameterized eigenvalues meet of infinite order at any $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ can be parameterized C^L in t.

If $t \in T = \mathbb{R}$ and all $A(t)$ are normal then the following holds:

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[68] We have to test along C^L curves for all L in $\mathcal{L}(Q)$, but for those L we do not have cartesian closedness in general. Testing along Banach plots is a workable replacement.

[69] See the example in [**KM03**].

(G) If $A(t)$ is real analytic in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue λ of $A(t_0)$ there exists $N \in \mathbb{N}_{>0}$ such that the eigenvalues near λ of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized real analytically in s near $s = 0$.

(H) If $A(t)$ is C^Q in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue λ of $A(t_0)$ there exists $N \in \mathbb{N}_{>0}$ such that the eigenvalues near λ of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized C^Q in s near $s = 0$.

(I) If $A(t)$ is C^L in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue λ of $A(t_0)$ at which no two unequal continuously arranged eigenvalues meet of infinite order, there exists $N \in \mathbb{N}_{>0}$ such that the eigenvalues near λ of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized C^L in s near $s = 0$.

(J) If $A(t)$ is C^∞ in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue λ of $A(t_0)$ at which no two unequal continuously arranged eigenvalues meet of infinite order, there exists $N \in \mathbb{N}_{>0}$ such that the eigenvalues near λ of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized C^∞ in s near $s = 0$.

(K) If $A(t)$ is C^∞ in $t \in \mathbb{R}$ and no two unequal continuously parameterized eigenvalues meet of infinite order at any $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ can be parameterized by absolutely continuous functions, locally in t.

If $t \in T = \mathbb{R}^n$ and all $A(t)$ are normal then the following holds:

(L) If $A(t)$ is C^ω (resp. C^Q) in $t \in \mathbb{R}^n$, then for each $t_0 \in \mathbb{R}^n$ and for each eigenvalue λ of $A(t_0)$, there exists a finite covering $\{\pi_k : U_k \to W\}$ of a neighborhood W of t_0, where each π_k is a composite of finitely many mappings each of which is either a local blow-up with smooth center or a local power substitution, such that the eigenvalues and the eigenvectors of $A(\pi_k(s))$ can be chosen C^ω (resp. C^Q) in s. If A is self-adjoint, then we do not need power substitutions.

(M) If $A(t)$ is C^ω (resp. C^Q) in $t \in \mathbb{R}^n$, then the eigenvalues and their eigenvectors of $A(t)$ can be parameterized by SBV functions, locally in t.

If $t \in T \subseteq E$, a c^∞-open subset in an infinite dimensional convenient vector space then the following holds:

(N) For $0 < \alpha \leq 1$, if $A(t)$ is $C^{0,\alpha}$ in $t \in T$ and all $A(t)$ are self-adjoint, then the eigenvalues of $A(t)$ may be parameterized in a $C^{0,\alpha}$ way in t.

(O) For $0 < \alpha \leq 1$, if $A(t)$ is $C^{0,\alpha}$ in $t \in T$ and all $A(t)$ are normal, then we have: For each $t_0 \in T$ and each eigenvalue z_0 of $A(t_0)$ consider a simple closed C^1-curve γ in the resolvent set of $A(t_0)$ enclosing only z_0 among all eigenvalues of $A(t_0)$. Then for t near t_0 in the c^∞-topology on T, no eigenvalue of $A(t)$ lies on γ. Let $\lambda(t) = (\lambda_1(t), \dots, \lambda_N(t))$ be the N-tuple of all eigenvalues (repeated according to their multiplicity) of $A(t)$ inside γ. Then $t \mapsto \lambda(t)$ is $C^{0,\alpha}$ for t near t_0 with respect to the non-separating metric

$$d(\lambda, \mu) = \min_{\sigma \in \mathcal{S}_N} \max_{1 \leq i \leq N} |\lambda_i - \mu_{\sigma(i)}|$$

on the space of N-tuples.

(G), (J), and (K) were proved in [**Rai09d**], p. 23. Part (N) was shown in [**KMR09d**], p. 153. The remaining parts (E), (F), (H), (I), (L), (M), and (O) were established in [**KMR09c**], p. 157. Except for (O), they became possible only after the convenient settings of C^L and C^Q mappings were developed; in particular, the uniform boundedness principles. The condition that no two unequal continuously parameterized eigenvalues have infinite order of contact cannot be dropped.[70]

The proofs follow a general scheme: Let C^* denote one of the classes C^ω, C^L, C^Q, C^∞, or $C^{k,\alpha}$. Thanks to the corresponding uniform boundedness principles, the assumption that $t \mapsto \langle A(t)u, v\rangle$ is C^*, for each $u \in V$ and $v \in H$, implies that $t \mapsto A(t)u$ is C^* (as a mapping into H), for each $u \in V$. We proved (again using uniform boundedness principles) that: If $A(t)$ is normal (resp. self-adjoint) and C^* in t, then the resolvent[71] $(t,z) \mapsto (A(t)-z)^{-1} \in L(H,H)$ is C^* on its natural domain, the global resolvent set $\{(t,z) \in T \times \mathbb{C} : (A(t)-z) : V \to H \text{ is invertible}\}$ which is open (and even connected).

Let z be an N-fold eigenvalue of $A(t_0)$. Choose a simple closed C^1 curve γ in the resolvent set of $A(t_0)$ for fixed t_0 enclosing only z among all eigenvalues of $A(t_0)$. Since the global resolvent set is open, no eigenvalue of $A(t)$ lies on γ, for t near t_0. It turns out that

$$t \mapsto -\frac{1}{2\pi i}\int_\gamma (A(t)-z)^{-1}\, dz =: P(t,\gamma) = P(t)$$

is a C^* mapping. Each $P(t)$ is a projection, namely onto the direct sum of all eigenspaces corresponding to eigenvalues of $A(t)$ in the interior of γ, with finite constant rank. So for t in a neighborhood U of t_0 there are equally many eigenvalues in the interior of γ.

The family of N-dimensional complex vector spaces $t \mapsto P(t)(H) \subseteq H$, for $t \in U$, form a C^* Hermitian vector subbundle over U of $U \times H \to U$. Now $A(t)$ maps $P(t)(H)$ to itself; in a C^* local frame it is given by a normal (resp. Hermitian) $N \times N$ matrix parameterized C^* by $t \in U$. Thus the (local) assertions follow from the corresponding results for matrices (cf. 1.3). For (M) and (O) we used results due to [**Wey12**] and [**BDM83**].[72]

Let us conclude with two applications: Let X be a compact C^Q manifold and let $t \mapsto g_t$ be a C^Q curve of C^Q Riemannian metrics on X. Then we get the corresponding C^Q curve $t \mapsto \Delta(g_t)$ of Laplace-Beltrami operators on $L^2(X)$. By (E) the eigenvalues and eigenvectors can be arranged C^Q.

Let Ω be a bounded region in $\mathbb{R}^n$ with C^Q boundary, and let $H(t) = -\Delta + V(t)$ be a C^Q curve of Schrödinger operators with varying C^Q potential and Dirichlet boundary conditions. Then the eigenvalues and eigenvectors can be arranged C^Q.

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[70] See the example in [**KMR09c**], p. 157.

[71] Note that the resolvent $(A(t)-z)^{-1} : H \to H$ is a compact operator for some (equivalently any) (t,z) if and only if the inclusion $\iota : V \to H$ is compact, since $\iota = (A(t)-z)^{-1} \circ (A(t)-z) : V \to H \to H$.

[72] Let A, B be Hermitian $N \times N$ matrices with increasingly ordered eigenvalues $\lambda_i(A)$ and $\lambda_i(B)$, for $1 \le i \le N$. Then (due to [**Wey12**], see also [**Bha97**, III.2.6])

$$\max_j |\lambda_j(A) - \lambda_j(B)| \le \|A - B\|.$$

Here $\| \ \|$ is the operator norm. If A, B are just normal matrices with eigenvalues $\lambda_i(A)$ and $\lambda_i(B)$, then (due to [**BDM83**], see also [**Bha97**, VII.4.1])

$$\min_{\sigma \in \mathcal{S}_N} \max_j |\lambda_j(A) - \lambda_{\sigma(j)}(B)| \le C\|A - B\|$$

for a universal constant C with $1 < C < 3$.

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Part 1

Perturbation theory for polynomials and matrices

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CHOOSING ROOTS OF POLYNOMIALS WITH SYMMETRIES SMOOTHLY

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Abstract. The roots of a smooth curve of hyperbolic polynomials may not in general be parameterized smoothly, even not $C^{1,\alpha}$ for any $\alpha > 0$. A sufficient condition for the existence of a smooth parameterization is that no two of the increasingly ordered continuous roots meet of infinite order. We give refined sufficient conditions for smooth solvability if the polynomials have certain symmetries. In general a C^{3n} curve of hyperbolic polynomials of degree n admits twice differentiable parameterizations of its roots. If the polynomials have certain symmetries we are able to weaken the assumptions in that statement.

1. Introduction

Consider a smooth curve of monic hyperbolic (i.e. all roots real) polynomials with fixed degree n:

$$P(t)(x) = x^n - a_1(t)x^{n-1} + a_2(t)x^{n-2} - \cdots + (-1)^n a_n(t) \qquad (t \in \mathbb{R}).$$

Is it possible to find n smooth functions $x_1(t), \ldots, x_n(t)$ which parameterize the roots of $P(t)$ for each t? It has been shown in [Rel37] that real analytic curves $P(t)$ allow real analytic parameterizations of its roots, and in [AKLM98] that the roots of smooth curves $P(t)$ may be chosen smoothly if no two of the increasingly ordered continuous roots meet of infinite order. In general, as shown in [KLM04], the roots of a C^{3n} curve $P(t)$ of hyperbolic polynomials can be parameterized twice differentiable. That regularity of the roots is best possible: In general no $C^{1,\alpha}$ parameterizations of the roots for any $\alpha > 0$ exist which is shown by examples in [AKLM98], [BBCP06], and [Gla63]. Further references related to that topic are [Bro79], [Man85], and [Wak86].

The space Hyp^n of monic hyperbolic polynomials P of fixed degree n may be identified with a semialgebraic subset in $\mathbb{R}^n$, the coefficients of P being the coordinates. Then $P(t)$ is a smooth curve in $\mathrm{Hyp}^n \subseteq \mathbb{R}^n$. If the curve $P(t)$ lies in some semialgebraic subset of Hyp^n, then it is evident that in general the conditions which guarantee smooth parameterizations of the roots of $P(t)$ are weaker than those mentioned in the previous paragraph. In the present paper we are going to study that phenomenon.

In section 3 we present a class of semialgebraic subsets in spaces of hyperbolic polynomials for which we are able to apply the described strategy. The construction of that class is based on results due to [SS87].

Our main goal is to investigate the problem of finding smooth roots of P under the assumption that the polynomials $P(t)$ satisfy certain symmetries. More precisely, we shall assume that the roots $x_1(t), \ldots, x_n(t)$ of $P(t)$ fulfill some linear relations, i.e., there is a linear subspace U of $\mathbb{R}^n$ such that $(x_1(t), \ldots, x_n(t)) \in U$ for all

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t. Then the curve $P(t)$ lies in the semialgebraic subset $E(U)$ of the space of hyperbolic polynomials $\mathrm{Hyp}^n = E(\mathbb{R}^n) = \mathbb{R}^n/\mathrm{S}_n$ of degree n, where $E = (E_1, \ldots, E_n)$ and E_i denotes the i-th elementary symmetric function. The symmetries of the roots of $P(t)$ are represented by the action of the group W on U which is inherited from the action of the symmetric group S_n on $\mathbb{R}^n$ by permuting the coordinates:

$$W = W(U) := N(U)/Z(U),$$

where $N(U) := \{\tau \in \mathrm{S}_n : \tau.U = U\}$ and $Z(U) := \{\tau \in \mathrm{S}_n : \tau.x = x \text{ for all } x \in U\}$.

Under the additional assumption that the restrictions $E_i|_U$, $1 \le i \le n$, generate the algebra $\mathbb{R}[U]^W$ of W-invariant polynomials on U, we will show that the conditions imposed on $P(t)$ in order to guarantee the existence of a smooth parameterization of its roots may be weakened. These conditions will be formulated in terms of the two natural stratifications carried by U and $E(U) = U/W$: the orbit type stratification with respect to W and the restriction of the orbit type stratification with respect to S_n. The latter will be called ambient stratification. See section 4. It will turn out (section 5) that we may find global smooth parameterizations of the roots of $P(t)$, provided that $P(t)$ is normally nonflat with respect to the orbit type stratification of $E(U) = U/W$ at any t. This condition is in general weaker than the condition found in [AKLM98], since we prove in section 4 that normal nonflatness with respect to the ambient stratification implies normal nonflatness with respect to the orbit type stratification. For a definition of 'normally nonflat' see 2.5.

These improvements are essentially applications of the lifting problem tackled in [AKLM00]. See also [KLMR05] and [KLMR06]. This generalization of the above problem studies the question whether it is possible to lift smoothly a smooth curve in the orbit space V/G of an orthogonal finite dimensional representation of a compact Lie group G into the representation space V. Here the orbit space V/G is identified with the semialgebraic subset $\sigma(V)$ in $\mathbb{R}^n$ given by the image of the orbit map $\sigma = (\sigma_1, \ldots, \sigma_n) : V \to \mathbb{R}^n$, where $\sigma_1, \ldots, \sigma_n$ constitute a system of homogeneous generators of the algebra $\mathbb{R}[V]^G$ of G-invariant polynomials on V. See section 2 for details.

As mentioned before a C^{3n} curve $P(t)$ of hyperbolic polynomials of degree n allows twice differentiable parameterizations of its roots. Using results found for the general lifting problem in [KLMR06], we are able to lower the degree of regularity in the assumption of that statement, if the polynomials $P(t)$ satisfy certain symmetries. See section 6.

A class of examples for which the described refinements apply will be constructed in section 7. For illustration we consider the case when W is a finite reflection group in section 8. Moreover, explicit examples will be treated.

The problem of finding regular roots of families of hyperbolic polynomials has relevance in the perturbation theory of selfadjoint operators (e.g. [Kat76], [KM03], [Rel37]) and in the theory of partial differential equations for the well-posedness of hyperbolic Cauchy problems (e.g. [Bro80], [Hör83]).

2. Preliminaries

2.1. **Representations of compact Lie groups.** Let G be a compact Lie group and let $\rho : G \to \mathrm{O}(V)$ be an orthogonal representation in a real finite dimensional Euclidean vector space V with inner product $\langle \quad | \quad \rangle$. By a classical theorem of Hilbert and Nagata, the algebra $\mathbb{R}[V]^G$ of invariant polynomials on V is finitely generated. So let $\sigma_1, \ldots, \sigma_n$ be a system of homogeneous generators of $\mathbb{R}[V]^G$ of positive degrees $d_1, \ldots, d_n$. Consider the *orbit map* $\sigma = (\sigma_1, \ldots, \sigma_n) : V \to \mathbb{R}^n$. The image $\sigma(V)$ is a semialgebraic set in $Z := \{y \in \mathbb{R}^n : P(y) = 0 \text{ for all } P \in I\}$

where I is the ideal of relations between $\sigma_1, \dots, \sigma_n$. Since G is compact, σ is proper and separates orbits of G, it thus induces a homeomorphism between V/G and $\sigma(V)$, by the following lemma.

Lemma. *Suppose that X and Y are locally compact, Hausdorff spaces and that $f : X \to Y$ is bijective, continuous, and proper. Then f is a homeomorphism.*

Proof. (E.g. [Bre93]) By defining $\tilde{f}(\infty) = \infty$, f extends to a continuous map $\tilde{f} : X \cup \{\infty\} \to Y \cup \{\infty\}$ between the one point compactifications, since it is proper. If $A \subseteq X$ is closed in X, then $A \cup \{\infty\}$ is closed in $X \cup \{\infty\}$ and hence compact. Then, $\tilde{f}(A \cup \{\infty\})$ is compact and hence closed in $Y \cup \{\infty\}$. Consequently, $f(A) = \tilde{f}(A \cup \{\infty\}) \cap Y$ is closed in Y. □

2.2. **Description of $\sigma(V)$.** Let $\langle \quad | \quad \rangle$ denote also the G-invariant dual inner product on V^*. The differentials $d\sigma_i : V \to V^*$ are G-equivariant, and the polynomials $v \mapsto \langle d\sigma_i(v) \mid d\sigma_j(v) \rangle$ are in $\mathbb{R}[V]^G$ and are entries of an $n \times n$ symmetric matrix valued polynomial

$$B(v) := \begin{pmatrix} \langle d\sigma_1(v) \mid d\sigma_1(v) \rangle & \cdots & \langle d\sigma_1(v) \mid d\sigma_n(v) \rangle \\ \vdots & \ddots & \vdots \\ \langle d\sigma_n(v) \mid d\sigma_1(v) \rangle & \cdots & \langle d\sigma_n(v) \mid d\sigma_n(v) \rangle \end{pmatrix}.$$

There is a unique matrix valued polynomial $\tilde{B}$ on Z such that $B = \tilde{B} \circ \sigma$. The following theorem is due to Procesi and Schwarz [PS85].

Theorem. $\sigma(V) = \{z \in Z : \tilde{B}(z)$ *positive semidefinite*$\}$.

This theorem provides finitely many equations and inequalities describing $\sigma(V)$. Changing the choice of generators may change the equations and inequalities, but not the set they describe.

For each $1 \le i_1 < \cdots < i_s \le n$ and $1 \le j_1 < \cdots < j_s \le n$ $(s \le n)$ consider the matrix with entries $\langle d\sigma_{i_p} \mid d\sigma_{j_q} \rangle$ for $1 \le p, q \le s$. Denote its determinant by $\Delta_{i_1,\dots,i_s}^{j_1,\dots,j_s}$. Then, $\Delta_{i_1,\dots,i_s}^{j_1,\dots,j_s}$ is a G-invariant polynomial on V, and thus there is a unique polynomial $\tilde{\Delta}_{i_1,\dots,i_s}^{j_1,\dots,j_s}$ on Z such that $\Delta_{i_1,\dots,i_s}^{j_1,\dots,j_s} = \tilde{\Delta}_{i_1,\dots,i_s}^{j_1,\dots,j_s} \circ \sigma$.

2.3. **The problem of lifting curves.** Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a smooth curve in the orbit space; smooth as curve in $\mathbb{R}^n$. A curve $\bar{c} : \mathbb{R} \to V$ is called lift of c to V, if $c = \sigma \circ \bar{c}$ holds. *The problem of lifting smooth curves over invariants is independent of the choice of a system of homogeneous generators of* $\mathbb{R}[V]^G$ in the following sense: Suppose $\sigma_1, \dots, \sigma_n$ and $\tau_1, \dots, \tau_m$ both generate $\mathbb{R}[V]^G$. Then for all i and j we have $\sigma_i = p_i(\tau_1, \dots, \tau_m)$ and $\tau_j = q_j(\sigma_1, \dots, \sigma_n)$ for polynomials p_i and q_j. If $c^\sigma = (c_1, \dots, c_n)$ is a curve in $\sigma(V)$, then $c^\tau = (q_1(c^\sigma), \dots, q_m(c^\sigma))$ defines a curve in $\tau(V)$ of the same regularity. Any lift $\bar{c}$ to V of the curve c^σ, i.e., $c^\sigma = \sigma \circ \bar{c}$, is a lift of c^τ as well (and conversely):

$$c^\tau = (q_1(c^\sigma), \dots, q_m(c^\sigma)) = (q_1(\sigma(\bar{c})), \dots, q_m(\sigma(\bar{c}))) = (\tau_1(\bar{c}), \dots, \tau_m(\bar{c})) = \tau \circ \bar{c}.$$

2.4. **Stratification of the orbit space.** Let $H = G_v$ be the isotropy group of $v \in V$ and (H) the conjugacy class of H in G which is called the type of an orbit $G.v$. The union $V_{(H)}$ of orbits of type (H) is called an orbit type submanifold of the representation ρ and $V_{(H)}/G$ is called an orbit type submanifold of the orbit space V/G. The collection of connected components of the manifolds $\{V_{(H)}/G\}$ forms a stratification of V/G called orbit type stratification, see [Pfl01], [Sch80]. The semialgebraic subset $\sigma(V) \subseteq \mathbb{R}^n$ is naturally Whitney stratified ([Loj65]). The homeomorphism of V/G and $\sigma(V)$ induced by σ provides an isomorphism between the orbit type stratification of V/G and the primary Whitney stratification of $\sigma(V)$, see [Bie75]. These facts are essentially consequences of the slice theorem, see e.g. [Sch80].

The inclusion relation on the set of subgroups of G induces a partial ordering on the family of conjugacy classes. There is a unique minimum orbit type, the principal orbit type, corresponding to the open and dense submanifold V_{reg} (respectively V_{reg}/G) consisting of regular points, i.e., points where the isotropy representation is trivial. The points in the complement V_{sing} (respectively V_{sing}/G) are called singular.

Theorem ([PS85])**.** *Let $\tilde{B}$ be as in 2.2. The k-dimensional primary strata of $\sigma(V)$ are the connected components of the set $\{z \in \sigma(V) : \operatorname{rank} \tilde{B}(z) = k\}$.*

2.5. **Smooth lifts.** Let us recall some results from [AKLM00].

Let $s \in \mathbb{N}_0$. Denote by A_s the union of all strata X of the orbit space V/G with $\dim X \leq s$, and by I_s the ideal of $\mathbb{R}[Z] = \mathbb{R}[V]^G$ consisting of all polynomials vanishing on A_{s-1}. Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a smooth curve, $t \in \mathbb{R}$, and $s = s(c,t)$ a minimal integer such that, for a neighborhood J of t in $\mathbb{R}$, we have $c(J) \subseteq A_s$. The curve c is called *normally nonflat at t* if there is $f \in I_s$ such that $f \circ c$ is nonflat at t, i.e., the Taylor series of $f \circ c$ at t is not identically zero. A smooth curve $c : \mathbb{R} \to \sigma(V) \subseteq \mathbb{R}^n$ is called *generic*, if c is normally nonflat at t for each $t \in \mathbb{R}$.

It is easy to see, that c is normally nonflat at $t \in \mathbb{R}$ if there is some integer $1 \leq r \leq n$ such that:

(1) The functions $\tilde{\Delta}_{i_1,\ldots,i_k}^{j_1,\ldots,j_k} \circ c$ vanish in a neighborhood of t whenever $k > r$.
(2) There exists a minor $\tilde{\Delta}_{i_1,\ldots,i_r}^{j_1,\ldots,j_r}$ such that $\tilde{\Delta}_{i_1,\ldots,i_r}^{j_1,\ldots,j_r} \circ c$ is nonflat at t.

Theorem. *Let $c : \mathbb{R} \to \sigma(V) \subseteq \mathbb{R}^n$ be a smooth curve which is normally nonflat at $t \in \mathbb{R}$. Then there exists a smooth lift $\bar{c}$ in V of c, locally near t. If c is generic then there exists a global smooth lift $\bar{c}$ of c.*

2.6. **Smooth roots.** In the special case that the symmetric group S_n is acting on $\mathbb{R}^n$ by permuting the coordinates there is the following interpretation of the described lifting problem. As generators of $\mathbb{R}[\mathbb{R}^n]^{\mathrm{S}_n}$ we may take the elementary symmetric functions

$$E_j(x) = \sum_{1 \leq i_1 < \cdots < i_j \leq n} x_{i_1} \cdots x_{i_j} \qquad (1 \leq j \leq n),$$

which constitute the coefficients a_j of a monic polynomial

$$P(x) = x^n - a_1 x^{n-1} + \cdots + (-1)a_n$$

with roots $x_1, \ldots, x_n$ via Vieta's formulas. Then a curve in the orbit space $\mathbb{R}^n/\mathrm{S}_n = E(\mathbb{R}^n)$ corresponds to a curve $P(t)$ of monic polynomials of degree n with only real roots (such polynomials are called *hyperbolic*), and a lift of $P(t)$ may be interpreted as a parameterization of the roots of $P(t)$.

The first n Newton polynomials

$$N_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j^i$$

which are related to the elementary symmetric functions by

$$N_k - N_{k-1}E_1 + N_{k-2}E_2 + \cdots + (-1)^{k-1} N_1 E_{k-1} + (-1)^k k E_k = 0 \quad (k \geq 1) \tag{2.1}$$

constitute a different system of generators of $\mathbb{R}[\mathbb{R}^n]^{\mathrm{S}_n}$. For convenience we shall switch from elementary symmetric functions to Newton polynomials and conversely, if it seems appropriate.

Let us choose $\frac{1}{j}N_j$, $1 \le j \le n$, as generators of $\mathbb{R}[\mathbb{R}^n]^{\mathrm{S}_n}$ and put $\Delta_k := \Delta^{1,\dots,k}_{1,\dots,k}$ and $\tilde\Delta_k := \tilde\Delta^{1,\dots,k}_{1,\dots,k}$. Then ([AKLM98])

$$\Delta_k(x) = \sum_{i_1<\cdots<i_k} (x_{i_1}-x_{i_2})^2 \cdots (x_{i_1}-x_{i_k})^2 \cdots (x_{i_{k-1}}-x_{i_k})^2. \tag{2.2}$$

Theorem ([AKLM98])**.** *Consider a smooth curve $P(t)$, $t \in \mathbb{R}$, of monic hyperbolic polynomials of fixed degree n. Let one of the following two equivalent conditions be satisfied:*

(1) *If two of the increasingly ordered continuous roots meet of infinite order at t_0 then their germs at t_0 are equal.*
(2) *Let k be maximal with the property that the germ at t_0 of $\tilde\Delta_k(P)$ is not 0. Then $\tilde\Delta_k(P)$ is not infinitely flat at t_0.*

Then $P(t)$ is smoothly solvable near $t = t_0$. If (1) or (2) are satisfied for any $t_0 \in \mathbb{R}$, then the roots of P may be chosen smoothly globally, and any two choices differ by a permutation.

Lemma. *Condition* (1) *(and thus condition* (2)*) in the above theorem is satisfied if and only if P is normally nonflat at t_0 as curve in $E(\mathbb{R}^n) = \mathbb{R}^n/\,\mathrm{S}_n$.*

Proof. Let P be normally nonflat at t_0. Let s be a minimal integer such that $P(t)$ lies in A_s for t near t_0 and let $f \in I_s$ be such that $f \circ P$ is not infinitely flat at t_0. Denote by $\bar I_s$ the ideal in $\mathbb{R}[\mathbb{R}^n]$ defining the closed subset $\pi^{-1}(A_{s-1}) \subseteq \mathbb{R}^n$, where $\pi : \mathbb{R}^n \to \mathbb{R}^n/\,\mathrm{S}_n$ is the quotient projection. It is easy to see that the polynomials

$$f_{i_1\dots i_s} = (x_{i_1}-x_{i_2})\cdots(x_{i_1}-x_{i_s})\cdots(x_{i_{s-1}}-x_{i_s}),$$

where $1 \le i_1 < \cdots < i_s \le n$, generate $\bar I_s$. So there exist polynomials $Q_{i_1\dots i_s} \in \mathbb{R}[\mathbb{R}^n]$ such that

$$f \circ \pi = \sum_{i_1<\cdots<i_s} Q_{i_1\dots i_s} f_{i_1\dots i_s}.$$

Denote by $\bar P(t)$ the lift of $P(t)$ given by the increasingly ordered continuous roots $x_1(t), \dots, x_n(t)$ of the polynomial $P(t)$. Then we have

$$f \circ P(t) = \sum_{i_1<\cdots<i_s} Q_{i_1\dots i_s} \circ \bar P(t) \cdot f_{i_1\dots i_s} \circ \bar P(t).$$

Since $f \circ P$ is not infinitely flat at t_0, at least one of the summands in this sum is not infinitely flat at t_0 and thus there is a polynomial $f_{i_1\dots i_s}$ such that $f_{i_1\dots i_s} \circ \bar P$ is not infinitely flat at t_0. By assumption, among the roots $x_1(t), \dots, x_n(t)$ there are precisely s distinct for t near t_0. Hence the germs at t_0 of the roots $x_{i_1}(t), \dots, x_{i_s}(t)$ are distinct, and no two of them meet of infinite order at t_0. Therefore, condition (1) in the above theorem is satisfied.

The other direction is evident by (2.2). □

3. Lifting smooth curves in spaces of hyperbolic polynomials

3.1. **The problem.** Let us denote by Hyp^n the space of hyperbolic polynomials of degree n

$$P(x) = x^n + \sum_{j=1}^n (-1)^j a_j x^{n-j}.$$

We may naturally view Hyp^n as a semialgebraic subset of $\mathbb{R}^n$ by identifying P with $(a_1, \dots, a_n)$. We have $\mathrm{Hyp}^n = E(\mathbb{R}^n) = \mathbb{R}^n/\,\mathrm{S}_n$, and, by means of 2.2, we may calculate explicitly a set of inequalities defining Hyp^n (no equalities since the ring $\mathbb{R}[\mathbb{R}^n]^{\mathrm{S}_n}$ is polynomial).

Suppose X is a semialgebraic subset of Hyp^n. Let $c : \mathbb{R} \to X$ be a smooth curve in X; smooth as curve in $\mathbb{R}^n$. We may view c as a curve in Hyp^n, i.e., as a smooth curve of monic hyperbolic polynomials of degree n. In 2.6 sufficient conditions for the existence of a smooth lift $\bar{c}$ to $\mathbb{R}^n$, i.e., a smooth parameterization of its roots, are presented. It is evident that a smooth curve c in X in order to be liftable smoothly over E to $E^{-1}(X)$ must in general fulfill weaker genericity conditions. Our purpose is to investigate that phenomenon.

3.2. **Orbit spaces embedded in spaces of hyperbolic polynomials.** We recall a construction due to L. Smith and R.E. Stong [SS87] (see also [BR83]) related to E. Noether's [Noe16] proof of Hilbert's finiteness theorem as recounted by H. Weyl [Wey39].

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a finite group G in a finite dimensional vector space V. Consider its induced representation in the dual V^*. For an orbit $B \subseteq V^*$ set

$$\phi_B(X) = \prod_{b \in B} (X + b)$$

which we regard as an element of the ring $\mathbb{R}[V][X]$, with X a new variable. The polynomial $\phi_B(X)$ is called the *orbit polynomial* of B. Evidently, $\phi_B \in \mathbb{R}[V]^G[X]$. If $|B|$ denotes the cardinality of the orbit B, we may expand $\phi_B(X)$ to a polynomial of degree $|B|$ in X,

$$\phi_B(X) = \sum_{i+j=|B|} C_i(B) X^j,$$

defining classes $C_i(B) \in \mathbb{R}[V]^G$ called the *orbit Chern classes* of B.

Theorem (L. Smith and R.E. Stong [SS87])**.** *Let $\rho : G \hookrightarrow \mathrm{GL}(V)$ be a faithful representation of a finite group G. Then there exist orbits $B_1, \ldots, B_l \subseteq V^*$ such that the associated orbit Chern classes $C_i(B_j)$, $1 \le i \le |B_j|$, $1 \le j \le l$, generate $\mathbb{R}[V]^G$.*

The field of real numbers may be replaced by any field of either characteristic zero or characteristic larger than the order of G. For our purpose the reals will suffice.

The Chern classes of the orbit are exactly the elementary symmetric functions in the elements of the orbit. If $B \subseteq V^*$ is an orbit and V_B^* is a vector space with basis identified with the elements of B, then there is a natural map $V_B^* \to V^*$ given by the identification. This map induces a map $\mathbb{R}[V_B]^{\mathrm{S}_{|B|}} \to \mathbb{R}[V]^G$ which sends the k-th elementary symmetric function to the k-th orbit Chern class of B.

In this notation the above theorem says that there exist orbits $B_1, \ldots, B_l \subseteq V^*$ such that the induced map

$$\bigotimes_{i=1}^{l} \mathbb{R}[V_{B_i}]^{\mathrm{S}_{|B_i|}} \longrightarrow \mathbb{R}[V]^G$$

is surjective.

The orbit Chern classes $C_i(B)$ of an orbit B, viewed as invariant polynomials on V, define a G-invariant map

$$C(B) = (C_1(B), \ldots, C_{|B|}(B)) : V \longrightarrow \mathbb{R}^{|B|}$$

whose image $C(B)(V)$ is a semialgebraic subset of the space $\mathrm{Hyp}^{|B|}$ of hyperbolic polynomials of degree $|B|$.

According to 2.1 and the above theorem, for any faithful representation $\rho : G \hookrightarrow \mathrm{GL}(V)$ of a finite group G there exist orbits $B_1, \ldots, B_l \subseteq V^*$ such that the map

$$C(\rho) = (C(B_1), \ldots, C(B_l)) : V \longrightarrow \mathrm{Hyp}^{|B_1|} \times \cdots \times \mathrm{Hyp}^{|B_l|} \subseteq \mathbb{R}^{|B_1|+\cdots+|B_l|}$$

induces a homeomorphism between the orbit space V/G and the image $C(\rho)(V)$ which is a semialgebraic subset of $\mathrm{Hyp}^{|B_1|} \times \cdots \times \mathrm{Hyp}^{|B_l|}$. By increasing the number of orbits B_i if necessary, we may assume that each irreducible subspace of V contributes at least one orbit B_i. Then, the linear forms $b \in B_1 \cup \cdots \cup B_l$ induce an injective inclusion $V \hookrightarrow \mathbb{R}^{|B_1|+\cdots+|B_l|}$.

Let $c : \mathbb{R} \to C(\rho)(V)$ be a smooth curve. Then $c = (c_1, \ldots, c_l)$ where each $c_i : \mathbb{R} \to C(B_i)(V)$ is smooth. Since $C(B_i)(V) \subseteq \mathrm{Hyp}^{|B_i|}$ we may view c_i as a curve in $\mathrm{Hyp}^{|B_i|}$. If there exist smooth lifts $\bar{c}_i : \mathbb{R} \to \mathbb{R}^{|B_i|}$ with respect to the representations $\mathrm{S}_{|B_i|} : \mathbb{R}^{|B_i|}$, then $\bar{c} = (\bar{c}_1, \ldots, \bar{c}_l) : \mathbb{R} \to \mathbb{R}^{|B_1|+\cdots+|B_l|}$ is a smooth lift with respect to $\mathrm{S}_{|B_1|} \times \cdots \times \mathrm{S}_{|B_l|} : \mathbb{R}^{|B_1|+\cdots+|B_l|}$. Consequently, it suffices to study the case when there is given a smooth curve in a semialgebraic subset of some Hyp^n. That is exactly the problem introduced in 3.1.

Suppose $\tilde{c} : \mathbb{R} \to V$ is a smooth lift of c with respect to ρ. Then, there exists a smooth lift $\bar{c} : \mathbb{R} \to \mathbb{R}^{|B_1|+\cdots+|B_l|}$ of c with respect to the representation of $\mathrm{S}_{|B_1|} \times \cdots \times \mathrm{S}_{|B_l|}$ on $\mathbb{R}^{|B_1|+\cdots+|B_l|}$, namely

$$\begin{tikzcd} & V \arrow[r, hook] \arrow[d] & \mathbb{R}^{|B_1|+\cdots+|B_l|} \arrow[d] \\ \mathbb{R} \arrow[ur, "\tilde{c}"] \arrow[r, "c"'] & C(\rho)(V) \arrow[r, hook] & \mathrm{Hyp}^{|B_1|} \times \cdots \times \mathrm{Hyp}^{|B_l|} \end{tikzcd}$$

It follows, by 2.5, that conditions which guarantee that c is generic as curve in the orbit space V/G suffice to imply the existence of a smooth lift of c with respect to $\mathrm{S}_{|B_1|} \times \cdots \times \mathrm{S}_{|B_l|} : \mathbb{R}^{|B_1|+\cdots+|B_l|}$.

We have seen that the above construction provides a class of semialgebraic subsets of spaces of hyperbolic polynomials, namely orbit spaces of faithful finite group representations, for which we are able to apply the strategy described in 3.1, thanks to the results of 2.5.

In the remaining sections we shall change the point of view. Assume we are given a curve of hyperbolic polynomials with certain symmetries. We will investigate whether we can weaken the conditions in 2.6 which guarantee the existence of smooth parameterizations of the roots. This will be performed in section 5. The following section provides the necessary preparation.

4. Orbit type and ambient stratification

Suppose U is a linear subspace of $\mathbb{R}^n$. Let the symmetric group S_n act on $\mathbb{R}^n$ by permuting the coordinates and endow U with the induced effective action of

$$W = W(U) := N(U)/Z(U),$$

where $N(U) := \{\tau \in \mathrm{S}_n : \tau.U = U\}$ and $Z(U) := \{\tau \in \mathrm{S}_n : \tau.x = x \text{ for all } x \in U\}$. Then U carries two natural stratifications: the orbit type stratification with respect to the W-action and the restriction to U of the orbit type stratification of $\mathbb{R}^n$ with respect to the S_n-action. It is easily seen that the latter indeed provides a Whitney stratification of U. Let us denote it as the *ambient stratification of* U.

4.1. **Proposition.** *Let U be a linear subspace in $\mathbb{R}^n$ endowed with the induced action by $W = W(U)$. Then for the ambient and orbit type stratification of U we have:*

1. *Each ambient stratum is contained in a unique orbit type stratum.*
2. *Each orbit type stratum contains at least one ambient stratum of the same dimension and is the union of all contained ambient strata.*

Proof. To (1): Let S be an ambient stratum, i.e., S is a component of $\mathrm{S}_n . \mathbb{R}^n_H \cap U$, where $H = (\mathrm{S}_n)_x$ for a $x \in U$ and $\mathbb{R}^n_H = \{y \in \mathbb{R}^n : (\mathrm{S}_n)_y = H\}$. Since S_n is finite and the manifolds $\tau.\mathbb{R}^n_H$ for $\tau \in \mathrm{S}_n$ either coincide or are pairwise disjoint, the components of $\mathrm{S}_n . \mathbb{R}^n_H$ are open subsets of $\tau.\mathbb{R}^n_H$ for $\tau \in \mathrm{S}_n$. Thus, we may assume that S is a component of $\mathbb{R}^n_H \cap U$.

Denote by π the quotient projection $N(U) \to N(U)/Z(U) = W$. For any $u \in U$ we have $W_u = \pi(N(U) \cap (\mathrm{S}_n)_u)$ and thus $\mathbb{R}^n_H \cap U \subseteq \{u \in U : W_u = W_x\}$. By definition and a similar argument as above, the components of the subset $\{u \in U : W_u = W_x\}$ are orbit type strata of U. So the ambient stratum S is contained in a unique isotropy type stratum R_S.

To (2): Let R be an orbit type stratum and let $\mathfrak{S}$ be the set of all ambient strata S such that $R_S = R$, where R_S is the unique orbit type stratum from (1). Clearly, $R = \bigcup \mathfrak{S}$ and for each $S \in \mathfrak{S}$ we have $\dim S \le \dim R$. Since the set $\mathfrak{S}$ is finite, there is a stratum $S \in \mathfrak{S}$ such that $\dim S = \dim R$. $\square$

4.2. **Remarks.** (1) It is easy to see that proposition 4.1 is true if one replaces the S_n-module $\mathbb{R}^n$ by any finite dimensional G-module V, where G is a finite group.

(2) Proposition 4.1 implies that the orbit type stratification of U is coarser than its ambient stratification. That means, following [Pfl01], that for each ambient stratum S there exists an orbit type stratum R_S such that $S \subseteq R_S$, $\mathrm{id}|_S : S \to R_S$ is smooth, and for all $S \subseteq \overline{S'}$ we have $R_S \subseteq \overline{R_{S'}}$. It remains to check the last condition: Assume that $S \subseteq \overline{S'}$. Since $S \subseteq R_S$ and $S \subseteq \overline{S'} \subseteq \overline{R_{S'}}$, we obtain $R_S \cap \overline{R_{S'}} \neq \emptyset$, and, by the frontier condition, $R_S \subseteq \overline{R_{S'}}$.

Assume that the restrictions $E_i|_U$, $1 \le i \le n$, generate the algebra $\mathbb{R}[U]^W$. It follows that $E|_U = (E_1|_U, \ldots, E_n|_U)$ induces a homeomorphism between U/W and the semialgebraic subset $E(U)$ of $\mathbb{R}^n/\mathrm{S}_n = E(\mathbb{R}^n) = \mathrm{Hyp}^n$, by 2.1. It is well-known that $U_{(H)} \to U_{(H)}/W$, where $H = W_u$ for some $u \in U$, is a Riemannian submersion. Since W is finite, it is even a local diffeomorphism. By proposition 4.1, this implies that for any ambient stratum S in U the image $E(S)$ is a smooth manifold. The collection $\mathcal{T} = \{E(S) : S \text{ ambient stratum in } U\}$ obviously coincides with the collection obtained by restricting to $E(U)$ the orbit type stratification of $\mathbb{R}^n/\mathrm{S}_n = E(\mathbb{R}^n) = \mathrm{Hyp}^n$. It is easily verified that the frontier condition for the orbit type stratification of $\mathbb{R}^n/\mathrm{S}_n = E(\mathbb{R}^n) = \mathrm{Hyp}^n$ implies the frontier condition for $\mathcal{T}$. Consequently, $\mathcal{T}$ provides a stratification of $E(U)$. Let us denote this stratification as the *ambient stratification of* $E(U)$.

Consider a smooth curve $c : \mathbb{R} \to E(U) = U/W$ in the sense of 2.3. It may then be also viewed as a smooth curve in $\mathbb{R}^n/\mathrm{S}_n = E(\mathbb{R}^n) = \mathrm{Hyp}^n$. Thus it makes sense to speak about the normal nonflatness of c at some point t_0 with respect to the orbit type stratification of U/W on the one hand and with respect to the orbit type stratification of $\mathbb{R}^n/\mathrm{S}_n$ on the other hand. To shorten notation we shall say that c is normally nonflat at t_0 with respect to the ambient stratification of U/W iff it is normally nonflat at t_0 with respect to the orbit type stratification of $\mathbb{R}^n/\mathrm{S}_n$.

4.3. **Proposition.** *Let U be a linear subspace in $\mathbb{R}^n$ endowed with the induced action by $W = W(U)$ and assume that the restrictions $E_i|_U$, $1 \le i \le n$, generate $\mathbb{R}[U]^W$. Consider a smooth curve $c : \mathbb{R} \to E(U) = U/W$. If c is normally nonflat at t_0 with respect to the ambient stratification of U/W, then it is normally nonflat at t_0 with respect to the orbit type stratification of U/W.*

Proof. The set of reflection hyperplanes H of the reflection group S_n is in bijective correspondence with the set of linear functionals ω_H on $\mathbb{R}^n$ of the form $x_j - x_i$ for $1 \le i < j \le n$, namely H is the kernel of ω_H. Let us consider the restrictions $\omega_H|_U$ to U. If c is normally nonflat at t_0 with respect to the ambient stratification, then, by lemma 2.6, any two of the increasingly ordered continuous roots of the polynomial $c(t) \in E(U) \subseteq \mathrm{Hyp}^n$ either coincide identically near t_0 or do not meet at t_0 of infinite order. Then for the continuous lift $\bar{c}$ of c defined by such a choice of roots any function $\omega_H \circ \bar{c}$ either vanishes identically near t_0 or does not vanish at t_0 of infinite order.

Let s be a minimal integer such that $c(t)$ lies in $A_{s,\mathrm{orb}}$ for t near t_0, where $A_{s,\mathrm{orb}}$ is the union of all orbit type strata of U/W of dimension $\leq s$.

Denote by π_U the projection $U \to U/W$. Let R be an orbit type stratum contained in $\pi_U^{-1}(A_{s-1,\mathrm{orb}})$ and let $S_1, \ldots, S_k$ be the ambient strata of the same dimension as R contained in R (see proposition 4.1). For each $1 \leq j \leq k$ denote by $\mathcal{H}_j$ the set of reflection hyperplanes for reflections in S_n fixing S_j pointwise. Let Ω_j be the set of linear functionals $\omega_H|_U$ for $H \in \mathcal{H}_j$. Put $f_{R,j} = \sum_{\omega \in \Omega_j} \omega^2$. By definition the equation $f_{R,j} = 0$ defines a linear subspace of U in which S_j is an open subset. Let $f_R = \prod_{j=1}^k f_{R,j}$. Consider the natural action of W on $\mathbb{R}[U]$ and let $W.f_R = \{f_R^1, \ldots, f_R^l\}$ be the orbit through f_R with respect to this action. Define $F_R = f_R^1 \cdots f_R^l$. By construction $F_R \in \mathbb{R}[U]^W$ and the set Z_R of zeros of F_R viewed as a function on U/W is contained in $A_{s-1,\mathrm{orb}}$. Moreover, $A_{s-1,\mathrm{orb}}$ is the union of the Z_R, where R ranges over all orbit type strata (of maximal dimension) contained in $\pi_U^{-1}(A_{s-1,\mathrm{orb}})$. Thus $F = \prod_R F_R$, where the product is taken over all orbit type strata (of maximal dimension) R contained in $\pi_U^{-1}(A_{s-1,\mathrm{orb}})$, is a regular function on U/W whose set of zeros equals $A_{s-1,\mathrm{orb}}$. By construction, the function $F \circ c$ is nonflat at t_0.

This proves the statement. □

We define $F_{\mathrm{amb}}(c)$ (resp. $F_{\mathrm{orb}}(c)$) to be the set of all $t \in \mathbb{R}$ such that c is normally flat at t with respect to the ambient (resp. orbit type) stratification of $E(U)$. It follows that in the situation of proposition 4.3 we have $F_{\mathrm{orb}}(c) \subseteq F_{\mathrm{amb}}(c)$.

5. Choosing roots of polynomials with symmetries smoothly

Consider a smooth curve of hyperbolic polynomials

$$P(t)(x) = x^n - a_1(t)x^{n-1} + a_2(t)x^{n-2} - \cdots + (-1)^n a_n(t) \qquad (t \in \mathbb{R}).$$

We are interested in conditions that guarantee the existence of a smooth parameterization of the roots of P. Such conditions have been found in [AKLM98], see 2.6. There no additional assumptions on the polynomials $P(t)$ have been made.

In this section we are going to improve those results if the set of roots $x_1(t), \ldots, x_n(t)$ of $P(t)$ has symmetries additional to its invariance under permutations.

Let as assume that the additional symmetries of $P(t)$ are given by linear relations between the roots of $P(t)$. Otherwise put, there is a linear subspace U of $\mathbb{R}^n$ such that $(x_1(t), \ldots, x_n(t)) \in U$ for all $t \in \mathbb{R}$. Then, the curve $P(t)$ lies in the semialgebraic subset $E(U)$ of $\mathrm{Hyp}^n = E(\mathbb{R}^n) = \mathbb{R}^n/\mathrm{S}_n$, the space of hyperbolic polynomials of degree n.

The linear subspace $U \subseteq \mathbb{R}^n$ inherits an effective action by the group $W = W(U)$.

Let us suppose that the restrictions $E_i|_U$, $1 \leq i \leq n$, generate the algebra $\mathbb{R}[U]^W$. Then $E|_U = (E_1|_U, \ldots, E_n|_U)$ induces a homeomorphism between U/W and the semialgebraic subset $E(U)$ of Hyp^n, by 2.1.

5.1. **Lemma.** *Consider a continuous curve of hyperbolic polynomials*

$$P(t)(x) = x^n - a_1(t)x^{n-1} + a_2(t)x^{n-2} - \cdots + (-1)^n a_n(t) \qquad (t \in \mathbb{R}).$$

Let U be some linear subspace of $\mathbb{R}^n$ and assume that the restrictions $E_i|_U$, $1 \leq i \leq n$, generate the algebra $\mathbb{R}[U]^{W(U)}$. Then the following two conditions are equivalent:

(1) *There exists a continuous parameterization $x(t)$ of the roots $x_1(t), \dots, x_n(t)$ of $P(t)$ such that $x(t) \in U$ for all $t \in \mathbb{R}$.*
(2) *$P(t) \in E(U)$ for all $t \in \mathbb{R}$.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial. Suppose that $P(t)$ is a continuous curve in $E(U)$. By assumption, we may view $P(t)$ as a curve in the orbit space $U/W(U) \cong E(U)$. It allows a continuous lift $x(t)$ into U, by [KLMR05] or [MY57], which constitutes a parameterization of the roots of $P(t)$. $\square$

The smooth curve of polynomials $P(t)$ which lies in $E(U)$ may be viewed as a smooth curve in the orbit space U/W in the sense of 2.3. A smooth lift of $P(t)$ over the orbit map $E|_U$ to the W-module U provides a smooth parameterization of the roots of the polynomials $P(t)$.

By theorem 2.5, we may conclude: If $P(t)$ is normally nonflat at $t = t_0$ with respect to the orbit type stratification of $E(U)$, then $P(t)$ is smoothly solvable near $t = t_0$.

Consider the closed sets $F_{\text{amb}}(P)$ and $F_{\text{orb}}(P)$, as defined in section 4. By proposition 4.3, the set $F_{\text{orb}}(P)$ is contained in $F_{\text{amb}}(P)$. We have found that that $P(t)$ is smoothly solvable locally near any $t_0 \in \mathbb{R}\backslash F_{\text{orb}}(P)$. Any two smooth parameterizations of the roots of $P(t)$ near such a t_0 differ by a constant permutation, see theorem 2.6. Thus the local solutions may be glued to a smooth solution on $\mathbb{R}\backslash F_{\text{orb}}(P)$.

It follows from a result in [KLM04] (see also [KLMR06]) that any smooth curve of monic hyperbolic polynomials of fixed degree allows a global twice differentiable parameterization of its roots. By the methods used in [KLM04], it is easy to combine this with the result above in order to get the following theorem.

5.2. **Theorem.** *Consider a smooth curve of hyperbolic polynomials*

$$P(t)(x) = x^n - a_1(t)x^{n-1} + a_2(t)x^{n-2} - \cdots + (-1)^n a_n(t) \qquad (t \in \mathbb{R}).$$

Let U be some linear subspace of $\mathbb{R}^n$ such that:

(1) *The restrictions $E_i|_U$, $1 \le i \le n$, generate the algebra $\mathbb{R}[U]^{W(U)}$.*
(2) *$P(t) \in E(U)$ for all $t \in \mathbb{R}$.*

Then: There exists a global twice differentiable parameterization of the roots of $P(t)$ on $\mathbb{R}$ which is smooth on $\mathbb{R}\backslash F_{\text{orb}}(P)$. $\square$

5.3. **Remark.** The orbit type stratification and the ambient stratification of $E(U)$ do in general not coincide, whence theorem 5.2 provides an actual improvement of the statement of theorem 2.6. In other words, in general we have $F_{\text{orb}}(P) \subsetneq F_{\text{amb}}(P)$. It may, for instance, happen that $P(0)$ is regular in $E(U) = U/W$ but singular in $\text{Hyp}^n = \mathbb{R}^n/\mathrm{S}_n$ and $P(t)$ is normally flat at $t = 0$ with respect to the ambient stratification. See examples in section 8.

Let us suppose that a linear subspace U of $\mathbb{R}^n$ is given. It is then a purely computational problem to check whether the assumptions we have made in the forgoing discussion are satisfied. There are algorithms in computational invariant theory (e.g. [DK02], [Stu93]) which allow to decide whether the restrictions $E_i|_U$, $1 \le i \le n$, generate the algebra $\mathbb{R}[U]^{W(U)}$. If the answer is yes, theorem 2.2 provides an explicit way to describe the semialgebraic subset $E(U) \subseteq \text{Hyp}^n$ by a finite set of polynomial equations and inequalities. So the condition that the curve P lies in $E(U)$ may again be check computationally. The orbit type stratification and the ambient stratification of $E(U)$ can be determined explicitly using theorem 2.4. Then all ingredients are supplied in order to decide whether the curve $P(t)$ is normally nonflat at some $t = t_0$ with respect to the one or the other stratification of $E(U)$.

Note that there are refined approaches and algorithms for computing the orbit space V/G and its orbit type stratification of a G-module V (when identified with the image of its orbit map). In [SV03] rational parameterizations of the strata are obtained, while [Bay04] provides an algorithm yielding a description of each stratum in terms of a minimal number of polynomial equations and inequalities, if G is finite.

We shall carry out that procedure explicitly in example 8.8.

6. Choosing roots of polynomials with symmetries differentiably

Consider a curve of hyperbolic polynomials

$$P(t)(x) = x^n - a_1(t)x^{n-1} + a_2(t)x^{n-2} - \cdots + (-1)^n a_n(t) \qquad (t \in \mathbb{R}).$$

Then the following results are known:

6.1. **Result.** *We have:**

(1) *If all a_i are of class C^n, then there exists a differentiable parameterization of the roots of $P(t)$ with locally bounded derivative,* [Bro79], [Wak86].
(2) *If all a_i are of class C^{2n}, then any differentiable parameterization of the roots of $P(t)$ is actually C^1,* [KLM04], [Man85].
(3) *If all a_i are of class C^{3n}, then there exists a twice differentiable parameterization of the roots of $P(t)$,* [KLM04].

In [KLMR06] we have proved the following generalizations:

6.2. **Result.** *Let $\rho : G \to \mathrm{O}(V)$ be a finite dimensional representation of a finite group G. Let $d = d(\rho)$ be the maximum of the degrees of a minimal system of homogeneous generators $\sigma_1, \dots, \sigma_m$ of $\mathbb{R}[V]^G$. Write $V = V_1 \oplus \cdots \oplus V_l$ as orthogonal direct sum of irreducible subspaces V_i. Define $k_i := \min\{|G.v| : v \in V_i \backslash \{0\}\}$, $1 \le i \le l$, and $k := \max\{d(\rho), k_1, \dots, k_l\}$. Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^m$ be a curve in the orbit space. Then:*†

(1) *If c is of class C^k, then there exists a differentiable lift of c to V with locally bounded derivative.*
(2) *If c is of class C^{k+d}, then any differentiable lift of c is actually of class C^1.*
(3) *If c is of class C^{k+2d}, then there exists a twice differentiable lift of c to V.*

Again we may use these facts in order to improve the results for curves $P(t)$ of hyperbolic polynomials with symmetries.

Let U be some linear subspace of $\mathbb{R}^n$ such that the restrictions $E_i|_U$, $1 \le i \le n$, generate the algebra $\mathbb{R}[U]^{W(U)}$, and $P(t) \in E(U)$ for all $t \in \mathbb{R}$. It follows that we may view $P(t)$ as a curve in the orbit space $U/W(U) = E(U)$, and any lift of $P(t)$ over the orbit map $E|_U$ to U gives a parameterization of the roots of $P(t)$ of the same regularity.

Provided that the integer k, associated to the $W(U)$-module U as above, is less than the degree n of the polynomials in $P(t)$, we are able, using 6.2, to lower the degree of regularity in the assumptions of the statements in 6.1. We shall give examples in section 8.

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* Due to [COP08], for the existence of C^1 (resp. twice differentiable) roots it actually suffices that the coefficients a_j are C^n (resp. C^{2n}).

† Due to [KLMR08], for the existence of a C^1 (resp. twice differentiable) lift it actually suffices that c is C^k (resp. C^{k+d}).

7. Construction of a class of examples

We will present a class of examples which our considerations apply to.

Let $G \subseteq \mathrm{O}(V)$ be a finite group whose action on the vector space V is irreducible and effective. Choose some non-zero orbit $G.v$. Introducing some numbering we can write $G.v = \{g_1.v, \ldots, g_n.v\}$, where $|G.v| = n$ and $g_i \in G$. We define a mapping $F_{G,v} : V \to \mathbb{R}^n$ by

$$F_{G,v}(x) := (\langle g_1.v \mid x\rangle, \ldots, \langle g_n.v \mid x\rangle).$$

Since the linear span of $G.v$ spans V, the mapping $F_{G,v}$ is a linear isomorphism onto its image $F_{G,v}(V) =: U_{G,v}$. The linear space $U_{G,v} \subseteq \mathbb{R}^n$ carries the action of $W_{G,v} := W(U_{G,v})$ and a natural G-action given by transformations from $W_{G,v}$. Since the G-action is irreducible, so is the $W_{G,v}$-action. Hence $U_{G,v} \subseteq \{y \in \mathbb{R}^n : y_1+\cdots+y_n = 0\}$. Irreducibility and effectiveness of the G-action induce an injection $G \hookrightarrow W_{G,v}$. Thus we may consider G as a subgroup of $W_{G,v}$, and in this picture $F_{G,v}$ is G-equivariant.

7.1. **Remark.** The linear space $U_{G,v}$ always intersects the submanifold of regular points in the S_n-module $\mathbb{R}^n$. Namely: For $1 \le i < j \le n$ we define $U_{i,j} = \{F_{G,v}(x) : \langle g_i.v \mid x\rangle = \langle g_j.v \mid x\rangle, x \in V\}$. By definition, $U_{i,j}$ is a linear subspace of $U_{G,v}$ and $\bigcup_{i<j} U_{i,j}$ is the set of singular points of the S_n-module $\mathbb{R}^n$ contained in $U_{G,v}$. Since, by definition, $g_i.v \ne g_j.v$ for any $i < j$, we have $\dim U_{i,j} = n-1$. Thus, $\bigcup_{i<j} U_{i,j} \ne U_{G,v}$, which gives the assertion.

Put $P_{G,v} := E \circ F_{G,v}$. Then $P_{G,v}$ is proper, since E and $F_{G,v}$ are proper.

7.2. **Lemma.** *Suppose that $P_{G,v}$ separates G-orbits. Then we have $G = W_{G,v}$.*

Proof. The groups G and $W_{G,v}$ have the same orbits in $U_{G,v}$. For: Suppose that $\tau \in W_{G,v}$ and $x, y \in V$ such that $F_{G,v}(y) = \tau.F_{G,v}(x)$. Since $P_{G,v}$ separates orbits, it follows that there exists some $g \in G$ such that $y = g.x$, whence $g.F_{G,v}(x) = \tau.F_{G,v}(x)$.

Now choose $x \in V$ such that $F_{G,v}(x)$ is a regular point of the $W_{G,v}$-module $U_{G,v}$. The regular points of any effective linear finite group representation are precisely those with trivial isotropy groups. We may conclude that x is a regular point of the G-module V. So $|W_{G,v}| = |W_{G,v}.F_{G,v}(x)| = |G.x| = |G|$, and thus $G = W_{G,v}$. □

If $P_{G,v}$ separates G-orbits, then, by lemma 7.2, the $G = W_{G,v}$-modules V and $U_{G,v}$ are equivalent. In particular, it follows that the restriction $E|_{U_{G,v}}$ separates $W_{G,v}$-orbits, $F_{G,v}$ induces a homeomorphism between V/G and $U_{G,v}/W_{\rho,v}$, and $F^*_{G,v} : \mathbb{R}[U_{G,v}]^{W_{G,v}} \to \mathbb{R}[V]^G$ is an algebra isomorphism.

7.3. **Proposition.** *The following conditions are equivalent:*

1. *$P_{G,v}$ separates G-orbits.*
2. *For all $x \in V$ we have $F_{G,v}(G.x) = \mathrm{S}_n .F_{G,v}(x) \cap U_{G,v}$.*
3. *$P_{G,v}$ induces a homeomorphism between V/G and $P_{G,v}(V)$.*

Proof. Since E separates S_n-orbits, for each $x \in V$ there exists a $z \in \mathbb{R}^n$ such that $E^{-1}(z) = \mathrm{S}_n .F_{G,v}(x)$. Then the equivalence of (1) and (2) follows from

$$P^{-1}_{G,v}(z) = F^{-1}_{G,v}(\mathrm{S}_n .F_{G,v}(x)) = F^{-1}_{G,v}(\mathrm{S}_n .F_{G,v}(x) \cap U_{G,v}).$$

The equivalence of (1) and (3) follows easily from lemma 2.1. □

Note that the introduced construction of $F_{G,v}$ and $P_{G,v}$ essentially coincides with the construction of orbit Chern classes as described in 3.2.

Let us discuss uniqueness of the above construction. Suppose $G \subseteq \mathrm{O}(V)$ is a finite group. Denote by $\mathrm{Aut}(G)$ the group of automorphisms of G. Let S be the set of all reflections belonging to G. Denote by $\mathrm{Aut}(G, S)$ the group of automorphisms of G preserving the set S. Let $a \in \mathrm{Aut}(G, S)$. A diffeomorphism $T : V \to V$ is called a-equivariant, if $T \circ g = a(g) \circ T$ for any $g \in G$ (cf. [Los01]).

7.4. **Lemma.** *Suppose $G \subseteq \mathrm{O}(V)$ is a finite group. Let $a \in \mathrm{Aut}(G, S)$ and let $T : V \to V$ be an a-equivariant diffeomorphism. Then the isotropy groups of x and $T(x)$ are isomorphic, for all $x \in V$, T maps orbits onto orbits, and T induces an automorphism of the orbit type stratification of V.*

Proof. It is easily seen that $G_{T(x)} = a(G_x)$ and $T(G.x) = G.T(x)$ for all $x \in V$. Further, it is evident that $G_x = gHg^{-1}$ if and only if $G_{T(x)} = a(g)a(H)a(g)^{-1}$. The statement follows. □

Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a smooth curve and $\bar{c} : \mathbb{R} \to V$ a smooth lift of c. The orbit space V/G has a smooth structure given by the sheaf $C^\infty(V/G) = C^\infty(V)^G$ of smooth G-invariant functions on V. Then c induces a continuous algebra morphism $c^* : C^\infty(V/G) \to C^\infty(\mathbb{R})$ and $\bar{c}$ induces a continuous algebra morphism $\bar{c}^* : C^\infty(V) \to C^\infty(\mathbb{R})$ such that $c^* = \bar{c}^* \circ \sigma^*$. This algebraic lifting problem is equivalent to the geometrical one. It is evident that to determine $\bar{c}^*$ it suffices to know the images under $\bar{c}^*$ of some system of global coordinate functions $x_1, \ldots, x_m$, where $m = \dim V$. The same is true for c^*, and in this case we may take the basic invariants $\sigma_1, \ldots, \sigma_n$ as global coordinates functions, by Schwarz's theorem [Sch75]. If $f : V/G \to V/G$ is a smooth diffeomorphism one can take instead of the σ_i the functions $f^*(\sigma_i)$ with the same result. Thus, the problem of smooth lifting is invariant with respect to the group of diffeomorphisms of V/G. Each such diffeomorphism has a smooth lift to V which is an a-equivariant diffeomorphism, for some $a \in \mathrm{Aut}(G, S)$, see [Los01]. Conversely, any smooth a-equivariant diffeomorphism of V induces a smooth diffeomorphism of V/G, by lemma 7.4.

Therefore, we may regard two constructions as described above, carried out for distinct points v and w in V, as equivalent with respect to our lifting problem, if there exists a smooth a-equivariant diffeomorphism $T : V \to V$ with $v = T(w)$, for some $a \in \mathrm{Aut}(G, S)$.

If T is of a particular form, we can even say more.

7.5. **Proposition.** *Suppose $G \subseteq \mathrm{O}(V)$ is a finite group. Let $v, w \in V \backslash \{0\}$. If there exists a homothety or an a-equivariant linear orthogonal map $T : V \to V$, for some $a \in \mathrm{Aut}(G, S)$, such that $v = T(w)$, then $P_{G,v}(V)$ and $P_{G,w}(V)$ are homeomorphic, and $\mathbb{R}[E_1 \circ F_{G,v}, \ldots, E_n \circ F_{G,v}]$ and $\mathbb{R}[E_1 \circ F_{G,w}, \ldots, E_n \circ F_{G,w}]$ are isomorphic.*

Moreover, in both cases, the ambient stratifications of $U_{G,v}$ and $U_{G,w}$ are isomorphic, i.e., there exists a linear isomorphism $U_{G,v} \to U_{G,w}$ mapping strata onto strata.

Proof. If T is a homothety, then it is equivariant ($a = \mathrm{id}$) and $U_{G,v} = U_{G,w}$. If T is a-equivariant linear orthogonal, then, by lemma 7.4, the linear subspaces $U_{G,v}$ and $U_{G,w}$ of $\mathbb{R}^n$ differ only by a permutation from S_n. In both cases $P_{G,v}(V)$ and $P_{G,w}(V)$ are homeomorphic, and $T^* : \mathbb{R}[E_1 \circ F_{G,v}, \ldots, E_n \circ F_{G,v}] \to \mathbb{R}[E_1 \circ F_{G,w}, \ldots, E_n \circ F_{G,w}]$ is an algebra isomorphism.

The supplement in the lemma follows immediately from the fact that $U_{G,v}$ and $U_{G,w}$ differ only by a permutation of S_n. □

If $P(t)$ is a smooth curve of hyperbolic polynomials lying in $P_{G,v}(V)$ and provided that the polynomials $E_i \circ F_{G,v}$, $1 \le i \le n$, generate $\mathbb{R}[V]^G$, we may apply the results of sections 5 and 6.

We will investigate the case of finite reflection groups in the next section.

8. Finite reflection groups

Suppose U is a linear subspace of $\mathbb{R}^n$. Let the symmetric group S_n act on $\mathbb{R}^n$ by permuting the coordinates and endow U with the induced action of $W = W(U)$. We shall assume in this section that W is a finite reflection group.

8.1. **Remark.** If W is a finite reflection group, proposition 4.1 reduces to the following statement: *Any reflection hyperplane of W in U is the intersection with U of some reflection hyperplane of S_n in $\mathbb{R}^n$.* For: Let H be a reflection hyperplane of W in U. By proposition 4.1, there exists a ambient stratum S of U such that $S \subseteq H$ and $\dim S = \dim H$. Obviously, $S \subseteq (\mathbb{R}^n)_{\mathrm{sing}} \cap U$, and so there are reflection hyperplanes $P_1, \ldots, P_l$ of S_n in $\mathbb{R}^n$ which contain S. Since $\dim S = \dim U - 1$, there is a $1 \le i \le n$ such that $P_i \cap U$ is a hyperplane in U. Since S is contained in both H and $P_i \cap U$, we have $H = P_i \cap U$.

For any finite reflection group $W \subseteq \mathrm{O}(U)$ we may write U as the orthogonal direct sum of W-invariant subspaces $U_0 = U^W, U_1, \ldots, U_l$ such that W is isomorphic to $W_0 \times W_1 \times \cdots \times W_l$, where $W_i = \{\tau|_{U_i} : \tau \in W\}$. Each W_i $(i \ge 1)$ is one of the groups (e.g. [Hum90])

$$\mathrm{A}_m, m \ge 1; \mathrm{B}_m, m \ge 2; \mathrm{D}_m, m \ge 4; \mathrm{I}_2^m, m \ge 5, m \ne 6;$$
$$\mathrm{G}_2; \mathrm{H}_3; \mathrm{H}_4; \mathrm{F}_4; \mathrm{E}_6; \mathrm{E}_7; \mathrm{E}_8 .$$

It follows that $\mathbb{R}[U]^W \cong \mathbb{R}[U_1]^{W_1} \otimes \cdots \otimes \mathbb{R}[U_l]^{W_l}$ and $U/W \cong U_1/W_1 \times \cdots \times U_l/W_l$. A smooth curve $c = (c_1, \ldots, c_l)$ in the orbit space U/W is then smoothly liftable to U if and only if, for all $1 \le i \le l$, c_i is smoothly liftable to U_i. Note that the orbit type stratification of U/W coincides with the product stratification of the orbit type stratifications $\mathcal{Z}_i$ of the factors U_i/W_i, i.e., the strata of U/W are $S_1 \times \cdots \times S_l$, where $S_i \in \mathcal{Z}_i$. Consequently, in order to apply the results of section 5 and section 6 we may consider each factor U_i/W_i separately. So let us assume that U is an irreducible W-module.

To this end we have to check whether the restrictions $E_i|_U$, $1 \le i \le n$, generate the algebra $\mathbb{R}[U]^W$. In practice this is easily accomplishable: The unique degrees $d_1, \ldots, d_m$, where $m = \dim U$, of the elements in a minimal system of homogeneous generators of $\mathbb{R}[U]^W$ are well known. It suffices to compute the Jacobian J of the polynomials $E_{d_i}|_U$, $1 \le i \le m$. If $J \ne 0 \in \mathbb{R}[U]$ then they generate $\mathbb{R}[U]^W$. Note that a necessary condition for the $E_i|_U$, $1 \le i \le n$, to generate $\mathbb{R}[U]^W$ is that the degrees $d_1, \ldots, d_m$ must be pairwise distinct, see remark 8.4.

Let us carry out the construction presented in section 7 for finite irreducible reflection groups $G \subseteq \mathrm{O}(V)$. Let $v \in V \backslash \{0\}$. If the polynomials $E_i \circ F_{G,v}$ generate the algebra $\mathbb{R}[V]^G$, then $W_{G,v}$ is a finite irreducible reflection group as well, by lemma 7.2.

Fix a system Π of simple roots of G. For any v in the fundamental domain $C = \{x \in V : \langle x \mid r \rangle \ge 0 \text{ for all } r \in \Pi\}$, the isotropy group G_v is generated by the simple reflections it contains (e.g. [Hum90]).

8.2. **Lemma.** *Let $G \subseteq \mathrm{O}(V)$ be a finite reflection group. Each automorphism of the corresponding Coxeter diagram $\Gamma(G)$ induces an a-equivariant orthogonal automorphism of V for some $a \in \mathrm{Aut}(G, S)$.*

Proof. ([Los01]) Since the vertices in the Coxeter diagram $\Gamma(G)$ represent the simple roots of G, an automorphism φ of $\Gamma(G)$, defines uniquely an automorphism $a_\varphi \in \mathrm{Aut}(G, S)$. Suppose the simple roots have unit length. Since they form a basis for V the automorphism φ defines naturally an orthogonal automorphism T_φ of V. It is easily checked that T_φ is a_φ-equivariant. □

8.3. **Theorem.** *Suppose $G \subseteq \mathrm{O}(V)$ is a finite irreducible reflection group. Let $v \in V \backslash \{0\}$ such that the cardinality of G_v is maximal. Then: The polynomials $E_i \circ F_{G,v}$, $1 \le i \le n$, generate $\mathbb{R}[V]^G$ and $P_{G,v}$ induces a homeomorphism between V/G and $P_{G,v}(V)$ if and only if $G \neq \mathrm{D}_m$, $m \ge 4$.*

Proof. By proposition 7.5 and lemma 8.2 it suffices to check the statement for one single $v \neq 0$ with maximal G_v. Choosing $e_1 + \cdots + e_m - m e_{m+1}$, e_1, and e_1 for A_m, B_m, and I_2^m, respectively, one obtains the usual systems of basic invariants. The choice e_1 for D_m yields $F_{\mathrm{D}_m,e_1} = F_{\mathrm{B}_m,e_1}$, whence the polynomials $E_i \circ F_{\mathrm{D}_m,e_1}$, $1 \le i \le n = 2m$, cannot separate D_m-orbits. For the remaining irreducible reflection groups the necessary computations have been carried out by Mehta [Meh88]. $\square$

8.4. **Remark.** If for D_m with m odd one chooses $v = e_1 + \cdots + e_m$, then the polynomials $E_i \circ F_{\mathrm{D}_m,v}$, $1 \le i \le n = 2^{m-1}$, generate $\mathbb{R}[\mathbb{R}^m]^{\mathrm{D}_m}$, since the Jacobian of the polynomials $N_i \circ F_{\mathrm{D}_m,w}$, $i = 2, 4, \cdots, 2n-2, n$, is up to a constant factor given by $\prod_{i<j}(x_i^2 - x_j^2)$. If $m(\ge 4)$ is even, this cannot be true since there have to be two basic invariants of degree $m/2$.

The following theorem is a corollary of theorem 8.3 and theorem 5.2.

8.5. **Theorem.** *Suppose $G \subseteq \mathrm{O}(V)$ is a finite irreducible reflection group and $G \neq \mathrm{D}_m$, $m \ge 4$. Let $v \in V \backslash \{0\}$ such that the cardinality of G_v is maximal. Let*

$$P(t)(x) = x^n - a_1(t)x^{n-1} + a_2(t)x^{n-2} - \cdots + (-1)^n a_n(t) \qquad (t \in \mathbb{R})$$

be a smooth curve of hyperbolic polynomials of degree $n = |G.v|$ lying in $P_{G,v}(V)$ for all $t \in \mathbb{R}$. Then there exists a global twice differentiable parameterization of the roots of $P(t)$ on $\mathbb{R}$ which is smooth on $\mathbb{R} \backslash F_{\mathrm{orb}}$. $\square$

8.6. **Remark.** It is easy to see that, under the assumption that the cardinality of G_v is maximal, the orbit type stratification and the ambient stratification of $U_{G,v}$ coincide only for $G = \mathrm{A}_m, \mathrm{B}_m, \mathrm{I}_2^m$. In general, if $|G_v|$ is not maximal, the orbit type stratification of $U_{G,v}$ will be strictly coarser than its ambient stratification.

It is easy to compute the integer k, associated to orthogonal representations of finite groups G in 6.2, if G is a finite irreducible reflection group. See figure 1.

G	A_m	B_m	D_m	I_2^m	G_2	H_3	H_4	F_4	E_6	E_7	E_8
k	$m+1$	$2m$	$2m$	m	6	12	120	24	27	56	240

FIGURE 1. Irreducible Coxeter groups with associated integer k.

In the situation of theorem 8.5 the strategy discussed in section 6 will lead to no improvement, since $k = n$ by definition. But, if we choose $v \in V \backslash \{0\}$ such that $|G_v|$ is not maximal, then $k < n$ and the methods of section 6 will yield refinements.

In many cases the following theorem provides an improvement of 6.1.

8.7. **Theorem.** *Suppose $G \subseteq \mathrm{O}(V)$ is a finite irreducible reflection group. Choose some $v \in V \backslash \{0\}$. Put $n = |G.v|$ and let k be as in figure 1. Suppose that the restrictions $E_i|_{U_{G,v}}$, $1 \le i \le n$, generate $\mathbb{R}[U_{G,v}]^{W_{G,v}}$. Let*

$$P(t)(x) = x^n - a_1(t)x^{n-1} + a_2(t)x^{n-2} - \cdots + (-1)^n a_n(t) \qquad (t \in \mathbb{R})$$

be a curve of hyperbolic polynomials lying in $P_{G,v}(V)$ for all $t \in \mathbb{R}$. Then:‡

‡ Cf. footnotes on page 13.

(1) *If all a_i are of class C^k, then there exists a differentiable parameterization of the roots of $P(t)$ with locally bounded derivative.*
(2) *If all a_i are of class C^{k+d}, then any differentiable parameterization of the roots of $P(t)$ is actually C^1.*
(3) *If all a_i are of class C^{k+2d}, then there exists a twice differentiable parameterization of the roots of $P(t)$.* □

8.8. **Example.** Consider the Coxeter group B_3 and choose $v = e_1 + e_2 + e_3$. We find

$$F_{\mathrm{B}_3,v}(x) = (x_1 + x_2 + x_3, -x_1 + x_2 + x_3, x_1 - x_2 + x_3, x_1 + x_2 - x_3, \\ - x_1 - x_2 + x_3, -x_1 + x_2 - x_3, x_1 - x_2 - x_3, -x_1 - x_2 - x_3)$$

and $U_{\mathrm{B}_3,v} = \{y \in \mathbb{R}^8 : y_i + y_j = 0 \text{ for } i + j = 9, y_1 = y_2 + y_3 + y_4\}$. It is easy to check that $N_{2i} \circ F_{\mathrm{B}_3,v}$, $1 \le i \le 3$, generate $\mathbb{R}[\mathbb{R}^3]^{\mathrm{B}_3}$, by computing their Jacobian. It is readily verified that the set of all reflection hyperplanes of $W_{\mathrm{B}_3,v}$ is given by intersecting the following hyperplanes in $\mathbb{R}^8$ with $U_{\mathrm{B}_3,v}$ (compare with remark 8.1):

$$\{y_1 = y_2, y_1 = y_3, y_1 = y_4, y_1 = y_5, y_1 = y_6, y_1 = y_7, y_2 = y_3, y_2 = y_4, y_3 = y_4\}.$$

Furthermore, the intersections with $U_{\mathrm{B}_3,v}$ of the following hyperplanes in $\mathbb{R}^8$,

$$\{y_1 = y_8, y_2 = y_7, y_3 = y_6, y_4 = y_5\},$$

are not among the set of reflection hyperplanes of $W_{\mathrm{B}_3,v}$. Therefore, the orbit type stratification of $U_{\mathrm{B}_3,v}$ is strictly coarser than its ambient stratification.

We follow the recipe for computing orbit type and ambient stratification of $E(U_{\mathrm{B}_3,v}) = N(U_{\mathrm{B}_3,v})$ given at the end of section 5. We will present only the outcome of the calculations. Using $N_{2i} \circ F_{\mathrm{B}_3,v}$, $1 \le i \le 3$, as basic invariants of $\mathbb{R}[\mathbb{R}^3]^{\mathrm{B}_3}$, we find that the symmetric matrix $\tilde{B} = (\tilde{b}_{ij})$ from 2.2 has entries

$$\tilde{b}_{11} = 32z_2,\ \tilde{b}_{12} = 64z_4,\ \tilde{b}_{13} = 96z_6,\ \tilde{b}_{22} = -3z_2^3 + 36z_2z_4 + 32z_6,$$
$$\tilde{b}_{23} = \frac{1}{8}(5z_2^4 - 108z_2^2z_4 + 192z_4^2 + 544z_2z_6),$$
$$\tilde{b}_{33} = \frac{1}{64}(27z_2^5 - 300z_2^3z_4 - 1140z_2z_4^2 + 1140z_2^2z_6 + 7680z_4z_6).$$

Put $\tilde{\Delta}_{ij} = \det \begin{pmatrix} \tilde{b}_{ii} & \tilde{b}_{ij} \\ \tilde{b}_{ji} & \tilde{b}_{jj} \end{pmatrix}$ where $i < j$. Then $N(U_{\mathrm{B}_3,v})$ is the subset in $\mathbb{R}^8$ defined by the following relations

$$z_2 \ge 0, \tilde{\Delta}_{12} \ge 0, \det \tilde{B} \ge 0$$
$$z_1 = z_3 = z_5 = z_7 = 0,$$
$$384z_8 = 5z_2^4 - 72z_2^2z_4 + 48z_4^2 + 256z_2z_6.$$

The 3-dimensional principal orbit type stratum is given by

$$R^{(3)} = N(U_{\mathrm{B}_3,v}) \cap \{z_2 > 0, \tilde{\Delta}_{12} > 0, \det \tilde{B} > 0\}.$$

Put

$$\tilde{f}_1 = 53z_2^6 - 840z_2^4z_4 + 1680z_2^2z_4^2 + 6144z_4^3 + 2752z_2^3z_6 - 16128z_2z_4z_6 + 9216z_6^2,$$
$$\tilde{f}_2 = z_2^3 - 12z_2z_4 + 32z_6.$$

There are three 2-dimensional orbit type strata

$$R_1^{(2)} = N(U_{\mathrm{B}_3,v}) \cap \{z_2 > 0, \tilde{\Delta}_{12} > 0, \tilde{f}_1 = 0\}$$
$$R_2^{(2)} = N(U_{\mathrm{B}_3,v}) \cap \{z_2 > 0, \tilde{\Delta}_{12} = 0, \tilde{\Delta}_{23} > 0, \tilde{f}_1 = 0\}$$
$$R_3^{(2)} = N(U_{\mathrm{B}_3,v}) \cap \{z_2 > 0, \tilde{\Delta}_{13} > 0, \tilde{f}_2 = 0\},$$

the three 1-dimensional orbit type strata $R_1^{(1)}$, $R_2^{(1)}$, $R_3^{(1)}$ are the connected components of

$$N(U_{\mathrm{B}_3,v}) \cap \{z_2 > 0, \tilde{\Delta}_{12} = \tilde{\Delta}_{13} = \tilde{\Delta}_{23} = 0\},$$

and $R^{(0)} = \{0\}$ is the only 0-dimensional stratum.

The ambient stratification of $N(U_{\mathrm{B}_3,v})$ is obtained by cutting with the surface $\{z_2^2 - 4z_4 = 0\}$. There are two 3-dimensional ambient strata

$$S_1^{(3)} = R^{(3)} \cap \{z_2^2 - 4z_4 > 0\} \quad \text{and} \quad S_2^{(3)} = R^{(3)} \cap \{z_2^2 - 4z_4 < 0\},$$

five 2-dimensional ambient strata

$$\begin{aligned} S_1^{(2)} &= R^{(3)} \cap \{z_2^2 - 4z_4 = 0\},\ S_2^{(2)} = R_1^{(2)} \cap \{z_2^2 - 4z_4 > 0\}, \\ S_3^{(2)} &= R_1^{(2)} \cap \{z_2^2 - 4z_4 < 0\},\ S_4^{(2)} = R_2^{(2)},\ S_5^{(2)} = R_3^{(2)}, \end{aligned}$$

four 1-dimensional ambient strata $S_1^{(1)} = R_1^{(1)}$, $S_2^{(1)} = R_2^{(1)}$, $S_3^{(1)} = R_3^{(1)}$, $S_4^{(1)} = R_1^{(2)} \cap \{z_2^2 - 4z_4 = 0\}$, and $S^{(0)} = R^{(0)} = \{0\}$ is the only 0-dimensional ambient stratum. See figure 2.

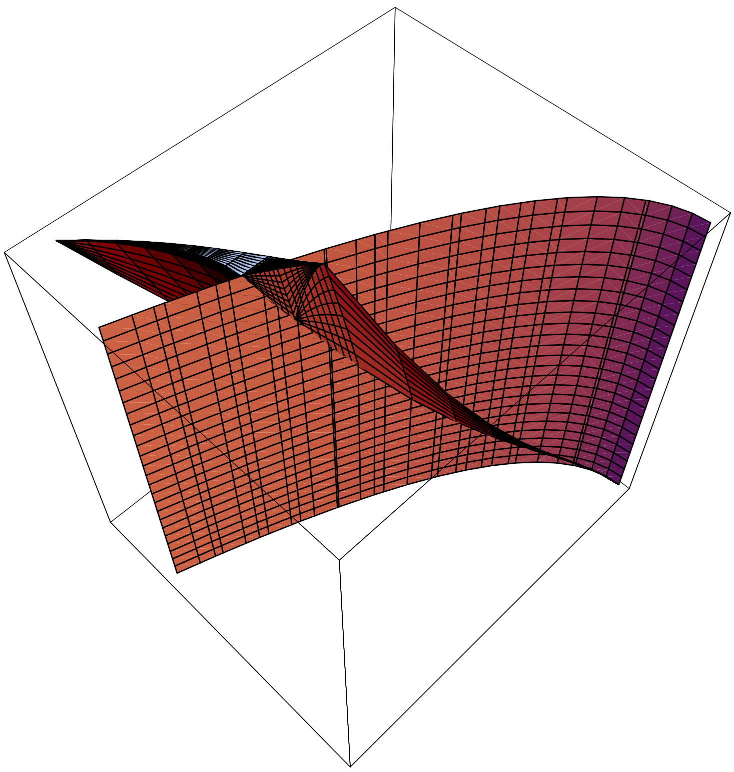

FIGURE 2. The projection of $N(U_{\mathrm{B}_3,v})$ to the $\{z_2, z_4, z_6\}$-subspace and intersection with the surface $\{z_2^2 - 4z_4 = 0\}$.

Let f, g, h be functions defined in some neighborhood of $0 \in \mathbb{R}$. Suppose that f and g are infinitely flat at 0 and $h(0) = 0$. For t near 0 consider the curve of polynomials $P(t)(x) = x^8 + \sum_{j=1}^{8}(-1)^j a_j(t)x^{8-j}$ where

$$a_1 = a_3 = a_5 = a_7 = 0,$$
$$a_2 = -56 + f,\ a_4 = 784 + g,\ a_6 = -2304 + h,$$
$$1024a_8 = 16a_2^4 - 128a_2^2 a_4 + 256a_4^2.$$

Then, for t near 0, $P(t)$ is a curve in $N(U_{\mathrm{B}_3,v})$ with $P(0) \in S_1^{(2)}$. At $t = 0$ it is normally flat with respect to the ambient stratification but normally nonflat with respect to the orbit type stratification.

If f, g and h are smooth, then $P(t)$ is smoothly solvable near $t = 0$, by theorem 5.2. Note that in this example we have $d = k = 6 < 8 = n$ and thus theorem 8.7 provides an actual improvement, too.

The following example shows that $W(U)$ must not necessarily be a finite reflection group, even though the $E_i|_U$ generate $\mathbb{R}[U]^{W(U)}$.

8.9. **Example.** Let U be the subspace of $\mathbb{R}^6$ defined by the following equations

$$x_1 + x_2 + x_3 = 0, \quad x_4 + x_5 + x_6 = 0.$$

The subgroup $N(U)$ of S_6 is generated by all permutations of x_1, x_2, x_3, all permutations of x_4, x_5, x_6, and the simultaneous transpositions of x_1 and x_4, x_2 and x_5, x_3 and x_6. The subgroup $Z(U)$ is trivial. Thus $W(U)$ is isomorphic to the semidirect product of $\mathrm{S}_3 \times \mathrm{S}_3$ and S_2.

One can get the subspace U above as follows. Consider the point $v = (x, x, x, y, y, y) \in \mathbb{R}^6$, where $x, y \neq 0$ and $x \neq y$. The isotropy group $H = (\mathrm{S}_6)_v$ of v is evidently isomorphic to $\mathrm{S}_3 \times \mathrm{S}_3$. Then $U = ((\mathbb{R}^6)^H)^\perp$. The group H is the normal subgroup of $W(U)$ generated by reflections.

First consider the action of H on U. It is clear that the algebra $\mathbb{R}[U]^H$ is a polynomial algebra generated by the basic generators

$$y_1 = x_1^2 + x_2^2 + x_1x_2,\ z_1 = x_1x_2(x_1 + x_2),$$
$$y_2 = x_4^2 + x_5^2 + x_4x_5,\ z_2 = x_4x_5(x_4 + x_5).$$

Consider the space $\mathbb{R}^4$ with the coordinates y_1, z_1, y_2, z_2 and the action of the group S_2 on it induced by the action of $\mathrm{S}_2 = W(U)/(\mathrm{S}_3 \times \mathrm{S}_3)$ on the above basic generators. It is easy to check that this action coincides with the diagonal action of S_2 on $(\mathbb{R}^2)^2$ for the standard action of S_2 on $\mathbb{R}^2$. Since the algebra of S_2-invariant polynomials on $(\mathbb{R}^2)^2$ is generated by the polarizations of basic invariants for the standard action of S_2 ob $\mathbb{R}^2$ we get the following system of generators of $\mathbb{R}[U]^{W(U)}$:

$$f_1 = y_1 + y_2,\ f_2 = z_1 + z_2,\ f_3 = y_1^2 + y_2^2,\ f_4 = y_1z_1 + y_2z_2,\ f_5 = z_1^2 + z_2^2.$$

Simple calculations for the restrictions of the Newton polynomials N_i on $\mathbb{R}^6$ to U gives the following result:

$$N_1|_U = 0,\ N_2|_U = 2f_1,\ N_3|_U = -3f_2,$$
$$N_4|_U = 2f_3,\ N_5|_U = -5f_4,\ N_6|_U = 3f_5 + 3f_1f_3 - f_1^3.$$

This proves that the morphism $\mathbb{R}[\mathbb{R}^6]^{\mathrm{S}_6} \to \mathbb{R}[U]^{W(U)}$ defined by restriction is surjective.

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PERTURBATION OF COMPLEX POLYNOMIALS AND NORMAL OPERATORS

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Abstract. We study the regularity of the roots of complex monic polynomials $P(t)$ of fixed degree depending smoothly on a real parameter t. We prove that each continuous parameterization of the roots of a generic C^∞ curve $P(t)$ (which always exists) is locally absolutely continuous. Generic means that no two of the continuously chosen roots meet of infinite order of flatness. Simple examples show that one cannot expect a better regularity than absolute continuity. This result will follow from the proposition that for any t_0 there exists a positive integer N such that $t \mapsto P(t_0 \pm (t - t_0)^N)$ admits smooth parameterizations of its roots near t_0. We show that C^n curves $P(t)$ (where $n = \deg P$) admit differentiable roots if and only if the order of contact of the roots is ≥ 1. We give applications to the perturbation theory of normal matrices and unbounded normal operators with compact resolvents and common domain of definition: The eigenvalues and eigenvectors of a generic C^∞ curve of such operators can be arranged locally in an absolutely continuous way.

1. Introduction

Let us consider a curve of polynomials

$$P(t)(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j(t) z^{n-j},$$

where the coefficients $a_j : I \to \mathbb{C}$, $1 \leq j \leq n$, are complex valued functions defined on an interval $I \subseteq \mathbb{R}$. Given that the coefficients a_j possess some regularity, it is natural to ask whether the roots of P can be arranged in a regular way as well, i.e., whether it is possible to find n regular functions $\lambda_j : I \to \mathbb{C}$, $1 \leq j \leq n$, such that $\lambda_1(t), \ldots, \lambda_n(t)$ represent the roots of $P(t)$ for each $t \in I$.

This problem has been extensively studied under the additional assumption that the polynomials $P(t)$ are hyperbolic, i.e., all roots of $P(t)$ are real. By a classical theorem due to Rellich [Rel37a], there exist real analytic parameterizations of the roots of P if its coefficients are real analytic. Bronshtein [Bro79] proved that if all a_j are of class C^n, then there exists a differentiable parameterization of the roots of P with locally bounded derivative (see also Wakabayashi [Wak86] for a different proof). It has been shown in [AKLM98] that if all a_j are smooth (C^∞) and no two of the increasingly ordered (thus continuous) roots meet of infinite order of flatness, then there exist smooth parameterizations of the roots. Moreover, by [KLM04], the roots may always be chosen twice differentiable provided that the a_j are C^{3n}. The conclusion in this statement is best possible as shown by an example in [BBCP06]. Recently, also the best possible assumptions have been found by [COP08]: If the coefficients a_j are C^n (resp. C^{2n}), the roots allow C^1 (resp. twice differentiable) parameterizations. For further results on this problem see also [Gla63], [Die70], [Man85], [CC04], [LR07], [KP08].

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If the hyperbolicity assumption is dropped, then there is the following general result (e.g. [Kat76, II §5 5.2]): There exist continuous functions $\lambda_j : I \to \mathbb{C}$, $1 \le j \le n$, which parameterize the roots of a curve of polynomials P with continuous coefficients $a_j : I \to \mathbb{C}$, $1 \le j \le n$. Note that, in contrast to the hyperbolic case, there is no hope that the roots of a polynomial P which depends regularly on more than one parameter might be parameterized even continuously; just take $P(t,s)(z) = z^2 - (t + is)$, where $t, s \in \mathbb{R}$ and $i = \sqrt{-1}$. Of course, in that example the roots are given as 2-valued analytic function with branching point 0 in terms of a Puiseux series, e.g. [Bau72, Appendix], but we do not go into that in this note. Here we restrict our attention to the one parameter case. In the absence of hyperbolicity the roots of a Lipschitz curve $t \mapsto P(t)$ of polynomials of degree n may still be parameterized in a $C^{0,1/n}$ way, locally, which follows from a result of Ostrowski [Ost40], but we cannot expect that the roots of P are locally Lipschitz continuous even when the coefficients a_j are real analytic; for instance, consider $P(t)(z) = z^2 - t$, $t \in \mathbb{R}$. However the roots of P may possess a weaker regularity: They may be parameterized by locally absolutely continuous functions. In fact, Spagnolo [Spa99] proved that there exist absolutely continuous parameterizations of the roots of P on compact intervals I if one of the following conditions holds:

(1) $n = 2$ and the coefficients a_j belong to C^5,
(2) $n = 3$ and the coefficients a_j belong to C^{25} (the case $n = 4$ is announced),
(3) $P(t)(z) = z^n - f(t)$ and f belongs to C^{2n+1}.

The proof makes essential use of the explicit formulas for the roots of P available in those cases.

In this paper we extend this result to generic smooth curves of polynomials P of arbitrary degree n. We say that P is generic if no two of the continuously arranged roots of P meet of infinite order of flatness. We show in section 3 that, if the a_j are smooth, then there exists an absolutely continuous parameterization of the roots of P on each compact interval I; actually, any continuous parameterization of the roots is locally absolutely continuous. In particular, these conditions are satisfied if the coefficients a_j are real analytic or, more generally, belong to a quasianalytic class of C^∞ functions. The main ingredient in the proof is the proposition 3.2 that for any $t_0 \in I$ there exists a positive integer N such that $t \mapsto P(t_0 \pm (t - t_0)^N)$ admits smooth parameterizations of its roots near t_0. It is not known whether the roots of P may be arranged in a locally absolutely continuous way if P is not generic. That problem requires different methods.

In section 4 we find conditions for the existence of differentiable parameterizations of the roots of P. Evidently, a necessary condition is that there exists a continuous choice of the roots such that whenever two of them meet they meet of order ≥ 1. We show that this condition is also sufficient, provided that the coefficients a_j of P belong to C^n.

In section 5 we discuss a reformulation of the problem in terms of a lifting problem which has been discussed in [AKLM00] and [KLMR05, KLMR06, KLMR08a]. This yields implicit sufficient conditions for a curve of polynomials P to allow smooth, C^1, or twice differentiable parameterizations of its roots. As application we discuss the quadratic case.

Applications to the perturbation theory of normal matrices are given in section 6. The eigenvalues and eigenvectors of a generic smooth curve $t \mapsto A(t)$ of normal complex matrices may be parameterized locally in an absolutely continuous way. The curve $t \mapsto A(t)$ is called generic if the associated characteristic polynomial $t \mapsto \chi_{A(t)}$ is generic. Examples show that without genericity or normality of $A(t)$ the eigenvectors need not admit continuous arrangements. We also prove that, for each t_0 there exists a positive integer N such that $t \mapsto A(t_0 \pm (t - t_0)^N)$ allows a smooth parameterization of its eigenvalues and eigenvectors near t_0. If A is real analytic, then the eigenvalues and eigenvectors of $t \mapsto A(t_0 \pm (t-t_0)^N)$ may be arranged real analytic as well.

In section 7 we obtain analogous results for curves $t \mapsto A(t)$ of unbounded normal operators in a Hilbert space with common domain of definition and with compact resolvents.

For more on the perturbation theory of linear operators consider Rellich [Rel37a, Rel37b, Rel39, Rel40, Rel42, Rel69], Kato [Kat76], Baumgärtel [Bau72], and also [AKLM98], [KM03], and [KMR09].

2. Preliminaries

2.1. Let

$$P(z) = z^n + \sum_{j=1}^n (-1)^j a_j z^{n-j} = \prod_{j=1}^n (z - \lambda_j)$$

be a monic polynomial with coefficients $a_1, \ldots, a_n \in \mathbb{C}$ and roots $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. By Vieta's formulas, $a_i = \sigma_i(\lambda_1, \ldots, \lambda_n)$, where $\sigma_1, \ldots, \sigma_n$ are the elementary symmetric functions in n variables:

$$\sigma_i(\lambda_1, \ldots, \lambda_n) = \sum_{1 \le j_1 < \cdots < j_i \le n} \lambda_{j_1} \cdots \lambda_{j_i}. \tag{2.1.1}$$

Denote by s_i, $i \in \mathbb{N}$, the Newton polynomials $\sum_{j=1}^n \lambda_j^i$ which are related to the elementary symmetric functions by

$$s_k - s_{k-1}\sigma_1 + s_{k-2}\sigma_2 - \cdots + (-1)^{k-1} s_1 \sigma_{k-1} + (-1)^k k \sigma_k = 0, \quad (k \ge 1). \tag{2.1.2}$$

Let us consider the so-called Bezoutiant

$$B := \begin{pmatrix} s_0 & s_1 & \ldots & s_{n-1} \\ s_1 & s_2 & \ldots & s_n \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & \ldots & s_{2n-2} \end{pmatrix} = (s_{i+j-2})_{1 \le i,j \le n}.$$

Since the entries of B are symmetric polynomials in $\lambda_1, \ldots, \lambda_n$, we find a unique symmetric $n \times n$ matrix $\tilde{B}$ with $B = \tilde{B} \circ \sigma$, where $\sigma = (\sigma_1, \ldots, \sigma_n)$.

Let B_k denote the minor formed by the first k rows and columns of B. Then we find

$$\Delta_k(\lambda) := \det B_k(\lambda) = \sum_{i_1 < i_2 < \cdots < i_k} (\lambda_{i_1} - \lambda_{i_2})^2 \cdots (\lambda_{i_1} - \lambda_{i_k})^2 \cdots (\lambda_{i_{k-1}} - \lambda_{i_k})^2. \tag{2.1.3}$$

Since the polynomials Δ_k are symmetric, we have $\Delta_k = \tilde{\Delta}_k \circ \sigma$ for unique polynomials $\tilde{\Delta}_k$.

From (2.1.3) follows that the number of distinct roots of P equals the maximal k such that $\tilde{\Delta}_k(P) \ne 0$.

2.2. **Multiplicity.** For a continuous real or complex valued function f defined near 0 in $\mathbb{R}$ let the *multiplicity* (or *order of flatness*) $m(f)$ at 0 be the supremum of all integers p such that $f(t) = t^p g(t)$ near 0 for a continuous function g. We define in the obvious way the *multiplicity* $m_{t_0}(f)$ of f at any $t_0 \in \mathbb{R}$ (if f is defined near t_0). Note that, if f is of class C^n and $m(f) < n$, then $f(t) = t^{m(f)} g(t)$ near 0, where now g is $C^{n-m(f)}$ and $g(0) \ne 0$.

If f is a continuous function on the space of polynomials, then for a fixed continuous curve P of polynomials we will denote by $m(f)$ the multiplicity at 0 of $t \mapsto f(P(t))$.

We shall say that two functions f and g *meet of order* $\ge p$ *at* 0 or *have order of contact* $\ge p$ *at* 0 iff $m(f-g) \ge p$.

Lemma. *Let $I \subseteq \mathbb{R}$ be an interval containing* 0. *Consider a curve of polynomials*

$$P(t)(z) = z^n + \sum_{j=2}^{n} (-1)^j a_j(t) z^{n-j},$$

with $a_j : I \to \mathbb{C}$, $2 \le j \le n$, smooth. Then, for integers r, the following conditions are equivalent:

1. *$m(a_k) \ge kr$, for all $2 \le k \le n$;*
2. *$m(\tilde{\Delta}_k) \ge k(k-1)r$, for all $2 \le k \le n$.*

Proof. Without loss of generality we can assume $r > 0$.

(1) $\Rightarrow$ (2): From (2.1.2) we deduce $m(\tilde{s}_k) \ge kr$ for all k, where $s_k = \tilde{s}_k \circ \sigma$. By the definition of $\tilde{\Delta}_k = \det(\tilde{B}_k)$ we obtain (2).

(2) $\Rightarrow$ (1): It is easy to see that $\tilde{\Delta}_k(0) = 0$ for all $2 \le k \le n$ implies $\tilde{s}_k(0) = 0$ for all $2 \le k \le n$. Then by (2.1.2) we have $a_k(0) = 0$ for all $2 \le k \le n$. So near 0 we have $a_2(t) = t^{2r} a_{2,2r}(t)$ and $a_k(t) = t^{m_k} a_{k,m_k}(t)$ for $3 \le k \le n$, where the m_k are positive integers and $a_{2,2r}, a_{3,m_3}, \ldots, a_{n,m_n}$ are smooth functions. Let us suppose for contradiction that for some $k > 2$ we have $m_k = m(a_k) < kr$. We put

$$m := \min\left\{r, \frac{m_3}{3}, \ldots, \frac{m_n}{n}\right\} < r, \tag{2.2.1}$$

and consider the following continuous curve of polynomials for (small) $t \ge 0$:

$$P_{(m)}(t)(z) := z^n + a_{2,2r}(t) t^{2r-2m} z^{n-2} - \cdots + (-1)^n a_{n,m_n}(t) t^{m_n - nm}.$$

We have $\tilde{\Delta}_k(P_{(m)}(t)) = t^{-k(k-1)m} \tilde{\Delta}_k(P(t))$. By (2.2.1), $r - m > 0$, whence $t \mapsto \tilde{\Delta}_k(P_{(m)}(t))$, $2 \le k \le n$, vanish at $t = 0$. We may conclude as before that all coefficients of $t \mapsto P_{(m)}(t)$ vanish for $t = 0$. But this is a contradiction for those k with $m(a_k) = m_k = km$. $\square$

Remark. If the coefficients a_j of P in lemma 2.2 are just of class C^n, the conclusion remains true for $r = 1$. The proof is the same with the slight modification that we define $m_k := \min\{k, m(a_k)\}$ for all k.

2.3. **Genericity condition.** Let $I \subseteq \mathbb{R}$ be an interval. We call a curve of monic polynomials

$$P(t)(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j(t) z^{n-j}$$

with continuous coefficients $a_j : I \to \mathbb{C}$, $1 \le j \le n$, *generic* if the following equivalent conditions are satisfied at any $t_0 \in I$:

1. If two of the continuously parameterized roots of P meet of infinite order of flatness at t_0, then their germs at t_0 are equal.
2. Let k be maximal with the property that the germ at t_0 of $t \mapsto \tilde{\Delta}_k(P(t))$ is not 0. Then $t \mapsto \tilde{\Delta}_k(P(t))$ is not infinitely flat at t_0.

The equivalence of (1) and (2) follows easily from (2.1.3). For instance, P is generic, if its coefficients are real analytic, or more generally, belong to a quasianalytic class of C^∞ functions.

2.4. **Lemma** (Splitting lemma [AKLM98, 3.4]). *Let $P_0 = z^n + \sum_{j=1}^{n} (-1)^j a_j z^{n-j}$ be a polynomial satisfying $P_0 = P_1 \cdot P_2$, where P_1 and P_2 are polynomials without common root. Then for P near P_0 we have $P = P_1(P) \cdot P_2(P)$ for real analytic mappings of monic polynomials $P \mapsto P_1(P)$ and $P \mapsto P_2(P)$, defined for P near P_0, with the given initial values.*

2.5. **Absolutely continuous functions.** Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{C}$ is called *absolutely continuous*, or $f \in AC(I)$, if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $\sum_{i=1}^{N}(b_i - a_i) < \delta$ implies $\sum_{i=1}^{N} |f(b_i) - f(a_i)| < \epsilon$, for all sequences of pairwise disjoint subintervals $(a_i, b_i) \subseteq I$, $1 \le i \le N$. By the fundamental theorem of calculus for the Lebesgue integral, $f \in AC([a, b])$ if and only if there is a function $g \in L^1([a, b])$ such that

$$f(t) = f(a) + \int_a^t g(s)ds \quad \text{for all } t \in [a, b].$$

Then $f' = g$ almost everywhere. Every Lipschitz function is absolutely continuous.

Gluing finitely many absolutely continuous functions provides an absolutely continuous function: Let $f_1 \in AC([a, b])$, $f_2 \in AC([b, c])$, and $f_1(b) = f_2(b)$. Then $f : [a, c] \to \mathbb{C}$, defined by $f(t) = f_1(t)$ if $t \in [a, b]$ and $f(t) = f_2(t)$ if $t \in [b, c]$, belongs to $AC([a, c])$. Similarly for more than two functions.

Let $\varphi : I \to I$ be bijective, strictly increasing, and Lipschitz continuous. If $f \in AC(I)$ then also $f \circ \varphi \in AC(I)$. Furthermore:

Lemma. *Let $r > 0$ and $n \in \mathbb{N}_{>0}$. Let $f \in AC([0, r])$ (resp. $f \in AC([-r, 0])$) and set $h(t) = f(\sqrt[n]{t})$ (resp. $h(t) = f(-\sqrt[n]{|t|})$). Then $h \in AC([0, r^n])$ (resp. $h \in AC([-r^n, 0])$).*

Proof. There exists a function $g \in L^1([0, r])$ such that

$$f(t) = f(0) + \int_0^t g(s)ds$$

for all $t \in [0, r]$. The function $(0, r^n] \to (0, r], t \mapsto \sqrt[n]{t}$, is smooth and bijective, so

$$\int_0^{r^n} |g(\sqrt[n]{s})|(\sqrt[n]{s})'ds = \int_0^r |g(s)|ds$$

and $t \mapsto g(\sqrt[n]{t})(\sqrt[n]{t})'$ belongs to $L^1([0, r^n])$. Thus $h(t) = f(\sqrt[n]{t})$ is in $AC([0, r^n])$.

For the second statement consider the absolutely continuous function $f \circ S|_{[0,r]}$, where $S : \mathbb{R} \to \mathbb{R}, t \mapsto -t$. By the above, $h_S(t) = (f \circ S|_{[0,r]})(\sqrt[n]{t})$ is in $AC([0, r^n])$, and so $h(t) = h_S(S^{-1}|_{[-r^n,0]}(t)) = f(-\sqrt[n]{-t}) = f(-\sqrt[n]{|t|})$ is in $AC([-r^n, 0])$. $\square$

3. Absolutely continuous parameterization of the roots

3.1. **Lemma.** *Let $I \subseteq \mathbb{R}$ be an interval. Consider a curve of monic polynomials*

$$P(t)(z) = z^n + \sum_{j=1}^{n}(-1)^j a_j(t) z^{n-j}$$

such that the coefficients $a_j : I \to \mathbb{C}$, $1 \le j \le n$, are continuous. If there is a Lipschitz parameterization of the roots of $P(t)$, then any continuous parameterization is Lipschitz.

Proof. Let $\mu_1, \ldots, \mu_n$ be a Lipschitz parameterization of the roots of P on I with common Lipschitz constant C. Assume that $t \mapsto \lambda(t)$ is any continuous root of $t \mapsto P(t)$ for $t \in I$. Let $t < s$ be in I. Then there is an i_0 such that $\lambda(t) = \mu_{i_0}(t)$. Now let t_1 be the maximum of all $r \in [t, s]$ such that $\lambda(r) = \mu_{i_0}(r)$. If $t_1 < s$ then $\mu_{i_0}(t_1) = \mu_{i_1}(t_1)$ for some $i_1 \ne i_0$. Let t_2 be the maximum of all $r \in [t_1, s]$ such that $\lambda(r) = \mu_{i_1}(r)$. If $t_2 < s$ then $\mu_{i_1}(t_2) = \mu_{i_2}(t_2)$ for some $i_2 \notin \{i_0, i_1\}$. And so on until $s = t_k$ for some $k \le n$. Then we have (where $t_0 = t$)

$$\frac{|\lambda(s) - \lambda(t)|}{s - t} \le \sum_{j=0}^{k-1} \frac{|\mu_{i_j}(t_{j+1}) - \mu_{i_j}(t_j)|}{t_{j+1} - t_j} \cdot \frac{t_{j+1} - t_j}{s - t} \le C. \quad \square$$

3.2. **Proposition.** *Let $I \subseteq \mathbb{R}$ be an interval. Consider a generic curve of monic polynomials*

$$P(t)(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j(t) z^{n-j},$$

with smooth coefficients $a_j : I \to \mathbb{C}$, $1 \le j \le n$. For any $t_0 \in I$, there exists a positive integer N such that the roots of $t \mapsto P(t_0 \pm (t - t_0)^N)$ can be parameterized smoothly near t_0. If the coefficients a_i are real analytic, then the roots of $t \mapsto P(t_0 \pm (t - t_0)^N)$ can be parameterized real analytically near t_0.

Proof. It is no restriction to assume that $0 \in I$ and $t_0 = 0$.

We use the following:

Algorithm. (1) If all roots of $P(0)$ are pairwise different, the roots of $t \to P(\pm t)$ may be parameterized smoothly near 0, by the implicit function theorem. Then $N = 1$.

(2) If there are distinct roots of $P(0)$, we put them into two subsets which factors $P(t) = P_1(t)P_2(t)$ by the splitting lemma 2.4. Suppose that $t \mapsto P_1(\pm t^{N_1})$ and $t \mapsto P_2(\pm t^{N_2})$ are smoothly solvable near 0, then $t \mapsto P(\pm t^{N_1 N_2})$ is smoothly solvable near 0 as well.

(3) If all roots of $P(0)$ are equal, we reduce to the case $a_1 = 0$, by replacing z with $z - a_1(t)/n$. Then all roots of $P(0)$ are equal to 0, hence, $a_k(0) = 0$ for all k. Let $m := \min\{m(a_k)/k : 2 \le k \le n\}$ which exists since P is generic (by lemma 2.2). Let d be a minimal integer such that $dm \ge 1$. Then for the multiplicity of $t \mapsto a_k(\pm t^d)$, $2 \le k \le n$, we find

$$m(a_k(\pm t^d)) = dm(a_k) \ge dmk \ge k.$$

Hence we may write $a_k(\pm t^d) = t^k \tilde{a}_k^{\pm}(t)$ near 0 with $\tilde{a}_k^{\pm}$ smooth, for all k. Consider

$$\tilde{P}^{\pm}(t)(z) = z^n + \sum_{j=2}^{n} (-1)^j \tilde{a}_j^{\pm}(t) z^{n-j}.$$

If $t \to \tilde{P}^{\pm}(t)$ is smoothly solvable and $t \mapsto \lambda_j^{\pm}(t)$ are its smooth roots, then $t \mapsto t\lambda_j^{\pm}(t)$ are smooth parameterizations of the roots of $t \mapsto P(\pm t^d)$.

Note that $m(\tilde{a}_k^{\pm}) = dm(a_k) - k$, for $2 \le k \le n$, and thus

$$\tilde{m} := \min_{2 \le k \le n} \frac{m(\tilde{a}_k^{\pm})}{k} = dm - 1 < m, \tag{3.2.1}$$

by the minimality of d.

If $\tilde{m} = 0$ there exists some k such that $\tilde{a}_k^{\pm}(0) \ne 0$, and not all roots of $\tilde{P}^{\pm}(0)$ are equal. We feed $\tilde{P}^{\pm}$ into step (2). Otherwise we feed $\tilde{P}^{\pm}$ into step (3).

Step (1) and (2) either provide a required parameterization or reduce the problem to a lower degree n. Since $\tilde{m}$ is of the form p/k where $2 \le k \le n$ and $p \in \mathbb{N}$ and by (3.2.1), also step (3) is visited only finitely many times. So the algorithm stops after finitely many steps and it provides an integer N and a smooth parameterization of the roots of $t \mapsto P(\pm t^N)$ near 0. The real analytic case is analogous. □

3.3. **Theorem.** *Let $I \subseteq \mathbb{R}$ be an interval. Consider a generic curve of monic polynomials*

$$P(t)(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j(t) z^{n-j},$$

with smooth coefficients $a_j : I \to \mathbb{C}$, $1 \le j \le n$. Any continuous parameterization $\lambda = (\lambda_1, \ldots, \lambda_n) : I \to \mathbb{C}^n$ of the roots of P is locally absolutely continuous.

Proof. It suffices to show that each $t_0 \in I$ has a neighborhood on which λ is absolutely continuous. Without restriction we assume that $0 \in I$ and $t_0 = 0$. By proposition 3.2, there is an integer N and a neighborhood J_N of 0 such that $t \mapsto P(\pm t^N)$ allows a smooth parameterization $\mu^\pm = (\mu_1^\pm, \ldots, \mu_n^\pm)$ of its roots on J_N. Another continuous parameterization is provided by $t \mapsto \lambda(\pm t^N) = (\lambda_1(\pm t^N), \ldots, \lambda_n(\pm t^N))$. By lemma 3.1, the parameterization $t \mapsto \lambda(\pm t^N)$ is actually Lipschitz (by shrinking J_N if necessary), in particular, absolutely continuous. Let $J = \{t \in I : \pm \sqrt[N]{|t|} \in J_N\}$, $J_{\geq 0} = \{t \in J : t \geq 0\}$, and $J_{\leq 0} = \{t \in J : t \leq 0\}$. By lemma 2.5, we find that λ is absolutely continuous on $J_{\geq 0}$. In order to see that λ is absolutely continuous on $J_{\leq 0}$ we apply lemma 2.5 to $t \mapsto \lambda(-t^N)$, if N is even, and to $t \mapsto \lambda(t^N)$, if N is odd. Hence λ is absolutely continuous on J. This completes the proof. $\square$

3.4. **Corollary.** *Any continuous parameterization of the roots of a real analytic, or more generally quasianalytic, curve $I \ni t \mapsto P(t)$ of monic polynomials is locally absolutely continuous.* $\square$

3.5. **Remark.** The conclusion in theorem 3.3 is best possible. In general the roots cannot be chosen with first derivative in L^p_{loc} for any $1 < p \leq \infty$. A counter example is given by

$$P(t)(z) = z^n - t, \quad t \in \mathbb{R},$$

if $n \geq \frac{p}{p-1}$, for $1 < p < \infty$, and if $n \geq 2$, for $p = \infty$.

On the other hand, finding the optimal assumptions on P for admitting locally absolutely continuous roots is an open problem.

4. Differentiable parameterization of the roots

4.1. **Lemma** ([KLMR05, 4.3]). *Consider a continuous curve $c : (a, b) \to X$ in a compact metric space X. Then the set of all accumulation points of $c(t)$ as $t \searrow a$ is connected.*

4.2. **Proposition.** *Let $I \subseteq \mathbb{R}$ be an interval containing* 0. *Consider a curve of monic polynomials*

$$P(t)(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j(t) z^{n-j},$$

such that the coefficients $a_j : I \to \mathbb{C}$, $1 \leq j \leq n$, are of class C^n. Then the following conditions are equivalent:

(1) *There exists a local continuous parameterization of the roots of P near* 0 *which is differentiable at* 0.
(2) *There exists a local continuous parameterization λ_i of the roots of P near* 0 *such that $\lambda_i(0) = \lambda_j(0)$ implies $m(\lambda_i - \lambda_j) \geq 1$, for all $i \neq j$.*
(3) *Split $P(t) = P_1(t) \cdots P_l(t)$ according to lemma 2.4, where l is the number of distinct roots of $P(0)$. Then $m(\tilde{\Delta}_k(P_i)) \geq k(k-1)$, for all $1 \leq i \leq l$ and $2 \leq k \leq \deg P_i$.*

Proof. (1) $\Rightarrow$ (2) is obvious and (2) $\Rightarrow$ (3) follows immediately from (2.1.3).

(3) $\Rightarrow$ (1): Using the splitting $P(t) = P_1(t) \cdots P_l(t)$, we may suppose that all roots of $P(0)$ coincide. We can reduce to the case $a_1 = 0$ by replacing the variable z with $z - a_1(t)/n$. Then all roots of $P(0)$ are equal to 0. By assumption and remark 2.2, we find $m(a_k) \geq k$, for all $2 \leq k \leq n$. So, for t near 0, we can write $a_k(t) = t^k a_{k,k}(t)$ for continuous $a_{k,k}$ and $2 \leq k \leq n$. The continuous curve of polynomials $P_{(1)}(t)(z) := z^n + \sum_{j=2}^n (-1)^j a_{j,j}(t) z^{n-j}$ admits a continuous parameterization $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$ of its roots near 0. Then $\lambda(t) := t\tilde{\lambda}(t)$ parameterizes the roots of P, locally near 0, and is differentiable at 0. $\square$

4.3. **Theorem.** *Let $I \subseteq \mathbb{R}$ be an open interval. Consider a curve of monic polynomials*

$$P(t)(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j(t) z^{n-j},$$

such that the coefficients $a_j : I \to \mathbb{C}$, $1 \le j \le n$, are of class C^n. Then the following conditions are equivalent:

(1) *There exists a global differentiable parameterization of the roots of P.*
(2) *There exists a global continuous parameterization of the roots of P with order of contact ≥ 1 (i.e. if any two roots meet they meet of order ≥ 1).*
(3) *Let $t_0 \in I$. Split $P(t) = P_1(t) \cdots P_l(t)$ near t_0 according to lemma 2.4, where l is the number of distinct roots of $P(t_0)$. Then $m_{t_0}(\tilde{\Delta}_k(P_i)) \ge k(k-1)$, for all $1 \le i \le l$ and $2 \le k \le \deg P_i$.*

Proof. By proposition 4.2, it just remains to check (3) $\Rightarrow$ (1).

We use induction on n. There is nothing to prove if $n = 1$. So let us assume that (3) $\Rightarrow$ (1) holds for degrees strictly less than n.

We may suppose that $a_1 = 0$ by replacing z with $z - a_1(t)/n$. Consider the set F of all $t \in I$ such that all roots of $P(t)$ coincide. Then F is closed and its complement $I \backslash F$ is a countable union of open subintervals whose boundary points lie in F.

Let J denote one such interval. For each $t_0 \in J$, the polynomial $P(t_0)$ has distinct roots which may be put into distinct subsets, and, by lemma 2.4, we obtain a local splitting $P(t) = P_1(t)P_2(t)$ near t_0, where both P_1 and P_2 have degree less than n. Clearly, P_1 and P_2 satisfy (3) as well. By induction hypothesis, we find differentiable parameterizations of the roots of P, locally near any $t_0 \in J$.

Let λ be a differentiable parameterization of the roots of P defined on a maximal subinterval $J' \subseteq J$. Suppose that the right (say) endpoint t_1 of J' belongs to J. Then there exists a differentiable parameterization $\bar{\lambda}$ of the roots of P, locally near t_1, and there is a $t_0 < t_1$ such that both λ and $\bar{\lambda}$ are defined near t_0. Let (t_m) be a sequence with $t_m \to t_0$. For each m there exists a permutation τ_m such that $\lambda(t_m) = \tau_m.\bar{\lambda}(t_m)$. By passing to a subsequence, again denoted by (t_m), we have $\lambda(t_m) = \tau.\bar{\lambda}(t_m)$ for a fixed permutation τ and for all m. Then $\lambda(t_0) = \lim_{t_m \to t_0} \lambda(t_m) = \tau.(\lim_{t_m \to t_0} \bar{\lambda}(t_m)) = \tau.\bar{\lambda}(t_0)$ and

$$\lambda'(t_0) = \lim_{t_m \to t_0} \frac{\lambda(t_m) - \lambda(t_0)}{t_m - t_0} = \lim_{t_m \to t_0} \frac{\tau.\bar{\lambda}(t_m) - \tau.\bar{\lambda}(t_0)}{t_m - t_0} = \tau.\bar{\lambda}'(t_0).$$

Hence, the differentiable parameterization λ of the roots of P was not maximal: we can extend it differentiably by defining $\tilde{\lambda}(t) := \lambda(t)$ for $t \le t_0$ and $\tilde{\lambda}(t) := \tau.\bar{\lambda}(t)$ for $t \ge t_0$. This shows that there exists a differentiable parameterization λ of the roots of P defined on J.

Let us extend λ to the closure of J, by setting it 0 at the endpoints of J. Since $a_1 = 0$, then λ still parameterizes the roots of P on the closure of J. Let t_0 denote the right (say) endpoint of J. By proposition 4.2, there exists a local continuous parameterization $\bar{\lambda}$ of the roots of P near t_0 which is differentiable at t_0. Let (t_m) be a sequence with $t_m \nearrow t_0$. By passing to a subsequence, we may assume that $\lambda(t_m) = \tau.\bar{\lambda}(t_m)$ for a fixed permutation τ and for all m. Then $\lim_{t_m \nearrow t_0} \lambda(t_m) = \tau.(\lim_{t_m \nearrow t_0} \bar{\lambda}(t_m)) = \tau.0 = 0$ and

$$\lim_{t_m \nearrow t_0} \frac{\lambda(t_m)}{t_m - t_0} = \lim_{t_m \nearrow t_0} \frac{\tau.\bar{\lambda}(t_m)}{t_m - t_0} = \tau.\bar{\lambda}'(t_0).$$

It follows that the set of accumulation points of $\lambda(t)/(t - t_0)$, as $t \nearrow t_0$, lies in the S_n-orbit through $\bar{\lambda}'(t_0)$ of the symmetric group S_n. Since this orbit is finite, we may conclude from lemma 4.1 that the limit $\lim_{t \nearrow t_0} \lambda(t)/(t - t_0)$ exits. Thus the one-sided derivative of λ at t_0 exists.

For isolated points in F, it follows from the discussion in the previous paragraph that we can apply a fixed permutation to one of the neighboring differentiable parameterizations of the roots in order to glue them differentiably. Therefore, we have found a differentiable parameterization λ of the roots of P defined on $I \backslash F'$, where F' denotes the set of accumulation points of F.

Let us extend λ by 0 on F'. Then it provides a global differentiable parameterization of the roots of P, since any parameterization is differentiable at points $t' \in F'$. For: It is clear that the derivative at t' of any differentiable parameterization has to be 0. Let $\bar{\lambda}$ be the local parameterization near t', provided by proposition 4.2. As above we may conclude that the set of accumulation points of $\lambda(t)/(t - t')$, as $t \to t'$, lies in the S_n-orbit through $\bar{\lambda}'(t') = 0$. □

5. Reformulation of the problem

5.1. **Lifting curves over invariants.** Let G be a compact Lie group and let $\rho : G \to \mathrm{O}(V)$ be an orthogonal representation in a real finite dimensional Euclidean vector space V. By a classical theorem of Hilbert and Nagata, the algebra $\mathbb{R}[V]^G$ of invariant polynomials on V is finitely generated. So let $\sigma_1, \ldots, \sigma_n$ be a system of homogeneous generators of $\mathbb{R}[V]^G$ of positive degrees $d_1, \ldots, d_n$. Consider the orbit map $\sigma = (\sigma_1, \ldots, \sigma_n) : V \to \mathbb{R}^n$. The image $\sigma(V)$ is a semialgebraic subset of $\{y \in \mathbb{R}^n : P(y) = 0 \text{ for all } P \in I\}$, where I is the ideal of relations between $\sigma_1, \ldots, \sigma_n$. Since G is compact, σ is proper and separates orbits of G, it thus induces a homeomorphism between V/G and $\sigma(V)$.

Let $H = G_v$ be the isotropy group of $v \in V$ and (H) the conjugacy class of H in G which is called the type of the orbit $G.v$. The union $V_{(H)}$ of orbits of type (H) is called an orbit type submanifold of the representation ρ, and $V_{(H)}/G$ is called an orbit type submanifold of the orbit space V/G. The collection of connected components of the manifolds $\{V_{(H)}/G\}$ forms a stratification of V/G called orbit type stratification, see e.g. [Pfl01, 4.3].

Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a smooth curve in the orbit space; smooth as curve in $\mathbb{R}^n$. A curve $\bar{c} : \mathbb{R} \to V$ is called *lift* of c to V, if $c = \sigma \circ \bar{c}$ holds. The problem of lifting smooth curves over invariants is independent of the choice of a system of homogeneous generators of $\mathbb{R}[V]^G$, see [KLMR05, 2.2].

Let $s \in \mathbb{N}$. Denote by A_s the union of all strata X of the orbit space V/G with $\dim X \le s$, and by I_s the ideal of $\mathbb{R}[V]^G$ consisting of all polynomials vanishing on A_{s-1}. Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a smooth curve, $t \in \mathbb{R}$, and $s = s(c,t)$ a minimal integer such that, for a neighborhood J of t in $\mathbb{R}$, we have $c(J) \subseteq A_s$. The curve c is called *normally nonflat at* t if there is $f \in I_s$ such that $f \circ c$ is not infinitely flat at t. A smooth curve $c : \mathbb{R} \to \sigma(V) \subseteq \mathbb{R}^n$ is called *generic*, if c is normally nonflat at t for each $t \in \mathbb{R}$.

Let $G = \mathrm{S}_n$, the symmetric group, and let ρ be the standard representation of S_n in $\mathbb{R}^n$ by permuting the coordinates. The elementary symmetric functions σ_i in (2.1.1) generate the algebra of symmetric polynomials $\mathbb{R}[\mathbb{R}^n]^{\mathrm{S}_n}$. Hence the image $\sigma(\mathbb{R}^n)$ may be identified with the space of monic hyperbolic polynomials of degree n. Recall that a polynomial is called hyperbolic if all its roots are real. A lift to $\mathbb{R}^n$ of a curve P in $\sigma(\mathbb{R}^n)$ represents a parameterization of the roots of P. A curve P of hyperbolic polynomials is generic in the sense of the last paragraph if and only if it is generic in the sense of 2.3, see e.g. [LR07, 2.6].

The following theorem generalizes the main results on the one dimensional perturbation theory of hyperbolic polynomials. It collects the main results of [AKLM00] and [KLMR05, KLMR06, KLMR08a].

Theorem. *Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a curve in the orbit space and let $d = \max\{d_1, \dots, d_n\}$. Then:*

(1) *If c is real analytic, then it allows a real analytic lift, locally.*
(2) *If c is smooth and generic, then there exists a global smooth lift.*
(3) *If c is C^d, then there exists a global differentiable lift.*

If G is finite, write $V = V_1 \oplus \cdots \oplus V_l$ as orthogonal direct sum of irreducible subspaces V_i and define $k = \max\{d, k_1, \dots, k_l\}$, where $k_i = \min\{|G.v| : v \in V_i \setminus \{0\}\}$. Then:

(4) *If c is C^k, then each differentiable lift is C^1.*
(5) *If c is C^{d+k}, then there exists a global twice differentiable lift.*

5.2. Let us consider the standard action of the symmetric group S_n on $\mathbb{C}^n$ by permuting the coordinates and the diagonal action of S_n on $\mathbb{R}^n \times \mathbb{R}^n$ by permuting the coordinates in each factor. Write $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ where $z_k = x_k + i y_k$, $1 \le k \le n$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. The mapping

$$T : \mathbb{C}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n : z \longmapsto (x, y)$$

is an equivariant $\mathbb{R}$-linear homeomorphism. Consequently, it descends to a homeomorphism $\hat{T}$ such that the following diagram commutes

$$\begin{tikzcd} \mathbb{C}^n \arrow[rr, "T"] \arrow[d] & & \mathbb{R}^n \times \mathbb{R}^n \arrow[d] \\ \mathbb{C}^n/\mathrm{S}_n \arrow[rr, "\hat{T}"'] & & (\mathbb{R}^n \times \mathbb{R}^n)/\mathrm{S}_n \end{tikzcd} \tag{5.2.1}$$

Consider the respective orbit type stratifications of the S_n-modules $\mathbb{C}^n$ and $\mathbb{R}^n \times \mathbb{R}^n$ and of its orbit spaces. It is evident that T, and thus also $\hat{T}$, maps strata onto strata. Note that, while the orbit type stratification of $\mathbb{C}^n/\mathrm{S}_n \cong \mathbb{C}^n$ is finer than its stratification as affine variety, the orbit type stratification of $(\mathbb{R}^n \times \mathbb{R}^n)/\mathrm{S}_n$ is its coarsest stratification, e.g. [Pfl01, 4.4.6].

Let $P : \mathbb{R} \to \mathbb{C}^n/\mathrm{S}_n = \mathbb{C}^n$ be a curve of monic polynomials of degree n. Then $\hat{T} \circ P$ is a curve in $(\mathbb{R}^n \times \mathbb{R}^n)/\mathrm{S}_n \subseteq \mathbb{R}^N$. It follows that P allows a regular lift to $\mathbb{C}^n$, i.e., a regular parameterization of its roots, if and only if $\hat{T} \circ P$ allows a lift of the same regularity to $\mathbb{R}^n \times \mathbb{R}^n$. Theorem 5.1 provides sufficient conditions for $\hat{T} \circ P$ to be liftable regularly, and hence for P to admit regular parameterizations of its roots.

As generators for the algebra $\mathbb{C}[\mathbb{C}^n]^{\mathrm{S}_n}$ we may choose the Newton polynomials $s_i(z) = \sum_{j=1}^n z_j^i$, for $1 \le i \le n$. By the first fundamental theorem of invariant theory for S_n (e.g. [Smi95, 3.4.1]), the algebra $\mathbb{R}[\mathbb{R}^n \times \mathbb{R}^n]^{\mathrm{S}_n}$ is generated by the polarizations of the s_i:

$$\tau_{i,j}(x, y) = \sum_{k=1}^n x_k^i y_k^j, \quad (i, j \in \mathbb{N}, 1 \le i + j \le n).$$

We may then identify the orbit projections

$$\mathbb{C}^n \longrightarrow \mathbb{C}^n/\mathrm{S}_n \quad \text{and} \quad \mathbb{R}^n \times \mathbb{R}^n \longrightarrow (\mathbb{R}^n \times \mathbb{R}^n)/\mathrm{S}_n$$

with the mappings

$$s = (s_i) : \mathbb{C}^n \longrightarrow s(\mathbb{C}^n) = \mathbb{C}^n \quad \text{and} \quad \tau = (\tau_{i,j}) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \tau(\mathbb{R}^n \times \mathbb{R}^n) \subseteq \mathbb{R}^N,$$

respectively. Here $N = \binom{n+2}{n} - 1 = \frac{1}{2}n(n+3)$. The image $\tau(\mathbb{R}^n \times \mathbb{R}^n)$ is a semialgebraic subset of $\mathbb{R}^N$. Since it is homeomorphic with $s(\mathbb{C}^n) = \mathbb{C}^n$, its dimension is $2n$. It follows that there are at least $\frac{1}{2}n(n-1)$ independent non-trivial polynomial relations between the $\tau_{i,j}$.

The homeomorphism $\hat{T}$ from the diagram (5.2.1) is then determined by:

$$\hat{T}^{-1} : \mathbb{R}^N \supseteq \tau(\mathbb{R}^n \times \mathbb{R}^n) \longrightarrow \mathbb{C}^n : (\tau_{i,j}) \longmapsto \left(\sum_{k=0}^{m} \binom{m}{k} i^k \tau_{m-k,k} \right)_{1 \le m \le n}.$$

5.3. **The quadratic case.** Without loss it suffices to consider $P(t)(z) = z^2 - f(t)$ with $f : I \to \mathbb{C}$. Let us consider the curve $\hat{T} \circ P$ in $(\mathbb{R}^2 \times \mathbb{R}^2)/\,\mathrm{S}_2$ whose coordinates $\tau_{i,j}(P)$ have to satisfy:

$$\tau_{1,0}(P) = \tau_{0,1}(P) = 0,\ \tau_{2,0}(P) - \tau_{0,2}(P) = 2\,\mathrm{Re}(f),\ \tau_{1,1}(P) = \mathrm{Im}(f),$$
$$\tau_{2,0}(P)\tau_{0,2}(P) = \tau_{1,1}^2.$$

It is easy to compute

$$\hat{T} \circ P = (0, 0, |f| + \mathrm{Re}(f), |f| - \mathrm{Re}(f), \mathrm{Im}(f)). \tag{5.3.1}$$

In the following a square root of f is any function g satisfying $g^2 = f$. Applying 5.1 and 5.2, we obtain:

(1) *If f is smooth and nowhere infinitely flat and $|f|$ is smooth, then there exist smooth square roots of f.*
(2) *If f and $|f|$ are of class C^4, then there exist twice differentiable square roots of f.*

Theorem 3.3 and theorem 4.3 give:

(3) *If f is smooth and nowhere infinitely flat, then any continuous choice of the square roots of f is locally absolutely continuous.*
(4) *Assume that f is C^2. Then there exist differentiable square roots of f if and only if f vanishes of order ≥ 2 at all its zeros.*

Let us assume that f is real valued. Then (5.3.1) reduces to:

$$(\hat{T} \circ P)(t) = \begin{cases} (0, 0, 2f(t), 0, 0) & \text{if } f(t) \geq 0, \\ (0, 0, 0, -2f(t), 0) & \text{if } f(t) \leq 0. \end{cases}$$

Suppose further that f is C^2 and that $f(t_0) = 0$ implies $f'(t_0) = f''(t_0) = 0$. It follows that $\hat{T} \circ P$ is of class C^2. By 5.1 and 5.2, there exist C^1 parameterizations of the square root of f. So:

(5) *If f is real valued, C^2, and $f(t_0) = 0$ implies $f'(t_0) = f''(t_0) = 0$, then there exist C^1 square roots of f.*

Combining (3) and (5) we obtain:

(6) *If f is real valued and smooth, then each continuous choice of square roots of f is locally absolutely continuous.*

6. Regular diagonalization of normal matrices

6.1. Let $I \subseteq \mathbb{R}$ be an interval. A smooth curve of normal complex $n \times n$ matrices $I \ni t \mapsto A(t) = (A_{ij}(t))_{1 \le i,j \le n}$ is called *generic*, if $I \ni t \mapsto \chi_{A(t)}$ is generic, where $\chi_{A(t)}(\lambda) = \det(A(t) - \lambda \mathbb{I})$ is the characteristic polynomial of $A(t)$.

6.2. **Theorem.** *Let $I \subseteq \mathbb{R}$ be an interval. Let $I \ni t \mapsto A(t) = (A_{ij}(t))_{1 \le i,j \le n}$ be a generic smooth curve of normal complex matrices acting on a complex vector space $V = \mathbb{C}^n$. Then:*

(1) *For each $t_0 \in I$ there exists an integer N such that $t \mapsto A(t_0 \pm (t - t_0)^N)$ allows a smooth parameterization of its eigenvalues and eigenvectors near t_0. If A is real analytic, then the eigenvalues and eigenvectors of $t \mapsto A(t_0 \pm (t - t_0)^N)$ may be parameterized real analytically near t_0.*
(2) *There exist locally absolutely continuous parameterizations of the eigenvalues and the eigenvectors of A.*

Proof. We adapt the proof of [AKLM98, 7.6].

By theorem 3.3, the characteristic polynomial

$$\chi_{A(t)}(\lambda) = \det(A(t) - \lambda\mathbb{I}) = \sum_{j=0}^{n}(-1)^{n-j}\operatorname{Trace}(\Lambda^j A(t))\lambda^{n-j} \tag{6.2.1}$$
$$= (-1)^n\Big(\lambda^n + \sum_{j=1}^{n}(-1)^j a_j(t)\lambda^{n-j}\Big)$$

admits a continuous, locally absolutely continuous parameterization $\lambda_1, \ldots, \lambda_n$ of its roots. This shows the first part of (2).

Let us show (1). Without loss we may assume that $t_0 = 0$. By proposition 3.2, there is an integer N_0 such that the eigenvalues of $t \mapsto A(\pm t^{N_0})$ can be parameterized by smooth functions $t \mapsto \mu_j^{\pm}(t)$ near 0. Consider the following algorithm:

(a) Not all eigenvalues of $A(0)$ agree. Let $\nu_1, \ldots, \nu_l$ denote the pairwise distinct eigenvalues of $A(0)$ with respective multiplicities $m_1, \ldots, m_l$. Assume without loss that

$$\begin{aligned}
\nu_1 &= \mu_1^{\pm}(0) = \cdots = \mu_{m_1}^{\pm}(0),\\
\nu_2 &= \mu_{m_1+1}^{\pm}(0) = \cdots = \mu_{m_1+m_2}^{\pm}(0),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\nu_l &= \mu_{n-m_l}^{\pm}(0) = \cdots = \mu_n^{\pm}(0).
\end{aligned}$$

This defines a partition into subsets of smooth eigenvalues such that, for t near 0, they do not meet each other if they belong to different subsets. For $1 \le j \le l$ consider

$$\begin{aligned}
V_t^{(j),\pm} &:= \bigoplus_{\{i \,:\, \nu_j = \mu_i^{\pm}(0)\}} \ker(A(\pm t^{N_0}) - \mu_i^{\pm}(t))\\
&= \ker\big(\circ_{\{i \,:\, \nu_j = \mu_i^{\pm}(0)\}}\,(A(\pm t^{N_0}) - \mu_i^{\pm}(t))\big).
\end{aligned}$$

Note that the order of the compositions in the above expression is not relevant. So $V_t^{(j),\pm}$ is the kernel of a smooth vector bundle homomorphism $B^{\pm}(t)$ of constant rank, and thus is a smooth vector subbundle of the trivial bundle $(-\epsilon, \epsilon) \times V \to (-\epsilon, \epsilon)$. This can be seen as follows: Choose a basis of V such that $A(0)$ is diagonal. By the elimination procedure one can construct a basis for the kernel of $B^{\pm}(0)$. For t near 0, the elimination procedure (with the same choices) gives then a basis of the kernel of $B^{\pm}(t)$. The elements of this basis are then smooth in t near 0.

It follows that it suffices to find smooth eigenvectors in each subbundle $V^{(j),\pm}$ separately, expanded in the constructed smooth frame field. But in this frame field the vector subbundle looks again like a constant vector space. So feed each of these parts ($t \to A(\pm t^{N_0})$ restricted to $V^{(j),\pm}$, as matrix with respect to the frame field) into step (b) below.

(b) All eigenvalues of $A(0)$ coincide and are equal to $a_1(0)/n$, according to (6.2.1). Eigenvectors of $A(t)$ are also eigenvectors of $A(t) - (a_1(t)/n)\mathbb{I}$, thus we may replace $A(t)$ by $A(t) - (a_1(t)/n)\mathbb{I}$ and assume that the first coefficient of the characteristic polynomial (6.2.1) vanishes identically. Then $A(0) = 0$.

If $A(t) = 0$ for t near 0 we choose the eigenvectors constant.

Otherwise write $A_{ij}(t) = t^m A_{ij}^{(m)}(t)$, where $m := \min\{m(A_{ij}) : 1 \le i, j \le n\}$ which exists by assumption. It follows from (6.2.1) that the characteristic polynomial of $A^{(m)}(t)$ is

$$\chi_{A^{(m)}(t)}(\lambda) = (-1)^n\Big(\lambda^n + \sum_{j=2}^{n}(-1)^j t^{-mj} a_j(t)\lambda^{n-j}\Big),$$

Hence $m(a_k) \geq mk$ for all k. By proposition 3.2, there exists an integer N_1 such that $t \mapsto \chi_{A^{(m)}(\pm t^{N_1})}$ admits smooth parameterizations of its roots (eigenvalues of $t \mapsto A^{(m)}(\pm t^{N_1})$) for t near 0. Eigenvectors of $A^{(m)}(\pm t^{N_1})$ are also eigenvectors of $A(\pm t^{N_1})$. There exist $1 \leq i, j \leq n$ such that $A^{(m)}_{ij}(0) \neq 0$ and thus not all eigenvalues of $A^{(m)}(0)$ are equal. Feed $t \mapsto A^{(m)}(\pm t^{N_1})$ into (a).

By assumption, this algorithm stops after finitely many steps and shows (1). The real analytic case is analogous.

Now we finish the proof of (2). By (1), we find an integer N such that $t \mapsto A(\pm t^N)$ allows smooth parameterizations $t \mapsto \mu_j^\pm(t)$ and $t \mapsto v_j^\pm(t)$ of its eigenvalues and eigenvectors near 0. In a similar way as in the proof of theorem 3.3, we can compose $t \mapsto \mu_j^\pm(t)$ and $t \mapsto v_j^\pm(t)$ with $t \mapsto \sqrt[N]{t}$ and $t \mapsto -\sqrt[N]{|t|}$ in order to obtain absolutely continuous parameterizations of the eigenvalues and eigenvectors of A near 0. $\square$

Remark. The condition that $A(t)$ is normal cannot be omitted. Any choice of eigenvectors of the following real analytic curve A of 2×2 matrices has a pole at 0. Hence there does not exist an integer N such that $t \mapsto A(\pm t^N)$ allows regular eigenvectors near 0.

$$A(t) = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}.$$

The following smooth curve A of symmetric real matrices allows smooth eigenvalues, but the eigenvectors cannot be chosen continuously. This example (due to [Rel37a, §2]) shows that the assumption that A is generic is essential in theorem 6.2.

$$A(t) = e^{-\frac{1}{t^2}} \begin{pmatrix} \cos \frac{2}{t} & \sin \frac{2}{t} \\ \sin \frac{2}{t} & -\cos \frac{2}{t} \end{pmatrix}, \quad A(0) = 0.$$

7. Perturbation of unbounded normal operators

7.1. **Theorem.** *Let $I \subseteq \mathbb{R}$ be an interval. Let $I \ni t \mapsto A(t)$ be a generic smooth curve of unbounded normal operators in a Hilbert space with common domain of definition and with compact resolvents. Let $t_0 \in I$ and let z_0 be an eigenvalue of $A(t_0)$. Let n be the multiplicity of z_0. Then:*

(1) *There exists an integer N such that the n eigenvalues of $t \mapsto A(t_0 \pm (t-t_0)^N)$ passing through z_0 and the corresponding eigenvectors allow smooth parameterizations, locally near t_0. If A is real analytic, then the n eigenvalues of $t \mapsto A(t_0 \pm (t-t_0)^N)$ passing through z_0 and its eigenvectors may be arranged real analytically, locally near t_0.*
(2) *There exist locally absolutely continuous parameterizations of the n eigenvalues of A passing through z_0 and its eigenvectors, locally near t_0.*

That $A(t)$ is a smooth (resp. real analytic) curve of unbounded operators means the following: There is a dense subspace V of the Hilbert space H such that V is the domain of definition of each $A(t)$, and such that each $A(t)$ is closed and $A(t)^*A(t) = A(t)A(t)^*$, where the adjoint operator $A(t)^*$ is defined as usual by $\langle A(t)u, v\rangle = \langle u, A(t)^*v\rangle$ for all v for which the left-hand side is bounded as function in $u \in H$. Note that the domain of definition of $A(t)^*$ is V. Moreover, we require that $t \mapsto \langle A(t)u, v\rangle$ is smooth (resp. real analytic) for each $u \in V$ and $v \in H$. This implies that $t \mapsto A(t)u$ is of the same class $\mathbb{R} \to H$ for each $u \in V$, by [KM97, 2.3] or [FK88, 2.6.2].

We call the curve $I \ni t \mapsto A(t)$ *generic*, if no two unequal continuously parameterized eigenvalues meet of infinite order at any $t \in I$.

Proof. We use the resolvent lemma in [KM03] (see also [AKLM98]): *If $A(t)$ is smooth (resp. real analytic), then also the resolvent $(A(t)-z)^{-1}$ is smooth (resp. real analytic) into $L(H,H)$ in t and z jointly.*

Let z be an eigenvalue of $A(s)$ of multiplicity n for s fixed. Choose a simple closed curve γ in the resolvent set of $A(s)$ enclosing only z among all eigenvalues of $A(s)$. Since the global resolvent set $\{(t,z) \in \mathbb{R} \times \mathbb{C} : (A(t)-z) : V \to H \text{ is invertible}\}$ is open, no eigenvalue of $A(t)$ lies on γ, for t near s. Consider

$$t \mapsto -\frac{1}{2\pi i} \int_\gamma (A(t)-z)^{-1} dz =: P(t),$$

a smooth (resp. real analytic) curve of projections (on the direct sum of all eigenspaces corresponding to eigenvalues in the interior of γ) with finite dimensional ranges and constant ranks (see [AKLM98] or [KM03]). So for t near s, there are equally many eigenvalues in the interior of γ. Let us call them $\lambda_i(t)$, $1 \le i \le n$, (repeated with multiplicity) and let us denote by $e_i(t)$, $1 \le i \le n$, a corresponding system of eigenvectors of $A(t)$. Then by the residue theorem we have

$$\sum_{i=1}^n \lambda_i(t)^p e_i(t) \langle e_i(t), \ \rangle = -\frac{1}{2\pi i} \int_\gamma z^p (A(t)-z)^{-1} dz$$

which is smooth (resp. real analytic) in t near s, as a curve of operators in $L(H,H)$ of rank n.

Recall claim 2 in [AKLM98, 7.8]: *Let $t \mapsto T(t) \in L(H,H)$ be a smooth (resp. real analytic) curve of operators of rank n in Hilbert space such that $T(0)T(0)(H) = T(0)(H)$. Then $t \mapsto \operatorname{Trace}(T(t))$ is smooth (resp. real analytic) near 0.*

We conclude that the Newton polynomials

$$\sum_{i=1}^n \lambda_i(t)^p = -\frac{1}{2\pi i} \operatorname{Trace} \int_\gamma z^p (A(t)-z)^{-1} dz,$$

are smooth (resp. real analytic) for t near s, and thus also the elementary symmetric functions

$$\sum_{i_1 < \cdots < i_p} \lambda_{i_1}(t) \cdots \lambda_{i_p}(t).$$

It follows that $\{\lambda_i(t) : 1 \le i \le n\}$ represents the set of roots of a polynomial of degree n with smooth (resp. real analytic) coefficients. The statement of the theorem follows then from proposition 3.2, theorem 3.3, and theorem 6.2, since the image of $t \mapsto P(t)$, for t near s describes a finite dimensional smooth (resp. real analytic) vector subbundle of $\mathbb{R} \times H \to \mathbb{R}$ and the $\lambda_i(t)$, $1 \le i \le n$, form the set of eigenvalues of $P(t)A(t)|_{P(t)(H)}$. □

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SMOOTH ROOTS OF HYPERBOLIC POLYNOMIALS WITH DEFINABLE COEFFICIENTS

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Dedicated to Peter W. Michor on the occasion of his 60th birthday

ABSTRACT. We prove that the roots of a definable C^∞ curve of monic hyperbolic polynomials admit a definable C^∞ parameterization, where 'definable' refers to any fixed o-minimal structure on $(\mathbb{R}, +, \cdot)$. Moreover, we provide sufficient conditions, in terms of the differentiability of the coefficients and the order of contact of the roots, for the existence of C^p (for $p \in \mathbb{N}$) arrangements of the roots in both the definable and the non-definable case. These conditions are sharp in the definable and under an additional assumption also in the non-definable case. In particular, we obtain a simple proof of Bronshtein's theorem in the definable setting. We prove that the roots of definable C^∞ curves of complex polynomials can be desingularized by means of local power substitutions $t \mapsto \pm t^N$. For a definable continuous curve of complex polynomials we show that any continuous choice of roots is actually locally absolutely continuous.

1. Introduction

A monic polynomial $P(x) = x^n + \sum_{j=1}^n (-1) a_j x^{n-j}$ is called hyperbolic if all its roots are real. The study of the regularity of its roots, when P depends smoothly on a real parameter, is a classical topic with important applications in PDE and perturbation theory. Rellich [Rel37] showed that a real analytic curve of hyperbolic polynomials P admits real analytic roots. However, the roots of a C^∞ curve P do in general not allow C^∞ (more precisely, $C^{1,\alpha}$ for any $\alpha > 0$) parameterizations. All counter-examples (e.g. in [Gla63], [AKLM98], [BBCP06]) are *oscillating*, meaning that some derivative switches sign infinitely often near some point, where the multiplicity of the roots changes. By [AKLM98], P allows C^∞ roots, if no two roots meet of infinite order.

We show in this note that definability of the coefficients guarantees C^∞ solvability of C^∞ curves of hyperbolic polynomials. By 'definable' we mean definable in some fixed, but arbitrary, o-minimal structure $\mathcal{M}$ on $(\mathbb{R}, +, \cdot)$. Definability excludes oscillation, however, infinitely flat functions may be definable in some $\mathcal{M}$. We also provide sufficient conditions, in terms of the differentiability of the coefficients and the order of contact of the roots, for the existence of C^p (for $p \in \mathbb{N}$) arrangements of the roots in both the definable and the non-definable case. These conditions are sharp in the definable and under an additional assumption (automatically satisfied if $n \le 4$) also in the non-definable case. In particular, we give a simple proof of Bronshtein's theorem in the special case of definable coefficients: C^n curves P admit C^1 roots (see [Bro79], [Wak86], and [COP08]). As a consequence C^{2n} curves P admit twice differentiable roots (see [KLM04] and [COP08]). Bronshtein's theorem is quite delicate and only poorly understood.

Our results complete the perturbation theory of hyperbolic polynomials. Analogous questions for several parameters require additional assumptions and are not

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treated in this paper: The roots of $P(t_1,t_2)(x) = x^2 - (t_1^2 + t_2^2)$, for $t_1, t_2 \in \mathbb{R}$, cannot be differentiable at $t_1 = t_2 = 0$.

If the hyperbolicity assumption is dropped, we cannot hope for parameterizations of the roots satisfying a local Lipschitz condition, even if the coefficients are real analytic. We prove that the roots of definable C^∞ curves of complex polynomials can be desingularized by means of local power substitutions $t \mapsto \pm t^N$. For definable continuous curves of complex polynomials, we show that any continuous choice of roots is actually locally absolutely continuous (not better!). This extends results in [Rai09].

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2. Definable functions and smoothness

2.1. **Multiplicity.** For a continuous real or complex valued function f defined near 0 in $\mathbb{R}$, let the *multiplicity* $m_0(f)$ at 0 be the supremum of all integers p such that $f(t) = t^p g(t)$ near 0 for a continuous function g. Note that, if f is of class C^n and $m_0(f) < n$, then $f(t) = t^{m_0(f)} g(t)$ near 0, where now g is $C^{n-m_0(f)}$ and $g(0) \neq 0$. Similarly, one can define the multiplicity $m_t(f)$ of a function f at any $t \in \mathbb{R}$.

2.2. **Lemma.** *Let $I \subseteq \mathbb{R}$ be an open interval containing 0. Let $f \in C^0(I,\mathbb{R})$ and $p \in \mathbb{N}$ such that:*

(1) $m_0(f) \geq p$
(2) $f|_{I\setminus\{0\}} \in C^{p+1}(I \setminus \{0\})$
(3) *0 is not an accumulation point of $\partial\{t \in I \setminus \{0\} : f^{(p+1)}(t) = 0\}$ (where $\partial A := \overline{A} \setminus A^\circ$ denotes the boundary of A) .*

Then $f \in C^p(I)$.

Proof. We use induction on p. Let us assume that the assertion is proved for non-negative integers $< p$. Note that (3) implies:

(3') 0 is not an accumulation point of $\partial\{t \in I\setminus\{0\} : f^{(q)}(t) = 0\}$, for any integer $0 \leq q \leq p+1$.

So we may suppose that $f \in C^{p-1}(I)$, and, by (1), $f^{(q)}(0) = 0$ for $0 \leq q \leq p-1$. We will show that $f \in C^p(I)$.

Let $t > 0$. By (3'), either $f^{(p)} = 0$ identically, or $f^{(p-1)}$ is strictly monotonic for small t, say, $t < \delta$. In the first case $f^{(p)}$ extends continuously to 0. Consider the second case. Without loss we may suppose that

$$f^{(p-1)}(s) > f^{(p-1)}(t) \quad \text{if } 0 < s < t < \delta \tag{2.2.1}$$

(otherwise consider $-f^{(p-1)}$). Then $f^{(p-1)}(s)/s > f^{(p-1)}(t)/t$ if $0 < s < t < \delta$. So

$$\lim_{t \searrow 0} \frac{f^{(p-1)}(t)}{t} = \sup_{0<t<\delta} \frac{f^{(p-1)}(t)}{t} =: a \in \mathbb{R} \cup \{+\infty\}.$$

By Taylor's formula, for each $t > 0$ there is a $0 < \xi(t) < t$ such that

$$f(t) = t^{p-1} \cdot \frac{f^{(p-1)}(\xi(t))}{(p-1)!}.$$

By (2.2.1), we have $f^{(p-1)}(\xi(t)) > f^{(p-1)}(t)$, and, thus,

$$\frac{f^{(p-1)}(t)}{t} < (p-1)! \cdot \frac{f(t)}{t^p}.$$

By (1), the right-hand side is convergent as $t \searrow 0$. So $a < +\infty$.

By (3), $f^{(p)}$ is strictly monotonic for small t, say, $t < \epsilon$. We may conclude that $\lim_{t\searrow 0} f^{(p)}(t)$ is given by either $\sup_{0<t<\epsilon} f^{(p)}(t)$ or $\inf_{0<t<\epsilon} f^{(p)}(t)$. By Taylor's formula, for each $n \in \mathbb{N}_{>0}$, there is a $0 < \nu(n) < 1/n$ such that

$$f^{(p)}(\nu(n)) = p! \cdot \frac{f(\frac{1}{n})}{(\frac{1}{n})^p} = p! \cdot g(\tfrac{1}{n}) \to p! \cdot g(0) \quad \text{as } n \to \infty,$$

where $g(t) := f(t)/t^p$ is continuous by (1). Hence, $\lim_{t\searrow 0} f^{(p)}(t) = p! \cdot g(0)$. By the mean value theorem, we obtain

$$a = \lim_{n\to\infty} \frac{f^{(p-1)}(\frac{1}{n})}{\frac{1}{n}} = \lim_{n\to\infty} f^{(p)}(\zeta(n)) = p! \cdot g(0),$$

where $0 < \zeta(n) < 1/n$. (Note that, if $f^{(p)} = 0$ identically, then $g(0) = 0$.)

In a similar way one proves that $\lim_{t\nearrow 0} f^{(p-1)}(t)/t = \lim_{t\nearrow 0} f^{(p)}(t) = p! \cdot g(0)$. So $f \in C^p(I)$. □

2.3. **Example.** Note that condition (3) in lemma 2.2 is necessary: The function $f(t) := e^{-1/t^2} \sin^2(e^{1/t^4})$, $f(0) := 0$, satisfies $m_0(f) = \infty$ and is C^∞ off 0, but it is not C^1 in any neighborhood of 0.

2.4. **Definable functions.** Cf. [vdD98]. Let $\mathcal{M} = \bigcup_{n\in\mathbb{N}_{>0}} \mathcal{M}_n$, where each $\mathcal{M}_n$ is a family of subsets of $\mathbb{R}^n$. We say that $\mathcal{M}$ is an *o-minimal structure* on $(\mathbb{R}, +, \cdot)$ if the following conditions are satisfied:

(1) Each $\mathcal{M}_n$ is closed under finite set-theoretical operations.
(2) If $A \in \mathcal{M}_n$ and $B \in \mathcal{M}_m$, then $A \times B \in \mathcal{M}_{n+m}$.
(3) If $A \in \mathcal{M}_{n+m}$ and $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is the projection on the first n coordinates, then $\pi(A) \in \mathcal{M}_m$.
(4) If $f, g_1, \dots, g_l \in \mathbb{R}[X_1, \dots, X_n]$, then $\{x \in \mathbb{R}^n : f(x) = 0, g_1(x) > 0, \dots, g_l(x) > 0\} \in \mathcal{M}_n$.
(5) $\mathcal{M}_1$ consists of all finite unions of open intervals and points.

For a fixed o-minimal structure $\mathcal{M}$ on $(\mathbb{R}, +, \cdot)$, we say that A is *$\mathcal{M}$-definable* if $A \in \mathcal{M}_n$ for some n. A mapping $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$, is called *$\mathcal{M}$-definable* if its graph is $\mathcal{M}$-definable.

From now on let $\mathcal{M}$ be some fixed, but arbitrary, o-minimal structure on $(\mathbb{R}, +, \cdot)$. If we write *definable* we will always mean $\mathcal{M}$-definable.

2.5. **Lemma.** *Let $I \subseteq \mathbb{R}$ be an open interval containing* 0*, let $f : I \to \mathbb{R}$ be definable, and $p, m \in \mathbb{N}$.*

(1) *If $f \in C^0(I)$ and $m_0(f) \geq p$, then f is C^p near* 0*.*
(2) *If $f \in C^p(I)$, then $h(t) := t^m f(t)$ is C^{p+m} near* 0*.*

Proof. (1) follows from lemma 2.2 and the Monotonicity theorem (e.g. [vdD98]).

(2) We use induction on m. The statement for $m = 0$ is trivial. Suppose that $m > 0$. By induction hypothesis, $g(t) := t^{m-1}f(t)$ belongs to $C^{p+m-1}(I)$ and $h^{(p+m-1)}(t) = tg^{(p+m-1)}(t) + (p + m - 1)g^{(p+m-2)}(t)$. Thus

$$\lim_{t\to 0} \frac{h^{(p+m-1)}(t) - h^{(p+m-1)}(0)}{t} = (p + m)g^{(p+m-1)}(0).$$

Let $t > 0$. By definability, $h^{(p+m)}(t)$ exists and is either a constant a or strictly monotonic for small t, say, $t < \epsilon$. Hence, $\lim_{t\searrow 0} h^{(p+m)}(t)$ is given by either a, $\sup_{0<t<\epsilon} h^{(p+m)}(t)$, or $\inf_{0<t<\epsilon} h^{(p+m)}(t)$. By the mean value theorem, for each $n \in \mathbb{N}_{>0}$, there is a $0 < \nu(n) < 1/n$ such that

$$h^{(p+m)}(\nu(n)) = \frac{h^{(p+m-1)}(\frac{1}{n}) - h^{(p+m-1)}(0)}{\frac{1}{n}} \to (p + m)g^{(p+m-1)}(0) \quad \text{as } n \to \infty.$$

So $\lim_{t\searrow 0} h^{(p+m)}(t) = (p+m)g^{(p+m-1)}(0)$. Similarly for $t<0$. $\square$

2.6. **Examples.** The conditions in lemma 2.5 are sharp: Let

$$f_p(t) := \begin{cases} t^{p+1} & \text{for } t \ge 0 \\ 0 & \text{for } t < 0 \end{cases}. \tag{2.6.1}$$

Then $m_0(f_p) = p$, and f_p is $C^{p,1}$ but not C^{p+1}. Moreover, $f_{p+m}(t) = t^m f_p(t)$ is $C^{p+m,1}$ but not C^{p+m+1}.

3. Smooth square roots

3.1. Let $I \subseteq \mathbb{R}$ be an open interval. If $f : I \to \mathbb{R}_{\ge 0}$ is definable and continuous, then $\{t \in I : 0 < m_t(f) < \infty\} \subseteq \partial\{t \in I : f(t) = 0\}$. So

$$2\overline{m}(f) := \sup\{m_t(f) < \infty : t \in I\}$$

is a well-defined integer. If f is C^n and $n > 2\overline{m}(f)$, then $\overline{m}(f)$ is the maximal *finite* order of vanishing of the square roots of f.

3.2. **Theorem.** *Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}_{\ge 0}$ a non-negative definable function, and $p \in \mathbb{N}_{>0}$. Consider $P(t)(x) = x^2 - f(t)$. Then we have:*

1. *If f is C^∞, then the roots of P admit definable C^∞ parameterizations.*
2. *If f is $C^{p+2\overline{m}(f)}$, then the roots of P admit definable $C^{p+\overline{m}(f)}$ parameterizations.*

Proof. We prove (1) and (2) simultaneously and indicate differences arising.

Note that any continuous choice of roots is definable (cf. lemma 4.4).

Let $t_0 \in I$. If $0 \le m_{t_0}(f) < \infty$, then $m_{t_0}(f) = 2m$ for some $m \in \mathbb{N}$, since $p + 2\overline{m}(f) - m_{t_0}(f) \ge 1$ and $f \ge 0$. So $f(t) = (t-t_0)^{2m} f_{(m)}(t)$, where

$$f_{(m)}(t) = \int_0^1 \frac{(1-r)^{2m-1}}{(2m-1)!} f^{(2m)}(t_0 + r(t-t_0))dr$$

is C^∞ (resp. $C^{p+2\overline{m}(f)-2m}$), definable, and $f_{(m)}(t_0) > 0$. . Then the functions $g_\pm(t) := \pm(t-t_0)^m \sqrt{f_{(m)}(t)}$ are C^∞ (resp. $C^{p+2\overline{m}(f)-m}$, by lemma 2.5(2)) and represent the roots of P near t_0.

Now assume that $m_{t_0}(f) = \infty$. In a neighborhood of t_0, consider the continuous functions $g_\pm(t) := \pm\sqrt{f(t)}$. Then $m_{t_0}(g_\pm) = \infty$. By lemma 2.5(1), for each p, there is a neighborhood I_p of t_0 such that the roots $g_\pm$ are C^p on I_p. Now t_0 belongs to $\partial(f^{-1}(0))$ which is finite, by definability. Thus, in case (1), $g_\pm$ is C^∞ off t_0, and, hence, near t_0.

So for each $t_0 \in I$ we have found local C^∞ (resp. $C^{p+\overline{m}(f)}$) parameterizations of the roots of P near t_0. One can glue these to a global parameterization, see 4.12(4) below. $\square$

3.3. **Examples.** The condition in theorem 3.2(2) is sharp: The non-negative function $f(t) := t^{2m}(1 + f_p(t))$, where f_p is defined in (2.6.1), is $C^{p+2m,1}$ but not C^{p+2m+1}. Its square roots $g_\pm(t) := \pm t^m \sqrt{1+f_p(t)}$ are C^{p+m} but not C^{p+m+1}.

4. Smooth roots of hyperbolic polynomials

4.1. Let

$$P(z) = z^n + \sum_{j=1}^n (-1)^j a_j z^{n-j} = \prod_{j=1}^n (z - \lambda_j)$$

be a monic polynomial with complex coefficients $a_1, \dots, a_n$ and roots $\lambda_1, \dots, \lambda_n$. By Vieta's formulas, $a_i = \sigma_i(\lambda_1, \dots, \lambda_n)$, where $\sigma_1, \dots, \sigma_n$ are the elementary symmetric functions in n variables:

$$\sigma_i(\lambda_1, \dots, \lambda_n) = \sum_{1 \le j_1 < \dots < j_i \le n} \lambda_{j_1} \cdots \lambda_{j_i}. \tag{4.1.1}$$

Denote by s_i, $i \in \mathbb{N}$, the Newton polynomials $\sum_{j=1}^n \lambda_j^i$ which are related to the elementary symmetric functions by

$$s_k - s_{k-1}\sigma_1 + s_{k-2}\sigma_2 - \dots + (-1)^{k-1} s_1 \sigma_{k-1} + (-1)^k k \sigma_k = 0, \quad (k \ge 1). \tag{4.1.2}$$

Let us consider the so-called Bezoutiant

$$B := \begin{pmatrix} s_0 & s_1 & \dots & s_{n-1} \\ s_1 & s_2 & \dots & s_n \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & \dots & s_{2n-2} \end{pmatrix} = (s_{i+j-2})_{1 \le i,j \le n}.$$

Since the entries of B are symmetric polynomials in $\lambda_1, \dots, \lambda_n$, we find a unique symmetric $n \times n$ matrix $\tilde{B}$ with $B = \tilde{B} \circ \sigma$, where $\sigma = (\sigma_1, \dots, \sigma_n)$.

Let B_k denote the minor formed by the first k rows and columns of B. Then we have

$$\Delta_k(\lambda) := \det B_k(\lambda) = \sum_{i_1 < \dots < i_k} (\lambda_{i_1} - \lambda_{i_2})^2 \cdots (\lambda_{i_1} - \lambda_{i_k})^2 \cdots (\lambda_{i_{k-1}} - \lambda_{i_k})^2. \tag{4.1.3}$$

Since the polynomials Δ_k are symmetric, we have $\Delta_k = \tilde{\Delta}_k \circ \sigma$ for unique polynomials $\tilde{\Delta}_k$.

By (4.1.3), the number of distinct roots of P equals the maximal k such that $\tilde{\Delta}_k(P) \ne 0$. (Abusing notation we identify P with the n-tupel $(a_1, \dots, a_n)$ of its coefficients when convenient.)

If all roots λ_j (and thus all coefficients a_j) of P are real, we say that P is *hyperbolic.*

Theorem (Sylvester's version of Sturm's theorem, see e.g. [Pro78] for a modern proof)**.** *Suppose that all coefficients of P are real. Then P is hyperbolic if and only if $\tilde{B}(P)$ is positive semidefinite. The rank of $\tilde{B}(P)$ equals the number of distinct roots of P and its signature equals the number of distinct real roots.*

4.2. **Lemma** (Splitting lemma [AKLM98, 3.4])**.** *Let $P_0 = z^n + \sum_{j=1}^n (-1)^j a_j z^{n-j}$ be a polynomial satisfying $P_0 = P_1 \cdot P_2$, where P_1 and P_2 are polynomials without common root. Then for P near P_0 we have $P = P_1(P) \cdot P_2(P)$ for analytic mappings of monic polynomials $P \mapsto P_1(P)$ and $P \mapsto P_2(P)$, defined for P near P_0, with the given initial values.*

4.3. For the rest of the section, let $I \subseteq \mathbb{R}$ be an open interval and consider a (continuous) curve of hyperbolic polynomials

$$P(t)(x) = x^n + \sum_{j=1}^n (-1)^j a_j(t) x^{n-j}, \quad (t \in I).$$

Then the roots of P admit a continuous parameterization, e.g., ordering them by size, $\lambda_1 \le \lambda_2 \le \dots \le \lambda_n$.

4.4. **Lemma.** *If the coefficients a_j of P are definable, then every continuous parameterization λ_j of the roots of P is definable.*

Proof. Ordering the roots of P by size, $\mu_1 \le \mu_2 \le \cdots \le \mu_n$, gives a continuous parameterization which is evidently definable. Since all $\tilde{\Delta}_k \circ P$ are definable, the set E of $t \in I$ where the multiplicity of the roots changes is finite. The complement of E consists of finitely many intervals, on each of which the parameterizations λ_j and μ_j differ only by a constant permutation. Thus each λ_j is definable. $\square$

4.5. **Lemma** (Multiplicity lemma [AKLM98, 3.7])**.** *Suppose that* $0 \in I$ *and that* $a_1 = 0$ *identically. Let* $r \in \mathbb{N}$. *If each* $a_j \in C^{nr}(I)$, *then the following conditions are equivalent:*

(1) $m_0(a_k) \ge kr$, *for all* $2 \le k \le n$.
(2) $m_0(\tilde{\Delta}_k) \ge k(k-1)r$, *for all* $2 \le k \le n$.
(3) $m_0(a_2) \ge 2r$.

Proof. Obvious modification of the proof of [AKLM98, 3.7]. $\square$

4.6. Let $E^{(\infty)}(P)$ denote the set of all $t \in I$ which satisfy following condition:

(#) Let $s = s(t, P)$ be maximal with the property that the germ at t of $\tilde{\Delta}_s \circ P$ is not 0. Then $m_t(\tilde{\Delta}_s \circ P) = \infty$.

Consider the condition:

(#') There exists a continuous parameterization λ_j of the roots of P such that distinct λ_j meet of infinite order at t, i.e., there exist $i \ne j$ such that the germs of λ_i and λ_j at t do not coincide and $m_t(\lambda_i - \lambda_j) = \infty$.

By (4.1.3), (#') implies (#).

If the coefficients of P (and thus the $\tilde{\Delta}_k \circ P$) are definable, then $E^{(\infty)}(P)$ is finite and the family of continuous parameterizations of the roots of P is finite. Then (#) and (#') are equivalent.

4.7. Let $t_0 \in I$. Choose a continuous parameterization λ_j of the roots of P. We denote by $\overline{m}_{t_0}(P, \lambda)$ the maximal *finite* order of contact of the λ_j at t_0, i.e.,

$$\overline{m}_{t_0}(P, \lambda) = \max\{m_{t_0}(\lambda_i - \lambda_j) < \infty : 1 \le i < j \le n\}.$$

The integer $\overline{m}_{t_0}(P, \lambda)$ depends on the choice of the λ_j.

If $t_0 \notin E^{(\infty)}(P)$ and $s = s(t_0, P)$ is the integer defined in (#), then, by (4.1.3),

$$\overline{m}_{t_0}(P, \lambda) \le \frac{m_{t_0}(\tilde{\Delta}_s \circ P)}{2}.$$

If the coefficients of P are definable, then the family of continuous parameterizations of the roots of P is finite.

Hence,

$$\overline{m}_{t_0}(P) := \sup_{\lambda} \overline{m}_{t_0}(P, \lambda),$$

where λ is any continuous arrangement of the roots of P, is a well-defined integer, if either $t_0 \notin E^{(\infty)}(P)$ or the coefficients of P are definable. It is the *maximal finite order of contact of the roots of* P.

4.8. **Lemma.** *Suppose that either* $t_0 \notin E^{(\infty)}(P)$ *or the coefficients of* P *are definable. We have:*

(1) *If* $P = P_1 \cdot P_2$ *as provided by the splitting lemma 4.2, then* $\overline{m}_{t_0}(P) = \max\{\overline{m}_{t_0}(P_1), \overline{m}_{t_0}(P_2)\}$.

Assume that all roots of $P(t_0)$ *coincide. Then:*

(2) *Replacing the variable* x *with* $x - a_1(t)/n$, *leaves* $\overline{m}_{t_0}(P)$ *unchanged.*
(3) *If* $a_1 = 0$, *then* $m_{t_0}(a_2) \le 2\overline{m}_{t_0}(P) + 1$.

(4) *Suppose that* $a_1 = 0$ *and* $a_k(t) = (t - t_0)^{kr} a_{(r),k}(t)$ *for continuous* $a_{(r),k}$, $2 \le k \le n$, *and some* $r \in \mathbb{N}_{>0}$. *Consider*

$$P_{(r)}(t)(x) := x^n + \sum_{j=2}^{n} (-1)^j a_{(r),k}(t) x^{n-j}.$$

Then $\overline{m}_{t_0}(P_{(r)}) \le \overline{m}_{t_0}(P) - r$.

Proof. (1) and (2) are immediate from the definition. (3) is a consequence of $-2na_2 = \sum_{i<j}(\lambda_i - \lambda_j)^2$ and the fact that, for a continuous function f, we have $m_{t_0}(f^2) \le 2m_{t_0}(f) + 1$. (4) follows from the observation that, if $t \mapsto \lambda_i(t)$ parameterize the roots of $t \mapsto P_{(r)}(t)$, then $t \mapsto (t - t_0)^r \lambda_i(t)$ represent the roots of $t \mapsto P(t)$. $\square$

4.9. **Example.** Note that in 4.8(3) equality can occur: Let $f(t) := t^{3+1/3}$ for $t \ge 0$ and $f(t) := 0$ for $t < 0$, and consider $P(t)(x) = x^2 - f(t)$. Then $m_0(f) = 3$ and $\overline{m}_0(P) = 1$.

4.10. If the coefficients a_j of P (and thus the $\Delta_k \circ P$) are definable, then the set $\{t \in I : \overline{m}_t(P) > 0\}$ is finite and

$$\overline{m}(P) = \overline{m}_I(P) := \sup\{\overline{m}_t(P) : t \in I\}$$

is a well-defined integer.

4.11. **Lemma.** *For* $n \in \mathbb{N}_{>0}$ *let* $\mathcal{R}(n)$ *denote the family of all rooted trees* T *with vertices labeled in the following way: the root is labeled* n*, the labels of the successors of a vertex labeled* m *form a partition of* m*, the leaves (vertices with no successors) are all labeled* 1. *Define* $d(n) := \max_{T \in \mathcal{R}(n)}\{$*sum over all labels* ≥ 2 *in* $T\}$. *Then*

$$d(n) = \frac{1}{2} n(n+1) - 1. \tag{4.11.1}$$

Proof. Observe that $d(1) = 0$. Then (4.11.1) is equivalent to $d(n+1) = n+1+d(n)$ for $n \ge 1$. We use induction on n. It suffices to show $d(n) \ge d(n_1) + \cdots + d(n_p)$ for $n_1 + \cdots + n_p = n+1$, where $p \ge 2$ and $n_i \in \mathbb{N}_{>0}$. By induction hypothesis, this inequality is equivalent to

$$\begin{aligned}
&\frac{1}{2}n(n+1) - 1 \ge \frac{1}{2}n_1(n_1+1) + \cdots + \frac{1}{2}n_p(n_p+1) - p \\
\iff \quad & \frac{1}{2}((n_1 + \cdots + n_p)^2 - (n_1^2 + \cdots + n_p^2)) \ge n_1 + \cdots + n_p - p + 1 \\
\Longleftarrow \quad & (n_1 + \cdots + n_{p-1})n_p \ge n_1 + \cdots + n_p - 1.
\end{aligned}$$

The last inequality has the form $ab \ge a + b - 1$ for $a, b \in \mathbb{N}_{>0}$, which is easily verified. $\square$

Note that $d(n) + n$ computes the maximal sum of all degrees occurring in a repeated splitting of a polynomial of degree n into a product of polynomials of strictly smaller degree until each factor has degree one.

4.12. **Theorem.** *Let* $I \subseteq \mathbb{R}$ *be an open interval. Consider a curve of hyperbolic polynomials*

$$P(t)(x) = x^n + \sum_{j=1}^{n} (-1)^j a_j(t) x^{n-j}, \quad (n \ge 2),$$

with definable coefficients a_j. *Let* $p \in \mathbb{N}_{>0}$ *and* $d(n) = n(n+1)/2 - 1$. *Then:*

(1) *If the* a_j *are* C^∞, *then the roots of* P *can be parameterized by definable* C^∞ *functions, globally.*

(2) *If the a_j are $C^{p+1+d(n)\overline{m}(P)}$, then the roots of P can be parameterized by definable C^p functions, globally.*

The condition in 4.12(2) is not best possible. However, it is convenient to prove this preliminary result parallel to the C^∞ case and strengthen it in theorem 5.2 below.

Proof. We prove (1) and (2) simultaneously and indicate differences arising. Any continuous parameterization of the roots of P is definable, by lemma 4.4.

We proceed by induction on n. The case $n = 2$ is covered by theorem 3.2 (since we may always assume $a_1 = 0$, see (II) below). Suppose the assertion is proved for degrees $< n$.

Claim (3)**.** *There exists local C^∞ (resp. C^p) parameterization λ_i of the roots of P near each $t_0 \in I$. The local C^∞ choices λ_i of the roots are unique in the following sense:*

($\star$) *On the set $\{\lambda_1, \dots, \lambda_n\}$ consider the equivalence relation $\lambda_i \sim \lambda_j$ iff $m_{t_0}(\lambda_i - \lambda_j) = \infty$. If μ_i is a different local C^∞ parameterization of the roots of P near t_0, then $\{\lambda_1, \dots, \lambda_n\}/\sim = \{\mu_1, \dots, \mu_n\}/\sim$.*

Note that ($\star$) is trivially satisfied if $n = 2$. Without loss we may assume that $0 \in I$ and $t_0 = 0$. We distinguish different cases:

(I) If there are distinct roots at 0, we may factor $P(t) = P_1(t) \cdot P_2(t)$ in an open subinterval $I_0 \ni 0$ such that P_1 and P_2 have no common roots, by the splitting lemma 4.2. The coefficients of each P_i are definable, since its roots are. By lemma 4.8(1), we have

$$\overline{m}_{I_0}(P) = \max\{\overline{m}_{I_0}(P_1), \overline{m}_{I_0}(P_2)\}.$$

By the induction hypothesis, P_1 and P_2 (and hence P) admit C^∞ (resp. C^p) parameterizations of its roots on I_0 which are unique in the sense of ($\star$) in case (1).

(II) If all roots of $P(0)$ coincide, then we first reduce P to the case $a_1 = 0$, by replacing x with $x - a_1(t)/n$ (which leaves $\overline{m}(P)$ and ($\star$) unchanged, by lemma 4.8(2)). Then all roots of $P(0)$ are equal to 0. So $a_2(0) = 0$. Clearly, the new coefficients are still definable.

(IIa) If $m_0(a_2)$ is finite, then $p + 1 + d(n)\overline{m}(P) - m_0(a_2) \geq 1$, by lemma 4.8(3). So $m_0(a_2) = 2r$ for some $r \in \mathbb{N}_{>0}$, since $0 \leq \tilde{\Delta}_2 = -2na_2$. Let

$$q := p + 1 + d(n)\overline{m}(P) - nr.$$

By the multiplicity lemma 4.5, we obtain $a_k(t) = t^{kr}a_{(r),k}(t)$ for definable C^∞ (resp. C^q) functions $a_{(r),k}$ and $2 \leq k \leq n$. Consider the C^∞ (resp. C^q) curve of hyperbolic polynomials

$$P_{(r)}(t)(x) := x^n + \sum_{j=2}^{n} (-1)^j a_{(r),k}(t) x^{n-j}. \tag{4.12.1}$$

Since $a_{(r),2}(0) \neq 0$, not all roots of $P_{(r)}(0)$ coincide. We have $d(n) - n = d(n-1)$ and, by lemma 4.8, $\overline{m}(P_{(r)}) \leq \overline{m}(P) - r$. Thus, the splitting lemma 4.2 and the induction hypothesis provide C^∞ (resp. C^p) parameterizations λ_j of the roots of $P_{(r)}$ near 0 which are unique in the sense of ($\star$). But then the C^∞ (resp. C^p) functions $t \mapsto t^r\lambda_j(t)$ represent the roots of $t \mapsto P(t)$ near 0 and they are unique in the sense of ($\star$) in case (1).

(IIb) If $m_0(a_2) = \infty$ and $a_2 = 0$, then all roots of P are identically 0.

(IIc) Finally, if $m_0(a_2) = \infty$ and $a_2 \neq 0$, then, since $-a_2 = \sum_{j=1}^n \lambda_j^2$, for any continuous choice of the roots λ_j we find $m_0(\lambda_j) = \infty$ (for all j). By lemma 2.5(1), for each p, there is a neighborhood I_p of 0 such that the roots λ_j are C^p on I_p. Since a_2 is definable, for small $t \neq 0$ either not all $\lambda_j(t)$ coincide or all λ_j are identically

0 (to the left or the right of 0). So, in case (1), all λ_j are C^∞ off 0, by the splitting lemma 4.2 and the induction hypothesis, and hence also near 0.

Claim (4). *We may glue the local C^∞ (resp. C^p) parameterizations of the roots to form a global parameterization.*

In case (1) the local C^∞ choices of the roots of P can be glued by their uniqueness in the sense of ($\star$). If C^∞ roots meet of infinite order at t_0, any permutation on one side of t_0 preserves smoothness.

For (2): Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a C^p parameterization of the roots of P defined on a maximal open interval $I_1 \subseteq I$. For contradiction, assume that the right (say) endpoint t_1 of I_1 belongs to I. By claim (3), there exists a local C^p parameterization $\mu = (\mu_1 \dots, \mu_n)$ of the roots of P near t_1. Let t_0 be in the common domain of λ and μ. Consider a sequence $t_k \searrow t_0$. For each k, there is a permutation τ_k of $\{1, \dots, n\}$ such that $\lambda(t_k) = \tau_k.\mu(t_k)$. By passing to a subsequence, we can assume that $\lambda(t_k) = \tau.\mu(t_k)$ for all k and a fixed permutation τ. Thus, $\lambda(t) = \tau.\mu(t)$ for all $t \ge t_0$, by definability. So $\tilde\lambda(t) := \lambda(t)$ for $t \le t_0$ and $\tilde\lambda(t) := \tau.\mu(t)$ for $t \ge t_0$ defines a C^p parameterization on a larger interval, a contradiction. $\square$

4.13. **Remark.** Suppose that $\overline{m}(P) = 0$. Then the roots of P do not meet or they meet *slowly*, i.e., $(\lambda_i(t) - \lambda_j(t))/t$ is not continuous at $t = 0$. In the latter case $a_2 \notin C^2$, by 4.8(3), and so 4.12(2) is empty.

5. Sharp sufficient conditions for C^p roots

The conditions in theorem 4.12(2) are not sharp. We shall obtain sharp sufficient conditions for C^p roots, given that the coefficients are definable. In the non-definable case we still get sharp sufficient conditions, if P is of a special type. The proof of 4.12(2) was not for nothing, since it is needed in the definition of Γ and γ below.

5.1. **The definable case.** Let $P(t)$, $t \in I$, be a curve of monic hyperbolic polynomials of degree n with definable $C^{d(n)\overline{m}(P)+2}$ coefficients a_j. For each $t_0 \in I$, let us define two integers $\Gamma_{t_0}(P)$ and $\gamma_{t_0}(P)$ inductively:

(I) If $P(t) = P_1(t) \cdot P_2(t)$ near t_0, and $P_i(t_0)$, $i = 1, 2$, have distinct roots,

$$\Gamma_{t_0}(P) := \max\{\Gamma_{t_0}(P_1), \Gamma_{t_0}(P_2)\}, \tag{5.1.1}$$

$$\gamma_{t_0}(P) := \Gamma_{t_0}(P) - \max\{\Gamma_{t_0}(P_1) - \gamma_{t_0}(P_1), \Gamma_{t_0}(P_2) - \gamma_{t_0}(P_2)\}. \tag{5.1.2}$$

(II) If $\deg(P) > 1$ and all roots of $P(t_0)$ coincide, reduce to the case $a_1 = 0$ (without changing $\Gamma_{t_0}(P)$ and $\gamma_{t_0}(P)$). If $m_{t_0}(a_2) = 2r < \infty$, consider $P_{(r)}$ as in (4.12.1) (for t_0 instead of 0), and set

$$\Gamma_{t_0}(P) := \Gamma_{t_0}(P_{(r)}) + \deg(P)r, \tag{5.1.3}$$

$$\gamma_{t_0}(P) := \gamma_{t_0}(P_{(r)}) + r. \tag{5.1.4}$$

If $m_{t_0}(a_2) = \infty$, set $\Gamma_{t_0}(P) := 0$ and $\gamma_{t_0}(P) := 0$.

(III) If $\deg(P) = 1$, set $\Gamma_{t_0}(P) := 0$ and $\gamma_{t_0}(P) := 0$.

Note that, (by the proof of 4.12(2)) the coefficients of P being in $C^{d(n)\overline{m}(P)+2}$, guarantees that $\Gamma_{t_0}(P)$ and $\gamma_{t_0}(P)$ are well-defined. With hindsight it suffices to assume that the coefficients of P belong to $C^{\Gamma_{t_0}(P)+1}$ near t_0.

Since the coefficients of P are definable, the set of $t_0 \in I$ such that $\Gamma_{t_0}(P) > 0$ or $\gamma_{t_0}(P) > 0$ is finite and

$$\Gamma(P) := \sup\{\Gamma_{t_0}(P) : t_0 \in I\}, \tag{5.1.5}$$

$$\gamma(P) := \Gamma(P) - \sup\{\Gamma_{t_0}(P) - \gamma_{t_0}(P) : t_0 \in I\} \tag{5.1.6}$$

are well-defined integers. By construction, we have

$$\gamma(P) \le \Gamma(P) \le d(n)\overline{m}(P) + 1.$$

If $P(t)(x) = x^2 - f(t)$ (where $f \ge 0$), then $\Gamma(P) = 2\overline{m}(f)$ and $\gamma(P) = \overline{m}(f)$.

5.2. **Theorem.** *Let $I \subseteq \mathbb{R}$ be an open interval. Consider a curve of hyperbolic polynomials*

$$P(t)(x) = x^n + \sum_{j=1}^{n} (-1)^j a_j(t) x^{n-j},$$

with definable coefficients a_j. For each $p \in \mathbb{N}_{>0}$, we have:

(1) *If the a_j are $C^{p+\Gamma(P)}$, then the roots of P can be parameterized by definable $C^{p+\gamma(P)}$ functions, globally.*

Proof. By 4.12(4) it suffices to show the local assertion. Let $t_0 \in I$ be fixed.

Claim (2)**.** *If the a_j are $C^{p+\Gamma_{t_0}(P)}$, then the roots of P can be chosen in $C^{p+\gamma_{t_0}(P)}$, locally near t_0.*

We use induction on n and follow the steps in 4.12. The case $n = 1$ is trivial and $n = 2$ is treated in theorem 3.2(2). Without loss assume that $0 \in I$ and $t_0 = 0$.

(I) If $P(0)$ has distinct roots, we have a factorization $P(t) = P_1(t) \cdot P_2(t)$ near 0, by the splitting lemma 4.2. The coefficients of each factor P_i belong to $C^{p+\Gamma_0(P)}$. Let $p_i := p + \Gamma_0(P) - \Gamma_0(P_i)$. Then $p_i \ge p$, by (5.1.1). By induction hypothesis, the roots of P_i admit a local parameterization in $C^{p_i+\gamma_0(P_i)}$. By (5.1.2), we obtain $p_i + \gamma_0(P_i) \ge p + \gamma_0(P)$, hence claim (2).

(II) If all roots of $P(0)$ coincide, we reduce to the case $a_1 = 0$. So $a_2(0) = 0$.

(IIa) If $m_0(a_2) = 2r < \infty$, consider $P_{(r)}$ as in (4.12.1). The coefficients of $P_{(r)}$ are in $C^{p+\Gamma_0(P)-nr}$ and $a_{(r),2}(0) \ne 0$. By (5.1.3) and (I), there are $C^{p+\gamma_0(P_{(r)})}$ functions λ_j which represent the roots of $P_{(r)}$ near 0. Then $t \mapsto t^r \lambda_j(t)$ form a local parameterization of the roots of P which is $C^{p+\gamma_0(P)}$, by lemma 2.5(2) and (5.1.4).

(IIb/c) If $m_0(a_2) = \infty$, then $m_0(\lambda_j) = \infty$ for each continuous choice of roots λ_j, and we are done, by lemma 2.5(1).

Claim (3)**.** *If the a_j are $C^{p+\Gamma(P)}$, then the roots of P can be chosen in $C^{p+\gamma(P)}$, locally near t_0.*

By claim (2), the roots of P can be chosen in $C^{p+\Gamma(P)-\Gamma_{t_0}(P)+\gamma_{t_0}(P)}$, locally near t_0. By (5.1.6), we have $p + \Gamma(P) - \Gamma_{t_0}(P) + \gamma_{t_0}(P) \ge p + \gamma(P)$. □

5.3. **Examples.** The condition in theorem 5.2 is sharp: Let $p \in \mathbb{N}_{>3}$ and let f_p be the function defined in (2.6.1). Consider the $C^{p,1}$ curve of polynomials

$$P_p(t)(x) = x^3 - f_p(t)x^2 + (2f_p(t) - t^2)x - f_p(t).$$

For the discriminant of P_p we find $\tilde{\Delta}_3(P_p(t)) = t^6(4 + o(1))$ if $t \ge 0$ (as $p \ge 3$) and $\tilde{\Delta}_3(P_p(t)) = 4t^6$ if $t < 0$. Thus, for small t, $P_p(t)$ is hyperbolic. It is easy to compute $\Gamma(P_p) = 3(< p)$ and $\gamma(P_p) = 1$. By theorem 5.2, P_p admits C^{p-2} roots. Suppose, for contradiction, that P_p has C^{p-1} roots λ_j. Since $m_0(\lambda_j) \ge 1$, we have $\lambda_j(t) = t\mu_j(t)$ for C^{p-2} functions μ_j. But then $f_p(t) = t^3\mu_1(t)\mu_2(t)\mu_3(t)$ is C^{p+1}, by lemma 2.5(2), a contradiction.

5.4. **The non-definable case.** Let $P(t)$, $t \in I$, be a curve of monic hyperbolic polynomials of degree n (not necessarily definable). Assume $E^{(\infty)}(P) = \emptyset$. We will prove analogs of theorem 4.12(2) and, if P is of a special type, of theorem 5.2. Without the assumption $E^{(\infty)}(P) = \emptyset$, we cannot hope for $C^{1,\alpha}$ roots (for any $\alpha > 0$), even if the coefficients are C^∞ (e.g. [Gla63], [AKLM98], [BBCP06]).

Let $J \subseteq I$ be a compact subinterval of I. Define

$$\overline{m}_J(P) := \sup\{\overline{m}_t(P) : t \in J\} \in \mathbb{N} \cup \{+\infty\}.$$

The interesting case is $\overline{m}_J(P) < \infty$, but what follows is also true for $\overline{m}_J(P) = \infty$.

Assume that P has $C^{d(n)\overline{m}_J(P)+2}$ coefficients a_j. For each $t_0 \in I$, we can define the two integers $\Gamma_{t_0}(P)$ and $\gamma_{t_0}(P)$ in the same way as in 5.1. Again it is enough to assume that the a_j belong to $C^{\Gamma_{t_0}(P)+1}$ near t_0. Define $\Gamma_J(P), \gamma_J(P) \in \mathbb{N} \cup \{+\infty\}$ by

$$\Gamma_J(P) := \sup\{\Gamma_{t_0}(P) : t_0 \in J\}, \tag{5.4.1}$$

$$\gamma_J(P) := \Gamma_J(P) - \sup\{\Gamma_{t_0}(P) - \gamma_{t_0}(P) : t_0 \in J\}. \tag{5.4.2}$$

By construction,

$$\gamma_J(P) \le \Gamma_J(P) \le d(n)\overline{m}_J(P) + 1.$$

The interesting case is when $\Gamma_J(P)$ and $\gamma_J(P)$ are finite, but what follows is true in any case.

5.5. **Theorem.** *If the coefficients a_j of P are $C^{p+\Gamma_J(P)}$, then the roots of P can be parameterized by C^p functions, globally near J.*

Proof. By the definition of $\Gamma_J(P)$, the coefficients a_j have the right differentiability for the proof of 4.12(2) to work. Definability was used in the proof of 4.12(2) only in (IIc) and in claim 4.12(4). The case (IIc) does not occur, since $E^{(\infty)}(P) = \emptyset$. In claim 4.12(4), the use of definability can be replaced by the following argument: If a real valued C^p function f vanishes on $t_k \searrow t_0$, then $f^{(q)}(t_0) = 0$ for all $0 \le q \le p$. This follows from a repeated application of Rolle's theorem. □

5.6. **Lemma.** *Let $I \subseteq \mathbb{R}$ be an open interval containing 0. Let $p, r \in \mathbb{N}_{>0}$. Suppose that $a_k(t) = t^{kr} a_{(r),k}(t) \in C^{p+nr}(I)$, for $2 \le k \le n$, $a_{(r),2}(0) \ne 0$, and consider*

$$P_{(r)}(t)(x) = x^n + \sum_{k=2}^{n} (-1)^k a_{(r),k}(t) x^{n-k}.$$

Factorize $P_{(r)} = \prod_{j=1}^{l} P_{(r),j}$ near 0, according to the splitting lemma 4.2, such that

$$P_{(r),j}(t)(x) = x^{n_j} + \sum_{k=1}^{n_j} (-1)^k a_{(r),j,k}(t) x^{n_j-k}, \quad 1 \le j \le l,$$

and the $P_{(r),j}$ have mutually distinct roots. Then, for all $1 \le j \le l$ and $1 \le k \le n_j$, $a_{j,k}(t) := t^{kr} a_{(r),j,k}(t)$ belongs to C^{p+kr} near 0.

Proof. By assumption, $t^m a_{(r),k}(t) \in C^{p+(n-k)r+m}$, for all $2 \le k \le n$ and $0 \le m \le kr$. We assert that

$$t^m a_{(r),k}^{(m)}(t) \in C^{p+(n-k)r}, \quad \text{for all } 2 \le k \le n \text{ and } 0 \le m \le kr, \tag{5.6.1}$$

(where $a_{(r),k}^{(m)}$ is understood as distributional derivative). This follows from

$$\partial_t^m (t^m a_{(r),k}(t)) = \sum_{j=0}^{m} \binom{m}{j} \frac{m!}{j!} t^j a_{(r),k}^{(j)}(t)$$

and from induction on m. From (5.6.1) we can deduce in a similar way that

$$t^q a_{(r),k}^{(q)}(t) \in C^p, \quad \text{for all } 2 \le k \le n \text{ and } 0 \le q \le nr. \tag{5.6.2}$$

Let $a_{(r)} := (a_{(r),2}, \ldots, a_{(r),n})$. By assumption, there exist C^ω functions $\Phi_{j,k}$ defined in a neighborhood of $a_{(r)}(0) \in \mathbb{R}^{n-1}$ such that $a_{(r),j,k} = \Phi_{j,k} \circ a_{(r)}$, for all $1 \le j \le l$ and $1 \le k \le n_j$. Then

$$a_{j,k}^{(kr)}(t) = \sum_{m=0}^{kr} \binom{kr}{m} \frac{(kr)!}{m!} A_{j,k}^m(t),$$

where (by Faà di Bruno, [FdB55] for the 1-dimensional version)

$$A_{j,k}^m(t) = \sum_{l \ge 0} \sum_{\substack{\alpha \in \mathbb{N}_{>0}^l \\ \alpha_1 + \cdots + \alpha_l = m}} \frac{m!}{l!} d^l \Phi_{j,k}(a_{(r)}(t)) \Big(\frac{t^{\alpha_1} a_{(r)}^{(\alpha_1)}(t)}{\alpha_1!}, \ldots, \frac{t^{\alpha_l} a_{(r)}^{(\alpha_l)}(t)}{\alpha_l!} \Big).$$

So, by (5.6.2), we find $a_{j,k}^{(kr)} \in C^p$ and, thus, $a_{j,k} \in C^{p+kr}$. □

5.7. **Lemma.** *Adopt the setting of lemma 5.6. However, assume that $a_k(t) = t^{kr} a_{(r),k}(t) \in C^{p+kr}(I)$, for $2 \le k \le n$, and that all roots of $P_{(r)}(0)$ are distinct. If λ_j are C^p functions representing the roots of $P_{(r)}$, then $\Lambda_j(t) := t^r \lambda_j(t)$ are C^{p+r} functions representing the roots of P.*

Proof. Instead of (5.6.1) we obtain

$$t^m a_{(r),k}^{(m)}(t) \in C^p, \quad \text{for all } 2 \le k \le n \text{ and } 0 \le m \le kr. \tag{5.7.1}$$

The second part of the proof is the same as in 5.6, where now $l = n$ and $n_j = 1$ for all j. In the end we use (5.7.1) instead of (5.6.2). □

5.8. Let $P(t)$, $t \in I$, be a curve of monic hyperbolic polynomials of degree n (not necessarily definable). Assume $E^{(\infty)}(P) = \emptyset$. Let $t_0 \in I$ and suppose that the coefficients of P belong to $C^{\Gamma_{t_0}(P)+1}$ near t_0. The gradual splitting of, firstly, P near t_0 into factors P_i with mutually distinct roots such that all roots of $P_i(t_0)$ coincide, then, secondly, of each $(P_i)_{(r_i)}$ (defined in (4.12.1)) and so on, determines a well-defined mapping $(P, t_0) \mapsto T(P, t_0)$, where $T(P, t_0)$ is a rooted tree in $\mathcal{R}(n)$ (cf. 4.11).

By the *height* $h(T)$ of a tree T we mean the maximal length (number of edges) of paths connecting the root with a leaf in T. The *k-level* of T is the set of all vertices whose distance (length of the connecting path) from the root is k.

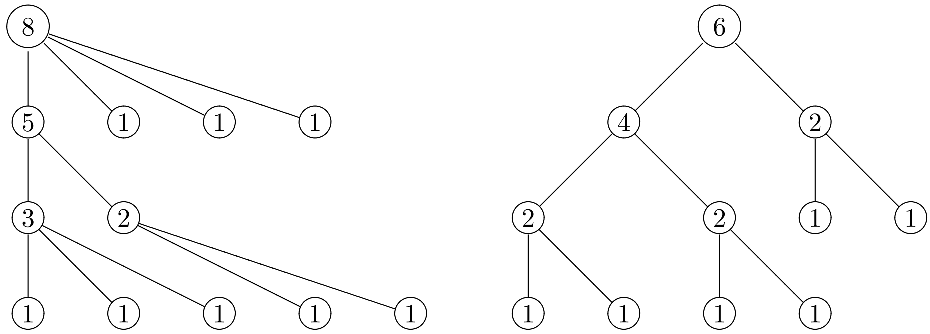

Figure 1. The first rooted tree is of type (A), the second is not.

5.9. **Theorem.** *Let $I \subseteq \mathbb{R}$ be an open interval and let $J \subseteq I$ be a compact subinterval. Consider a curve of hyperbolic polynomials*

$$P(t)(x) = x^n + \sum_{j=1}^{n} (-1)^j a_j(t) x^{n-j}, \quad (t \in I)$$

such that $E^{(\infty)}(P) = \emptyset$. Assume that the following condition is satisfied for all $t \in J$:

(A) *For all $k \le h(T(P,t)) - 2$, the k-level of $T(P,t)$ contains at most one vertex with label ≥ 2.*

For each $p \in \mathbb{N}_{>0}$ we have:

(1) *If the a_j are $C^{p+\Gamma_J(P)}$, then the roots of P can be parameterized by $C^{p+\gamma_J(P)}$ functions, globally near J.*

Proof. By 4.12(4) and the argument in the proof of 5.5, it suffices to show the local assertion. Let $t_0 \in J$ be fixed.

Claim (2)**.** *If the a_j are $C^{p+\Gamma_{t_0}(P)}$, then the roots of P can be chosen in $C^{p+\gamma_{t_0}(P)}$, locally near t_0.*

Without loss assume that $0 \in J$ and $t_0 = 0$. We proceed by induction on n. If $n = 1$ then $\Gamma_0(P) = \gamma_0(P) = 0$ and we are done. Suppose $n > 1$ and the claim is proved for degrees $\le n - 1$.

(I) If $P(0)$ has distinct roots, we have a factorization $P(t) = P_1(t) \cdot P_2(t)$ near 0, by the splitting lemma 4.2. The coefficients of each factor P_i belong to $C^{p+\Gamma_0(P)}$. Let $p_i := p + \Gamma_0(P) - \Gamma_0(P_i)$. Then $p_i \ge p$, by (5.1.1). Clearly, each $T(P_i, 0)$ is of type (A). By induction hypothesis, the roots of P_i admit a local parameterization in $C^{p_i+\gamma_0(P_i)}$. By (5.1.2), $p_i + \gamma_0(P_i) \ge p + \gamma_0(P)$, hence claim (2).

(II) If all roots of $P(0)$ coincide, we reduce to the case $a_1 = 0$. So $a_2(0) = 0$. If $a_2 = 0$ identically, then all roots are 0 identically, and claim (2) is satisfied. Suppose that $a_2 \ne 0$. Since $E^{(\infty)}(P) = \emptyset$ and since $\Gamma_0(P) \ge m_0(a_2)$ by definition, we have $m_0(a_2) = 2r < \infty$. Consider $P_{(r)}$ as in (4.12.1). The coefficients of $P_{(r)}$ are in $C^{p+\Gamma_0(P)-nr}$, and $a_{(r),2}(0) \ne 0$. By (5.1.3) and (I), there are $C^{p+\gamma_0(P_{(r)})}$ functions λ_j which represent the roots of $P_{(r)}$ near 0. Then the functions $\Lambda_j(t) = t^r \lambda_j(t)$ form a local parameterization of the roots of P. The proof of claim (2) is complete once claim (3) below is shown.

Claim (3)**.** *Each Λ_j belongs to $C^{p+\gamma_0(P)}$.*

We treat the following cases separately:

(3a) Suppose that $h(T(P,0)) \le 2$.

If all $\lambda_j(0)$ are distinct, then claim (3) follows from (5.1.4) and lemma 5.7.

Otherwise, we can assume (after possibly reordering the λ_j) that

$$\lambda_1(0) = \cdots = \lambda_{n_1}(0) < \lambda_{n_1+1}(0) = \cdots = \lambda_{n_1+n_2}(0) < \cdots < \lambda_{n-n_l+1}(0) = \cdots = \lambda_n(0).$$

Set $N(1) := 0$ and $N(j) := n_1 + \cdots + n_{j-1}$ for $2 \le j \le l$. By the splitting lemma 4.2, for each $1 \le j \le l$,

$$P_{(r),j}(t)(x) = x^{n_j} + \sum_{k=1}^{n_j} (-1)^k a_{(r),j,k}(t) x^{n_j-k} := \prod_{i=1}^{n_j} (x - \lambda_{N(j)+i}(t))$$

has $C^{p+\Gamma_0(P_{(r)})}$ coefficients $a_{(r),j,k}$ near 0. By replacing the variable x with $x - a_{(r),j,1}(t)/n_j$, we obtain

$$\bar{P}_{(r),j}(t)(x) = x^{n_j} + \sum_{k=2}^{n_j} (-1)^k \bar{a}_{(r),j,k}(t) x^{n_j-k} = \prod_{i=1}^{n_j} (x - (\lambda_{N(j)+i}(t) - \tfrac{a_{(r),j,1}(t)}{n_j})),$$

where the $\bar{a}_{(r),j,k}$ are still $C^{p+\Gamma_0(P_{(r)})}$ near 0. All roots of $\bar{P}_{(r),j}(0)$ are equal to 0. As above we may conclude that there is a $q_j \in \mathbb{N}_{>0}$ such that $\bar{a}_{(r),j,k}(t) = t^{kq_j}\bar{a}_{(r,q_j),j,k}(t)$, for $2 \le k \le n_j$, $\bar{a}_{(r,q_j),j,2}(0) \neq 0$, and

$$\bar{P}_{(r,q_j),j}(t)(x) := x^{n_j} + \sum_{k=2}^{n_j}(-1)^k \bar{a}_{(r,q_j),j,k}(t)x^{n_j-k}$$

has $C^{p+\Gamma_0(\bar{P}_{(r,q_j),j})}$ coefficients $\bar{a}_{(r,q_j),j,k}(t)$ and $C^{p+\gamma_0(\bar{P}_{(r,q_j),j})}$ roots $\mu_{j,i}$. Then

$$t^{q_j}\mu_{j,i}(t) = \lambda_{N(j)+i}(t) - \tfrac{a_{(r),j,1}(t)}{n_j}, \quad \text{for } 1 \le i \le n_j. \tag{5.9.1}$$

Thus,

$$\Lambda_{N(j)+i}(t) = t^{r+q_j}\mu_{j,i}(t) + t^r \tfrac{a_{(r),j,1}(t)}{n_j}, \quad \text{for } 1 \le i \le n_j. \tag{5.9.2}$$

By lemma 5.6,

$$t^{kr}a_{(r),j,k}(t) \in C^{p+\Gamma_0(P_{(r)})+kr}, \quad \text{for all } 1 \le j \le l \text{ and } 1 \le k \le n_j.$$

In particular, $t^r a_{(r),j,1}(t) \in C^{p+\Gamma_0(P_{(r)})+r}$. So, in order to show claim (3), it remains to prove that the first summand on the right-hand side of (5.9.2) belongs to $C^{p+\gamma_0(P)}$.

The mapping $(a_{(r),j,1}, \ldots, a_{(r),j,n_j}) \mapsto (\bar{a}_{(r),j,2}, \ldots, \bar{a}_{(r),j,n_j})$ is polynomial. Thus, there exist C^ω functions $\bar{\Phi}_{j,k}$ defined in a neighborhood of $a_{(r)}(0) \in \mathbb{R}^{n-1}$ such that $\bar{a}_{(r),j,k} = \bar{\Phi}_{j,k} \circ a_{(r)}$, for all $1 \le j \le l$ and $2 \le k \le n_j$. Hence, by (the proof of) lemma 5.6, we also obtain

$$t^{kr}\bar{a}_{(r),j,k}(t) \in C^{p+\Gamma_0(P_{(r)})+kr}, \quad \text{for all } 1 \le j \le l \text{ and } 2 \le k \le n_j,$$

and thus

$$t^{k(r+q_j)}\bar{a}_{(r,q_j),j,k}(t) \in C^{p+\Gamma_0(P_{(r)})+kr} \subseteq C^{p+\Gamma_0(\bar{P}_{(r,q_j),j})+k(r+q_j)}, \quad \text{for all } 1 \le j \le l \text{ and } 2 \le k \le n_j.$$

By the assumption $h(T(P,0)) \le 2$, all $\mu_{j,i}(0)$ are distinct. Then claim (3) follows from (5.1.4) and lemma 5.7.

(3b) Suppose that $h(T(P,0)) > 2$. Let us use the notation of (3a). Since $T(P,0)$ is of type (A), we may assume $n_2 = n_3 = \cdots = n_l = 1$, and the roots λ_j, for $n_1 + 1 \le j \le n$, belong to $C^{p+\Gamma_0(P_{(r)})}$.

Consider the Newton polynomials $s_{(r),k} = \sum_{j=1}^n \lambda_j^k$ and $\bar{s}_{(r,q_1),1,k} = \sum_{j=1}^{n_1} \mu_{1,j}^k$, associated to $P_{(r)}$ and $\bar{P}_{(r,q_1),1}$, respectively. (In the following argument it is convenient to work with the Newton polynomials of the roots instead of the elementary symmetric functions (coefficients). They are related to each other by the polynomial diffeomorphism defined in (4.1.2).) Note that $s_{(r),1} = \bar{s}_{(r,q_1),1,1} = 0$ and $\bar{s}_{(r,q_1),1,0} = n_1$. We have, by (5.9.1),

$$0 = s_{(r),1} = a_{(r),1,1}(t) + \sum_{i=n_1+1}^{n} \lambda_i(t), \tag{5.9.3}$$

(5.9.4)

$$s_{(r),k}(t) = \sum_{i=0}^{k}\binom{k}{i} t^{iq_1}\bar{s}_{(r,q_1),1,i}(t)\big(\tfrac{a_{(r),1,1}(t)}{n_1}\big)^{k-i} + \sum_{i=n_1+1}^{n}\lambda_i(t)^k, \quad 2 \le k \le n_1.$$

By lemma 5.6 and (5.1.3), $\Lambda_i(t) = t^r\lambda_i(t) \in C^{p+\Gamma_0(P)}$, for $n_1 + 1 \le i \le n$. Thus, by (5.9.3), $t^r a_{(r),1,1}(t) \in C^{p+\Gamma_0(P)}$. By (4.1.2), we have $t^{kr}s_{(r),k}(t) \in C^{p+\Gamma_0(P)}$, for $2 \le k \le n$ (since the same is true when the $s_{(r),k}$ are replaced by the $a_{(r),k}$). Hence, (5.9.4) implies inductively that $t^{i(r+q_1)}\bar{s}_{(r,q_1),1,i}(t) \in C^{p+\Gamma_0(P)}$, for $2 \le i \le n_1$, and equivalently,

$$t^{i(r+q_1)}\bar{a}_{(r,q_1),1,i}(t) \in C^{p+\Gamma_0(P)}, \quad \text{for } 2 \le i \le n_1.$$

Let us repeat this procedure with

$$\tilde{P}(t)(x) := x^{n_1} + \sum_{j=2}^{n_1}(-1)^j t^{j(r+q_1)}\bar{a}_{(r,q_1),1,j}(t)x^{n_1-j} = \prod_{i=1}^{n_1}(x - t^{r+q_1}\mu_{1,i}(t))$$

instead of P. Evidently, $T(\tilde{P},0)$ is of type (A). After finitely many steps the situation is reduced to case (3a). This completes the proof of claim (3).

Claim (4). *If the a_j are $C^{p+\Gamma_J(P)}$, then the roots of P can be chosen in $C^{p+\gamma_J(P)}$, locally near t_0.*

By claim (2), the roots of P can be chosen in $C^{p+\Gamma_J(P)-\Gamma_{t_0}(P)+\gamma_{t_0}(P)}$, locally near t_0. By definition, $p + \Gamma_J(P) - \Gamma_{t_0}(P) + \gamma_{t_0}(P) \ge p + \gamma_J(P)$. □

5.10. **Remark.** We do not know whether or not theorem 5.9 holds, if $T(P,t)$ is not of type (A). Note that each $T \in \bigcup_{n=1}^4 \mathcal{R}(n)$ is automatically of type (A). Thus, theorem 5.9 is true for all P with degree at most 4.

6. Definable version of Bronshtein's theorem

6.1. **Theorem.** *Let $I \subseteq \mathbb{R}$ be an open interval. Consider a curve of hyperbolic polynomials*

$$P(t)(x) = x^n + \sum_{j=1}^{n}(-1)^j a_j(t)x^{n-j}$$

with definable C^n coefficients a_j. Then the roots of P can be parameterized by definable C^1 functions, globally.

If 'definable' is omitted in the formulation of theorem 6.1, then we obtain Bronshtein's theorem [Bro79] (see also [Wak86]). Actually we obtain the refinement of Bronshtein's theorem due to [COP08]. The proof of Bronshtein's theorem is very delicate and only poorly understood. In the definable case it becomes remarkably simple.

Proof. By 4.12(4), it suffices to show the local statement. We follow the proof of theorem 4.12(3) and indicate the necessary modifications. Let us begin the induction on n with the case $n = 1$, which is trivial. (I) and (II) can be adopted with obvious minor changes. So assume that $a_1 = 0$ identically and $a_2(0) = 0$. Since $0 \le \tilde{\Delta}_2 \circ P = -2na_2$, we have $m_0(a_2) \ge 2$. By the multiplicity lemma 4.5 (for $r = 1$), $m_0(a_k) \ge k$ for $2 \le k \le n$, and $P_{(1)}$ (defined in (4.12.1)) is a continuous curve of hyperbolic polynomials. Let μ_j be a continuous parameterization of the roots of $P_{(1)}$ near 0. Then the functions $\lambda_j(t) := t\mu_j(t)$ form a definable continuous parameterization of the roots of P near 0 such that $m_0(\lambda_j) \ge 1$ for each j. By lemma 2.5(1), each λ_j is C^1 near 0. □

6.2. **Examples.** (1) The function $f(t) = t^2|t|$ is in $C^{2,1}$ (but not three times differentiable). The square roots of f may be chosen C^1 but not $C^{1,1}$.

(2) Let $g(t) = 1/3$ for $t \ge 0$ and $g(t) = 0$ otherwise. Consider the following $C^{2,1}$ curve of cubic polynomials (cf. [COP08, Example 4.6]):

$$P(t)(x) = x^3 - t^3g(t)x^2 + (2t^3g(t) - t^2)x - t^3g(t).$$

Its discriminant is $\tilde{\Delta}_3(P(t)) = t^6(1 + o(1))$ if $t \geq 0$ and $\tilde{\Delta}_3(P(t)) = 4t^6$ if $t < 0$. Thus, for small t, $P(t)$ is hyperbolic. The roots of P cannot be chosen differentiable at 0: Note that 0 is a triple root of $P(0)$. Consider, for $t \neq 0$,

$$Q(t)(y) = t^{-3}P(t)(ty) = y^3 - t^2 g(t)y^2 + (2tg(t) - 1)y - g(t).$$

Then $\lim_{t \searrow 0} Q(t)(y) = y^3 - y - 1/3$ and $\lim_{t \nearrow 0} Q(t)(y) = y^3 - y$. Thus, the roots of P cannot be differentiable at 0.

7. Complex polynomials

7.1. In this section let $I \subseteq \mathbb{R}$ be an open interval and consider a curve of complex polynomials

$$P(t)(x) = x^n + \sum_{j=1}^{n} (-1)^j a_j(t) x^{n-j},$$

i.e., each coefficient $a_j : I \to \mathbb{C}$ is a continuous complex valued function. Then the roots of P admit a continuous parameterization (e.g. [Kat76, II 5.2]).

A complex valued function $f : I \to \mathbb{C}$ is called *definable* if $(\operatorname{Re} f, \operatorname{Im} f) : I \to \mathbb{R}^2$ is definable. We will assume that the coefficients a_j of P are definable.

The set $E^{(\infty)}(P)$ can be defined and has the same properties as in the hyperbolic case (cf. 4.6).

7.2. **Lemma.** *If the coefficients a_j of P are definable, then every continuous parameterization λ_j of the roots of P is definable.*

Proof. The real and imaginary parts $\operatorname{Re} \lambda_j$, $\operatorname{Im} \lambda_j$, $1 \leq j \leq n$, parameterize the solutions of the $2n$ algebraic equations with definable coefficients $\operatorname{Re} P(t)(\lambda_j(t)) = 0$, $\operatorname{Im} P(t)(\lambda_j(t)) = 0$, $1 \leq j \leq n$. The family of continuous parameterizations of the solutions of these equations is finite. □

7.3. **Theorem.** *Let $I \subseteq \mathbb{R}$ be an open interval. Consider a curve of polynomials*

$$P(t)(x) = x^n + \sum_{j=1}^{n} (-1)^j a_j(t) x^{n-j},$$

with definable C^∞ coefficients a_j. Then, for each $t_0 \in I$, there is an $N \in \mathbb{N}_{>0}$ such that $t \mapsto P(t_0 \pm (t - t_0)^N)$ admits definable C^∞ parameterizations of its roots, locally near t_0.

Proof. Since the coefficients of $t \mapsto P(t_0 \pm (t - t_0)^N)$ are definable, we need not care about the definability of its roots, by lemma 7.2. Without loss assume that $0 \in I$ and $t_0 = 0$. We proceed by induction on n. The case $n = 1$ is trivial.

(I) If $P(0)$ has distinct roots, we are done, by the splitting lemma 4.2 and the induction hypothesis. (Here we use that, if $t \mapsto P_i(\pm t^{N_i})$, $i = 1, 2$, admit C^∞ roots then so does $t \mapsto P_1(\pm t^{N_1 N_2}) P_2(\pm t^{N_1 N_2})$.)

(II) If all roots of $P(0)$ coincide, we reduce to the case $a_1 = 0$. Then all roots of $P(0)$ are equal to 0.

(IIa) If $m_0(a_k) < \infty$ for some $2 \leq k \leq n$, there exist $N, r \in \mathbb{N}_{>0}$ such that $(t \mapsto P(\pm t^N))_{(r)}$ (the reduced curve of polynomials defined in (4.12.1) associated to $t \mapsto P(\pm t^N)$) has distinct roots at $t = 0$ (see [Rai09]). By the splitting lemma 4.2 and the induction hypothesis, we are done.

(IIb) If all $a_k = 0$ identically, then all roots of P are identically 0.

(IIc) If $m_0(a_k) = \infty$ for all $2 \leq k \leq n$, then for any continuous choice λ_j of the roots of P we find $m_0(\lambda_j) = \infty$ (for all j). For: Let $\lambda(t)$ be any continuous root of $P(t)$ and $r \in \mathbb{N}_{>0}$. Then, for $t \neq 0$, $\mu(t) = t^{-r}\lambda(t)$ is a root of $P_{(r)}(t)$ (defined in (4.12.1)), hence bounded in t. So $\lambda(t) = t^{r-1} \cdot t\mu(t)$, and $t \mapsto t\mu(t)$ is continuous.

Thus $m_0(\lambda) = \infty$, since r was arbitrary. By lemma 2.5(1) (applied to $\operatorname{Re}\lambda_j$ and $\operatorname{Im}\lambda_j$), for each p, there is a neighborhood I_p of 0 such that each λ_j is C^p on I_p. Since the coefficients a_j (and hence the $\tilde{\Delta}_k \circ P$) are definable, for small $t \neq 0$ the multiplicity of the $\lambda_j(t)$ is constant. So all λ_j are C^∞ off 0 (by the splitting lemma 4.2) and hence also near 0. □

7.4. In [Rai09] we have deduced from the analog of theorem 7.3 that any continuous parameterizations of the roots of a C^∞ curve P of complex polynomials with $E^\infty(P) = \emptyset$ is locally actually absolutely continuous (not better!, see 7.7 below). The optimal conditions for absolutely continuous roots are unknown.

However, in the definable case we have the following best possible result:

7.5. **Theorem.** *Any continuous choice of the roots of a curve of monic complex polynomials with definable continuous coefficients is locally absolutely continuous.*

Proof. This follows from lemma 7.2 and lemma 7.6 below. □

7.6. **Lemma.** *Let $I \subseteq \mathbb{R}$ be an interval. A definable continuous function $f : I \to \mathbb{C}$ is locally absolutely continuous.*

Proof. We show that a continuous definable function $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is a compact interval, is absolutely continuous. By the Monotonicity theorem [vdD98], f is C^1 on the complement of finitely many points $J = I \setminus \{a_1, \ldots, a_n\}$. Let J_i be some connected component of J. By definability, we can partition J_i into finitely many subintervals J_{ij} on each of which either $f' > 0$ or $f' \leq 0$. Then it is easy to see that $f'|_{J_{ij}}$ belongs to L^1 for every J_{ij}, thus $f'|_{J_i}$ belongs to L^1 (here we use that f is continuous). Let $[a, b] := \overline{J}_i$ denote the closure of J_i. Then we have $f(x) = f(a) + \int_a^x f'(t)dt$ for $x \in [a, b]$. So $f|_{\overline{J}_i}$ is absolutely continuous. Since J_i was arbitrary, the proof is complete. □

7.7. **Examples.** Absolute continuity is the best we can hope for: In general the roots cannot be chosen with first derivative in L^p_{loc} for any $1 < p \leq \infty$. This is demonstrated by

$$P(t)(z) = z^n - t, \quad t \in \mathbb{R},$$

for $1 < p < \infty$ if $n \geq \frac{p}{p-1}$ and for $p = \infty$ if $n \geq 2$.

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QUASIANALYTIC MULTIPARAMETER PERTURBATION OF POLYNOMIALS AND NORMAL MATRICES

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Abstract. We study the regularity of the roots of multiparameter families of complex univariate monic polynomials $P(x)(z) = z^n + \sum_{j=1}^n (-1)^j a_j(x) z^{n-j}$ with fixed degree n whose coefficients belong to a certain subring $\mathcal{C}$ of C^∞-functions. We require that $\mathcal{C}$ includes polynomial but excludes flat functions (quasianalyticity) and is closed under composition, derivation, division by a coordinate, and taking the inverse. Examples are quasianalytic Denjoy–Carleman classes, in particular, the class of real analytic functions C^ω.

We show that there exists a locally finite covering $\{\pi_k\}$ of the parameter space, where each π_k is a composite of finitely many $\mathcal{C}$-mappings each of which is either a local blow-up with smooth center or a local power substitution (in coordinates given by $x \mapsto (\pm x_1^{\gamma_1}, \ldots, \pm x_q^{\gamma_q})$, $\gamma_i \in \mathbb{N}_{>0}$), such that, for each k, the family of polynomials $P \circ \pi_k$ admits a $\mathcal{C}$-parameterization of its roots. If P is hyperbolic (all roots real), then local blow-ups suffice.

Using this desingularization result, we prove that the roots of P can be parameterized by SBV_{loc}-functions whose classical gradients exist almost everywhere and belong to L^1_{loc}. In general the roots cannot have gradients in L^p_{loc} for any $1 < p \le \infty$. Neither can the roots be in $W^{1,1}_{\text{loc}}$ or VMO.

We obtain the same regularity properties for the eigenvalues and the eigenvectors of $\mathcal{C}$-families of normal matrices. A further consequence is that every continuous subanalytic function belongs to SBV_{loc}.

1. Introduction

Let us consider a family of univariate monic polynomials

$$P(x)(z) = z^n + \sum_{j=1}^n (-1)^j a_j(x) z^{n-j}$$

where the coefficients $a_j : U \to \mathbb{C}$ (for $1 \le j \le n$) are complex valued functions defined in an open subset $U \subseteq \mathbb{R}^q$. If the coefficients a_j are regular (of some kind) it is natural to ask whether the roots of P can be arranged regularly as well, i.e., whether it is possible to find n regular functions $\lambda_j : U \to \mathbb{C}$ (for $1 \le j \le n$) such that $\lambda_1(x), \ldots, \lambda_n(x)$ represent the roots of $P(x)(z) = 0$ for each $x \in U$.

This perturbation problem has been intensively studied under the following additional assumptions:

(1) The parameter space is one dimensional: $q = 1$.
(2) The polynomials $P(x)$ are hyperbolic, i.e., all roots of $P(x)$ are real.

If both of these conditions are satisfied, there exist real analytic parameterizations of the roots of P if its coefficients a_j are real analytic, by a classical theorem due to Rellich [Rel37a]. If all a_j are smooth (C^∞) and no two of the increasingly ordered (hence) continuous roots meet of infinite order of flatness, then there exist smooth parameterizations of the roots, by [AKLM98]. Without additional condition we cannot hope for smooth roots. By [Rai09b], smooth roots exist if the

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coefficients are smooth and definable in some o-minimal expansion of the real field, which implies that not flat contact but oscillatory behavior is responsible for the loss of smoothness. The roots may always be chosen C^1 (resp. twice differentiable) provided that the a_j are in C^{2n} (resp. C^{3n}), see [Man85] and [KLM04]. Recently, the assumptions in this statement have been refined to C^n (resp. C^{2n}) by [COP08]. It is then best possible in both hypothesis and conclusion as shown by examples (e.g. in [COP08] and [BBCP06]). Sharp sufficient conditions, in terms of the differentiability of the coefficients and the order of contact of the roots, for the existence of C^p-roots ($p \in \mathbb{N}$) are found in [Rai09b].

If the polynomials $P(x)$ are hyperbolic and all a_j are in C^n, but the parameter space is multidimensional ($q > 1$), then the roots of P may still be parameterized by locally Lipschitz functions (by ordering them increasingly for instance). This follows from the fundamental results of Bronshtein [Bro79] and (alternatively) Wakabayashi [Wak86] (which also constitute the main part in the proof of all but the last of the finite differentiability statements above). For a detailed presentation of those see [Rai]. A different and easier proof for the partial case that the coefficients a_j are real analytic was recently given by Kurdyka and Paunescu [KP08]. In that paper the real analytic multiparameter perturbation theory of hyperbolic polynomials P and symmetric matrices A is studied. It is shown that there exists a modification $\Phi : W \to U$, namely a locally finite composition of blow-ups with smooth centers, such that the roots of $P \circ \Phi$ can be locally parameterized by real analytic functions, and $A \circ \Phi$ is real analytically diagonalizable. For further results on the perturbation problem of hyperbolic polynomials see (among others) [Gla63], [Die70], [CC04], and [LR07].

The one parameter case $q = 1$, but with the hyperbolicity assumption dropped, was treated in [Rai09a]. In that case continuous parameterizations of the roots still exist given that the coefficients a_j are continuous (e.g. Kato [Kat76, II 5.2]). If all a_j are smooth and no two of the continuously chosen roots meet of infinite order of flatness, then any continuous parameterization of the roots is locally absolutely continuous. Absolute continuity is the best one can expect, see 7.13. This theorem follows from the (Puiseux type) proposition that for any x_0 there exists an integer N such that $x \mapsto P(x_0 \pm (x - x_0)^N)$ admits smooth parameterizations of its roots near x_0. It seems unknown whether the roots still can be arrange locally absolutely continuously if the condition on the order of contact is omitted. Spagnolo [Spa99] gave an affirmative answer for degree 2 and 3 polynomials (degree 4 is announced).

In the present paper we study smooth multiparameter perturbations of complex polynomials, i.e., without the restrictions (1) and (2). It is easy to see that every choice of the roots of a bounded family P of polynomials is bounded as well (proposition 2.4). By a theorem due to Ostrowski [Ost40], for a continuous family P of polynomials, the set of all roots still is continuous and satisfies a Hölder condition of order $1/n$. But in general there may not exist continuous parameterizations of the single roots as in the one dimensional or hyperbolic case. For instance, $P(x_1, x_2)(z) = z^2 - (x_1 + ix_2)$, with $x_1, x_2 \in \mathbb{R}$ and $i = \sqrt{-1}$. Nevertheless, the roots of P may have some other regularity properties.

We show the following (theorem 6.7): Let $\mathcal{C}$ be a certain class of C^∞-functions (specified below). If the coefficients a_j of P are $\mathcal{C}$-functions on a $\mathcal{C}$-manifold M, then for each compact subset $K \subseteq M$ there exist:

(a) a neighborhood W of K, and
(b) a finite covering $\{\pi_k : U_k \to W\}$ of W by $\mathcal{C}$-mappings, where each π_k is a composite of finitely many mappings each of which is either a local blow-up with smooth center or a local power substitution,

such that, for all k, the family of polynomials $P \circ \pi_k$ allows a $\mathcal{C}$-parameterization of its roots on U_k. If P is hyperbolic, then local blow-ups suffice (theorem 6.10). A local blow-up over an open subset $U \subseteq M$ is a blow-up over U composed with the inclusion of U in M. A local power substitution is the composite of the inclusion of a coordinate chart W in M and a mapping $V \to W$ given in local coordinates by

$$(x_1, \dots, x_q) \mapsto ((-1)^{\epsilon_1} x_1^{\gamma_1}, \dots, (-1)^{\epsilon_q} x_q^{\gamma_q})$$

for some $\gamma \in (\mathbb{N}_{>0})^q$ and all $\epsilon \in \{0,1\}^q$. (See 6.1 for a precise explanation of these notions.)

The proof uses resolution of singularities. Accordingly, $\mathcal{C}$ is a class of C^∞-functions admitting resolution of singularities. Due to Bierstone and Milman [BM04] (and [BM97]), it suffices that $\mathcal{C}$ is a subring of C^∞ that includes polynomial but excludes flat functions (quasianalyticity) and is closed under composition, differentiation, division by a coordinate, and taking the inverse (see section 3). For instance, $\mathcal{C}$ may be any quasianalytic Denjoy–Carleman class C^M, where the weight sequence M satisfies some mild conditions (see section 4). In particular, $\mathcal{C}$ can be the class of real analytic functions C^ω. Hence, in the hyperbolic case, we recover a version of the aforementioned theorem due to Kurdyka and Paunescu [KP08].

The above result (theorem 6.7) enables us to investigate the regularity of the roots of $\mathcal{C}$-families of polynomials P. We show:

(i) The roots of P allow a parameterization by "piecewise Sobolev $W^{1,1}_{\text{loc}}$" functions. More precisely, the roots of P can locally be chosen of class $\mathcal{C}$ outside of a closed nullset of finite $(q-1)$-dimensional Hausdorff measure such that its classical gradient belongs to L^1_{loc} (theorem 7.11).

(ii) The roots of P allow a parameterization in SBV_{loc} (theorem 8.4).

Note that (i) implies (ii) (see section 8). Simple examples show that the conclusion in (i) is best possible: In general we cannot expect that the roots of P admit arrangements having gradients in L^p_{loc} for any $1 < p \le \infty$ (see 7.13). In contrast to the one parameter case (see [Rai09a] and 7.15), multiparameter families of polynomials do in general not allow roots in $W^{1,1}_{\text{loc}}$ (see the polynomial counter-example in 7.17) or in VMO (see 7.18).

The question for optimal assumptions is open. For instance, it is unknown whether (ii) still holds when the coefficients of P are just C^∞-functions. That problem requires different methods.

Table 1 on page 61 provides a summary of the most important results on the perturbation theory of polynomials.

In section 9 we deduce consequences for the perturbation theory of normal matrices. There will be applications to the perturbation theory of unbounded normal operators with compact resolvents and common domain of definition as well. It requires a differential calculus for quasianalytic classes beyond Banach spaces (see [KMR09a] for the case of non-quasianalytic Denjoy–Carleman classes). This will be taken up elsewhere (see [KMR09b] and [KMR09c]). Our results generalize theorems obtained in [KP08] and [Rai09a]. For more on the perturbation theory of linear operators consider Rellich [Rel37a, Rel37b, Rel39, Rel40, Rel42, Rel69], Kato [Kat76], Baumgärtel [Bau72], and also [AKLM98], [KM03], and [KMR09d].

We prove the following (theorem 9.1): Let $A = (A_{ij})_{1 \le i,j \le n}$ be a family of normal complex matrices, where the entries A_{ij} are $\mathcal{C}$-functions on a $\mathcal{C}$-manifold M. Then, for each compact subset $K \subseteq M$, there exist a neighborhood W of K and a finite covering $\{\pi_k : U_k \to W\}$ of W of the type described in (b), such that, for all k, the family of normal matrices $A \circ \pi_k$ allows $\mathcal{C}$-parameterizations of its eigenvalues and its eigenvectors. If A is a family of Hermitian matrices, then local blow-ups suffice. Both a nonflatness condition (such as quasianalyticity) and normality of the matrices $A(x)$ are necessary for the desingularization of the eigenvectors (see 9.4 and 9.5).

We conclude that the eigenvalues and the eigenvectors of a $\mathcal{C}$-family of normal complex matrices A locally admit parameterizations by "piecewise Sobolev $W^{1,1}_{\text{loc}}$" functions (in the sense of (i)) and, thus, by SBV_{loc}-functions (theorem 9.6).

A further application of the method developed in this paper is given in section 10: Any continuous subanalytic function belongs to SBV_{loc}.

Notation. We use $\mathbb{N} = \mathbb{N}_{>0} \cup \{0\}$. Let $\alpha = (\alpha_1, \ldots, \alpha_q) \in \mathbb{N}^q$ and $x = (x_1, \ldots, x_q) \in \mathbb{R}^q$. We write $\alpha! = \alpha_1! \cdots \alpha_q!$, $|\alpha| = \alpha_1 + \cdots + \alpha_q$, $x^\alpha = x_1^{\alpha_1} \cdots x_q^{\alpha_q}$, and $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_q^{\alpha_q}$. We shall also use $\partial_i = \partial/\partial x_i$. If $\alpha, \beta \in \mathbb{N}^q$, then $\alpha \le \beta$ means $\alpha_i \le \beta_i$ for all $1 \le i \le q$.

Let $U \subseteq \mathbb{R}^q$ be an open subset. For a function $f \in C^\infty(U)$ we denote by $\hat{f}_a \in \mathcal{F}_q$ its Taylor series at $a \in U$, i.e.,

$$\hat{f}_a(x) = \sum_{\alpha \in \mathbb{N}^q} \frac{1}{\alpha!} \partial^\alpha f(a) x^\alpha,$$

where $\mathcal{F}_q$ denotes the ring of power series in q variables.

S_n denotes the symmetric group on $\{1, 2, \ldots, n\}$.

We denote by $\mathcal{H}^q$ (resp. $\mathcal{L}^q$) the q-dimensional Hausdorff (resp. Lebesgue) measure. We also use $|X| = \mathcal{L}^q(X)$ and $\int_X f(x)dx = \int_X f(x) d\mathcal{L}^q(x)$. We write $\mathbf{1}_X$ for the indicator function of a set X. For $x \in \mathbb{R}^q$, $B_r(x) = \{y \in \mathbb{R}^q : |x - y| < r\}$ is the open ball with center x and radius r with respect to the Euclidean metric.

All manifolds in this paper are assumed to be Hausdorff, paracompact, and finite dimensional.

2. Preliminaries on polynomials

2.1. **Coefficients and roots.** Let

$$P(z) = z^n + \sum_{j=1}^n (-1)^j a_j z^{n-j} = \prod_{j=1}^n (z - \lambda_j)$$

be a univariate monic complex polynomial with coefficients $a_1, \ldots, a_n \in \mathbb{C}$ and roots $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. By Vieta's formulas, $a_i = \sigma_i(\lambda_1, \ldots, \lambda_n)$, where $\sigma_1, \ldots, \sigma_n$ denote the elementary symmetric functions in n variables:

$$\sigma_i(\lambda_1, \ldots, \lambda_n) = \sum_{1 \le j_1 < \cdots < j_i \le n} \lambda_{j_1} \cdots \lambda_{j_i}. \tag{2.1.1}$$

It is well-known that each symmetric polynomial in n variables can be written as a polynomial in $\sigma_1, \ldots, \sigma_n$, i.e., $\mathbb{C}[\lambda_1, \ldots, \lambda_n]^{\mathrm{S}_n} = \mathbb{C}[\sigma_1, \ldots, \sigma_n]$, where S_n denotes the symmetric group on $\{1, 2, \ldots, n\}$.

Denote by s_i (for $i \in \mathbb{N}$) the Newton polynomials

$$s_i(\lambda_1, \ldots, \lambda_n) = \sum_{j=1}^n \lambda_j^i \tag{2.1.2}$$

which are related to the elementary symmetric functions by

$$s_k - s_{k-1}\sigma_1 + s_{k-2}\sigma_2 - \cdots + (-1)^{k-1} s_1 \sigma_{k-1} + (-1)^k k \sigma_k = 0, \quad (k \ge 1). \tag{2.1.3}$$

These relations define a polynomial diffeomorphism Ψ^n such that:

$$\begin{aligned} \sigma^n &= (\sigma_1, \ldots, \sigma_n) : \mathbb{C}^n \to \mathbb{C}^n, \\ s^n &= (s_1, \ldots, s_n) : \mathbb{C}^n \to \mathbb{C}^n, \\ s^n &= \Psi^n \circ \sigma^n. \end{aligned}$$

Table 1: Let $P(x)(z) = z^n + \sum_{j=1}^n (-1)^j a_j(x) z^{n-j}$ be a family of polynomials with coefficients $a_j : \mathbb{R}^q \to \mathbb{C}$ (for $1 \le j \le n$). The table provides a (by no means exhaustive) summary of the most important results concerning the existence of parameterizations of the roots of P of some regularity, given that P fulfills certain conditions. The regularity of the roots is in general best possible under the respective conditions on P, which might partly not be optimal. 'Definable' refers to an arbitrary but fixed o-minimal expansion of the real field. By $\mathcal{C}$ we mean a class of C^∞ functions satisfying (3.1.1)–(3.1.6). For a definition of $\mathcal{W}^{\mathcal{C}}$ see 7.2. Normal nonflatness is introduced in 7.15. And s is maximal with the property that $\tilde{\Delta}_s(P) \neq 0$, where $\tilde{\Delta}_s(P)$ is given by (2.1.5).

Extra conditions	&	**Coefficients**	$\Longrightarrow$	**Roots**	**Reference**
$q = 1$		continuous		continuous	[Kat76, II 5.2]
$q = 1$		continuous & definable		AC_{loc} & definable	[Rai09b]
$q = 1$		C^∞ & normally nonflat		local desingularization by $x \mapsto \pm x^\gamma$ ($\gamma \in \mathbb{N}_{>0}$), AC_{loc} & no two distinct roots meet ∞-flat	[Rai09a]
$q = 1$ & $n = 2, 3, (4)$		C^∞		AC_{loc}	[Spa99]
		bounded		bounded	proposition 2.4
		continuous		continuous as a set, fulfill a Hölder condition of order $1/n$	[Ost40]
		$\mathcal{C}$ (resp. continuous & subanalytic)		local desingularization by finitely many local blow-ups with smooth center and local power substitutions (in the sense of 6.1), $\mathcal{W}^{\mathcal{C}}_{\text{loc}}$ & SBV_{loc}	theorem 6.7 (resp. theorem 10.3) theorems 7.11 & 8.4
hyperbolic & $q = 1$		C^ω (resp. $\mathcal{C}$)		C^ω (resp. $\mathcal{C}$)	[Rel37a] (resp. corollary 6.11)
hyperbolic & $q = 1$		C^∞ & normally nonflat		C^∞ & no two distinct roots meet ∞-flat	[AKLM98]
hyperbolic & $q = 1$		C^∞ & definable		C^∞ & definable	[Rai09b]
hyperbolic & $q = 1$		C^n (resp. C^{2n})		C^1 (resp. twice differentiable)	[Bro79], [Wak86], [Man85], [KLM04], & [COP08]
hyperbolic		continuous		continuous (e.g. by ordering them increasingly)	e.g. [AKLM98, 4.1]
hyperbolic		C^n		locally Lipschitz	[Bro79] & [Wak86] (also [KP08])
hyperbolic		C^ω (resp. $\mathcal{C}$, resp. arc-analytic & subanalytic)		local desingularization by finitely many local blow-ups with smooth center	[KP08] (resp. theorem 6.10, resp. remark 10.4)
hyperbolic & $\tilde{\Delta}_s(P)$ has only normal crossings		C^ω		locally C^ω	[KP08, 5.4]

It is easy to compute the Jacobian determinants $\det(ds^n(\lambda)) = n!\prod_{i<j}(\lambda_j - \lambda_i)$, $\det(d\Psi^n(\sigma^n)) = (-1)^{n(n-1)/2}n!$, and, hence,

$$\det(d\sigma^n(\lambda)) = \prod_{i<j}(\lambda_i - \lambda_j). \tag{2.1.4}$$

Let us consider the so-called Bezoutiant

$$B := \begin{pmatrix} s_0 & s_1 & \dots & s_{n-1} \\ s_1 & s_2 & \dots & s_n \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & \dots & s_{2n-2} \end{pmatrix} = (s_{i+j-2})_{1\le i,j\le n}\,.$$

Since the entries of B are symmetric polynomials in $\lambda_1, \dots, \lambda_n$, there exists a unique symmetric $n \times n$ matrix $\tilde{B}$ with $B = \tilde{B} \circ \sigma^n$.

Let B_k denote the minor formed by the first k rows and columns of B. Then it is easy to see that

(2.1.5)

$$\Delta_k(\lambda) := \det B_k(\lambda) = \sum_{i_1<i_2<\dots<i_k} (\lambda_{i_1} - \lambda_{i_2})^2 \cdots (\lambda_{i_1} - \lambda_{i_k})^2 \cdots (\lambda_{i_{k-1}} - \lambda_{i_k})^2.$$

In particular, $\Delta_1(\lambda) = s_0 = n$. Since the polynomials Δ_k are symmetric, we have $\Delta_k = \tilde{\Delta}_k \circ \sigma^n$ for unique polynomials $\tilde{\Delta}_k$. By (2.1.5), the number of distinct roots of P equals the maximal k such that $\tilde{\Delta}_k(P) \neq 0$. (Abusing notation we identify P with the n-tuple $(a_1, \dots, a_n)$ of its coefficients when convenient.)

2.2. **Theorem** (Sylvester's version of Sturm's theorem, e.g. [Pro78])**.** *Suppose that all coefficients of P are real. Then all roots of P are real if and only if the symmetric $n \times n$ matrix $\tilde{B}(P)$ is positive semidefinite. The rank of $\tilde{B}(P)$ equals the number of distinct roots of P and its signature equals the number of distinct real roots.*

2.3. **Hyperbolic polynomials.** If all roots λ_j (and thus all coefficients a_j) of P are real, we say that P is *hyperbolic*.

The space of all hyperbolic polynomials P of fixed degree n can be identified with the semialgebraic subset $\sigma^n(\mathbb{R}^n) \subseteq \mathbb{R}^n$. Its structure is described in theorem 2.2. If the roots are ordered increasingly, i.e.,

$$\lambda_1(P) \le \lambda_2(P) \le \dots \le \lambda_n(P), \qquad \text{for all } P \in \sigma^n(\mathbb{R}^n),$$

then each root $\lambda_i : \sigma^n(\mathbb{R}^n) \to \mathbb{R}$ (for $1 \le i \le n$) is continuous (e.g. [AKLM98, 4.1]).

Note that all roots of a hyperbolic polynomial P with $a_1 = a_2 = 0$ are equal to 0, since

$$\sum \lambda_i^2 = s_2(\lambda) = \sigma_1(\lambda)^2 - 2\sigma_2(\lambda) = a_1^2 - 2a_2.$$

Replacing the variable z by $z - a_1(P)/n$ transforms any polynomial P to another polynomial $\bar{P}$ with $a_1(\bar{P}) = 0$. If all roots of $\bar{P}$ coincide, they have to be equal to 0. We use that fact repeatedly.

2.4. **Proposition** (Bounded roots)**.** *Let (P_m) be a sequence of univariate monic polynomials over $\mathbb{C}$ with fixed degree n and bounded coefficients. If $(\lambda_m) \subseteq \mathbb{C}$ such that $P_m(\lambda_m) = 0$ for all m, then (λ_m) is bounded.*

Proof. If $a_{m,j}$ denote the coefficients of P_m, we find

$$|\lambda_m|^n \le \sum_{j=1}^{n} |a_{m,j}||\lambda_m|^{n-j}. \tag{2.4.1}$$

Suppose that (λ_m) is unbounded. Without loss we may assume that $0 < |\lambda_m| \nearrow \infty$. Dividing (2.4.1) by $|\lambda_m|^{n-1}$ yields a contradiction. $\square$

3. C^∞ CLASSES THAT ADMIT RESOLUTION OF SINGULARITIES

Following [BM04, Section 3] we discuss classes of smooth functions that admit resolution of singularities.

3.1. Classes $\mathcal{C}$ of C^∞-functions. Let us assume that for every open $U \subseteq \mathbb{R}^q$, $q \in \mathbb{N}$, we have a subalgebra $\mathcal{C}(U)$ of $C^\infty(U) = C^\infty(U, \mathbb{R})$. Resolution of singularities in $\mathcal{C}$ (see 5.3) requires only the following assumptions (3.1.1)–(3.1.6) on $\mathcal{C}(U)$, for any open $U \subseteq \mathbb{R}^q$.

(3.1.1) $\mathcal{P}(U) \subseteq \mathcal{C}(U)$, where $\mathcal{P}(U)$ denotes the algebra of restrictions to U of polynomial functions on $\mathbb{R}^q$.
(3.1.2) $\mathcal{C}$ *is closed under composition.* If $V \subseteq \mathbb{R}^p$ is open and $\varphi = (\varphi_1, \ldots, \varphi_p) : U \to V$ is a mapping with each $\varphi_i \in \mathcal{C}(U)$, then $f \circ \varphi \in \mathcal{C}(U)$, for all $f \in \mathcal{C}(V)$.

A mapping $\varphi : U \to V$ is called a $\mathcal{C}$*-mapping* if $f \circ \varphi \in \mathcal{C}(U)$, for every $f \in \mathcal{C}(V)$. It follows from (3.1.1) and (3.1.2) that $\varphi = (\varphi_1, \ldots, \varphi_p)$ is a $\mathcal{C}$-mapping if and only if $\varphi_i \in \mathcal{C}(U)$, for all $1 \le i \le p$.

(3.1.3) $\mathcal{C}$ *is closed under derivation.* If $f \in \mathcal{C}(U)$ and $1 \le i \le q$, then $\partial_i f \in \mathcal{C}(U)$.
(3.1.4) $\mathcal{C}$ *is quasianalytic.* If $f \in \mathcal{C}(U)$ and $\hat{f}_a = 0$, for $a \in U$, then f vanishes in a neighborhood of a.

Since $\{x : \hat{f}_x = 0\}$ is closed in U, (3.1.4) is equivalent to the following property: If U is connected, then, for each $a \in U$, the Taylor series homomorphism $\mathcal{C}(U) \to \mathcal{F}_q$, $f \mapsto \hat{f}_a$, is injective.

(3.1.5) $\mathcal{C}$ *is closed under division by a coordinate.* If $f \in \mathcal{C}(U)$ is identically 0 along a hyperplane $\{x : x_i = a_i\}$, i.e., $f(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_q) \equiv 0$, then $f(x) = (x_i - a_i)h(x)$, where $h \in \mathcal{C}(U)$.
(3.1.6) $\mathcal{C}$ *is closed under taking the inverse.* Let $\varphi : U \to V$ be a $\mathcal{C}$-mapping between open subsets U and V in $\mathbb{R}^q$. Let $a \in U$, $\varphi(a) = b$, and suppose that the Jacobian matrix $(\partial\varphi/\partial x)(a)$ is invertible. Then there exist neighborhoods U' of a, V' of b, and a $\mathcal{C}$-mapping $\psi : V' \to U'$ such that $\psi(b) = a$ and $\varphi \circ \psi = \mathrm{id}_{V'}$.

Property (3.1.6) is equivalent to the *implicit function theorem in* $\mathcal{C}$: Let $U \subseteq \mathbb{R}^q \times \mathbb{R}^p$ be open. Suppose that $f_1, \ldots, f_p \in \mathcal{C}(U)$, $(a, b) \in U$, $f(a, b) = 0$, and $(\partial f/\partial y)(a, b)$ is invertible, where $f = (f_1, \ldots, f_p)$. Then there is a neighborhood $V \times W$ of (a, b) in U and a $\mathcal{C}$-mapping $g : V \to W$ such that $g(a) = b$ and $f(x, g(x)) = 0$, for $x \in V$.

It follows from (3.1.6) that $\mathcal{C}$ *is closed under taking the reciprocal*: If $f \in \mathcal{C}(U)$ vanishes nowhere in U, then $1/f \in \mathcal{C}(U)$.

A complex valued function $f : U \to \mathbb{C}$ is said to be a $\mathcal{C}$*-function*, or to belong to $\mathcal{C}(U, \mathbb{C})$, if $(\mathrm{Re} f, \mathrm{Im} f) : U \to \mathbb{R}^2$ is a $\mathcal{C}$-mapping. It is immediately verified that (3.1.3)–(3.1.5) hold for complex valued functions $f \in \mathcal{C}(U, \mathbb{C})$ as well.

From now on, unless otherwise stated, $\mathcal{C}$ will denote a fixed, but arbitrary, class of C^∞-functions satisfying the conditions (3.1.1)–(3.1.6).

3.2. Lemma (Splitting lemma in $\mathcal{C}$, cf. [AKLM98, 3.4])**.** *Let $P_0 = z^n + \sum_{j=1}^n (-1)^j a_j z^{n-j}$ be a complex polynomial satisfying $P_0 = P_1 \cdot P_2$, where P_1 and P_2 are monic polynomials without common root. Then for P near P_0 we have $P = P_1(P) \cdot P_2(P)$ for $\mathcal{C}$-mappings of monic polynomials $P \mapsto P_1(P)$ and $P \mapsto P_2(P)$, defined for P near P_0, with the given initial values. (Here $P \mapsto P_i(P)$ is understood as a mapping $\mathbb{R}^{2n} \to \mathbb{R}^{2\deg P_i}$.)*

Proof. Let the polynomial P_0 be represented as the product

$$P_0 = P_1 \cdot P_2 = \Big(z^p + \sum_{j=1}^{p} (-1)^j b_j z^{p-j}\Big) \cdot \Big(z^q + \sum_{j=1}^{q} (-1)^j c_j z^{q-j}\Big),$$

where $p + q = n$. Let $\lambda_1, \dots, \lambda_n$ be the roots of P_0, ordered in such a way that the first p are the roots of P_1 and the last q are those of P_2. There is a polynomial mapping $\Phi^{p,q}$ such that $(a_1, \dots, a_n) = \Phi^{p,q}(b_1, \dots, b_p, c_1, \dots, c_q)$. Let $b = (b_1, \dots, b_p)$ and $c = (c_1, \dots, c_q)$. Then

$$\sigma^n = \Phi^{p,q} \circ (\sigma^p \times \sigma^q),$$
$$\det(d\sigma^n) = \det(d\Phi^{p,q}(b, c)) \det(d\sigma^p) \det(d\sigma^q),$$

and, by (2.1.4),

$$\det(d\Phi^{p,q}(b, c)) = \prod_{1 \le i \le p < j \le n} (\lambda_i - \lambda_j) \neq 0,$$

since P_1 and P_2 do not have common roots.

If we view $\Phi^{p,q}$ as a mapping $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$, then its Jacobian determinant at (b, c) is still not 0, by lemma 3.3 below. So, by (3.1.1) and (3.1.6), $\Phi^{p,q}$ is a $\mathcal{C}$-diffeomorphism near (b, c). □

3.3. **Lemma.** *Let $A = (A_{ij}) \in \mathbb{C}^{n \times n}$. Consider the block matrix $B = (B_{ij}) \in \mathbb{R}^{2n \times 2n}$, where*

$$B_{ij} = \begin{pmatrix} \operatorname{Re} A_{ij} & -\operatorname{Im} A_{ij} \\ \operatorname{Im} A_{ij} & \operatorname{Re} A_{ij} \end{pmatrix}, \quad (1 \le i, j \le n).$$

Then $\det_{\mathbb{R}} B = |\det_{\mathbb{C}} A|^2$. □

3.4. **$\mathcal{C}$-manifolds.** One can use the open subsets $U \subseteq \mathbb{R}^q$ and the algebras of functions $\mathcal{C}(U)$ as local models to define a category $\underline{\mathcal{C}}$ of *$\mathcal{C}$-manifolds* and *$\mathcal{C}$-mappings*. The dimension theory of $\underline{\mathcal{C}}$ follows from that of C^∞-manifolds.

The implicit function property (3.1.6) implies that a *smooth* (not singular) subset of a $\mathcal{C}$-manifold is a $\mathcal{C}$-submanifold:

3.5. **Proposition.** *Let M be a $\mathcal{C}$-manifold. Suppose that U is open in M, $g_1, \dots, g_p \in \mathcal{C}(U)$, and the gradients ∇g_i are linearly independent at every point of the zero set $X := \{x \in U : g_i(x) = 0$ for all $i\}$. Then X is a closed $\mathcal{C}$-submanifold of U of codimension p.* □

4. Quasianalytic Denjoy–Carleman classes

4.1. **Denjoy–Carleman classes.** See [Thi08] and references therein. Let $U \subseteq \mathbb{R}^q$ be open. Let $M = (M_k)_{k \in \mathbb{N}}$ be a non-decreasing sequence of real numbers with $M_0 = 1$. We denote by $C^M(U)$ the set of all $f \in C^\infty(U)$ such that for every compact $K \subseteq U$ there are constants $C, \rho > 0$ with

$$|\partial^\alpha f(x)| \le C \rho^{|\alpha|} |\alpha|!\, M_{|\alpha|} \quad \text{for all } \alpha \in \mathbb{N}^q \text{ and } x \in K. \tag{4.1.1}$$

We call $C^M(U)$ a *Denjoy–Carleman class* of functions on U. If $M_k = 1$, for all k, then $C^M(U)$ coincides with the ring $C^\omega(U)$ of real analytic functions on U. In general, $C^\omega(U) \subseteq C^M(U) \subseteq C^\infty(U)$. Hence $\mathcal{C} = C^M$ satisfies property (3.1.1).

We assume that $M = (M_k)$ is *logarithmically convex*, i.e.,

$$M_k^2 \le M_{k-1}\, M_{k+1} \quad \text{for all } k, \tag{4.1.2}$$

or, equivalently, M_{k+1}/M_k is increasing. Using $M_0 = 1$, we obtain that also $(M_k)^{1/k}$ is increasing and

$$M_l\, M_k \le M_{l+k} \quad \text{for all } l, k \in \mathbb{N}. \tag{4.1.3}$$

Hypothesis (4.1.2) implies that $C^M(U)$ is a ring, for all open subsets $U \subseteq \mathbb{R}^q$, which can easily be derived from (4.1.3) by means of Leibniz' rule. Note that definition (4.1.1) makes sense also for mappings $U \to \mathbb{R}^p$. For C^M-mappings, (4.1.2) guarantees stability under composition ([Rou63], [BM04, 4.7]). So $\mathcal{C} = C^M$ satisfies property (3.1.2).

A further consequence of (4.1.2) is the inverse function theorem for C^M ([Kom79], [BM04, 4.10]). Thus $\mathcal{C} = C^M$ satisfies property (3.1.6).

Suppose that $M = (M_k)$ and $N = (N_k)$ satisfy

$$\sup_{k \in \mathbb{N}_{>0}} \Big(\frac{M_k}{N_k}\Big)^{\frac{1}{k}} < \infty. \tag{4.1.4}$$

Then, evidently $C^M(U) \subseteq C^N(U)$. The converse is true as well: There exists $f \in C^M(\mathbb{R})$ such that $|f^{(k)}(0)| \ge k!\, M_k$ for all k (see [Thi08, Theorem 1]). So the inclusion $C^M(U) \subseteq C^N(U)$ implies (4.1.4).

Setting $N_k = 1$ in (4.1.4) yields that $C^\omega(U) = C^M(U)$ if and only if

$$\sup_{k \in \mathbb{N}_{>0}} (M_k)^{\frac{1}{k}} < \infty.$$

As $(M_k)^{1/k}$ is increasing (by (4.1.2)), the strict inclusion $C^\omega(U) \subsetneq C^M(U)$ is equivalent to

$$\lim_{k \to \infty} (M_k)^{\frac{1}{k}} = \infty.$$

The class $\mathcal{C} = C^M$ is stable under derivation (property (3.1.3)) if and only if

$$\sup_{k \in \mathbb{N}_{>0}} \Big(\frac{M_{k+1}}{M_k}\Big)^{\frac{1}{k}} < \infty. \tag{4.1.5}$$

The first order partial derivatives of elements in $C^M(U)$ belong to $C^{M^{+1}}(U)$, where M^{+1} denotes the shifted sequence $M^{+1} = (M_{k+1})_{k \in \mathbb{N}}$. So the equivalence follows from (4.1.4), by replacing M with M^{+1} and N with M.

By the standard integral formula, stability under derivation implies that $\mathcal{C} = C^M$ fulfills property (3.1.5).

4.2. **Quasianalyticity.** Suppose that M is logarithmically convex (actually, logarithmic convexity of $k!M_k$ suffices). Then, by the Denjoy–Carleman theorem ([Den21], [Car26]), $\mathcal{C} = C^M$ is quasianalytic (satisfies (3.1.4)) if and only if

$$\sum_{k=1}^{\infty} \frac{1}{(k!M_k)^{1/k}} = \infty \quad \text{or, equivalently,} \quad \sum_{k=0}^{\infty} \frac{M_k}{(k+1)M_{k+1}} = \infty. \tag{4.2.1}$$

For contemporary proofs see for instance [Hör83, 1.3.8] or [Rud87, 19.11].

4.3. **Proposition.** *If M is a non-decreasing sequence of real numbers with $M_0 = 1$ satisfying* (4.1.2)*,* (4.1.5)*, and* (4.2.1)*, then the Denjoy–Carleman class $\mathcal{C} = C^M$ has the properties* (3.1.1)*–*(3.1.6)*. If C^M is not closed under derivation (i.e.,* (4.1.5) *fails), then $\mathcal{C} = \bigcup_{j \in \mathbb{N}} C^{M^{+j}}$ has the properties* (3.1.1)*–*(3.1.6)*.* □

5. Resolution of singularities in $\mathcal{C}$

5.1. **Blow-ups.** Let M be a smooth manifold and let C be a smooth closed subset of M. The *blow-up* of M with *center* C is a proper smooth mapping $\varphi : M' \to M$ from a smooth manifold M' that can be described in local coordinates as follows.

Let $U \subseteq \mathbb{R}^q$ be an open neighborhood of 0 and let $C = \{x_i = 0 \text{ for } i \in I\}$ be a coordinate subspace, where I is a subset of $\{1, \ldots, q\}$. The *blow-up* $\varphi : U' \to U$

with *center* C is a mapping where U' can be covered by coordinate charts U'_i, for $i \in I$, and each U'_i has a coordinate system $y_1, \ldots, y_q$ in which φ is given by

$$x_j = \begin{cases} y_i, & \text{for } j = i \\ y_i y_j, & \text{for } j \in I \setminus \{i\} \\ y_j, & \text{for } j \notin I \end{cases}.$$

Assuming (without loss) $I = \{1, \ldots, p\}$ and $x = (\bar{x}, \tilde{x}) \in \mathbb{R}^p \times \mathbb{R}^{q-p}$, we have

$$U' \cong \{(x, \xi) \in U \times \mathbb{RP}^{p-1} : \bar{x} \in \xi\},$$

and, if we use homogeneous coordinates $\xi = [\xi_1, \ldots, \xi_p]$,

$$U' = \{(x, \xi) \in U \times \mathbb{RP}^{p-1} : x_i \xi_j = x_j \xi_i \text{ for } 1 \le i, j \le p\}.$$

We can cover U' by coordinate charts $U'_i = \{(x, \xi) \in U' : \xi_i \neq 0\}$, for $i \in I$, with coordinates $y_1, \ldots, y_q$ where

$$y_j = \begin{cases} x_i, & \text{for } j = i \\ \frac{\xi_j}{\xi_i}, & \text{for } j \in I \setminus \{i\} \\ x_j, & \text{for } j \notin I \end{cases}.$$

The *blow-up* of a smooth manifold M with *center* a smooth closed subset C is a smooth mapping $\varphi : M' \to M$, where M' is a smooth manifold, such that:

(1) Every point of C admits a coordinate neighborhood U in which C is a coordinate subspace and over U the mapping $\varphi : M' \to M$ identifies with the mapping $U' \to U$ from above.
(2) φ restricts to a diffeomorphism over $M \setminus C$.

These conditions determine $\varphi : M' \to M$ uniquely up to a diffeomorphism of M' commuting with φ. If $\operatorname{codim} C = 1$ then the blow-up φ is the identity.

If M is a $\mathcal{C}$-manifold and $\varphi : M' \to M$ is the blow-up with center a closed $\mathcal{C}$-submanifold C of M, then M' is a $\mathcal{C}$-manifold and φ is a $\mathcal{C}$-mapping (cf. [BM04, 3.9]):

5.2. **Proposition.** *The category $\underline{\mathcal{C}}$ of $\mathcal{C}$-manifolds and $\mathcal{C}$-mappings is closed under blowing up with center a closed $\mathcal{C}$-submanifold.* □

5.3. **Resolution of singularities.** We shall use a simple version of the desingularization theorem of Hironaka [Hir64] for $\mathcal{C}$-function classes due to Bierstone and Milman [BM04]. We use the terminology therein.

Let us regard a $\mathcal{C}$-manifold M as local-ringed space $(|M|, \mathcal{O}_M^{\mathcal{C}})$ with $|M|$ the underlying topological space of M and $\mathcal{O}_M^{\mathcal{C}}$ the sheaf of germs of $\mathcal{C}$-functions at points of M. Let $\mathcal{I} \subseteq \mathcal{O}_M^{\mathcal{C}}$ be a sheaf of ideals of finite type, i.e., for each $a \in M$, there is an open neighborhood U of a and finitely many sections $f_1, \ldots, f_p \in \mathcal{O}_M^{\mathcal{C}}(U) = \mathcal{C}(U)$ such that, for all $b \in U$, the stalk $\mathcal{I}_b$ is generated by the germs of the f_i at b. Put $|X| := \operatorname{supp} \mathcal{O}_M^{\mathcal{C}}/\mathcal{I}$ and $\mathcal{O}_X^{\mathcal{C}} := (\mathcal{O}_M^{\mathcal{C}}/\mathcal{I})|_{|X|}$. Then $X = (|X|, \mathcal{O}_X^{\mathcal{C}})$ is called a *closed $\mathcal{C}$-subspace* of M, and we write $\mathcal{I} = \mathcal{I}_X$. It is a *hypersurface* if $\mathcal{I}_X$ is a sheaf of principal ideals. A closed $\mathcal{C}$-subspace X is *smooth* at $a \in X$ if $\mathcal{I}_{X,a}$ is generated by elements with linearly independent gradients at a. By proposition 3.4, a smooth $\mathcal{C}$-subspace is a $\mathcal{C}$-submanifold.

Let $\varphi : N \to M$ be a $\mathcal{C}$-mapping of $\mathcal{C}$-manifolds. If $\mathcal{I} \subseteq \mathcal{O}_M^{\mathcal{C}}$ is a sheaf of ideals of finite type, we denote by $\varphi^{-1}(\mathcal{I}) \subseteq \mathcal{O}_N^{\mathcal{C}}$ the ideal sheaf $\varphi^*(\mathcal{I}) \cdot \mathcal{O}_N^{\mathcal{C}}$ whose stalk at each $b \in N$ is generated by the ring of pullbacks $\varphi^*(\mathcal{I})_b$ of all elements in $\mathcal{I}_{\varphi(b)}$. If X is a closed $\mathcal{C}$-subspace of M, let $\varphi^{-1}(X)$ denote the closed $\mathcal{C}$-subspace of N determined by $\varphi^{-1}(\mathcal{I}_X)$.

Let M be a $\mathcal{C}$-manifold, C a $\mathcal{C}$-submanifold of M, and let $\varphi : M' \to M$ be the blow-up of M with center C. Then $\varphi^{-1}(C)$ is a smooth closed subspace in M'. We denote by y_{exc} a generator of $\mathcal{I}_{\varphi^{-1}(C),a'}$, at any $a' \in M'$.

Let $X \subseteq M$ be a hypersurface. The *strict transform* X' *of* X *by* φ is the hypersurface of M' determined by $\mathcal{I}_{X'}$, where $\mathcal{I}_{X'} \subseteq \mathcal{O}^{\mathcal{C}}_{M'}$ is defined as follows: If $a' \in M'$, $a = \varphi(a')$, and g is a generator of $\mathcal{I}_{X,a}$, then $\mathcal{I}_{X',a'}$ is the ideal generated by $g' := y_{\text{exc}}^{-d} g \circ \varphi$, where d is the largest power of y_{exc} that factors from $g \circ \varphi$. (If $a' \notin \varphi^{-1}(C)$, then we may take $y_{\text{exc}} = 1$.) See [BM04, 5.6] and [BM97, Section 3] for the difference between *weak* and *strict* transform (and the problems with the latter in $\mathcal{C}$) if X is *not* a hypersurface.

We say that a hypersurface X has only *normal crossings*, if locally there exist suitable coordinates in which $\mathcal{I}_X$ is generated by a monomial.

5.4. **Theorem** ([BM04, 5.12])**.** *Let M be a $\mathcal{C}$-manifold, X a closed $\mathcal{C}$-hypersurface in M, and K a compact subset of M. Then, there is a neighborhood W of K and a surjective mapping $\varphi : W' \to W$ of class $\mathcal{C}$, such that:*

(1) *φ is a composite of finitely many $\mathcal{C}$-mappings, each of which is either a blow-up with smooth center (that is nowhere dense in the smooth points of the strict transform of X) or a surjection of the form $\bigsqcup_j U_j \to \bigcup_j U_j$, where the latter is a finite covering of the target space by coordinate charts.*
(2) *The final strict transform X' of X is smooth, and $\varphi^{-1}(X)$ has only normal crossings. (In fact $\varphi^{-1}(X)$ and $\det d\varphi$ simultaneously have only normal crossings, where $d\varphi$ is the Jacobian matrix of φ with respect to any local coordinate system.)*

See [BM04, 5.9 & 5.10] and [BM97] for stronger desingularization theorems in $\mathcal{C}$.

6. Quasianalytic perturbation of polynomials

We prove in this section that the roots of a $\mathcal{C}$-family of polynomials P can be parameterized locally by $\mathcal{C}$-functions after modifying P in a precise way.

6.1. **Local blow-ups and local power substitutions.** We introduce notation following [BM88, Section 4].

Let M be a $\mathcal{C}$-manifold. A family of $\mathcal{C}$-mappings $\{\pi_j : U_j \to M\}$ is called a *locally finite covering* of M if the images $\pi_j(U_j)$ are subordinate to a locally finite open covering $\{W_j\}$ of M (i.e. $\pi_j(U_j) \subseteq W_j$ for all j) and if, for each compact $K \subseteq M$, there are compact $K_j \subseteq U_j$ such that $K = \bigcup_j \pi_j(K_j)$ (the union is finite).

Locally finite coverings can be *composed* in the following way (see [BM88, 4.5]): Let $\{\pi_j : U_j \to M\}$ be a locally finite covering of M, and let $\{W_j\}$ be as above. For each j, suppose that $\{\pi_{ji} : U_{ji} \to U_j\}$ is a locally finite covering of U_j. We may assume without loss that the W_j are relatively compact. (Otherwise, choose a locally finite covering $\{V_j\}$ of M by relatively compact open subsets. Then the mappings $\pi_j|_{\pi_j^{-1}(V_i)} : \pi_j^{-1}(V_i) \to M$, for all i and j, form a locally finite covering of M.) Then, for each j, there is a finite subset $I(j)$ of $\{i\}$ such that the $\mathcal{C}$-mappings $\pi_j \circ \pi_{ji} : U_{ji} \to M$, for all j and all $i \in I(j)$, form a locally finite covering of M.

We shall say that $\{\pi_j\}$ is a *finite covering*, if j varies in a finite index set.

A *local blow-up* Φ over an open subset U of M means the composition $\Phi = \iota \circ \varphi$ of a blow-up $\varphi : U' \to U$ with smooth center and of the inclusion $\iota : U \to M$.

We denote by *local power substitution* a mapping of $\mathcal{C}$-manifolds $\Psi : V \to M$ of the form $\Psi = \iota \circ \psi$, where $\iota : W \to M$ is the inclusion of a coordinate chart W of M and $\psi : V \to W$ is given by

$$(y_1, \ldots, y_q) = \psi_{\gamma,\epsilon}(x_1, \ldots, x_q) := ((-1)^{\epsilon_1} x_1^{\gamma_1}, \ldots, (-1)^{\epsilon_q} x_q^{\gamma_q}), \tag{6.1.1}$$

for some $\gamma = (\gamma_1, \ldots, \gamma_q) \in (\mathbb{N}_{>0})^q$ and all $\epsilon = (\epsilon_1, \ldots, \epsilon_q) \in \{0,1\}^q$, where $y_1, \ldots, y_q$ denote the coordinates of W (and $q = \dim M$).

6.2. We consider the natural partial ordering of multi-indices: If $\alpha, \beta \in \mathbb{N}^q$, then $\alpha \le \beta$ means $\alpha_i \le \beta_i$ for all $1 \le i \le q$.

6.3. **Lemma** ([BM04, 7.7] or [BM88, 4.7])**.** *Let $\alpha, \beta, \gamma \in \mathbb{N}^q$ and let $a(x), b(x), c(x)$ be non-vanishing germs of real or complex valued functions of class $\mathcal{C}$ at the origin of $\mathbb{R}^q$. If*
$$x^\alpha a(x) - x^\beta b(x) = x^\gamma c(x),$$
then either $\alpha \le \beta$ or $\beta \le \alpha$.

Proof. Let $\delta = (\delta_1, \dots, \delta_q)$ where $\delta_k = \min\{\alpha_k, \beta_k\}$. If $\delta = \alpha$ then $\alpha \le \beta$. Otherwise, $\delta_k \ne \alpha_k$ for some k. On $\{x_k = 0\}$ we have $x^{\alpha-\delta} = 0$ and $0 \ne -x^{\beta-\delta} b(x) = x^{\gamma-\delta} c(x)$. Since b and c are non-vanishing, we obtain $\beta = \gamma$, by (3.1.5). So $x^\alpha a(x) = x^\beta (b(x) + c(x))$ and hence $\alpha \ge \beta$, again by (3.1.5). $\square$

6.4. Let M be a $\mathcal{C}$-manifold and let f be a real or complex valued $\mathcal{C}$-function on M. We say that f has only *normal crossings* if each point in M admits a coordinate neighborhood U with coordinates $x = (x_1, \dots, x_q)$ such that
$$f(x) = x^\alpha g(x), \quad x \in U,$$
where g is a non-vanishing $\mathcal{C}$-function on U, and $\alpha \in \mathbb{N}^q$. Observe that, if a product of functions has only normal crossings, then each factor has only normal crossings. For: Let f_1, f_2, g be $\mathcal{C}$-functions defined near $0 \in \mathbb{R}^q$ such that $f_1(x) f_2(x) = x^\alpha g(x)$ and g is non-vanishing. By quasianalyticity (3.1.4), $f_1 f_2|_{\{x_j=0\}} = 0$ implies $f_1|_{\{x_j=0\}} = 0$ or $f_2|_{\{x_j=0\}} = 0$. So the assertion follows from (3.1.5).

6.5. Let M be a $\mathcal{C}$-manifold, $K \subseteq M$ be compact, and $f \in \mathcal{C}(M, \mathbb{C})$. Then the exists a neighborhood W of K and a finite covering $\{\pi_k : U_k \to W\}$ of W by $\mathcal{C}$-mappings π_k, each of which is a composite of finitely many local blow-ups with smooth center, such that, for each k, the function $f \circ \pi_k$ has only normal crossings. This follows from theorem 5.4 applied to the real valued $\mathcal{C}$-function $|f|^2 = f\overline{f}$ and the observation in 6.4.

6.6. **Reduction to smaller permutation groups.** In the proof of theorem 6.7 we shall reduce our perturbation problem in virtue of the splitting lemma 3.2:

The space Pol^n of polynomials $P(z) = z^n + \sum_{j=1}^n (-1)^j a_j z^{n-j}$ of fixed degree n naturally identifies with $\mathbb{C}^n$ (by mapping P to $(a_1, \dots, a_n)$). Moreover, Pol^n may be viewed as the orbit space $\mathbb{C}^n/\mathrm{S}_n$ with respect to the standard action of the symmetric group S_n on $\mathbb{C}^n$ by permuting the coordinates (the roots of P). In this picture the mapping $\sigma^n : \mathbb{C}^n \to \mathbb{C}^n$ identifies with the orbit projection $\mathbb{C}^n \to \mathbb{C}^n/\mathrm{S}_n$, since the elementary symmetric functions σ_i in (2.1.1) generate the algebra of symmetric polynomials on $\mathbb{C}^n$, i.e., $\mathbb{C}[\mathbb{C}^n]^{\mathrm{S}_n} = \mathbb{C}[\sigma_1, \dots, \sigma_n]$.

Consider a family of polynomials
$$P(x)(z) = z^n + \sum_{j=1}^n (-1)^j a_j(x) z^{n-j},$$
where the coefficients a_j are complex valued $\mathcal{C}$-functions defined in a $\mathcal{C}$-manifold M. Let $x_0 \in M$. If $P(x_0)$ has distinct roots $\nu_1, \dots, \nu_l$, the splitting lemma 3.2 provides a $\mathcal{C}$-factorization $P(x) = P_1(x) \cdots P_l(x)$ near x_0 such that no two factors have common roots and all roots of $P_h(x_0)$ are equal to ν_h, for $1 \le h \le l$. This factorization amounts to a reduction of the S_n-action on $\mathbb{C}^n$ to the $\mathrm{S}_{n_1} \times \cdots \times \mathrm{S}_{n_l}$-action on $\mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_l}$, where n_h is the multiplicity of ν_h.

We shall use the following notation:
$$\mathrm{S}(P(x_0)) := \mathrm{S}_{n_1} \times \cdots \times \mathrm{S}_{n_l},$$
iff $P(x_0)$ has l pairwise distinct roots with respective multiplicities $n_1, \dots, n_l$.

Further, we will *remove fixed points* of the $\mathrm{S}_{n_1} \times \cdots \times \mathrm{S}_{n_l}$-action on $\mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_l}$ or, equivalently, reduce each factor $P_h(x)(z) = z^{n_h} + \sum_{j=1}^{n_h}(-1)^j a_{h,j}(x) z^{n_h-j}$ to the case $a_{h,1} = 0$ by replacing z by $z - a_{h,1}(x)/n_h$. The effect on the roots of P_h is a shift by a $\mathcal{C}$-function.

If P is hyperbolic, we consider the S_n-module $\mathbb{R}^n$ instead of $\mathbb{C}^n$. In that case the orbit space $\mathbb{R}^n/\mathrm{S}_n$ identifies with the semialgebraic subset $\sigma^n(\mathbb{R}^n) \subseteq \mathbb{R}^n$, whose structure is described in theorem 2.1. Evidently, the splitting lemma 3.2 produces a $\mathcal{C}$-factorization $P = P_1 \cdots P_l$, where each factor P_h is hyperbolic again.

6.7. **Theorem** ($\mathcal{C}$-perturbation of polynomials)**.** *Let M be a $\mathcal{C}$-manifold. Consider a family of polynomials*

$$P(x)(z) = z^n + \sum_{j=1}^{n}(-1)^j a_j(x) z^{n-j},$$

with coefficients a_j (for $1 \le j \le n$) in $\mathcal{C}(M, \mathbb{C})$. Let $K \subseteq M$ be compact. Then there exist:

(1) *a neighborhood W of K, and*
(2) *a finite covering $\{\pi_k : U_k \to W\}$ of W, where each π_k is a composite of finitely many mappings each of which is either a local blow-up with smooth center or a local power substitution (in the sense of 6.1),*

such that, for all k, the family of polynomials $P \circ \pi_k$ allows a $\mathcal{C}$-parameterization of its roots on U_k, i.e., there exist $\lambda_i^k \in \mathcal{C}(U_k, \mathbb{C})$ (for $1 \le i \le n$) such that

$$P(\pi_k(x))(z) = z^n + \sum_{j=1}^{n}(-1)^j a_j(\pi_k(x)) z^{n-j} = \prod_{i=1}^{n}(z - \lambda_i^k(x)).$$

Proof. Since the statement is local, we may assume without loss that M is an open neighborhood of $0 \in \mathbb{R}^q$. In view of 6.6, we use induction on $|\,\mathrm{S}(P(0))|$, the order of the permutation group acting on the roots of $P(0)$.

If $|\,\mathrm{S}(P(0))| = 1$, all roots of $P(0)$ are pairwise different. Then the roots of P may be parameterized in a $\mathcal{C}$-way near 0, by the implicit function theorem (property (3.1.6)) or by the splitting lemma 3.2.

Suppose that $|\,\mathrm{S}(P(0))| > 1$. Let $\nu_1, \ldots, \nu_l$ denote the distinct roots of $P(0)$; some of them are multiple ($l = 1$ is allowed). The splitting lemma 3.2 provides a $\mathcal{C}$-factorization $P(x) = P_1(x) \cdots P_l(x)$ near 0, where, for $1 \le h \le l$,

$$P_h(x)(z) = z^{n_h} + \sum_{j=1}^{n_h}(-1)^j a_{h,j}(x) z^{n_h-j},$$

such that no two factors have common roots and all roots of $P_h(0)$ are equal to ν_h. As indicated in 6.6, we reduce to the $\mathrm{S}_{n_1} \times \cdots \times \mathrm{S}_{n_l}$-action on $\mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_l}$ and we remove fixed points. So we may assume that $a_{h,1} = 0$ for all h.

Then all roots of $P_h(0)$ are equal to 0, and so $a_{h,j}(0) = 0$, for all $1 \le h \le l$ and $1 \le j \le n_h$ (by Vieta's formulas). If all coefficients $a_{h,j}$ (for $1 \le j \le n_h$) of P_h are identically 0, we choose its roots $\lambda_{h,j} = 0$ for all $1 \le j \le n_h$ and remove the factor P_h from the product $P_1 \cdots P_l$. So we can assume that for each $1 \le h \le l$ there is a $2 \le j \le n_h$ such that $a_{h,j} \ne 0$.

Let us define the $\mathcal{C}$-functions

$$A_{h,j}(x) = a_{h,j}(x)^{\frac{n!}{j}}, \quad (\text{for } 1 \le h \le l \text{ and } 2 \le j \le n_h). \tag{6.7.1}$$

By theorem 5.4 (and 6.5), we find a finite covering $\{\pi_k : U_k \to U\}$ of a neighborhood U of 0 by $\mathcal{C}$-mappings π_k, each of which is a composite of finitely many local blow-ups with smooth center, such that, for each k, the non-zero $A_{h,j} \circ \pi_k$ (for $1 \le h \le l$ and $2 \le j \le n_h$) and its pairwise non-zero differences $A_{h,i} \circ \pi_k - A_{m,j} \circ \pi_k$ (for

$1 \le h \le m \le l$, $1 \le i \le n_h$, and $1 \le j \le n_m$) simultaneously have only normal crossings.

Let k be fixed and let $x_0 \in U_k$. Then x_0 admits a neighborhood W_k with suitable coordinates in which $x_0 = 0$ and such that (for $1 \le h \le l$ and $2 \le j \le n_h$) either $A_{h,j} \circ \pi_k = 0$ or

$$(A_{h,j} \circ \pi_k)(x) = x^{\alpha_{h,j}} A^k_{h,j}(x),$$

where $A^k_{h,j}$ is a non-vanishing $\mathcal{C}$-function on W_k, and $\alpha_{h,j} \in \mathbb{N}^q$. The collection of the multi-indices $\{\alpha_{h,j} : A_{h,j} \circ \pi_k \ne 0, 1 \le h \le l, 2 \le j \le n_h\}$ is totally ordered, by lemma 6.3. Let α denote its minimum.

If $\alpha = 0$, then $(A_{h,j} \circ \pi_k)(x_0) = A^k_{h,j}(x_0) \ne 0$ for some $1 \le h \le l$ and $2 \le j \le n_h$. So, by (6.7.1), not all roots of $(P_h \circ \pi_k)(x_0)$ coincide (since $a_{h,1} \circ \pi_k = 0$). Thus, $|\operatorname{S}((P \circ \pi_k)(x_0))| < |\operatorname{S}(P(0))|$, and, by the induction hypothesis, there exists a finite covering $\{\pi_{kl} : W_{kl} \to W_k\}$ of W_k (possibly shrinking W_k) of the type described in (2) such that, for all l, the family of polynomials $P \circ \pi_k \circ \pi_{kl}$ allows a $\mathcal{C}$-parameterization of its roots on W_{kl}.

Let us assume that $\alpha \ne 0$. Then there exist $\mathcal{C}$-functions $\tilde{A}^k_{h,j}$ (some of them 0) such that, for all $1 \le h \le l$ and $2 \le j \le n_h$,

$$(A_{h,j} \circ \pi_k)(x) = x^{\alpha} \tilde{A}^k_{h,j}(x), \tag{6.7.2}$$

and $\tilde{A}^k_{h,j}(x_0) = A^k_{h,j}(x_0) \ne 0$ for some $1 \le h \le l$ and $2 \le j \le n_h$. Let us write

$$\frac{\alpha}{n!} = \Big(\frac{\alpha_1}{n!}, \ldots, \frac{\alpha_q}{n!}\Big) = \Big(\frac{\beta_1}{\gamma_1}, \ldots, \frac{\beta_q}{\gamma_q}\Big),$$

where $\beta_i, \gamma_i \in \mathbb{N}$ are relatively prime (and $\gamma_i > 0$), for all $1 \le i \le q$. Put $\beta = (\beta_1, \ldots, \beta_q)$ and $\gamma = (\gamma_1, \ldots, \gamma_q)$. Then (by (6.7.1) and (6.7.2)), for each $1 \le h \le l$, $2 \le j \le n_h$, and $\epsilon \in \{0,1\}^q$, the $\mathcal{C}$-function $a_{h,j} \circ \pi_k \circ \psi_{\gamma,\epsilon}$ is divisible by $x^{j\beta}$ (where $\psi_{\gamma,\epsilon}$ is defined by (6.1.1)). By (3.1.5), there exist $\mathcal{C}$-functions $a^{k,\gamma,\epsilon}_{h,j}$ such that

$$(a_{h,j} \circ \pi_k \circ \psi_{\gamma,\epsilon})(x) = x^{j\beta} a^{k,\gamma,\epsilon}_{h,j}(x), \quad (\text{for } 1 \le h \le l \text{ and } 2 \le j \le n_h).$$

By construction, for some $1 \le h \le l$ and $2 \le j \le n_h$, we have $a^{k,\gamma,\epsilon}_{h,j}(0) \ne 0$, independently of ϵ. So there exist a local power substitution $\psi_k : V_k \to W_k$ given in local coordinates by $\psi_{\gamma,\epsilon}$ (for $\epsilon \in \{0,1\}^q$) and functions $a^k_{h,j}$ given in local coordinates by $a^{k,\gamma,\epsilon}_{h,j}$ (for $\epsilon \in \{0,1\}^q$) such that

$$(a_{h,j} \circ \pi_k \circ \psi_k)(x) = x^{j\beta} a^k_{h,j}(x), \quad (\text{for } 1 \le h \le l \text{ and } 2 \le j \le n_h).$$

Let us consider the $\mathcal{C}$-family of polynomials $P^k := P^k_1 \cdots P^k_l$, where

$$P^k_h(x)(z) := z^{n_h} + \sum_{j=2}^{n_h} (-1)^j a^k_{h,j}(x) z^{n_h - j}.$$

Let $y_0 := \psi_k^{-1}(x_0) \in V_k$. There exist $1 \le h \le l$ and $2 \le j \le n_h$ such that $a^k_{h,j}(y_0) \ne 0$, and, thus (as $a^k_{h,1} = 0$), not all roots of $P^k_h(y_0)$ coincide. Therefore, $|\operatorname{S}(P^k(y_0))| < |\operatorname{S}(P(0))|$, and, by the induction hypothesis, there exists a finite covering $\{\pi_{kl} : V_{kl} \to V_k\}$ of V_k (possibly shrinking V_k) of the type described in (2) such that, for all l, the family of polynomials $P^k \circ \pi_{kl}$ admits a $\mathcal{C}$-parameterization λ^{kl}_j (for $1 \le j \le n$) of its roots on V_{kl}. Since the roots of P^k and $P \circ \pi_k \circ \psi_k$ differ by the monomial factor $m(x) := x^{\beta}$, the $\mathcal{C}$-functions $x \mapsto m(\pi_{kl}(x)) \cdot \lambda^{kl}_j(x)$ form a choice of the roots of the family $x \mapsto (P \circ \pi_k \circ \psi_k \circ \pi_{kl})(x)$ for $x \in V_{kl}$.

Since k and x_0 were arbitrary, the assertion of the theorem follows (by 6.1). $\square$

6.8. **Hyperbolic version.** If P is hyperbolic, no local power substitutions are needed, see theorem 6.10.

6.9. **Lemma.** *Let $U \subseteq \mathbb{R}^q$ be an open neighborhood of 0. Consider a family of hyperbolic polynomials*

$$P(x)(z) = z^n + \sum_{j=2}^{n} (-1)^j a_j(x) z^{n-j},$$

with coefficients a_j (for $2 \le j \le n$) in $\mathcal{C}(U, \mathbb{R})$. Assume that $a_2 \ne 0$ and that, for all j, $a_j \ne 0$ implies $a_j(x) = x^{\alpha_j} b_j(x)$, where $b_j \in \mathcal{C}(U, \mathbb{R})$ is non-vanishing, and $\alpha_j \in \mathbb{N}^q$. Then there exists a $\delta \in \mathbb{N}^q$ such that $\alpha_2 = 2\delta$ and $\alpha_j \ge j\delta$, for those j with $a_j \ne 0$.

Proof. Since $0 \le \tilde{\Delta}_2(P) = -2na_2$ (by theorem 2.2), we have $\alpha_2 = 2\delta$ for some $\delta \in \mathbb{N}^q$. If $\delta = 0$, the assertion is trivial. Let as assume that $\delta \ne 0$.

Set $\mu = (\mu_1, \ldots, \mu_q)$, where

$$\mu_i := \min \Big\{ \frac{(\alpha_j)_i}{j} : a_j \ne 0 \Big\}. \tag{6.9.1}$$

For contradiction, assume that there is an i_0 such that $\mu_{i_0} < \delta_{i_0}$. Consider

$$\tilde{P}(x)(z) := z^n + \sum_{j=2}^{n} (-1)^j x^{-j\mu} a_j(x) z^{n-j}.$$

If all $x_i \ge 0$, then $\tilde{P}$ is continuous (by (6.9.1)), and if all $x_i > 0$, then $\tilde{P}$ is hyperbolic (its roots differ from those of P by the factor $x^{-\mu}$). Since the space of hyperbolic polynomials of fixed degree is closed (by theorem 2.2), $\tilde{P}$ is hyperbolic, if all $x_i \ge 0$. Since $(\alpha_2)_{i_0} - 2\mu_{i_0} = 2\delta_{i_0} - 2\mu_{i_0} > 0$, all roots (and thus all coefficients) of $\tilde{P}(x)$ vanish on $\{x_{i_0} = 0\}$ (as the first and second coefficient vanish, see 2.3). This is a contradiction for those j with $(\alpha_j)_{i_0} = j\mu_{i_0}$. □

6.10. **Theorem** ($\mathcal{C}$-perturbation of hyperbolic polynomials)**.** *Let M be a $\mathcal{C}$-manifold. Consider a family of hyperbolic polynomials*

$$P(x)(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j(x) z^{n-j},$$

with coefficients a_j (for $1 \le j \le n$) in $\mathcal{C}(M, \mathbb{R})$. Let $K \subseteq M$ be compact. Then there exist:

1. *a neighborhood W of K, and*
2. *a finite covering $\{\pi_k : U_k \to W\}$ of W, where each π_k is a composite of finitely many local blow-ups with smooth center,*

such that, for all k, the family of polynomials $P \circ \pi_k$ allows a $\mathcal{C}$-parameterization of its roots on U_k.

Proof. It suffices to modify the proof in 6.7 such that no local power substitution is needed. Suppose we have reduced the problem in virtue of 6.6.

So $a_{h,j}(0) = 0$ for all $1 \le h \le l$ and $1 \le j \le n_h$. Since $a_{h,1} = 0$, we can assume that $a_{h,2} \ne 0$ for all h (otherwise all roots of P_h are identically 0, see 2.3). By theorem 5.4, we find a finite covering $\{\pi_k : U_k \to U\}$ of a neighborhood U of 0 by $\mathcal{C}$-mappings π_k, each of which is a composite of finitely many local blow-ups with smooth center, such that, for each k, the non-zero $a_{h,j} \circ \pi_k$ (for $1 \le h \le l$ and $2 \le j \le n_h$) simultaneously have only normal crossings.

Let k be fixed and let $x_0 \in U_k$. Then x_0 admits a neighborhood W_k with suitable coordinates in which $x_0 = 0$ and such that (for $1 \le h \le l$ and $2 \le j \le n_h$) either $a_{h,j} \circ \pi_k = 0$ or

$$(a_{h,j} \circ \pi_k)(x) = x^{\alpha_{h,j}} a^k_{h,j}(x), \tag{6.10.1}$$

where $a^k_{h,j}$ is a non-vanishing $\mathcal{C}$-function on W_k, and $\alpha_{h,j} \in \mathbb{N}^q$. By lemma 6.9, for each h, there exists a $\delta_h \in \mathbb{N}^q$ such that $\alpha_{h,2} = 2\delta_h$.

If some $\delta_h = 0$, then $(a_{h,2} \circ \pi_k)(x_0) = a^k_{h,2}(x_0) \neq 0$ and so not all roots of $(P_h \circ \pi_k)(x_0)$ coincide. Thus, $|\operatorname{S}((P \circ \pi_k)(x_0))| < |\operatorname{S}(P(0))|$, and, by the induction hypothesis, there exists a finite covering $\{\pi_{kl} : W_{kl} \to W_k\}$ of W_k (possibly shrinking W_k) of the type described in (2) such that, for all l, the family of polynomials $P \circ \pi_k \circ \pi_{kl}$ allows a $\mathcal{C}$-parameterization of its roots on W_{kl}.

Let us assume that $\delta_h \neq 0$ for all $1 \le h \le l$. By lemma 6.9, we have $\alpha_{h,j} \ge j\delta_h$, for all $1 \le h \le l$ and those $2 \le j \le n_h$ with $a_{h,j} \circ \pi_k \neq 0$. Then

$$P^k_h(x)(z) := z^{n_h} + \sum_{j=2}^{n_h} (-1)^j x^{-j\delta_h} a_{h,j}(\pi_k(x)) z^{n_h - j}$$

is a $\mathcal{C}$-family of hyperbolic polynomials. Since $\alpha_{h,2} = 2\delta_h$ and $a^k_{h,2}(x_0) \neq 0$, not all roots of $P^k_h(x_0)$ coincide. Put $P^k := P^k_1 \cdots P^k_l$. Then, $|\operatorname{S}(P^k(x_0))| < |\operatorname{S}(P(0))|$, and, by the induction hypothesis, there exists a finite covering $\{\pi_{kl} : W_{kl} \to W_k\}$ of W_k (possibly shrinking W_k) of the type described in (2) such that, for all l, the family of polynomials $P^k \circ \pi_{kl}$ admits a $\mathcal{C}$-parameterization $\lambda^{kl}_{h,j}$ (for $1 \le h \le l$ and $1 \le j \le n_h$) of its roots on W_{kl}. Since the roots of P^k_h and $P_h \circ \pi_k$ differ by the monomial factor $m_h(x) := x^{\delta_h}$, the $\mathcal{C}$-functions $x \mapsto m_h(\pi_{kl}(x)) \cdot \lambda^{kl}_{h,j}(x)$ form a choice of the roots of the family $x \mapsto (P \circ \pi_k \circ \pi_{kl})(x)$ for $x \in W_{kl}$.

Since k and x_0 were arbitrary, the assertion of the theorem follows (by 6.1). $\square$

If the parameter space is one dimensional, we obtain a $\mathcal{C}$-version of Rellich's classical theorem [Rel37a, Hilfssatz 2] (see also [AKLM98, 5.1]):

6.11. **Corollary.** *Let $I \subseteq \mathbb{R}$ be an open interval. Consider a curve of hyperbolic polynomials*

$$P(x)(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j(x) z^{n-j},$$

with coefficients a_j (for $1 \le j \le n$) in $\mathcal{C}(I, \mathbb{R})$. Then there exists a global parameterization $\lambda_j \in \mathcal{C}(I, \mathbb{R})$ (for $1 \le j \le n$) of the roots of P.

Proof. The local statement follows immediately from theorem 6.10. (Each local blow-up is the identity map, and, in fact, each non-zero a_j automatically has only normal crossings.) We claim that a local choice of $\mathcal{C}$-roots is unique up to permutations. In view of this uniqueness property we may glue the local parameterizations of the roots of P to a global one.

For the proof of the claim let $\lambda^i = (\lambda^i_1, \ldots, \lambda^i_n) : J \to \mathbb{R}^n$ (for $i = 1, 2$) be two local $\mathcal{C}$-parameterizations of the roots of P. Let $x_k \to x_\infty \in J$ be a sequence converging in J. For each k there exists a permutation $\tau_k \in \operatorname{S}_n$ such that $\lambda^1(x_k) = \tau_k(\lambda^2(x_k))$. Passing to a subsequence, we may assume that $\lambda^1(x_k) = \tau(\lambda^2(x_k))$ for all k and a fixed $\tau \in \operatorname{S}_n$. By Rolle's theorem (applied repeatedly), the Taylor series at x_∞ of λ^1 and $\tau \circ \lambda^2$ coincide. Quasianalyticity (3.1.4) implies that $\lambda^1 = \tau \circ \lambda^2$. $\square$

6.12. **Real analytic perturbation of polynomials.** If $\mathcal{C} = C^\omega$, theorem 6.7 can be strengthened.

6.13. **Theorem** (C^ω-perturbation of polynomials). *Let M be a real analytic manifold. Consider a family of polynomials*

$$P(x)(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j(x) z^{n-j},$$

with coefficients a_j (for $1 \le j \le n$) in $C^\omega(M, \mathbb{C})$. Let $K \subseteq M$ be compact. Then there exist:

(1) *a neighborhood W of K,*
(2) *a finite covering $\{\pi_k : U_k \to W\}$ of W, where each π_k is a composite of finitely many local blow-ups with smooth center,*
(3) *a finite covering $\{\pi_{kl} : U_{kl} \to U_k\}$ of each U_k, where each π_{kl} is a single local power substitution.*

such that, for all k, l, the family of polynomials $P \circ \pi_k \circ \pi_{kl}$ allows a real analytic parameterization of its roots on U_{kl}.

Proof. Applying resolution of singularities (e.g. Hironaka's classical theorem [Hir64], or theorem 5.4 for $\mathcal{C} = C^\omega$), we obtain that $\tilde{\Delta}_s(P \circ \pi_k)$ has only normal crossings, where s is maximal with the property that $\tilde{\Delta}_s(P) \neq 0$ (locally). Note that $\tilde{\Delta}_s(P)$ is up to a constant factor the discriminant of the square-free reduction of P. Then the assertion follows from the Abhyankar–Jung theorem [Jun08], [Abh55] (see also [KV04], [Sus90, Section 5], and [Par94b, Lemma 2.8]). Here we used that the square-free reduction of a real analytic family of polynomials is real analytic again (see [KP08, 5.1]). □

6.14. **Remarks.** (1) Note that the hyperbolic version of this theorem, where no local power substitutions are needed, is due to Kurdyka and Paunescu [KP08, 5.8].

(2) It is unclear to me how to prove this stronger version of theorem 6.7 for arbitrary $\mathcal{C}$ (satisfying (3.1.1)–(3.1.6)). It seems that one can produce a proof of a $\mathcal{C}$-version of the Abhyankar–Jung theorem along the lines of Luengo's approach [Lue83]. Unfortunately, the proof in [Lue83] contains a gap as pointed out by Kiyek and Vicente [KV04].

(3) Compare this theorem with Parusinski's preparation theorem for subanalytic functions [Par94a, 7.5].

7. Roots with gradients in L^1_{loc}

Let M be a $\mathcal{C}$-manifold of dimension q equipped with a C^∞ Riemannian metric. Consider a family of polynomials

$$P(x)(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j(x) z^{n-j},$$

with coefficients a_j (for $1 \le j \le n$) in $\mathcal{C}(M, \mathbb{C})$. We show in this section that the roots of P admit a parameterization by "piecewise Sobolev $W^{1,1}_{\text{loc}}$" functions λ_i (for $1 \le i \le n$). That means, there exists a closed nullset $E \subseteq M$ of finite $(q-1)$-dimensional Hausdorff measure such that each λ_i belongs to $W^{1,1}(K \setminus E)$ for all compact subsets $K \subseteq M$. In particular, the classical derivative $\nabla \lambda_i$ exists almost everywhere and belongs to L^1_{loc}. The distributional derivatives of the λ_i may not be locally integrable. In fact, P does in general not allow roots in $W^{1,1}_{\text{loc}}$ (by example 7.17).

7.1. We denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure. It depends on the metric but not on the ambient space. Recall that for a Lipschitz mapping $f : U \to \mathbb{R}^p$, $U \subseteq \mathbb{R}^q$, we have

$$\mathcal{H}^k(f(E)) \le \big(\operatorname{Lip}(f)\big)^k \mathcal{H}^k(E), \quad \text{for all } E \subseteq U, \tag{7.1.1}$$

where $\operatorname{Lip}(f)$ denotes the Lipschitz constant of f. The q-dimensional Hausdorff measure $\mathcal{H}^q$ and the q-dimensional Lebesgue measure $\mathcal{L}^q$ coincide in $\mathbb{R}^q$. If B is a subset of a k-plane in $\mathbb{R}^q$ then $\mathcal{H}^k(B) = \mathcal{L}^k(B)$.

7.2. **The class $\mathcal{W}^{\mathcal{C}}$.** Let M be a $\mathcal{C}$-manifold of dimension q equipped with a C^∞ Riemannian metric g. We denote by $\mathcal{W}^{\mathcal{C}}(M)$ the class of all real or complex valued functions f with the following properties:

($\mathcal{W}_1$) f is defined and of class $\mathcal{C}$ on the complement $M \setminus E_{M,f}$ of a closed set $E_{M,f}$ with $\mathcal{H}^q(E_{M,f}) = 0$ and $\mathcal{H}^{q-1}(E_{M,f}) < \infty$.
($\mathcal{W}_2$) f is bounded on $M \setminus E_{M,f}$.
($\mathcal{W}_3$) ∇f belongs to $L^1(M \setminus E_{M,f}) = L^1(M)$.

For example, the Heaviside function belongs to $\mathcal{W}^{\mathcal{C}}((-1,1))$, but the function $f(x) := \sin 1/|x|$ does not. A $\mathcal{W}^{\mathcal{C}}$-function f may or may not be defined on $E_{M,f}$. Note that, if the volume of M is finite, then

$$f \in \mathcal{W}^{\mathcal{C}}(M) \Longrightarrow f \in L^\infty(M \setminus E_{M,f}) \cap W^{1,1}(M \setminus E_{M,f}). \tag{7.2.1}$$

We shall also use the notations $\mathcal{W}^{\mathcal{C}}_{\text{loc}}(M)$ and $\mathcal{W}^{\mathcal{C}}(M, \mathbb{C}^n) = (\mathcal{W}^{\mathcal{C}}(M,\mathbb{C}))^n$ with the obvious meanings.

In general $\mathcal{W}^{\mathcal{C}}(M)$ depends on the Riemannian metric g. It is easy to see that $\mathcal{W}^{\mathcal{C}}(U)$ is independent of g for any relatively compact open subset $U \subseteq M$. Thus also $\mathcal{W}^{\mathcal{C}}_{\text{loc}}(M)$ is independent of g. If (U,u) is a relatively compact coordinate chart and g^u_{ij} is the coordinate expression of g, then there exists a constant C such that $(1/C)\delta_{ij} \le g^u_{ij} \le C\delta_{ij}$ as bilinear forms.

From now on, given a $\mathcal{C}$-manifold M, we tacitly choose a C^∞ Riemannian metric g on M and consider $\mathcal{W}^{\mathcal{C}}(M)$ with respect to g.

7.3. Let $\rho = (\rho_1, \ldots, \rho_q) \in (\mathbb{R}_{>0})^q$, $\gamma = (\gamma_1, \ldots, \gamma_q) \in (\mathbb{N}_{>0})^q$, and $\epsilon = (\epsilon_1, \ldots, \epsilon_q) \in \{0,1\}^q$. Set

$$\begin{aligned} \Omega(\rho) &:= \{x \in \mathbb{R}^q : |x_j| < \rho_j \text{ for all } j\}, \\ \Omega_\epsilon(\rho) &:= \{x \in \mathbb{R}^q : 0 < (-1)^{\epsilon_j} x_j < \rho_j \text{ for all } j\}. \end{aligned}$$

Then $\Omega(\rho) \setminus \{\prod_j x_j = 0\} = \bigsqcup \{\Omega_\epsilon(\rho) : \epsilon \in \{0,1\}^q\}$. The power transformation

$$\psi_{\gamma,\epsilon} : \mathbb{R}^q \to \mathbb{R}^q : (x_1, \ldots, x_q) \mapsto ((-1)^{\epsilon_1} x_1^{\gamma_1}, \ldots, (-1)^{\epsilon_q} x_q^{\gamma_q})$$

maps $\Omega_\mu(\rho)$ onto $\Omega_\nu(\rho^\gamma)$, where $\nu = (\nu_1, \ldots, \nu_q)$ such that $\nu_j \equiv \epsilon_j + \gamma_j \mu_j \mod 2$ for all j. The range of the j-th coordinate behaves differently depending on whether γ_j is even or odd. So let us consider

$$\bar\psi_{\gamma,\epsilon} : \Omega_\epsilon(\rho) \to \Omega_\epsilon(\rho^\gamma) : (x_1, \ldots, x_q) \mapsto ((-1)^{\epsilon_1}|x_1|^{\gamma_1}, \ldots, (-1)^{\epsilon_q}|x_q|^{\gamma_q})$$

and its inverse mapping

$$\bar\psi^{-1}_{\gamma,\epsilon} : \Omega_\epsilon(\rho^\gamma) \to \Omega_\epsilon(\rho) : (x_1, \ldots, x_q) \mapsto ((-1)^{\epsilon_1}|x_1|^{\frac{1}{\gamma_1}}, \ldots, (-1)^{\epsilon_q}|x_q|^{\frac{1}{\gamma_q}}).$$

Then we have $\bar\psi_{\gamma,\epsilon} \circ \bar\psi^{-1}_{\gamma,\epsilon} = \mathrm{id}_{\Omega_\epsilon(\rho^\gamma)}$ and $\bar\psi^{-1}_{\gamma,\epsilon} \circ \bar\psi_{\gamma,\epsilon} = \mathrm{id}_{\Omega_\epsilon(\rho)}$ for all $\gamma \in (\mathbb{R}_{>0})^q$ and $\epsilon \in \{0,1\}^q$. Note that

$$\{\bar\psi_{\gamma,\epsilon} : \epsilon \in \{0,1\}^q\} \subseteq \{\psi_{\gamma,\mu}|_{\Omega_\epsilon(\rho)} : \epsilon, \mu \in \{0,1\}^q\}. \tag{7.3.1}$$

7.4. **Lemma.** *If $f \in \mathcal{W}^{\mathcal{C}}(\Omega_\epsilon(\rho))$ then $f \circ \bar\psi^{-1}_{\gamma,\epsilon} \in \mathcal{W}^{\mathcal{C}}(\Omega_\epsilon(\rho^\gamma))$.*

Proof. The mapping $\bar\psi_{\gamma,\epsilon} : \Omega_\epsilon(\rho) \to \Omega_\epsilon(\rho^\gamma)$ is a $\mathcal{C}$-diffeomorphism (by (3.1.1) and (3.1.6)), and it is Lipschitz. Hence, $E_{\Omega_\epsilon(\rho^\gamma), f\circ\bar\psi^{-1}_{\gamma,\epsilon}} = \bar\psi_{\gamma,\epsilon}(E_{\Omega_\epsilon(\rho),f})$ is closed, and we have $\mathcal{H}^q(E_{\Omega_\epsilon(\rho^\gamma), f\circ\bar\psi^{-1}_{\gamma,\epsilon}}) = 0$ and $\mathcal{H}^{q-1}(E_{\Omega_\epsilon(\rho^\gamma), f\circ\bar\psi^{-1}_{\gamma,\epsilon}}) < \infty$, by (7.1.1). This implies $(\mathcal{W}_1)$ and $(\mathcal{W}_2)$. Since $f \in \mathcal{W}^{\mathcal{C}}(\Omega_\epsilon(\rho))$, we have $\partial_i f \in L^1(\Omega_\epsilon(\rho))$. Thus

$$\begin{aligned}\infty > \int_{\Omega_\epsilon(\rho)} |\partial_i f(x)| dx &= \int_{\Omega_\epsilon(\rho^\gamma)} |\partial_i f(\bar\psi^{-1}_{\gamma,\epsilon}(x))| |\det d\bar\psi^{-1}_{\gamma,\epsilon}(x)| dx \\ &= \Big(\prod_{j\ne i} \tfrac{1}{\gamma_j}\Big) \int_{\Omega_\epsilon(\rho^\gamma)} |\partial_i(f\circ\bar\psi^{-1}_{\gamma,\epsilon})(x)| \prod_{j\ne i} |x_j|^{\frac{1-\gamma_j}{\gamma_j}} dx \\ &\ge \Big(\prod_{j\ne i} \frac{\rho_j^{1-\gamma_j}}{\gamma_j}\Big) \int_{\Omega_\epsilon(\rho^\gamma)} |\partial_i(f\circ\bar\psi^{-1}_{\gamma,\epsilon})(x)| dx.\end{aligned}$$

That shows $(\mathcal{W}_3)$. □

7.5. Let us define $\bar\psi^{-1}_\gamma : \Omega(\rho^\gamma) \to \Omega(\rho)$ by setting $\bar\psi^{-1}_\gamma|_{\Omega_\epsilon(\rho^\gamma)} := \bar\psi^{-1}_{\gamma,\epsilon}$, for $\epsilon \in \{0,1\}^q$, and by extending it continuously to $\Omega(\rho^\gamma)$. Analogously, define $\bar\psi_\gamma : \Omega(\rho) \to \Omega(\rho^\gamma)$ such that $\bar\psi_\gamma \circ \bar\psi^{-1}_\gamma = \mathrm{id}_{\Omega(\rho^\gamma)}$ and $\bar\psi^{-1}_\gamma \circ \bar\psi_\gamma = \mathrm{id}_{\Omega(\rho)}$.

Lemma 7.4 implies:

7.6. **Lemma.** *If $f \in \mathcal{W}^{\mathcal{C}}(\Omega(\rho))$ then $f \circ \bar\psi^{-1}_\gamma \in \mathcal{W}^{\mathcal{C}}(\Omega(\rho^\gamma))$.*

Proof. The set

$$E_{\Omega(\rho^\gamma), f\circ\bar\psi^{-1}_\gamma} = \bigcup_{\epsilon\in\{0,1\}^q} E_{\Omega_\epsilon(\rho^\gamma), f\circ\bar\psi^{-1}_{\gamma,\epsilon}} \cup \{x \in \Omega(\rho^\gamma) : \prod_j x_j = 0\}$$

obviously has the required properties. □

7.7. Let I be a subset of $\{1,\dots,q\}$ with $|I| \ge 2$. For $i \in I$ consider the mapping $\varphi_i : \mathbb{R}^q \to \mathbb{R}^q : x \mapsto y$ given by

$$y_j = \begin{cases} x_i, & \text{for } j = i \\ x_i x_j, & \text{for } j \in I\setminus\{i\} \\ x_j, & \text{for } j \notin I \end{cases}. \tag{7.7.1}$$

The image $\varphi_i(\Omega(\rho)\setminus\{x_i = 0\}) =: \tilde\Omega_i(\rho)$ has the form

$$\tilde\Omega_i(\rho) = \{x \in \mathbb{R}^q : 0 < |x_i| < \rho_i, |x_j| < \rho_j|x_i| \text{ for } j \in I\setminus\{i\}, |x_j| < \rho_j \text{ for } j \notin I\}.$$

If $\rho_i > 1$ for all $i \in I$, then $\Omega(\rho)\setminus\{x_i = 0 \text{ for all } i \in I\} \subseteq \bigcup_{i\in I} \tilde\Omega_i(\rho)$. Let us consider $\tilde\varphi_i := \varphi_i|_{\Omega(\rho)\setminus\{x_i=0\}}$ and its inverse mapping $\tilde\varphi_i^{-1} : \tilde\Omega_i(\rho) \to \Omega(\rho)\setminus\{x_i = 0\} : x \mapsto y$ given by

$$y_j = \begin{cases} x_i, & \text{for } j = i \\ \frac{x_j}{x_i}, & \text{for } j \in I\setminus\{i\} \\ x_j, & \text{for } j \notin I \end{cases}.$$

7.8. **Lemma.** *If $f \in \mathcal{W}^{\mathcal{C}}(\Omega(\rho))$ then $f \circ \tilde\varphi_i^{-1} \in \mathcal{W}^{\mathcal{C}}(\tilde\Omega_i(\rho))$.*

Proof. We may view f as a function in $\mathcal{W}^{\mathcal{C}}(\Omega(\rho)\setminus\{x_i = 0\})$, where $E_{\Omega(\rho)\setminus\{x_i=0\},f} = E_{\Omega(\rho),f}\setminus\{x_i = 0\}$. The mapping $\tilde\varphi_i : \Omega(\rho)\setminus\{x_i = 0\} \to \tilde\Omega_i(\rho)$ is a $\mathcal{C}$-diffeomorphism (by (3.1.1) and (3.1.6)), and it is Lipschitz. Hence, $E_{\tilde\Omega_i(\rho), f\circ\tilde\varphi_i^{-1}} = \tilde\varphi_i(E_{\Omega(\rho)\setminus\{x_i=0\},f})$ is closed, and we have $\mathcal{H}^q(E_{\tilde\Omega_i(\rho), f\circ\tilde\varphi_i^{-1}}) = 0$ and $\mathcal{H}^{q-1}(E_{\tilde\Omega_i(\rho), f\circ\tilde\varphi_i^{-1}}) < \infty$, by (7.1.1). This implies $(\mathcal{W}_1)$ and $(\mathcal{W}_2)$.

The following identities are consequences of the substitution formula (applied from right to left). The right-hand sides are finite, since $\partial_j f \in L^1(\Omega(\rho))$ for all j and since $|I| \ge 2$.

$$\begin{aligned}
\int_{\tilde\Omega_i(\rho)} \big|\partial_i f(\tilde\varphi_i^{-1}(x))\big|\, dx &= \int_{\Omega(\rho)} |\partial_i f(x)||x_i|^{|I|-1} dx < \infty,\\
\int_{\tilde\Omega_i(\rho)} \left|\partial_j f(\tilde\varphi_i^{-1}(x))\frac{x_j}{x_i^2}\right| dx &= \int_{\Omega(\rho)} |\partial_j f(x)||x_i|^{|I|-2}|x_j| dx < \infty, \quad j \in I\setminus\{i\},\\
\int_{\tilde\Omega_i(\rho)} \left|\partial_j f(\tilde\varphi_i^{-1}(x))\frac{1}{x_i}\right| dx &= \int_{\Omega(\rho)} |\partial_j f(x)||x_i|^{|I|-2} dx < \infty, \quad j \in I\setminus\{i\},\\
\int_{\tilde\Omega_i(\rho)} \big|\partial_j f(\tilde\varphi_i^{-1}(x))\big|\, dx &= \int_{\Omega(\rho)} |\partial_j f(x)||x_i|^{|I|-1} dx < \infty, \quad j \notin I.
\end{aligned}$$

It follows that the partial derivatives

$$\partial_j(f\circ\tilde\varphi_i^{-1})(x) = \begin{cases} \partial_i f(\tilde\varphi_i^{-1}(x)) - \sum_{k\in I\setminus\{i\}} \partial_k f(\tilde\varphi_i^{-1}(x))\frac{x_k}{x_i^2}, & \text{for } j = i\\ \partial_j f(\tilde\varphi_i^{-1}(x))\frac{1}{x_i}, & \text{for } j \in I\setminus\{i\}\\ \partial_j f(\tilde\varphi_i^{-1}(x)), & \text{for } j \notin I \end{cases}$$

belong to $L^1(\tilde\Omega_i(r))$. Thus $(\mathcal{W}_3)$ is shown. □

7.9. **Lemma.** *Let $\varphi : M' \to M$ be a blow-up of a $\mathcal{C}$-manifold M with center a closed $\mathcal{C}$-submanifold C of M. If $f \in \mathcal{W}^{\mathcal{C}}_{\mathrm{loc}}(M')$ then $f \circ (\varphi|_{M'\setminus\varphi^{-1}(C)})^{-1} \in \mathcal{W}^{\mathcal{C}}_{\mathrm{loc}}(M)$.*

Proof. Let $K \subseteq M$ be compact. Hence K can be covered by finitely many relatively compact coordinate neighborhoods (U, u) such that over U the mapping φ identifies with the mapping $U' \to U$ described in 5.1. Each U' is covered by charts (U_i', u_i') such that $u \circ \varphi|_{U_i'} \circ (u_i')^{-1} = \varphi_i$ (where φ_i is defined in (7.7.1)).

$$\begin{tikzcd}
M' \arrow[d, "\varphi"'] & U' \arrow[l, hook'] \arrow[d, "\varphi|_{U'}"'] & U_i' \arrow[l, hook'] \arrow[r, "u_i'"] \arrow[d, "\varphi|_{U_i'}"'] & \Omega(\rho) \arrow[d, "\varphi_i"'] & \Omega(\rho)\setminus\{x_i = 0\} \arrow[l, hook'] \arrow[d, "\tilde\varphi_i"'] \\
M & U \arrow[l, hook'] & \varphi(U_i') \arrow[l, hook'] \arrow[r, "u|_{\varphi(U_i')}"'] & \varphi_i(\Omega(\rho)) & \tilde\Omega_i(\rho) \arrow[l, hook']
\end{tikzcd}$$

Since φ is proper and U is relatively compact, U' is relatively compact as well. Thus $f|_{U'} \in \mathcal{W}^{\mathcal{C}}(U')$, and $\mathcal{W}^{\mathcal{C}}(U')$ is independent of the Riemannian metric. We may assume that there is a $\rho \in (\mathbb{R}_{>1})^q$ such that $u_i'(U_i') = \Omega(\rho)$. By lemma 7.8, $f|_{U_i'} \circ (u_i')^{-1} \circ \tilde\varphi_i^{-1} \in \mathcal{W}^{\mathcal{C}}(\tilde\Omega_i(\rho))$. Since $u_i'(U_i' \setminus \varphi^{-1}(C)) = \Omega(\rho)\setminus\{x_i = 0\}$ and $\tilde\varphi_i = \varphi_i|_{\Omega(\rho)\setminus\{x_i=0\}}$, we have

$$f|_{U_i'} \circ (u_i')^{-1} \circ \tilde\varphi_i^{-1} = f|_{U_i'} \circ (\varphi|_{U_i'\setminus\varphi^{-1}(C)})^{-1} \circ u^{-1}|_{\tilde\Omega_i(\rho)} \in \mathcal{W}^{\mathcal{C}}(\tilde\Omega_i(\rho)). \tag{7.9.1}$$

Let $\Upsilon(\rho) := \bigcup_{i\in I} \tilde\Omega_i(\rho)$. Note that $\Omega(\rho)\setminus\{x_i = 0 \text{ for all } i \in I\} \subseteq \Upsilon(\rho)$. Then

$$f|_{U'} \circ (\varphi|_{U'\setminus\varphi^{-1}(C)})^{-1} \circ u^{-1}|_{\Upsilon(\rho)} \in \mathcal{W}^{\mathcal{C}}(\Upsilon(\rho)), \tag{7.9.2}$$

where $E_{\Upsilon(\rho),\star} := \bigcup_{i\in I}\big(E_{\tilde\Omega_i(\rho),\star\star} \cup \partial\big(\tilde\Omega_i(\rho)\big)\big)$ and $\star$ and $\star\star$ represent the functions in (7.9.2) and (7.9.1), respectively. So we find (possibly shrinking U)

$$f \circ (\varphi|_{M'\setminus\varphi^{-1}(C)})^{-1}|_U = f|_{U'} \circ (\varphi|_{U'\setminus\varphi^{-1}(C)})^{-1} \in \mathcal{W}^{\mathcal{C}}(U),$$

where $\mathcal{W}^{\mathcal{C}}(U)$ is independent of the Riemannian metric. It follows immediately that

$$f \circ (\varphi|_{M'\setminus\varphi^{-1}(C)})^{-1}|_{\bigcup U} \in \mathcal{W}^{\mathcal{C}}\big(\bigcup U\big),$$

where the union in finite. This completes the proof. □

7.10. **Lemma.** *Let $K \subseteq M$ be compact, let $\{(U_j, u_j) : 1 \le j \le N\}$ be a finite collection of connected relatively compact coordinate charts covering K, and let $f_j \in \mathcal{W}^{\mathcal{C}}(U_j)$. Then, after shrinking the U_j slightly such that they still cover K, there exists a function $f \in \mathcal{W}^{\mathcal{C}}(\bigcup_j U_j)$ satisfying the following condition:*

(1) *If $x \in \bigcup_j U_j$ then either $x \in E_{\bigcup_j U_j}$ or $f(x) = f_j(x)$ for a $j \in \{i : x \in U_i\}$.*

Proof. We construct f step-by-step. Suppose that a function $f' \in \mathcal{W}^{\mathcal{C}}(\bigcup_{j=1}^{n-1} U_j)$ satisfying (1) has been found. If $(\bigcup_{j=1}^{n-1} U_j) \cap U_n = \emptyset$ then the function

$$f := f' \mathbf{1}_{\bigcup_{j=1}^{n-1} U_j} + f_n \mathbf{1}_{U_n} \in \mathcal{W}^{\mathcal{C}}(\bigcup_{j=1}^{n} U_j)$$

has property (1). Otherwise, consider the chart (U_n, u_n). We may assume that $u_n(U_n) = B_1(0)$, the open unit ball in $\mathbb{R}^q$. Choose $\epsilon > 0$ small, such that the collection $\{U_j : 1 \le j \le N, j \ne n\} \cup U_n'$, where $U_n' := u_n^{-1}(B_{1-\epsilon}(0))$, still covers K. The set $S := \partial B_{1-\epsilon}(0) \cap u_n((\bigcup_{j=1}^{n-1} U_j) \cap U_n)$ is closed in $u_n((\bigcup_{j=1}^{n-1} U_j) \cap U_n)$, $\mathcal{H}^q(S) = 0$, and $\mathcal{H}^{q-1}(S) < \infty$. So $u_n^{-1}(S)$ is closed in $\bigcup_{j=1}^{n-1} U_j \cup U_n'$, and, by (7.1.1), $\mathcal{H}^q(u_n^{-1}(S)) = 0$, and $\mathcal{H}^{q-1}(u_n^{-1}(S)) < \infty$. Thus

$$f := f' \mathbf{1}_{(\bigcup_{j=1}^{n-1} U_j) \setminus U_n'} + f_n \mathbf{1}_{U_n'} \in \mathcal{W}^{\mathcal{C}}(\bigcup_{j=1}^{n-1} U_j \cup U_n')$$

and satisfies (1). Repeating this procedure finitely many times, produces the required function. □

7.11. **Theorem** ($\mathcal{W}^{\mathcal{C}}$-roots). *Let M be a $\mathcal{C}$-manifold. Consider a family of polynomials*

$$P(x)(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j(x) z^{n-j},$$

with coefficients a_j (for $1 \le j \le n$) in $\mathcal{C}(M, \mathbb{C})$. For any compact subset $K \subseteq M$ there exists a relatively compact neighborhood W of K and a parameterization λ_j (for $1 \le j \le n$) of the roots of P on W such that $\lambda_j \in \mathcal{W}^{\mathcal{C}}(W)$ for all j. In particular, for each λ_j we have $\nabla \lambda_j \in L^1(W)$.

Proof. By theorem 6.7, there exists a neighborhood W of K and a finite covering $\{\pi_k : U_k \to W\}$ of W, where each π_k is a composite of finitely many mappings each of which is either a local blow-up Φ with smooth center or a local power substitution Ψ (cf. 6.1), such that, for all k, the family of polynomials $P \circ \pi_k$ allows a $\mathcal{C}$-parameterization λ_i^k (for $1 \le i \le n$) of its roots on U_k.

In view of lemma 7.10, the proof of the theorem will be complete once the following assertions are shown:

(1) Let $\Psi = \iota \circ \psi : V \to W \to M$ be a local power substitution. If the roots of $P \circ \Psi$ allow a parameterization in $\mathcal{W}^{\mathcal{C}}_{\text{loc}}$, then so do the roots of $P|_W$.
(2) Let $\Phi = \iota \circ \varphi : U' \to U \to M$ be local blow-up with smooth center. If the roots of $P \circ \Phi$ allow a parameterization in $\mathcal{W}^{\mathcal{C}}_{\text{loc}}$, then so do the roots of $P|_U$.

Assertion (2) is an immediate consequence of lemma 7.9. To prove (1), let $\lambda_i^\Psi = \lambda_i^{\psi_{\gamma,\epsilon}}$ (for some $\gamma \in (\mathbb{N}_{>0})^q$ and all $\epsilon \in \{0,1\}^q$, cf. 6.1) be functions in $\mathcal{W}^{\mathcal{C}}_{\text{loc}}(V)$ which parameterize the roots of $P \circ \Psi$. We can assume without loss (possibly shrinking V) that $V = \Omega(\rho)$, $W = \Omega(\rho^\gamma)$, and that each $\lambda_i^{\psi_{\gamma,\epsilon}} \in \mathcal{W}^{\mathcal{C}}(\Omega(\rho))$, for some $\rho \in (\mathbb{R}_{>0})^q$. Let us define $\lambda_i^{\bar\psi_\gamma} \in \mathcal{W}^{\mathcal{C}}(\Omega(\rho))$ by setting (in view of (7.3.1) and 7.5)

$$\lambda_i^{\bar\psi_\gamma}|_{\Omega_\epsilon(\rho)} := \lambda_i^{\bar\psi_{\gamma,\epsilon}}|_{\Omega_\epsilon(\rho)}, \quad \epsilon \in \{0,1\}^q.$$

On the set $\{x \in \Omega(\rho) : \prod_j x_j = 0\}$ we may define $\lambda_i^{\bar\psi_\gamma}$ (for $1 \le i \le n$) arbitrarily such that they form a parameterization of the roots of $P \circ \iota \circ \bar\psi_\gamma$. By lemma 7.6,

$$\lambda_i := \lambda_i^{\bar\psi_\gamma} \circ \bar\psi_\gamma^{-1} \in \mathcal{W}^{\mathcal{C}}(\Omega(\rho^\gamma)) = \mathcal{W}^{\mathcal{C}}(W).$$

Clearly, λ_i (for $1 \le i \le n$) constitutes a parameterization of the roots of $P|_W$. Thus the proof of (1) is complete. □

7.12. **Corollary** (Local $\mathcal{W}^{\mathcal{C}}$-sections)**.** *The mapping $\sigma^n : \mathbb{C}^n \to \mathbb{C}^n$ from roots to coefficients (cf.* (2.1.1)*) admits local $\mathcal{W}^{\mathcal{C}}$-sections, for $\mathcal{C}$ any class of C^∞-functions satisfying* (3.1.1)–(3.1.6).

Proof. Apply theorem 7.11 to the family

$$P(a)(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j z^{n-j}, \quad a = (a_1, \ldots, a_n) \in \mathbb{C}^n = \mathbb{R}^{2n}.$$

It is a $\mathcal{C}$-family by (3.1.1). □

In the following we show that the conclusion of theorem 7.11 is best possible.

7.13. **Example** (The derivatives of the roots are not in L^p_{loc} for any $1 < p \le \infty$)**.** In general the roots of a $\mathcal{C}$ (even polynomial) family of polynomials P do not allow parameterizations λ_j with $\nabla\lambda_j \in L^p_{\text{loc}}$ for any $1 < p \le \infty$. That is shown by the example

$$P(x)(z) = z^n - x_1 \cdots x_q, \quad x = (x_1, \ldots, x_q) \in \mathbb{R}^q,$$

if $n \ge \frac{p}{p-1}$, for $1 < p < \infty$, and if $n \ge 2$, for $p = \infty$.

7.14. **Remark.** Compare theorem 7.11 with the results obtained in [CJS83] and [CL03]: For a non-negative real valued function $f \in C^k(U)$, where $U \subseteq \mathbb{R}^q$ is open and $k \ge 2$, they find in [CJS83] that $\nabla(f^{1/k}) \in L^1_{\text{loc}}(U)$. Actually, for each compact $K \subseteq U$, one has $\nabla(f^{1/k}) \in L_w^{k/(k-2)}(K)$, due to [CL03], where L^p_w denotes the *weak* L^p space. By example 7.13, we can in general not expect that the derivatives of the roots of P belong to any $L^p_w(K)$ with $p > 1$, since $L^p(K) \subseteq L^p_w(K) \subseteq L^q(K)$ for $1 \le q < p < \infty$.

7.15. **The one dimensional case.** Let P be a curve of polynomials. Then the proof of lemma 7.4 actually shows that pullback by $\bar\psi_{\gamma,\epsilon}^{-1}(x) = (-1)^\epsilon |x|^{1/\gamma}$, ($x \in \mathbb{R}$, $\gamma \in \mathbb{N}_{>0}$, and $\epsilon = 0, 1$), preserves absolute continuity. So theorem 7.11 reproduces (for $\mathcal{C}$-coefficients) the following result proved in [Rai09a] (see also [Spa99]):

7.16. **Theorem.** *The roots of an everywhere normally nonflat C^∞-curve of polynomials P may be parameterized by locally absolutely continuous functions.*

A curve of polynomials P with C^∞-coefficients a_j is *normally nonflat* at x_0 if $x \mapsto \tilde\Delta_s(P(x))$ is not infinitely flat at x_0, where s is maximal with the property that the germ at x_0 of $x \mapsto \tilde\Delta_s(P(x))$ is not 0. Or, equivalently, no two of the continuously chosen roots (which is always possible in the one dimensional case, cf. [Kat76, II 5.2]) meet of infinite order of flatness.

On an interval $I \subseteq \mathbb{R}$ the space of locally absolutely continuous functions coincides with the Sobolev space $W^{1,1}_{\text{loc}}(I)$. However:

7.17. **Example** (The roots are not in $W^{1,1}_{\text{loc}}$)**.** Multiparameter $\mathcal{C}$ (even polynomial) families of polynomials do not allow roots in $W^{1,1}_{\text{loc}}$, as the following example shows:

$$P(x)(z) = z^2 - x, \quad x \in \mathbb{C} = \mathbb{R}^2.$$

The roots are $\lambda_{12} = \pm\sqrt{x}$ which must have a jump along some ray. The distributional derivative of $\sqrt{x}$ with respect to angle contains a delta distribution which is not in L^1_{loc}.

7.18. **Example** (The roots are not in VMO)**.** Let $U \subseteq \mathbb{R}^q$ be open. We say that a real or complex valued $f \in L^1_{\text{loc}}(U)$ has *vanishing mean oscillation*, or $f \in VMO(U)$, if, for cubes $Q \subseteq \mathbb{R}^q$ with closure $\overline{Q} \subseteq U$, we have

$$\|f\|_{BMO} := \sup\{\text{mo}(f,Q) : Q\} < \infty \quad \text{and} \quad \lim_{s\to 0} \sup\{\text{mo}(f,Q) : |Q| \le s\} = 0,$$

where

$$f_Q := \frac{1}{|Q|}\int_Q f(x)dx \quad \text{and} \quad \text{mo}(f,Q) := \frac{1}{|Q|}\int_Q |f(x) - f_Q|dx.$$

Functions $f \in L^1_{\text{loc}}(U)$ with $\|f\|_{BMO} < \infty$ are said to have *bounded mean oscillation* (or $f \in BMO(U)$). Cf. [Sar75] and [BN95, BN96].

By proposition 2.4, the roots of a family of polynomials P whose coefficients are bounded functions on U are bounded as well and hence in $BMO(U)$. Thus it makes sense to ask whether the roots of a $\mathcal{C}$-family P admit parameterizations in VMO. In general the answer is no: 7.17 provides a counter example.

Namely: Let $S = (-\infty, 0] \times \{0\} \subseteq \mathbb{R}^2$ be the left x-axis and let $f : \mathbb{R}^2 \setminus S \to \mathbb{C}$ be defined, in polar coordinates $(r, \phi) \in (0,\infty) \times (-\pi, \pi)$, by

$$f(r,\phi) = \sqrt{r}\big(\cos\tfrac{\phi}{2} + i\sin\tfrac{\phi}{2}\big).$$

For convenience of computation we use

$$Q(x_0,\epsilon) := \{(r,\phi) : |r - x_0| < \epsilon, -\pi < \phi < -\pi + \epsilon \text{ or } \pi - \epsilon < \phi < \pi\},$$

where $0 < \epsilon < x_0 < \pi/2$. Since $Q(x_0, \epsilon)$ is symmetric with respect to the x-axis, we find $\text{Im}f_{Q(x_0,\epsilon)} = (\text{Im}f)_{Q(x_0,\epsilon)} = 0$. It is easy to compute

$$\text{mo}(\text{Im}f, Q(x_0,\epsilon)) = \tfrac{2}{5}\sin\tfrac{\epsilon}{2} \cdot \frac{(x_0+\epsilon)^{\frac{5}{2}} - (x_0-\epsilon)^{\frac{5}{2}}}{x_0\epsilon^2} \xrightarrow{\epsilon\to 0} \sqrt{x_0}.$$

Since $\text{mo}(f, Q(x_0,\epsilon)) \ge \text{mo}(\text{Im}f, Q(x_0,\epsilon))$, we may conclude that $f \notin VMO(U)$, for each open $U \subseteq \mathbb{R}^2$ containing the origin.

8. Roots with locally bounded variation

The roots of a $\mathcal{C}$-family of polynomials admit a parameterization by functions having locally bounded variation, actually, even by SBV_{loc}-functions.

8.1. **Functions of bounded variation.** Cf. [AFP00]. Let $U \subseteq \mathbb{R}^q$ be open. A real valued function $f \in L^1(U)$ is said to have *bounded variation*, or to belong to $BV(U)$, if its distributional derivative is representable by a finite Radon measure in U, i.e.,

$$\int_U f\,\partial_i\phi\,dx = -\int_U \phi\,dD_i f, \qquad \text{for all } \phi \in C^\infty_c(U) \text{ and } 1 \le i \le q,$$

for some $\mathbb{R}^q$-valued measure $Df = (D_1 f, \cdots, D_q f)$ in U. Then $W^{1,1}(U) \subseteq BV(U)$: for any $f \in W^{1,1}(U)$ the distributional derivative is given by $(\nabla f)\mathcal{L}^q$. See [AFP00, Section 3.1] for equivalent definitions and properties of BV-functions.

A complex valued function $f : U \to \mathbb{C}$ is said to be of *bounded variation*, or to be in $BV(U,\mathbb{C})$, if $(\text{Re}f, \text{Im}f) \in (BV(U))^2$.

8.2. **Special functions of bounded variation.** This notion is due to [DGA88]. For a detailed treatment see [AFP00]. Let $U \subseteq \mathbb{R}^q$ be open and let $f \in BV(U)$. We may write

$$Df = D^a f + D^s f,$$

where $D^a f$ is the absolutely continuous part of Df and $D^s f$ is the singular part of Df with respect to $\mathcal{L}^q$.

We say that f has an *approximate limit at* $x \in U$ if there exists $a \in \mathbb{R}$ such that

$$\lim_{r \searrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - a| dy = 0.$$

The *approximate discontinuity set* S_f is the set of points where this property does not hold. A point $x \in U$ is called *approximate jump point of* f if there exist $a^\pm \in \mathbb{R}$ and $\nu \in S^{q-1}$ such that $a^+ \neq a^-$ and

$$\lim_{r \searrow 0} \frac{1}{|B_r^\pm(x,\nu)|} \int_{B_r^\pm(x,\nu)} |f(y) - a^\pm| dy = 0,$$

where $B_r^\pm(x,\nu) := \{y \in B_r(x) : \pm\langle y - x \mid \nu \rangle > 0\}$. The set of approximate jump points is denoted by J_f.

For any $f \in BV(U)$ the measures

$$D^j f := \mathbf{1}_{J_f} D^s f \quad \text{and} \quad D^c f := \mathbf{1}_{U \setminus S_f} D^s f$$

are called the *jump part* and the *Cantor part* of the derivative. Since Df vanishes on the $\mathcal{H}^{q-1}$-negligible set $S_f \setminus J_f$, we obtain the decomposition

$$Df = D^a f + D^j f + D^c f.$$

We say that $f \in BV(U)$ is a *special function of bounded variation*, and we write $f \in SBV(U)$, if the Cantor part of its derivative $D^c f$ is zero.

8.3. **Proposition** ([AFP00, 4.4]). *Let $U \subseteq \mathbb{R}^q$ be open and bounded, $E \subseteq \mathbb{R}^q$ closed, and $\mathcal{H}^{q-1}(E \cap U) < \infty$. Then, any function $f : U \to \mathbb{R}$ that belongs to $L^\infty(U \setminus E) \cap W^{1,1}(U \setminus E)$ belongs also to $SBV(U)$ and satisfies $\mathcal{H}^{q-1}(S_f \setminus E) = 0$.*

A complex valued function f belongs to $SBV(U, \mathbb{C})$ if $(\operatorname{Re} f, \operatorname{Im} f) \in (SBV(U))^2$.

8.4. **Theorem** (SBV-roots). *Let $U \subseteq \mathbb{R}^q$ be open. Consider a family of polynomials*

$$P(x)(z) = z^n + \sum_{j=1}^n (-1)^j a_j(x) z^{n-j},$$

with coefficients a_j (for $1 \leq j \leq n$) in $\mathcal{C}(U, \mathbb{C})$. For any compact subset $K \subseteq U$ there exists a relatively compact neighborhood W of K and a parameterization λ_j (for $1 \leq j \leq n$) of the roots of P on W such that $\lambda_j \in SBV(W, \mathbb{C})$ for all j.

Proof. It follows immediately from theorem 7.11, proposition 8.3, and (7.2.1). □

Combining corollary 7.12 with proposition 8.3 or applying theorem 8.4 to the family P in 7.12 gives:

8.5. **Corollary** (Local SBV-sections). *The mapping $\sigma^n : \mathbb{C}^n \to \mathbb{C}^n$ from roots to coefficients (see* (2.1.1)*) admits local SBV-sections, for $\mathcal{C}$ any class of C^∞-functions satisfying* (3.1.1)–(3.1.6). □

9. Perturbation of normal matrices

We investigate the consequences of our results in the perturbation theory of normal matrices. It is evident that the eigenvalues of a $\mathcal{C}$-family of normal matrices possess the regularity properties of the roots of a $\mathcal{C}$-family of polynomials. We prove that the same it true for the eigenvectors.

9.1. **Theorem** ($\mathcal{C}$-perturbation of normal matrices)**.** *Let M be a $\mathcal{C}$-manifold. Consider a family of normal complex matrices*

$$A(x) = (A_{ij}(x))_{1\le i,j\le n}$$

(acting on a complex vector space $V = \mathbb{C}^n$), where the entries A_{ij} (for $1 \le i, j \le n$) belong to $\mathcal{C}(M,\mathbb{C})$. Let $K \subseteq M$ be compact. Then there exist:

(1) *a neighborhood W of K, and*
(2) *a finite covering $\{\pi_k : U_k \to W\}$ of W, where each π_k is a composite of finitely many mappings each of which is either a local blow-up with smooth center or a local power substitution,*

such that, for all k, the family of normal complex matrices $A \circ \pi_k$ allows a $\mathcal{C}$-parameterization of its eigenvalues and eigenvectors.

If A is a family of Hermitian matrices, then the above statement holds with each π_k being a composite of finitely many local blow-ups with smooth center only.

Proof. By theorem 6.7 applied to the characteristic polynomial

$$\chi(A(x))(\lambda) = \det(A(x) - \lambda\mathbb{I}) = \sum_{j=0}^{n}(-1)^{n-j}\operatorname{Trace}(\Lambda^j A(x))\lambda^{n-j} \tag{9.1.1}$$
$$=: (-1)^n\Big(\lambda^n + \sum_{j=1}^{n}(-1)^j a_j(x)\lambda^{n-j}\Big),$$

there exist a neighborhood W of K and a finite covering $\{\pi_k : U_k \to W\}$ of W of the type described in (2) such that, for all k, the family of normal matrices $A\circ\pi_k$ admits a $\mathcal{C}$-parameterization λ_i (for $1 \le i \le n$) of its eigenvalues.

Let us prove the statement about the eigenvectors. We shall show that (for each k) there exists a finite covering $\{\pi_{kl} : U_{kl} \to U_k\}$ of U_k of the type described in (2) such that $A\circ\pi_k\circ\pi_{kl}$ admits a $\mathcal{C}$-parameterization of its eigenvectors (for all l). This assertion follows from the following claim. Composing the finite coverings in the sense of 6.1, will complete the proof.

Claim. *Let $A = A(x)$ be a family of normal complex $n \times n$ matrices, where the entries A_{ij} are $\mathcal{C}$-functions and the eigenvalues of A admit a $\mathcal{C}$-parameterization λ_j in a neighborhood of $0 \in \mathbb{R}^q$. Then there exists a finite covering $\{\pi_k : U_k \to U\}$ of a neighborhood U of 0 of the type described in (2) such that, for all k, $A\circ\pi_k$ admits a $\mathcal{C}$-parameterization of its eigenvectors.*

Proof of the claim. We use induction on $|\operatorname{S}(\chi(A(0)))|$ (cf. 6.6).

First consider the following reduction: Let $\nu_1,\ldots,\nu_l$ denote the pairwise distinct eigenvalues of $A(0)$ with respective multiplicities $m_1,\ldots,m_l$. The sets

$$\Lambda_h := \{\lambda_i : \lambda_i(0) = \nu_h\}, \quad 1 \le h \le l,$$

form a partition of the λ_i such that, for x near 0, $\lambda_i(x) \ne \lambda_j(x)$ if λ_i and λ_j belong to different Λ_h. Consider

$$V_x^{(h)} := \bigoplus_{\lambda\in\Lambda_h}\ker(A(x)-\lambda(x)) = \ker\big(\circ_{\lambda\in\Lambda_h}(A(x)-\lambda(x))\big), \quad 1 \le h \le l.$$

(The order of the compositions is not relevant.) So $V_x^{(h)}$ is the kernel of a vector bundle homomorphism $B(x)$ of class $\mathcal{C}$ with constant rank (even of constant dimension of the kernel), and thus it is a vector subbundle of class $\mathcal{C}$ of the trivial bundle $U \times V \to U$ (where $U \subseteq \mathbb{R}^q$ is a neighborhood of 0) which admits a $\mathcal{C}$-framing. This can be seen as follows: Choose a basis of V such that $A(0)$ is diagonal. By the elimination procedure one can construct a basis for the kernel of $B(0)$. For x near 0, the elimination procedure (with the same choices) gives then a basis of the kernel of $B(x)$. This clearly involves only operations which preserve the class $\mathcal{C}$. The elements of this basis are then of class $\mathcal{C}$ in x near 0.

Therefore, it suffices to find $\mathcal{C}$-eigenvectors in each subbundle $V^{(h)}$ separately, expanded in the constructed frame field of class $\mathcal{C}$. But in this frame field the vector subbundle looks again like a constant vector space. So we may treat each of these parts (A restricted to $V^{(h)}$, as matrix with respect to the frame field) separately. For simplicity of notation we suppress the index h.

Let us suppose that all eigenvalues of $A(0)$ coincide and are equal to $a_1(0)/n$, according to (9.1.1). Eigenvectors of $A(x)$ are also eigenvectors of $A(x)-(a_1(x)/n)\mathbb{I}$ (and vice versa), thus we may replace $A(x)$ by $A(x)-(a_1(x)/n)\mathbb{I}$ and assume that the first coefficient of the characteristic polynomial (9.1.1) vanishes identically. Then $A(0)=0$.

If $A=0$ identically, we choose the eigenvectors constant and we are done. Note that this proves the claim, if $|\operatorname{S}(\chi(A(0)))|=1$.

Assume that $A \neq 0$. By theorem 5.4 (and 6.5), there exists a finite covering $\{\pi_k : U_k \to U\}$ of a neighborhood U of 0 by $\mathcal{C}$-mappings π_k, each of which is a composite of finitely many local blow-ups with smooth center, such that, for each k, the non-zero entries $A_{ij} \circ \pi_k$ of $A \circ \pi_k$ and its pairwise non-zero differences $A_{ij} \circ \pi_k - A_{lm} \circ \pi_k$ simultaneously have only normal crossings.

Let k be fixed and let $x_0 \in U_k$. Then x_0 admits a neighborhood W_k with suitable coordinates in which $x_0 = 0$ and such that either $A_{ij} \circ \pi_k = 0$ or

$$(A_{ij} \circ \pi_k)(x) = x^{\alpha_{ij}} B_{ij}^k(x),$$

where B_{ij}^k is a non-vanishing $\mathcal{C}$-function on W_k, and $\alpha_{ij} \in \mathbb{N}^q$. The collection of multi-indices $\{\alpha_{ij} : A_{ij} \circ \pi_k \neq 0\}$ is totally ordered, by lemma 6.3. Let α denote its minimum.

If $\alpha = 0$, then $(A_{ij} \circ \pi_k)(x_0) = B_{ij}^k(x_0) \neq 0$ for some $1 \le i, j \le n$. Since the first coefficient of $\chi(A \circ \pi_k)$ vanishes, we may conclude that not all eigenvalues of $(A \circ \pi_k)(x_0)$ coincide. Thus, $|\operatorname{S}(\chi(A \circ \pi_k)(x_0))| < |\operatorname{S}(\chi(A(0)))|$, and, by the induction hypothesis, there exists a finite covering $\{\pi_{kl} : W_{kl} \to W_k\}$ of W_k (possibly shrinking W_k) of the type described in (2) such that, for all l, the family of normal matrices $A \circ \pi_k \circ \pi_{kl}$ allows a $\mathcal{C}$-parameterization of its eigenvectors on W_{kl}.

Assume that $\alpha \neq 0$. Then there exist $\mathcal{C}$-functions A_{ij}^k (some of them 0) such that, for all $1 \le i, j \le n$,

$$(A_{ij} \circ \pi_k)(x) = x^{\alpha} A_{ij}^k(x),$$

and $A_{ij}^k(x) = B_{ij}^k(x) \neq 0$ for some i, j and all $x \in W_k$. By (9.1.1), the characteristic polynomial of the $\mathcal{C}$-family of normal matrices $A^k(x) = (A_{ij}^k(x))_{1 \le i,j \le n}$ has the form

$$\chi(A^k(x))(\lambda) = (-1)^n \Big(\lambda^n + \sum_{j=2}^{n} (-1)^j x^{-j\alpha} a_j(\pi_k(x)) \lambda^{n-j}\Big).$$

By theorem 6.7, there exists a finite covering $\{\pi_{kl} : W_{kl} \to W_k\}$ of W_k (possibly shrinking W_k) of the type described in (2) such that, for all l, the family of polynomials $\chi(A^k \circ \pi_{kl})$ admits a $\mathcal{C}$-parameterization of its roots (the eigenvalues of

$A^k \circ \pi_{kl})$. Eigenvectors of $(A^k \circ \pi_{kl})(x)$ are also eigenvectors of $(A \circ \pi_k \circ \pi_{kl})(x)$ (and vice versa).

Let l be fixed and let $y_0 \in W_{kl}$. As there exist indices $1 \le i, j \le n$ such that $A^k_{ij}(x) \neq 0$ for all $x \in W_k$, and, thus, $(A^k_{ij} \circ \pi_{kl})(y_0) \neq 0$, not all eigenvalues of $(A^k \circ \pi_{kl})(y_0)$ coincide. Hence, $|\,\mathrm{S}(\chi(A^k \circ \pi_{kl})(y_0))| < |\,\mathrm{S}(\chi(A(0)))|$, and the induction hypothesis implies the claim.

The statement for Hermitian families A can be proved in the same way, using theorem 6.10 instead of theorem 6.7. □

9.2. **Remark.** The real analytic diagonalization of real analytic multiparameter families of symmetric matrices was treated by [KP08, 6.2]. A one parameter version of theorem 9.1 is proved in [Rai09a] for C^∞-curves of normal matrices A such that $\chi(A)$ is everywhere normally nonflat (see 7.15).

If the parameter space is one dimensional, we recover a $\mathcal{C}$-version of Rellich's classical perturbation result [Rel37a, Satz 1]:

9.3. **Corollary.** *Let $I \subseteq \mathbb{R}$ be an open interval. Consider a curve of Hermitian complex matrices*

$$A(x) = (A_{ij}(x))_{1 \le i,j \le n},$$

where the entries A_{ij} (for $1 \le i, j \le n$) belong to $\mathcal{C}(I, \mathbb{C})$. Then there exist global $\mathcal{C}$-parameterizations of the eigenvalues and the eigenvectors of A on I.

Proof. The global statement for the eigenvectors can be proved by the arguments in the end of [AKLM98, 7.6]. □

9.4. **Example** (A nonflatness condition is necessary)**.** The following simple example (due to Rellich [Rel37a], see also [Kat76, II 5.3]) shows that the above theorem is false if no nonflatness condition (such as quasianalyticity or normal nonflatness) is required: The eigenvectors of the smooth Hermitian family

$$A(x) := e^{-\frac{1}{x^2}} \begin{pmatrix} \cos \frac{2}{x} & \sin \frac{2}{x} \\ \sin \frac{2}{x} & -\cos \frac{2}{x} \end{pmatrix} \text{ for } x \in \mathbb{R} \setminus \{0\}, \text{ and } A(0) := 0,$$

cannot be chosen continuously near 0.

9.5. **Example** (Normality of A is necessary)**.** Neither can the condition that A is normal be omitted: Any choice of eigenvectors of the real analytic family

$$A(x) := \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \text{ for } x \in \mathbb{R}$$

has a pole at 0. The two parameter family

$$A(x, y) := \begin{pmatrix} 0 & x^2 \\ y^2 & 0 \end{pmatrix} \text{ for } x, y \in \mathbb{R}$$

has the eigenvalues $\pm xy$. But its eigenvectors cannot be chosen continuously near 0, even after applying blow-ups or power substitutions.

9.6. **Theorem** (Regularity of the eigenvalues and eigenvectors)**.** *Let M be a $\mathcal{C}$-manifold. Consider a family of normal complex matrices*

$$A(x) = (A_{ij}(x))_{1 \le i,j \le n}$$

(acting on a complex vector space $V = \mathbb{C}^n$), where the entries A_{ij} (for $1 \le i, j \le n$) belong to $\mathcal{C}(M, \mathbb{C})$. For any compact subset $K \subseteq M$ there exists a relatively compact neighborhood W of K and parameterizations of the eigenvalues λ_i and the eigenvectors v_i (for $1 \le i \le n$) of A on W such that for all i:

(1) $\lambda_i \in \mathcal{W}^{\mathcal{C}}(W, \mathbb{C})$ *and* $v_i \in \mathcal{W}^{\mathcal{C}}(W, \mathbb{C}^n)$.

If M is a open subset of $\mathbb{R}^q$, then:

(2) $\lambda_i \in SBV(W, \mathbb{C})$ *and* $v_i \in SBV(W, \mathbb{C}^n)$.

Proof. The assertions for the eigenvalues follow immediately from the theorems 7.11 and 8.4. The statements for the eigenvectors can be deduced from theorem 9.1 in an analogous way as theorem 7.11 and theorem 8.4 are deduced from theorem 6.7 (compare with section 7 and section 8). $\square$

9.7. **Example.** Consider the Hermitian family

$$A(x,y) := \begin{pmatrix} x & iy \\ -iy & -x \end{pmatrix} \text{ for } x, y \in \mathbb{R}.$$

Its eigenvalues $\pm\sqrt{x^2+y^2}$ are not differentiable at 0 and its eigenvectors cannot be arranged continuously near 0. Blowing up the origin, we end up with a family of Hermitian matrices which admits real analytic eigenvalues and eigenvectors; in coordinates:

$$A(x, xy) = x \begin{pmatrix} 1 & iy \\ -iy & -1 \end{pmatrix}$$

has eigenvalues $\pm x\sqrt{1+y^2}$ and eigenvectors

$$\begin{pmatrix} -1-\sqrt{1+y^2} \\ iy \end{pmatrix} \text{ and } \begin{pmatrix} iy \\ -1-\sqrt{1+y^2} \end{pmatrix};$$

likewise,

$$A(xy, y) = y \begin{pmatrix} x & i \\ -i & -x \end{pmatrix}$$

has eigenvalues $\pm y\sqrt{1+x^2}$ and eigenvectors

$$\begin{pmatrix} -x+\sqrt{1+x^2} \\ i \end{pmatrix} \text{ and } \begin{pmatrix} i \\ -x+\sqrt{1+x^2} \end{pmatrix}.$$

Setting

$$v_1(x,y) := \begin{pmatrix} -1-\sqrt{1+(\frac{y}{x})^2} \\ i\frac{y}{x} \end{pmatrix}, \; v_2(x,y) := \begin{pmatrix} i\frac{y}{x} \\ -1-\sqrt{1+(\frac{y}{x})^2} \end{pmatrix}, \quad \text{if } 0 < |y| \le |x|,$$

$$v_1(x,y) := \begin{pmatrix} -\frac{x}{y}+\sqrt{1+(\frac{x}{y})^2} \\ i \end{pmatrix}, \; v_2(x,y) := \begin{pmatrix} i \\ -\frac{x}{y}+\sqrt{1+(\frac{x}{y})^2} \end{pmatrix}, \quad \text{if } 0 < |x| < |y|,$$

$$v_1(x,y) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; v_2(x,y) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{if } y = 0,$$

$$v_1(x,y) := \begin{pmatrix} 1 \\ i \end{pmatrix}, \; v_2(x,y) := \begin{pmatrix} i \\ 1 \end{pmatrix}, \quad \text{if } x = 0 \ne y,$$

provides a choice of eigenvectors v_1, v_2 of A which, clearly, is not continuous, but belongs to $\mathcal{W}^{\mathcal{C}}_{\text{loc}}$ (for any $\mathcal{C}$ satisfying (3.1.1)–(3.1.6)) and, thus, also to SBV_{loc}.

10. Applications to subanalytic functions

10.1. **Subanalytic functions.** Cf. [BM88]. Let M be a real analytic manifold. A subset $X \subseteq M$ is called *subanalytic* if each point of M admits a neighborhood U such that $X \cap U$ is a projection of a relatively compact semianalytic set.

Let U be an open subanalytic subset of $\mathbb{R}^q$. Following [Par94b] we call a function $f : U \to \mathbb{R}$ *subanalytic* if the closure in $\mathbb{R}^q \times \mathbb{RP}^1$ of the graph of f is a subanalytic subset of $\mathbb{R}^q \times \mathbb{RP}^1$.

Any continuous subanalytic function $f : U \to \mathbb{R}$ admits rectilinearization: There exists a locally finite covering $\{\pi_k : U_k \to U\}$ of U, where each π_k is a composite of finitely many mappings each of which is either a local blow-up with smooth center or a local power substitution, such that, for all k, the function $f \circ \pi_k$ is real analytic [BM90, 1.4 & 1.7]. This result was improved in [Par94b, 2.7] to show that in the composition of the π_k it is enough to substitute powers at the last step after all local blow-ups.

10.2. **Theorem.** *Let U be an open subanalytic subset of $\mathbb{R}^q$. Any continuous subanalytic function $f : U \to \mathbb{R}$ belongs to $\mathcal{W}^{\mathcal{C}^\omega}_{\mathrm{loc}}(U)$, and, thus, to $SBV_{\mathrm{loc}}(U)$.*

Proof. This follows from rectilinearization and the reasoning in section 7 and section 8. $\square$

10.3. **Theorem.** *The roots of a family of polynomials P whose coefficients are continuous subanalytic functions admit a parameterization in $\mathcal{W}^{\mathcal{C}^\omega}_{\mathrm{loc}}$, and, thus, in SBV_{loc}.*

Proof. Apply rectilinearization to the coefficients of P and use theorem 6.13. $\square$

10.4. **Remark.** We cannot expect that for the rectilinearization of the roots of a continuous subanalytic hyperbolic family P no local power substitutions are needed. This is shown by the following example:

$$P(x)(z) := z^2 - |x|, \quad \text{for } x \in \mathbb{R}^q.$$

If we additionally require that all coefficients of a subanalytic hyperbolic family P are also arc-analytic, then indeed local blow-ups suffice, by [BM90, 1.4] (see also [Par94b, 3.1]) and theorem 6.10.

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Part 2

The convenient setting for Denjoy–Carleman classes

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THE CONVENIENT SETTING FOR NON-QUASIANALYTIC DENJOY–CARLEMAN DIFFERENTIABLE MAPPINGS

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Abstract. For Denjoy–Carleman differentiable function classes C^M where the weight sequence $M = (M_k)$ is logarithmically convex, stable under derivations, and non-quasianalytic of moderate growth, we prove the following: A mapping is C^M if it maps C^M-curves to C^M-curves. The category of C^M-mappings is cartesian closed in the sense that $C^M(E, C^M(F, G)) \cong C^M(E \times F, G)$ for convenient vector spaces. Applications to manifolds of mappings are given: The group of C^M-diffeomorphisms is a C^M-Lie group but not better.

1. Introduction

Denjoy–Carleman differentiable functions form spaces of functions between real analytic and C^∞. They are described by growth conditions on the Taylor expansions, see (2.1). Under appropriate conditions the fundamental results of calculus still hold: Stability under differentiation, composition, solving ODEs, applying the implicit function theorem. See section (2) for a review of Denjoy–Carleman differentiable functions, which is summarized in Table 1.

In [Kri82], [Kri83], [FK88], [KN85], [KM90], see [KM97a] for a comprehensive presentation, convenient calculus was developed for C^∞, holomorphic, and real analytic functions: see appendix (7), (8), (9) for a short overview of the essential results.

In this paper we develop the convenient calculus for Denjoy–Carleman classes C^M where the weight sequence $M = (M_k)$ is logarithmically convex, stable under derivations, and non-quasianalytic of moderate growth (this holds for all Gevrey differentiable functions $G^{1+\delta}$ for $\delta > 0$). By 'convenient calculus' we mean that the following theorems are proved: A mapping is C^M if it maps C^M-curves to C^M-curves, see (3.9); this is wrong in the quasianalytic case, see (3.12). The category of C^M-mappings is cartesian closed in the sense that $C^M(E, C^M(F, G)) \cong C^M(E \times F, G)$ for convenient vector spaces, see (5.3); this is wrong for weight sequences of non-moderate growth, see (5.4). The uniform boundedness principle holds for linear mappings into spaces of C^M-mappings.

For the quasianalytic case we hope for results similar to the real analytic case, but the methods have to be different. This will be taken up in another paper.

In chapter (6) some applications to manifolds of mappings are given: The group of C^M-diffeomorphisms is a C^M-Lie group but not better.

2. Review of Denjoy–Carleman differentiable functions

2.1. **Denjoy–Carleman classes $C^M(\mathbb{R}^n, \mathbb{R})$ of differentiable functions.** We mainly follow [Thi08] (see also the references therein). We use $\mathbb{N} = \mathbb{N}_{>0} \cup \{0\}$.

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For each multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we write $\alpha! = \alpha_1! \cdots \alpha_n!$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$.

Let $M = (M_k)_{k\in\mathbb{N}}$ be an increasing sequence ($M_{k+1} \ge M_k$) of positive real numbers with $M_0 = 1$. Let $U \subseteq \mathbb{R}^n$ be open. We denote by $C^M(U)$ the set of all $f \in C^\infty(U)$ such that, for all compact $K \subseteq U$, there exist positive constants C and ρ such that

$$|\partial^\alpha f(x)| \le C\,\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|} \tag{2.1.1}$$

for all $\alpha \in \mathbb{N}^n$ and $x \in K$. The set $C^M(U)$ is a *Denjoy–Carleman class* of functions on U. If $M_k = 1$, for all k, then $C^M(U)$ coincides with the ring $C^\omega(U)$ of real analytic functions on U. In general, $C^\omega(U) \subseteq C^M(U) \subseteq C^\infty(U)$.

We assume that $M = (M_k)$ is *logarithmically convex*, i.e.,

$$M_k^2 \le M_{k-1}\,M_{k+1} \quad \text{for all } k, \tag{2.1.2}$$

or, equivalently, M_{k+1}/M_k is increasing. Considering $M_0 = 1$, we obtain that also $(M_k)^{1/k}$ is increasing and

$$M_l\,M_k \le M_{l+k} \quad \text{for all } l, k \in \mathbb{N}. \tag{2.1.3}$$

We also get (see (2.9))

$$M_1^k\,M_k \ge M_j\,M_{\alpha_1} \cdots M_{\alpha_j} \quad \text{for all } \alpha_i \in \mathbb{N}_{>0}, \alpha_1 + \cdots + \alpha_j = k. \tag{2.1.4}$$

Let $M = (M_k)$ be logarithmically convex. Then $M'_k = M_k/M_0\,M_1^k \ge 1$ is increasing by (2.1.4), logarithmically convex, and $C^M(U) = C^{M'}(U)$ for all U open in $\mathbb{R}^n$ by (2.1.5). So without loss we assumed at the beginning that M is increasing.

Hypothesis (2.1.2) implies that $C^M(U)$ is a ring, for all open subsets $U \subseteq \mathbb{R}^n$, which can easily be derived from (2.1.3) by means of Leibniz's rule. Note that definition (2.1.1) makes sense also for mappings $U \to \mathbb{R}^p$. For C^M-mappings, (2.1.2) guarantees stability under composition ([Rou63], see also [BM04, 4.7]; a proof is also contained in the end of the proof of (3.9)).

A further consequence of (2.1.2) is the inverse function theorem for C^M ([Kom79]; for a proof see also [BM04, 4.10]): Let $f : U \to V$ be a C^M-mapping between open subsets $U, V \subseteq \mathbb{R}^n$. Let $x_0 \in U$. Suppose that the Jacobian matrix $(\partial f/\partial x)(x_0)$ is invertible. Then there are neighborhoods U' of x_0, V' of $y_0 := f(x_0)$ such that $f : U' \to V'$ is a C^M-diffeomorphism.

Moreover, (2.1.2) implies that C^M is closed under solving ODEs (due to [Kom80]): Consider the initial value problem

$$\frac{dx}{dt} = f(t, x), \quad x(0) = y,$$

where $f : (-T, T) \times \Omega \to \mathbb{R}^n$, $T > 0$, and $\Omega \subseteq \mathbb{R}^n$ is open. Assume that $f(t, x)$ is Lipschitz in x, locally uniformly in t. Then for each relative compact open subset $\Omega_1 \subseteq \Omega$ there exists $0 < T_1 \le T$ such that for each $y \in \Omega_1$ there is a unique solution $x = x(t, y)$ on the interval $(-T_1, T_1)$. If $f : (-T, T) \times \Omega \to \mathbb{R}^n$ is a C^M-mapping then the solution $x : (-T_1, T_1) \times \Omega_1 \to \mathbb{R}^n$ is a C^M-mapping as well.

Suppose that $M = (M_k)$ and $N = (N_k)$ satisfy $M_k \le C^k\,N_k$, for all k and a constant C, or equivalently,

$$\sup_{k\in\mathbb{N}_{>0}} \Big(\frac{M_k}{N_k}\Big)^{\frac{1}{k}} < \infty. \tag{2.1.5}$$

Then, evidently $C^M(U) \subseteq C^N(U)$. The converse is true as well (if (2.1.2) is assumed): One can prove that there exists $f \in C^M(\mathbb{R})$ such that $|f^{(k)}(0)| \ge k!\,M_k$ for all k (see [Thi08, Theorem 1]). So the inclusion $C^M(U) \subseteq C^N(U)$ implies (2.1.5).

Setting $N_k = 1$ in (2.1.5) yields that $C^\omega(U) = C^M(U)$ if and only if

$$\sup_{k\in\mathbb{N}_{>0}} (M_k)^{\frac{1}{k}} < \infty.$$

Since $(M_k)^{1/k}$ is increasing (by logarithmic convexity), the strict inclusion $C^\omega(U) \subsetneq C^M(U)$ is equivalent to

$$\lim_{k\to\infty} (M_k)^{\frac{1}{k}} = \infty.$$

We shall also assume that C^M is stable under derivation, which is equivalent to the following condition

$$\sup_{k\in\mathbb{N}_{>0}} \Big(\frac{M_{k+1}}{M_k}\Big)^{\frac{1}{k}} < \infty. \tag{2.1.6}$$

Note that the first order partial derivatives of elements in $C^M(U)$ belong to $C^{M^{+1}}(U)$, where M^{+1} denotes the shifted sequence $M^{+1} = (M_{k+1})_{k\in\mathbb{N}}$. So the equivalence follows from (2.1.5), by replacing M with M^{+1} and N with M.

Definition. By a *DC-weight sequence* we mean a sequence $M = (M_k)_{k\in\mathbb{N}}$ of positive numbers with $M_0 = 1$ which is monotone increasing ($M_{k+1} \ge M_k$), logarithmically convex (2.1.2), and satisfies (2.1.6). Then $C^M(U,\mathbb{R})$ is a differential ring, and the class of C^M-functions is stable under compositions. DC stands for Denjoy-Carleman and also for derivation closed.

2.2. **Quasianalytic function classes.** Let $\mathcal{F}_n$ denote the ring of formal power series in n variables (with real or complex coefficients). For a sequence $M_0 = 1, M_1, M_2, \dots > 0$, we denote by $\mathcal{F}_n^M$ the set of elements $F = \sum_{\alpha\in\mathbb{N}^n} F_\alpha\, x^\alpha$ of $\mathcal{F}_n$ for which there exist positive constants C and ρ such that

$$|F_\alpha| \le C\,\rho^{|\alpha|}\,M_{|\alpha|}$$

for all $\alpha \in \mathbb{N}^n$. A class C^M is called *quasianalytic* if, for open connected $U \subseteq \mathbb{R}^n$ and all $a \in U$, the Taylor series homomorphism

$$T_a : C^M(U) \to \mathcal{F}_n^M,\ f \mapsto T_a f(x) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\,\partial^\alpha f(a)\,x^\alpha$$

is injective. By the Denjoy–Carleman theorem ([Den21], [Car26]), *the following statements are equivalent:*

(1) C^M *is quasianalytic.*
(2) $\sum_{k=1}^\infty \frac{1}{m_k} = \infty$ *where* $m_k = \inf\{(j!\,M_j)^{1/j} : j \ge k\}$ *is the increasing minorant of* $(k!\,M_k)^{1/k}$.
(3) $\sum_{k=1}^\infty (\frac{1}{M_k^*})^{1/k} = \infty$ *where* $M_k^* = \inf\{(j!\,M_j)^{(l-k)/(l-j)}(l!\,M_l)^{(k-j)/(l-j)} : j \le k \le l, j < l\}$ *is the logarithmically convex minorant of* $k!\,M_k$.
(4) $\sum_{k=0}^\infty \frac{M_k^*}{M_{k+1}^*} = \infty$.

For contemporary proofs see for instance [Hör83, 1.3.8] or [Rud87, 19.11].

Suppose that $C^\omega(U) \subsetneq C^M(U)$ and $C^M(U)$ is quasianalytic and logarithmically convex. Then $T_a : C^M(U) \to \mathcal{F}_n^M$ is not surjective. This is due to Carleman [Car26]; an elementary proof can be found in [Thi08, Theorem 3].

2.3. **Non-quasianalytic function classes.** If M is a DC-weight sequence which is not quasianalytic, then there are C^M partitions of unity. Namely, there exists a C^M function f on $\mathbb{R}$ which does not vanish in any neighborhood of 0 but which has vanishing Taylor series at 0. Let $g(t) = 0$ for $t \le 0$ and $g(t) = f(t)$ for $t > 0$. From g we can construct C^M bump functions as usual.

2.4. **Strong non-quasianalytic function classes.** Let M be a DC-weight sequence with $C^\omega(U,\mathbb{R}) \subsetneq C^M(U,\mathbb{R})$. Then the mapping $T_a : C^M(U,\mathbb{R}) \to \mathcal{F}_n^M$ is surjective, for all $a \in U$, if and only if there is a constant C such that

$$\sum_{k=j}^{\infty} \frac{M_k}{(k+1)\,M_{k+1}} \le C \frac{M_j}{M_{j+1}} \quad \text{for any integer } j \ge 0. \tag{2.4.1}$$

See [Pet88] and references therein. (2.4.1) is called *strong non-quasianalyticity* condition.

2.5. **Moderate growth.** A DC-weight sequence M has *moderate growth* if

$$\sup_{j,k\in\mathbb{N}_{>0}} \Big(\frac{M_{j+k}}{M_j\,M_k}\Big)^{\frac{1}{j+k}} < \infty. \tag{2.5.1}$$

Moderate growth implies derivation closed.

Moderate growth together with strong non-quasianalyticity (2.4.1) is called *strong regularity*: Then a version of Whitney's extension theorem holds for the corresponding function classes (e.g. [BBMT91]).

2.6. **Gevrey functions.** Let $\delta > 0$ and put $M_k = (k!)^\delta$, for $k \in \mathbb{N}$. Then $M = (M_k)$ is strongly regular. The corresponding class C^M of functions is the *Gevrey class* $G^{1+\delta}$.

2.7. **More examples.** Let $\delta > 0$ and put $M_k = (\log(k+e))^{\delta\, k}$, for $k \in \mathbb{N}$. Then $M = (M_k)$ is quasianalytic for $0 < \delta \le 1$ and non-quasianalytic (but not strongly) for $\delta > 1$. In any case M is of moderate growth.

Let $q > 1$ and put $M_k = q^{k^2}$, for $k \in \mathbb{N}$. The corresponding C^M-functions are called *q-Gevrey regular*. Then $M = (M_k)$ is strongly non-quasianalytic but not of moderate growth, thus not strongly regular. It is derivation closed.

2.8. **Spaces of C^M-functions.** Let $U \subseteq \mathbb{R}^n$ be open and let M be a DC-weight sequence. For any $\rho > 0$ and $K \subseteq U$ compact with smooth boundary, define

$$C^M_\rho(K) := \{f \in C^\infty(K) : \|f\|_{\rho,K} < \infty\}$$

with

$$\|f\|_{\rho,K} := \sup\Big\{\frac{|\partial^\alpha f(x)|}{\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|}} : \alpha \in \mathbb{N}^n, x \in K\Big\}.$$

It is easy to see that $C^M_\rho(K)$ is a Banach space. In the description of $C^M_\rho(K)$, instead of compact K with smooth boundary, we may also use open $K \subset U$ with $\overline{K}$ compact in U, like [Thi08]. Or we may work with Whitney jets on compact K, like [Kom73b].

The space $C^M(U)$ carries the projective limit topology over compact $K \subseteq U$ of the inductive limit over $\rho \in \mathbb{N}_{>0}$:

$$C^M(U) = \varprojlim_{K\subseteq U} \big(\varinjlim_{\rho\in\mathbb{N}_{>0}} C^M_\rho(K)\big).$$

One can prove that, for $\rho < \rho'$, the canonical injection $C^M_\rho(K) \to C^M_{\rho'}(K)$ is a compact mapping; it is even nuclear (see [Kom73b], [Kom73a, p. 166]). Hence $\varinjlim_\rho C^M_\rho(K)$ is a Silva space, i.e., an inductive limit of Banach spaces such that the canonical mappings are compact; therefore it is complete, webbed, and ultrabornological, see [Flo71], [Jar81, 5.3.3], also [KM97a, 52.37]. We shall use this locally convex topology below only for $n = 1$ – in general it is stronger than the one which we will define in (3.1), but it has the same system of bounded sets, see (4.6).

2.9. **Lemma.** *For a logarithmically convex sequence M_k with $M_0 = 1$ we have*

$$M_1^k\, M_k \ge M_j\, M_{\alpha_1} \cdots M_{\alpha_j} \quad \textit{for all } \alpha_i \in \mathbb{N}_{>0}, \alpha_1 + \cdots + \alpha_j = k.$$

Proof. We use induction on k. The assertion is trivial for $k = j$. Assume that $j < k$. Then there exists i such that $\alpha_i \ge 2$. Put $\alpha_i' := \alpha_i - 1$. By induction hypothesis,

$$M_j\, M_{\alpha_1} \cdots M_{\alpha_i'} \cdots M_{\alpha_j} \le M_1^{k-1} M_{k-1}.$$

Since M_{k+1}/M_k is increasing by (2.1.2), we obtain

$$\begin{aligned} M_j\, M_{\alpha_1} \cdots M_{\alpha_j} &= M_j\, M_{\alpha_1} \cdots M_{\alpha_i'} \cdots M_{\alpha_j} \cdot \frac{M_{\alpha_i}}{M_{\alpha_i'}} \le \\ &\le M_1^{k-1}\, M_{k-1} \cdot \frac{M_k}{M_{k-1}} \le M_1^k\, M_k. \quad \square \end{aligned}$$

Table 1: Let $M = (M_k)$ and $N = (N_k)$ be increasing ($\le$) sequences of real numbers with $M_0 = N_0 = 1$. By U we denote an open subset of $\mathbb{R}^n$. The mapping $T_a : C^M(U) \to \mathcal{F}_n^M$ is the Taylor series homomorphism for $a \in U$ (see (2.2)). Recall that M is a DC-weight sequence if it is logarithmically convex and stable under derivation.

Properties of M		**Properties of** C^M
M increasing, $M_0 = 1$, (always assumed below this line)	$\Rightarrow$	$C^\omega(U) \subseteq C^M(U) \subseteq C^\infty(U)$
M is logarithmically convex (always assumed below this line), i.e., $M_k^2 \le M_{k-1}\, M_{k+1}$ for all k. Then: $(M_k)^{1/k}$ is increasing, $M_l\, M_k \le M_{l+k}$ for all l, k, and $M_1^k\, M_k \ge M_j\, M_{\alpha_1} \cdots M_{\alpha_j}$ for $\alpha_i \in \mathbb{N}_{>0}, \alpha_1 + \cdots + \alpha_j = k$.	$\Rightarrow$	$C^M(U)$ is a ring. C^M is closed under composition. C^M is closed under applying the inverse function theorem. C^M is closed under solving ODEs.
$\sup_{k\in\mathbb{N}_{>0}}(M_k/N_k)^{1/k} < \infty$	$\Leftrightarrow$	$C^M(U) \subseteq C^N(U)$
$\sup_{k\in\mathbb{N}_{>0}}(M_k)^{1/k} < \infty$	$\Leftrightarrow$	$C^\omega(U) = C^M(U)$
$\lim_{k\to\infty}(M_k)^{1/k} = \infty$	$\Leftrightarrow$	$C^\omega(U) \subsetneq C^M(U)$
$\sup_{k\in\mathbb{N}_{>0}}(M_{k+1}/M_k)^{1/k} < \infty$ (always assumed below this line)	$\Leftrightarrow$	C^M is closed under derivation.
$\sum_{k=0}^\infty \frac{M_k}{(k+1)M_{k+1}} = \infty$ or, equivalently, $\sum_{k=1}^\infty (\frac{1}{k!\, M_k})^{1/k} = \infty$	$\Leftrightarrow$	C^M is quasianalytic, i.e., $T_a : C^M(U) \to \mathcal{F}_n^M$ is injective (not surjective if $C^\omega(U) \subsetneq C^M(U)$).
$\sum_{k=0}^\infty \frac{M_k}{(k+1)M_{k+1}} < \infty$	$\Leftrightarrow$	C^M is non-quasianalytic. Then C^M partitions of unity exist.
$\lim_{k\to\infty}(M_k)^{1/k} = \infty$ and $\sum_{k=j}^\infty \frac{M_k}{(k+1)M_{k+1}} \le C \frac{M_j}{M_{j+1}}$ for all $j \in \mathbb{N}$ and some C	$\Leftrightarrow$	$C^\omega(U) \subsetneq C^M(U)$ and $T_a : C^M(U) \to \mathcal{F}_n^M$ is surjective, i.e., C^M is strongly non-quasianalytic.

M has moderate growth, i.e., $\sup_{j,k\in\mathbb{N}_{>0}}(\frac{M_{j+k}}{M_j\,M_k})^{1/(j+k)}<\infty$	$\Rightarrow$	C^M is cartesian closed will be proved in (5.3)
M is strongly regular, i.e., it is strongly non-quasianalytic and has moderate growth.	$\Rightarrow$	Whitney's extension theorem holds in C^M.
$\delta>0$ and $M_k=(k!)^\delta$ for $k\in\mathbb{N}$. Then M is strongly regular.	$\Leftrightarrow$	C^M is the Gevrey class $G^{1+\delta}$.

3. C^M-mappings

3.1. **Definition: C^M-mappings.** Let M be a DC-weight sequence, and let E be a locally convex vector space. A curve $c:\mathbb{R}\to E$ is called C^M if for each continuous linear functional $\ell\in E^*$ the curve $\ell\circ c:\mathbb{R}\to\mathbb{R}$ is of class C^M. The curve c is called *strongly* C^M if c is smooth and for all compact $K\subset\mathbb{R}$ there exists $\rho>0$ such that

$$\left\{\frac{c^{(k)}(x)}{\rho^k\,k!\,M_k}:k\in\mathbb{N},x\in K\right\}\text{ is bounded in }E.$$

The curve c is called *strongly uniformly* C^M if c is smooth and there exists $\rho>0$ such that

$$\left\{\frac{c^{(k)}(x)}{\rho^k\,k!\,M_k}:k\in\mathbb{N},x\in \mathbb{R}\right\}\text{ is bounded in }E.$$

Now let M be a non-quasianalytic DC-weight sequence. Let U be a c^∞-open subset of E, and let F be another locally convex vector space. *A mapping $f:U\to F$ is called C^M if f is smooth in the sense of* (7.3) *and if $f\circ c$ is a C^M-curve in F for every C^M-curve c in U.* Obviously, *the composite of C^M-mappings is again a C^M-mapping, and the chain rule holds.* This notion is equivalent to the expected one on Banach spaces, see (3.9) below.

We equip the space $C^M(U,F)$ with the initial locally convex structure with respect to the family of mappings

$$C^M(U,F)\xrightarrow{C^M(c,\ell)}C^M(\mathbb{R},\mathbb{R}),\quad f\mapsto\ell\circ f\circ c,\quad \ell\in E^*,c\in C^M(\mathbb{R},U)$$

where $C^M(\mathbb{R},\mathbb{R})$ carries the locally convex structure described in (2.8) and where E^* is the space of all continuous linear functionals on E.

For $U\subseteq\mathbb{R}^n$, this locally convex topology differs from the one described in (2.8), but they have the same bounded sets, see (4.6) below.

If F is convenient, then by standard arguments, the space $C^M(U,F)$ is c^∞-closed in the product $\prod_{\ell,c}C^M(\mathbb{R},\mathbb{R})$ and hence is *convenient.* If F is convenient, then a mapping $f:U\to F$ is C^M if and only if $\ell\circ f$ is C^M for all $\ell\in F^*$.

3.2. **Example: There are weak C^M-curves which are not strong.** By [Thi08, Theorem 1], for each DC-weight sequence M there exists $f\in C^M(\mathbb{R},\mathbb{R})$ such that $|f^{(k)}(0)|\geq k!\,M_k$ for all $k\in\mathbb{N}$. Then $g:\mathbb{R}\to\mathbb{R}^{\mathbb{N}}$ given by $g(t)_n=f(nt)$ is C^M but not strongly C^M. Namely, each bounded linear functional ℓ on $\mathbb{R}^{\mathbb{N}}$ depends only on finitely many coordinates, so we take the maximal ρ for the finitely many coordinates of g being involved. On the other hand, for each ρ and any compact neighborhood L of 0 the set

$$\left\{\frac{g^{(k)}(t)}{\rho^k\,k!\,M_k}:t\in L,k\in\mathbb{N}\right\}$$

has n-th coordinate unbounded if $n>\rho$.

3.3. **Lemma.** *Let E be a convenient vector space such that there exists a Baire vector space topology on the dual E^* for which the point evaluations ev_x are continuous for all $x \in E$. Then a curve $c : \mathbb{R} \to E$ is C^M if and only if c is strongly C^M, for any DC-weight sequence M.*

See (5.2) for a more general version.

Proof. Let K be compact in $\mathbb{R}$. We consider the sets

$$A_{\rho,C} := \Big\{\ell \in E^* : \frac{|(\ell \circ c)^{(k)}(x)|}{\rho^k\, k!\, M_k} \le C \text{ for all } k \in \mathbb{N}, x \in K\Big\}$$

which are closed subsets in E^* for the Baire topology. We have $\bigcup_{\rho,C} A_{\rho,C} = E^*$. By the Baire property there exists ρ and C such that the interior U of $A_{\rho,C}$ is non-empty. If $\ell_0 \in U$ then for all $\ell \in E^*$ there is an $\epsilon > 0$ such that $\epsilon\ell \in U - \ell_0$ and hence for all $x \in K$ and all k we have

$$|(\ell \circ c)^{(k)}(x)| \le \tfrac{1}{\epsilon}\Big(|((\epsilon\ell + \ell_0) \circ c)^{(k)}(x)| + |(\ell_0 \circ c)^{(k)}(x)|\Big) \le \tfrac{2C}{\epsilon}\, \rho^k\, k!\, M_k.$$

So the set

$$\Big\{\frac{c^{(k)}(x)}{\rho^k\, k!\, M_k} : k \in \mathbb{N}, x \in K\Big\}$$

is weakly bounded in E and hence bounded. □

3.4. **Lemma.** *Let M be a DC-weight sequence, and let E be a Banach space. For a curve $c : \mathbb{R} \to E$ the following are equivalent.*

(1) *c is C^M.*
(2) *For each sequence (r_k) with $r_k\, t^k \to 0$ for all $t > 0$, and each compact set K in $\mathbb{R}$, the set $\{\frac{1}{k!M_k}\, c^{(k)}(a)\, r_k : a \in K, k \in \mathbb{N}\}$ is bounded in E.*
(3) *For each sequence (r_k) satisfying $r_k > 0$, $r_k r_\ell \ge r_{k+\ell}$, and $r_k\, t^k \to 0$ for all $t > 0$, and each compact set K in $\mathbb{R}$, there exists an $\epsilon > 0$ such that $\{\frac{1}{k!M_k}\, c^{(k)}(a)\, r_k\, \epsilon^k : a \in K, k \in \mathbb{N}\}$ is bounded in E.*

Proof. (1) $\implies$ (2) For K, there exists $\rho > 0$ such that

$$\Big\|\frac{c^{(k)}(a)}{k!\, M_k} r_k\Big\|_E = \Big\|\frac{c^{(k)}(a)}{k!\, \rho^k\, M_k}\Big\|_E \cdot |r_k \rho^k|$$

is bounded uniformly in $k \in \mathbb{N}$ and $a \in K$ by (3.3).

(2) $\implies$ (3) Use $\epsilon = 1$.

(3) $\implies$ (1) Let $a_k := \sup_{a\in K} \|\frac{1}{k!\, M_k}\, c^{(k)}(a)\|_E$. Using [KM97a, 9.2.(4⇒1)] these are the coefficients of a power series with positive radius of convergence. Thus a_k/ρ^k is bounded for some $\rho > 0$. □

3.5. **Lemma.** *Let M be a DC-weight sequence. Let E be a convenient vector space, and let $\mathcal{S}$ be a family of bounded linear functionals on E which together detect bounded sets (i.e., $B \subseteq E$ is bounded if and only if $\ell(B)$ is bounded for all $\ell \in \mathcal{S}$). Then a curve $c : \mathbb{R} \to E$ is C^M if and only if $\ell \circ c : \mathbb{R} \to \mathbb{R}$ is C^M for all $\ell \in \mathcal{S}$.*

Proof. For smooth curves this follows from [KM97a, 2.1 and 2.11]. By (3.4), for any $\ell \in E'$, the function $\ell \circ c$ is C^M if and only if:

(1) For each sequence (r_k) with $r_k\, t^k \to 0$ for all $t > 0$, and each compact set K in $\mathbb{R}$, the set $\{\frac{1}{k!M_k}\, (\ell \circ c)^{(k)}(a)\, r_k : a \in K, k \in \mathbb{N}\}$ is bounded.

By (1) the curve c is C^M if and only if the set $\{\frac{1}{k!M_k}\, c^{(k)}(a)\, r_k : a \in K, k \in \mathbb{N}\}$ is bounded in E. By (1) again this is in turn equivalent to $\ell \circ c \in C^M$ for all $\ell \in \mathcal{S}$, since $\mathcal{S}$ detects bounded sets. □

3.6. C^M **curve lemma.** A sequence x_n in a locally convex space E is said to be *Mackey convergent* to x, if there exists some $\lambda_n \nearrow \infty$ such that $\lambda_n(x_n - x)$ is bounded. If we fix $\lambda = (\lambda_n)$ we say that x_n is λ-converging.

Lemma. *Let M be a non-quasianalytic DC-weight sequence. Then there exist sequences $\lambda_k \to 0$, $t_k \to t_\infty$, $s_k > 0$ in $\mathbb{R}$ with the following property: For $1/\lambda = (1/\lambda_n)$-converging sequences x_n and v_n in a convenient vector space E there exists a strongly uniformly C^M-curve $c : \mathbb{R} \to E$ with $c(t_k + t) = x_k + t.v_k$ for $|t| \le s_k$.*

Proof. Since C^M is not quasianalytic we have $\sum_k 1/(k!M_k)^{1/k} < \infty$. We choose another non-quasianalytic DC-weight sequence $\bar M = (\bar M_k)$ with $(M_k/\bar M_k)^{1/k} \to \infty$. By (2.3) there is a $C^{\bar M}$-function $\varphi : \mathbb{R} \to [0,1]$ which is 0 on $\{t : |t| \ge \frac12\}$ and which is 1 on $\{t : |t| \le \frac13\}$, i.e. there exist $\bar C, \rho > 0$ such that

$$|\varphi^{(k)}(t)| \le \bar C\,\rho^k\,k!\,\bar M_k \quad \text{for all } t \in \mathbb{R} \text{ and } k \in \mathbb{N}.$$

For x, v in a absolutely convex bounded set $B \subseteq E$ and $0 < T \le 1$ the curve $c : t \mapsto \varphi(t/T)\cdot(x + t\,v)$ satisfies (cf. [Bom67, Lemma 2]):

$$\begin{aligned}
c^{(k)}(t) &= T^{-k}\varphi^{(k)}(\tfrac{t}{T}).(x+t.v) + k\,T^{1-k}\,\varphi^{(k-1)}(\tfrac{t}{T}).v\\
&\in T^{-k}\bar C\,\rho^k\,k!\,\bar M_k(1+\tfrac{T}{2}).B + k\,T^{1-k}\,\bar C\,\rho^{k-1}\,(k-1)!\,\bar M_{k-1}.B\\
&\subseteq T^{-k}\bar C\,\rho^k\,k!\,\bar M_k(1+\tfrac{T}{2}).B + T\,T^{-k}\,\bar C\tfrac{1}{\rho}\,\rho^k\,k!\,\bar M_k.B\\
&\subseteq \bar C(\tfrac32+\tfrac1\rho)\,T^{-k}\,\rho^k\,k!\,\bar M_k.B
\end{aligned}$$

So there are $\rho, C := \bar C(\frac32 + \frac1\rho) > 0$ which do not depend on x, v and T such that $c^{(k)}(t) \in C\,T^{-k}\,\rho^k\,k!\,\bar M_k.B$ for all k and t.

Let $0 < T_j \le 1$ with $\sum_j T_j < \infty$ and $t_k := 2\sum_{j<k} T_j + T_k$. We choose the λ_j such that $0 < \lambda_j/T_j^k \le M_k/\bar M_k$ (note that $T_j^k\,M_k/\bar M_k \to \infty$ for $k \to \infty$) for all j and k, and that $\lambda_j/T_j^k \to 0$ for $j \to \infty$ and each k.

Without loss we may assume that $x_n \to 0$. By assumption there exists a closed bounded absolutely convex subset B in E such that $x_n, v_n \in \lambda_n \cdot B$. We consider $c_j : t \mapsto \varphi\big((t-t_j)/T_j\big)\cdot\big(x_j + (t-t_j)\,v_j\big)$ and $c := \sum_j c_j$. The c_j have disjoint support $\subseteq [t_j - T_j, t_j + T_j]$, hence c is C^∞ on $\mathbb{R}\setminus\{t_\infty\}$ with

$$c^{(k)}(t) \in C\,T_j^{-k}\,\rho^k k!\bar M_k\,\lambda_j \cdot B \quad \text{for } |t - t_j| \le T_j.$$

Then

$$\|c^{(k)}(t)\|_B \le C\,\rho^k\,k!\bar M_k\,\frac{\lambda_j}{T_j^k} \le C\rho^k k!\bar M_k\,\frac{M_k}{\bar M_k} = C\,\rho^k\,k!M_k$$

for $t \ne t_\infty$. Hence $c : \mathbb{R} \to E_B$ (see [KM97a, 2.14.6] or (7.1)) is smooth at t_∞ as well, and is strongly C^M by the following lemma. □

3.7. **Lemma.** *Let $c : \mathbb{R}\setminus\{0\} \to E$ be strongly C^M in the sense that c is smooth and for all bounded $K \subset \mathbb{R}\setminus\{0\}$ there exists $\rho > 0$ such that*

$$\left\{\frac{c^{(k)}(x)}{\rho^k\,k!\,M_k} : k \in \mathbb{N}, x \in K\right\} \text{ is bounded in } E.$$

Then c has a unique extension to a strongly C^M-curve on $\mathbb{R}$.

Proof. The curve c has a unique extension to a smooth curve by [KM97a, 2.9]. The strong C^M condition extends by continuity. □

3.8. **Corollary.** *Let M be a non-quasianalytic DC-weight sequence. Then we have:*

(1) *The final topology on E with respect to all strongly C^M-curves equals the Mackey closure topology.*

(2) *A locally convex space E is convenient* (7.2) *if and only if for any (strongly) C^M-curve $c : \mathbb{R} \to E$ there exists a (strongly) C^M-curve $c_1 : \mathbb{R} \to E$ with $c_1' = c$.*

Proof. (1) For any Mackey converging sequence there exists a C^M-curve passing through a subsequence in finite time by (3.6). So the final topologies generated by the Mackey converging sequences and by the C^M-curves coincide.

(2) In order to show that a locally convex space E is convenient, we have to prove that it is c^∞-closed in its completion. So let $x_n \in E$ converge Mackey to x_∞ in the completion. Then by (3.6) there exists a strongly C^M-curve c in the completion passing in finite time through a subsequence of the x_n with velocity $v_n = 0$. The form of c (in the proof of (3.6)) shows that its derivatives $c^{(k)}(t)$ for $k > 0$ are multiples of the x_n and hence have values in E. Then c' is a C^M-curve and so the antiderivative c of c' lies in E by assumption. In particular $x_\infty \in c(\mathbb{R}) \subseteq E$.

Conversely, if E is convenient, then every smooth curve c has a smooth antiderivative c_1 in E by [KM97a, 2.14]. Since

$$\frac{1}{\rho^{k+1}(k+1)!\, M_{k+1}} c_1^{(k+1)}(t) = \frac{M_k}{\rho(k+1)M_{k+1}} \frac{1}{\rho^k\, k!\, M_k} c^{(k)}(t)$$

and since

$$\frac{M_k}{\rho(k+1)M_{k+1}} \le \frac{1}{\rho M_1}$$

by (2.1.2) the antiderivative c_1 is (strongly) C^M if c is so. □

3.9. **Theorem.** *Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence. Let $U \subseteq E$ be c^∞-open in a convenient vector space, and let F be a Banach space. For a mapping $f : U \to F$, the following assertions are equivalent.*

(1) *f is C^M.*
(2) *f is C^M along strongly C^M curves.*
(3) *f is smooth, and for each closed bounded absolutely convex B in E and each $x \in U \cap E_B$ there are $r > 0$, $\rho > 0$, and $C > 0$ such that*

$$\frac{1}{k!\, M_k} \|d^k(f \circ i_B)(a)\|_{L^k(E_B,F)} \le C\, \rho^k$$

for all $a \in U \cap E_B$ with $\|a - x\|_B \le r$ and all $k \in \mathbb{N}$.

(4) *f is smooth, and for each closed bounded absolutely convex B in E and each compact $K \subseteq U \cap E_B$ there are $\rho > 0$ and $C > 0$ such that*

$$\frac{1}{k!\, M_k} \|d^k(f \circ i_B)(a)\|_{L^k(E_B,F)} \le C\, \rho^k$$

for all $a \in K$ and all $k \in \mathbb{N}$.

Proof. (1) $\implies$ (2) is clear.

(2) $\implies$ (3) Without loss let $E = E_B$ be a Banach space. For each $v \in E$ and $x \in U$ the iterated directional derivative $d_v^k f(x)$ exists since f is C^M along affine lines. To show that f is smooth it suffices to check that $d_{v_n}^k f(x_n)$ is bounded for each $k \in \mathbb{N}$ and each Mackey convergent sequences x_n and $v_n \to 0$, by [KM97a, 5.20]. For contradiction let us assume that there exist k and sequences x_n and v_n with $\|d_{v_n}^k f(x_n)\| \to \infty$. By passing to a subsequence we may assume that x_n and v_n are $(1/\lambda_n)$-converging for the λ_n from (3.6). Hence there exists a strongly C^M-curve c in E and with $c(t + t_n) = x_n + t.v_n$ for t near 0 for each n separately, and for t_n from (3.6). But then $\|(f \circ c)^{(k)}(t_n)\| = \|d_{v_n}^k f(x_n)\| \to \infty$, a contradiction. So f is smooth.

Assume for contradiction that the boundedness condition in (3) does not hold. Then there exists $x \in U$ such that for all $r, \rho, C > 0$ there is an $a = a(r, \rho, C) \in U$ and $k = k(r, \rho, C) \in \mathbb{N}$ with $\|a - x\| \le r$ but

$$\frac{1}{k!\, M_k} \|d^k f(a)\|_{L^k(E,F)} > C\, \rho^k.$$

By [KM97a, 7.13] we have

$$\|d^k f(a)\|_{L^k(E,F)} \le (2e)^k \sup_{\|v\|\le 1} \|d_v^k f(a)\|.$$

So for each ρ and n take $r = \frac{1}{n\rho}$ and $C = n$. Then there are $a_{n,\rho} \in U$ with $\|a_{n,\rho} - x\| \le \frac{1}{n\rho}$, moreover $v_{n,\rho}$ with $\|v_{n,\rho}\| = 1$, and $k_{n,\rho} \in \mathbb{N}$ such that

$$\frac{(2e)^{k_{n,\rho}}}{k_{n,\rho}!\, M_{k_{n,\rho}}\, \rho^{k_{n,\rho}}} \|d_{v_{n,\rho}}^{k_{n,\rho}} f(a_{n,\rho})\| > n.$$

Since $K := \{a_{n,\rho} : n, \rho \in \mathbb{N}\} \cup \{x\}$ is compact, this contradicts the following

Claim. *For each compact $K \subseteq E$ there are $C, \rho \ge 0$ such that for all $k \in \mathbb{N}$ and $x \in K$ we have $\sup_{\|v\|\le 1} \|d_v^k f(x)\| \le C\, \rho^k k! M_k$.*

Otherwise, there exists a compact set $K \subseteq E$ such that for each $n \in \mathbb{N}$ there are $k_n \in \mathbb{N}$, $x_n \in K$, and v_n with $\|v_n\| = 1$ such that

$$\|d_{v_n}^{k_n} f(x_n)\| > k_n!\, M_{k_n} \left(\frac{1}{\lambda_n^2}\right)^{k_n+1},$$

where we used $C = \rho := 1/\lambda_n^2$ with the λ_n from (3.6). By passing to a subsequence (again denoted n) we may assume that the x_n are $1/\lambda$-converging, thus there exists a strongly C^M-curve $c : \mathbb{R} \to E$ with $c(t_n + t) = x_n + t.\lambda_n.v_n$ for t near 0 by (3.6). Since

$$(f \circ c)^{(k)}(t_n) = \lambda_n^k d_{v_n}^k f(x_n),$$

we get

$$\left(\frac{\|(f \circ c)^{(k_n)}(t_n)\|}{k_n! M_{k_n}}\right)^{\frac{1}{k_n+1}} = \left(\lambda_n^{k_n} \frac{\|d_{v_n}^{k_n} f(x_n)\|}{k_n! M_{k_n}}\right)^{\frac{1}{k_n+1}} > \frac{1}{\lambda_n^{\frac{k_n+2}{k_n+1}}} \to \infty,$$

a contradiction to $f \circ c \in C^M$.

(3) $\implies$ (4) is obvious since the compact set K is covered by finitely many balls.

(4) $\implies$ (1) We have to show that $f \circ c$ is C^M for each C^M-curve $c : \mathbb{R} \to E$. By (3.4.2) it suffices to show that for each sequence (r_k) satisfying $r_k > 0$, $r_k r_\ell \ge r_{k+\ell}$, and $r_k\, t^k \to 0$ for all $t > 0$, and each compact interval I in $\mathbb{R}$, there exists an $\epsilon > 0$ such that $\{\frac{1}{k! M_k} (f \circ c)^{(k)}(a)\, r_k\, \epsilon^k : a \in I, k \in \mathbb{N}\}$ is bounded.

By (3.4.2) applied to $r_k 2^k$ instead of r_k, for each $\ell \in E^*$, each sequence (r_k) with $r_k\, t^k \to 0$ for all $t > 0$, and each compact interval I in $\mathbb{R}$ the set $\{\frac{1}{k! M_k} (\ell \circ c)^{(k)}(a)\, r_k\, 2^k : a \in I, k \in \mathbb{N}\}$ is bounded in $\mathbb{R}$. Thus $\{\frac{1}{k! M_k} c^{(k)}(a)\, r_k\, 2^k : a \in I, k \in \mathbb{N}\}$ is contained in some closed absolutely convex $B \subseteq E$. Consequently, $c^{(k)} : I \to E_B$ is smooth and hence $K_k := \{\frac{1}{k! M_k} c^{(k)}(a)\, r_k\, 2^k : a \in I\}$ is compact in E_B for each k. Then each sequence (x_n) in the set

$$K := \left\{\frac{1}{k! M_k} c^{(k)}(a)\, r_k : a \in I, k \in \mathbb{N}\right\} = \bigcup_{k \in \mathbb{N}} \frac{1}{2^k} K_k$$

has a cluster point in $K \cup \{0\}$: either there is a subsequence in one K_k, or $2^{k_n} x_{k_n} \in K_{k_n} \subseteq B$ for $k_n \to \infty$, hence $x_{k_n} \to 0$ in E_B. So $K \cup \{0\}$ is compact.

By Faà di Bruno ([FdB55] for the 1-dimensional version)

$$\frac{(f\circ c)^{(k)}(a)}{k!} = \sum_{j\ge 0} \sum_{\substack{\alpha\in\mathbb{N}_{>0}^j\\ \alpha_1+\dots+\alpha_j=k}} \frac{1}{j!} d^j f(c(a))\Big(\frac{c^{(\alpha_1)}(a)}{\alpha_1!},\dots,\frac{c^{(\alpha_j)}(a)}{\alpha_j!}\Big)$$

and (2.1.4) for $a\in I$ and $k\in\mathbb{N}$ we have

$$\begin{aligned}
&\left\| \frac{1}{k!M_k}\,(f\circ c)^{(k)}(a)\,r_k \right\| \le \\
&\le M_1^k \sum_{j\ge 0} \sum_{\substack{\alpha\in\mathbb{N}_{>0}^j\\ \alpha_1+\dots+\alpha_j=k}} \frac{\|d^j f(c(a))\|_{L^j(E_B,F)}}{j!M_j} \prod_{i=1}^j \frac{\|c^{(\alpha_i)}(a)\|_B\, r_{\alpha_i}}{\alpha_i!M_{\alpha_i}} \\
&\le M_1^k \sum_{j\ge 0} \binom{k-1}{j-1} C\,\rho^j\,\frac{1}{2^k} = M_1^k \rho(1+\rho)^{k-1} C\,\frac{1}{2^k}.
\end{aligned}$$

So $\left\{ \frac{1}{k!M_k}\,(f\circ c)^{(k)}(a) \left(\frac{2}{M_1(1+\rho)}\right)^k r_k : a\in I, k\in\mathbb{N}\right\}$ is bounded as required. $\square$

3.10. **Corollary.** *Let M and N be non-quasianalytic DC-weight sequences with* (2.1.5)

$$\sup_{k\in\mathbb{N}_{>0}} \Big(\frac{M_k}{N_k}\Big)^{\frac{1}{k}} < \infty.$$

Then $C^M(U,F)\subseteq C^N(U,F)$ for all convenient vector spaces E and F and each c^∞-open $U\subseteq E$. Moreover $C^\omega(U,F)\subseteq C^M(U,F)\subseteq C^\infty(U,F)$. All these inclusions are bounded.

Proof. The inclusions $C^M\subseteq C^N\subseteq C^\infty$ follow from (3.9) since this is true for condition (3.9.3) applied to $\ell\circ f$ for $\ell\in F^*$.

Without loss let $F=\mathbb{R}$. If f is C^ω then for each closed absolutely convex bounded $B\subseteq E$ the mapping $f\circ i_B : U\cap E_B\to\mathbb{R}$ is given by its locally converging Taylor series by [KM97a, 10.1]. So (3.9.3) is satisfied for $M_k=1$ and thus for each DC-weight sequence M. So f is C^M. All inclusions are bounded by the uniform boundedness principle (4.1) below for C^M and [KM97a, 5.26] for C^∞. $\square$

3.11. **Corollary.** *Let $M=(M_k)$ be a non-quasianalytic DC-weight sequence. Then we have:*

1. *Multilinear mappings between convenient vector spaces are C^M if and only if they are bounded.*
2. *If $f: E\supseteq U\to F$ is C^M, then the derivative $df: U\to L(E,F)$ is C^M, and also $\widehat{df}: U\times E\to F$ is C^M, where the space $L(E,F)$ of all bounded linear mappings is considered with the topology of uniform convergence on bounded sets.*
3. *The chain rule holds.*

Proof. (1) If f is multilinear and C^M then it is smooth by (3.9) and hence bounded by (7.3.2). Conversely, if f is multilinear and bounded then it is smooth by (7.3.2). Furthermore, $f\circ i_B$ is multilinear and continuous and all derivatives of high order vanish. Thus condition (3.9.3) is satisfied, so f is C^M.

(2) Since f is smooth, by (7.3.3) the map $df: U\to L(E,F)$ exists and is smooth. Let $c:\mathbb{R}\to U$ be a C^M-curve. We have to show that $t\mapsto df(c(t))\in L(E,F)$ is C^M. By [KM97a, 5.18] and (3.5) it suffices to show that $t\mapsto c(t)\mapsto \ell(df(c(t)).v)\in\mathbb{R}$ is C^M for each $\ell\in F^*$ and $v\in E$. We are reduced to show that $x\mapsto \ell(df(x).v)$ satisfies the conditions of (3.9). By (3.9) applied to $\ell\circ f$, for each closed bounded

absolutely convex B in E and each $x \in U \cap E_B$ there are $r > 0$, $\rho > 0$, and $C > 0$ such that

$$\frac{1}{k!\,M_k}\|d^k(\ell \circ f \circ i_B)(a)\|_{L^k(E_B,\mathbb{R})} \le C\,\rho^k$$

for all $a \in U \cap E_B$ with $\|a - x\|_B \le r$ and all $k \in \mathbb{N}$. For $v \in E$ and those B containing v we then have

$$\begin{aligned}
\|d^k(d(\ell \circ f)(\quad)(v)) \circ i_B)(a)\|_{L^k(E_B,\mathbb{R})} &= \|d^{k+1}(\ell \circ f \circ i_B)(a)(v, \dots)\|_{L^k(E_B,\mathbb{R})}\\
&\le \|d^{k+1}(\ell \circ f \circ i_B)(a)\|_{L^{k+1}(E_B,\mathbb{R})}\|v\|_{E_B} \le C\,\rho^{k+1}\,(k+1)!\,M_{k+1}\\
&\le C\,\rho^k k! M_k\Big((k+1)\rho\frac{M_{k+1}}{M_k}\Big)\\
&\le C\,\bar\rho^k k! M_k \qquad \text{for } \bar\rho > \rho \sup_{k\ge 1}\Big((k+1)\rho\frac{M_{k+1}}{M_k}\Big)^{1/k},
\end{aligned}$$

the latter quantity being finite by (2.1.6). By (4.2) below also $\widehat{df}$ is C^M.

(3) This is valid for all smooth f. $\square$

3.12. **Remark.** For a quasianalytic DC-weight sequence M, theorem (3.9) is wrong. In fact, take any rational function, e.g. $\frac{xy^2}{x^2+y^2}$. Let $t \mapsto x(t), y(t)$ be in $C^M(\mathbb{R},\mathbb{R})$ with $x(0) = 0 = y(0)$. Then $x(t) = t^r\bar x(t)$ and $y(t) = t^r\bar y(t)$ for $r > 0$ and for C^M-functions $\bar x$ and $\bar y$ since C^M is derivation closed. If (x, y) is not constant we may choose r such that $\bar x(0)^2 + \bar y(0)^2 \ne 0$, since C^M is quasianalytic. Then $t \mapsto \frac{x(t)y(t)^2}{x(t)^2+y(t)^2} = t^r\frac{\bar x(t)\bar y(t)^2}{\bar x(t)^2+\bar y(t)^2}$ is C^M near 0, but the rational function is not smooth.

4. C^M-uniform boundedness principles

4.1. **Theorem** (Uniform boundedness principle)**.** *Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence. Let E, F, G be convenient vector spaces and let $U \subseteq F$ be c^∞-open. A linear mapping $T : E \to C^M(U, G)$ is bounded if and only if $\operatorname{ev}_x \circ T : E \to G$ is bounded for every $x \in U$.*

This is the C^M-analogon of (7.3.7). Compare with [KM97a, 5.22–5.26] for the principles behind it. They will be used in the following proof and in (4.6) and (4.10).

Proof. For $x \in U$ and $\ell \in G^*$ the linear mapping $\ell \circ \operatorname{ev}_x = C^M(x, \ell) : C^M(U, G) \to \mathbb{R}$ is continuous, thus ev_x is bounded. So if T is bounded then so is $\operatorname{ev}_x \circ T$.

Conversely, suppose that $\operatorname{ev}_x \circ T$ is bounded for all $x \in U$. For each closed absolutely convex bounded $B \subseteq E$ we consider the Banach space E_B. For each $\ell \in G^*$, each C^M-curve $c : \mathbb{R} \to U$, each $t \in \mathbb{R}$, and each compact $K \subset \mathbb{R}$ the composite given by the following diagram is bounded.

$$\begin{tikzcd}
E \arrow[r, "T"] & C^M(U,G) \arrow[rr, "\operatorname{ev}_{c(t)}"] \arrow[d, "C^M(c,\ell)"] & & G \arrow[d, "\ell"] \\
E_B \arrow[u] \arrow[r] & C^M(\mathbb{R},\mathbb{R}) \arrow[r] & \varinjlim_\rho C^M_\rho(K,\mathbb{R}) \arrow[r, "\operatorname{ev}_t"] & \mathbb{R}
\end{tikzcd}$$

By [KM97a, 5.24 and 5.25] the map T is bounded. In more detail: Since $\varinjlim_\rho C^M_\rho(K,\mathbb{R})$ is webbed by (2.8), the closed graph theorem [KM97a, 52.10] yields that the mapping $E_B \to \varinjlim_\rho C^M_\rho(K,\mathbb{R})$ is continuous. Thus T is bounded. $\square$

4.2. **Corollary.** *Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence.*

(1) *For convenient vector spaces E and F, on $L(E,F)$ the following bornologies coincide which are induced by:*
- *The topology of uniform convergence on bounded subsets of E.*
- *The topology of pointwise convergence.*
- *The embedding $L(E,F) \subset C^\infty(E,F)$.*
- *The embedding $L(E,F) \subset C^M(E,F)$.*

(2) *Let E, F, G be convenient vector spaces and let $U \subset E$ be c^∞-open. A mapping $f : U \times F \to G$ which is linear in the second variable is C^M if and only if $f^\vee : U \to L(F,G)$ is well defined and C^M.*

Analogous results hold for spaces of multilinear mappings.

Proof. (1) That the first three topologies on $L(E,F)$ have the same bounded sets has been shown in [KM97a, 5.3 and 5.18]. The inclusion $C^M(E,F) \to C^\infty(E,F)$ is bounded by (3.10) and by the uniform boundedness principle in (7.3.7). It remains to show that the inclusion $L(E,F) \to C^M(E,F)$ is bounded, where the former space is considered with the topology of uniform convergence on bounded sets. This follows from the uniform boundedness principle (4.1).

(2) The assertion for C^∞ is true by (7.3.6).

If f is C^M let $c : \mathbb{R} \to U$ be a C^M-curve. We have to show that $t \mapsto f^\vee(c(t)) \in L(F,G)$ is C^M. By [KM97a, 5.18] and (3.5) it suffices to show that $t \mapsto \ell(f^\vee(c(t))(v)) = \ell(f(c(t),v)) \in \mathbb{R}$ is C^M for each $\ell \in G^*$ and $v \in F$; this is obviously true.

Conversely, let $f^\vee : U \to L(F,G)$ be C^M. We claim that $f : U \times F \to G$ is C^M. By composing with $\ell \in G^*$ we may assume that $G = \mathbb{R}$. By induction we have

$$\begin{aligned}
&d^k f(x,w_0)\big((v_k,w_k),\dots,(v_1,w_1)\big) = d^k(f^\vee)(x)(v_k,\dots,v_1)(w_0)+\\
&+\sum_{i=1}^k d^{k-1}(f^\vee)(x)(v_k,\dots,\widehat{v_i},\dots,v_1)(w_i)
\end{aligned}$$

We check condition (3.9.3) for f:

$$\begin{aligned}
&\|d^k f(x,w_0)\|_{L^k(E_B\times F_{B'},\mathbb{R})} \le\\
&\le \|d^k(f^\vee)(x)(\dots)(w_0)\|_{L^k(E_B,\mathbb{R})} + \sum_{i=1}^k \|d^{k-1}(f^\vee)(x)\|_{L^{k-1}(E_B,L(F_{B'},\mathbb{R}))}\\
&\le \|d^k(f^\vee)(x)\|_{L^k(E_B,L(F_{B'},\mathbb{R}))}\|w_0\|_{B'} + \sum_{i=1}^k \|d^{k-1}(f^\vee)(x)\|_{L^{k-1}(E_B,L(F_{B'},\mathbb{R}))}\\
&\le C\,\rho^k\,k!\,M_k\|w_0\|_{B'} + \sum_{i=1}^k C\,\rho^{k-1}\,(k-1)!\,M_{k-1} = C\,\rho^k\,k!\,M_k(\|w_0\|_{B'} + \tfrac{M_{k-1}}{\rho\,M_k})
\end{aligned}$$

where we used (3.9.3) for $L(i_{B'},\mathbb{R}) \circ f^\vee : U \to L(F_{B'},\mathbb{R})$. Thus f is C^M. □

4.3. **Proposition.** *Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence. Let E and F be convenient vector spaces and let $U \subseteq E$ be c^∞-open. Then we have the bornological identity*

$$C^M(U,F) = \varprojlim_s C^M(\mathbb{R},F),$$

where s runs through the strongly C^M-curves in U and the connecting mappings are given by g^* for all reparametrizations $g \in C^M(\mathbb{R},\mathbb{R})$ of curves s.

Proof. By (3.9) the linear spaces $C^M(U,F)$, $\varprojlim_s C^M(\mathbb{R},F)$ and $\varprojlim_c C^M(\mathbb{R},F)$ coincide, where c runs through the C^M-curves in U: Each element $(f_c)_c$ determines a unique function $f : U \to F$ given by $f(x) := (f \circ \mathrm{const}_x)(0)$ with $f \circ c = f_c$ for all such curves c, and $f \in C^M$ if and only if $f_c \in C^M$ for all such c, by (3.9).

Since $C^M(\mathbb{R}, F)$ carries the initial structure with respect to ℓ_* for all $\ell \in F^*$ we may assume $F = \mathbb{R}$. Obviously the identity $\varprojlim_c C^M(\mathbb{R},\mathbb{R}) \to \varprojlim_s C^M(\mathbb{R},\mathbb{R})$ is continuous. As projective limit the later space is convenient, so we may apply the uniform boundedness principle (4.1) to conclude that the identity in the converse direction is bounded. □

4.4. **Proposition.** *Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence. Let E and F be convenient vector spaces and let $U \subseteq E$ be c^∞-open. Then the bornology of $C^M(U, F)$ is initial with respect to each of the following families of mappings*

$$i_B^* = C^M(i_B, F) : C^M(U, F) \to C^M(U \cap E_B, F), \tag{1}$$

$$C^M(i_B, \pi_V) : C^M(U, F) \to C^M(U \cap E_B, F_V), \tag{2}$$

$$C^M(i_B, \ell) : C^M(U, F) \to C^M(U \cap E_B, \mathbb{R}), \tag{3}$$

where B runs through the closed absolutely convex bounded subsets of E and $i_B : E_B \to E$ denotes the inclusion, and where ℓ runs through the continuous linear functionals on F, and where V runs through the absolutely convex 0-neighborhoods of F and F_V is obtained by factoring out the kernel of the Minkowsky functional of V and then taking the completion with respect to the induced norm.

Warning: The structure in (2) gives a projective limit description of $C^M(U, F)$ if and only if F is complete since then $F = \varprojlim_V F_V$.

Proof. Since $i_B : E_B \to E$, $\pi_V : F \to F_V$ and $\ell : F \to \mathbb{R}$ are bounded linear the mappings i_B^*, $C^M(i_B, \pi_V)$ and $C^M(i_B, \ell)$ are bounded and linear.

The structures given by (1), (2) and (3) are successively weaker. So let, conversely, $C^M(i_B, \ell)(B)$ be bounded in $C^M(U \cap E_B, \mathbb{R})$ for all B and ℓ. By (4.3) $C^M(U, F)$ carries the initial structure with respect to all $c^* : C^M(U, F) \to C^M(\mathbb{R}, F)$, where $c : \mathbb{R} \to U$ are the strongly C^M curves and these factor locally as (strongly) C^M-curves into some E_B. By definition $C^M(\mathbb{R}, F)$ carries the initial structure with respect to $C^M(\iota_I, \ell) : C^M(\mathbb{R}, F) \to C^M(I, \mathbb{R})$ where $\iota_I : I \hookrightarrow \mathbb{R}$ are the inclusions of compact intervals into $\mathbb{R}$ and $\ell \in F^*$. Thus $C^M(U, F)$ carries the initial structure with respect to $C^M(c|_I, \ell) : C^M(U, F) \to C^M(I, \mathbb{R})$, which is coarser than that induced by $C^M(U, F) \to C^M(U \cap E_B, \mathbb{R})$. □

4.5. **Definition.** Let E and F be Banach spaces and $A \subseteq E$ convex. We consider the linear space $C^\infty(A, F)$ consisting of all sequences $(f^k)_k \in \prod_{k\in\mathbb{N}} C(A, L^k(E, F))$ satisfying

$$f^k(y)(v) - f^k(x)(v) = \int_0^1 f^{k+1}(x + t(y - x))(y - x, v)\, dt$$

for all $k \in \mathbb{N}$, $x, y \in A$, and $v \in E^k$. If A is open we can identify this space with that of all smooth functions $A \to F$ by passing to jets.

In addition, let $M = (M_k)$ be a non-quasianalytic DC-weight sequence and (r_k) a sequence of positive real numbers. Then we consider the normed spaces

$$C^M_{(r_k)}(A, F) := \Big\{ (f^k)_k \in C^\infty(A, F) : \|(f^k)\|_{(r_k)} < \infty \Big\}$$

where the norm is given by

$$\|(f^k)\|_{(r_k)} := \sup\Big\{ \frac{\|f^k(a)(v_1, \dots, v_k)\|}{k!\, r_k\, M_k\, \|v_1\| \cdot \dots \cdot \|v_k\|} : k \in \mathbb{N}, a \in A, v_i \in E \Big\}.$$

If $(r_k) = (\rho^k)$ for some $\rho > 0$ we just write ρ instead of (r_k) as indices. The spaces $C^M_{(r_k)}(A, F)$ are Banach spaces, since they are closed in $\ell^\infty(\mathbb{N}, \ell^\infty(A, L^k(E, F)))$ via $(f^k)_k \mapsto (k \mapsto \frac{1}{k!\, r_k\, M_k} f^k)$.

4.6. **Theorem.** *Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence. Let E and F be Banach spaces and let $U \subseteq E$ be open. Then the space $C^M(U, F)$ can be described bornologically in the following equivalent ways, i.e. these constructions give the same vector space and the same bounded sets.*

$$\varprojlim_K \varinjlim_{\rho, W} C^M_\rho(W, F) \tag{1}$$

$$\varprojlim_K \varinjlim_{\rho} C^M_\rho(K, F) \tag{2}$$

$$\varprojlim_{K,(r_k)} C^M_{(r_k)}(K, F) \tag{3}$$

$$\varprojlim_{c,I} \varinjlim_{\rho} C^M_\rho(I, F) \tag{4}$$

Moreover, all involved inductive limits are regular, i.e. the bounded sets of the inductive limits are contained and bounded in some step.

Here K runs through all compact convex subsets of U ordered by inclusion, W runs through the open subsets $K \subseteq W \subseteq U$ again ordered by inclusion, ρ runs through the positive real numbers, (r_k) runs through all sequences of positive real numbers for which $\rho^k / r_k \to 0$ for all $\rho > 0$, c runs through the C^M-curves in U ordered by reparametrization with $g \in C^M(\mathbb{R}, \mathbb{R})$ and I runs through the compact intervals in $\mathbb{R}$.

Proof. Note first that all four descriptions describe smooth functions $f : U \to F$, which are given by $x \mapsto f^0(x)$ in (1)–(3) for appropriately chosen K with $x \in K$ where $f^0 : K \to F$ and by $x \mapsto f_c(t)$ in (4) for c with $x = c(t)$, $t \in I$ and $f_c : I \to F$. Smoothness of f follows, since we may test with C^M-curves and these factor locally into some K.

By (3.9) all four descriptions describe $C^M(U, F)$ as vector space.

Obviously the identity is continuous from (1) to (2) and from (2) to (3).

The identity from (3) to (1) is continuous, since the space given by (3) is as inverse limit of Banach spaces convenient and the inductive limit in (1) is by construction an (LB)-space, hence webbed, and thus we can apply the uniform $\mathcal{S}$-boundedness principle [KM97a, 5.24], where $\mathcal{S} = \{\mathrm{ev}_x : x \in U\}$.

So the descriptions in (1)–(3) describe the same complete bornology on $C^M(U, F)$ and satisfy the uniform $\mathcal{S}$-boundedness principle.

Moreover, the inductive limits involved in (1) and (2) are regular: In fact the bounded sets $\mathcal{B}$ therein are also bounded in the structure of (3), i.e., for every compact $K \subseteq U$ and sequence (r_k) of positive real numbers for which $\rho^k / r_k \to 0$ for all $\rho > 0$:

$$\sup\Big\{ \frac{\|f^k(a)(v_1, \dots, v_k)\|}{k!\, r_k\, M_k\, \|v_1\| \cdot \dots \cdot \|v_k\|} : k \in \mathbb{N}, a \in A, v_i \in E, f \in \mathcal{B} \Big\} < \infty$$

and so the sequence

$$a_k := \sup\Big\{ \frac{\|f^k(a)(v_1, \dots, v_k)\|}{k!\, M_k\, \|v_1\| \cdot \dots \cdot \|v_k\|} : a \in A, v_i \in E, f \in \mathcal{B} \Big\} < \infty$$

satisfies $\sup_k a_k / r_k < \infty$ for all (r_k) as above. By [KM97a, 9.2] these are the coefficients of a power series with positive radius of convergence. Thus a_k / ρ^k is bounded for some $\rho > 0$. This means that $\mathcal{B}$ is contained and bounded in $C^M_\rho(K, F)$.

That also (4) describes the same bornology follows again by the $\mathcal{S}$-uniform boundedness principle, since the inductive limit in (4) is regular by what we said before for the special case $E = \mathbb{R}$ and hence the structure of (4) is convenient. $\square$

4.7. **Lemma.** *Let M be a non-quasianalytic DC-weight sequence. For any convenient vector space E the flip of variables induces an isomorphism $L(E, C^M(\mathbb{R}, \mathbb{R})) \cong C^M(\mathbb{R}, E')$ as vector spaces.*

Proof. For $c \in C^M(\mathbb{R}, E')$ consider $\tilde{c}(x) := \mathrm{ev}_x \circ c \in C^M(\mathbb{R}, \mathbb{R})$ for $x \in E$. By the uniform boundedness principle (4.1) the linear mapping $\tilde{c}$ is bounded, since $\mathrm{ev}_t \circ \tilde{c} = c(t) \in E'$.

If conversely $\ell \in L(E, C^M(\mathbb{R}, \mathbb{R}))$, we consider $\tilde{\ell}(t) = \mathrm{ev}_t \circ \ell \in E' = L(E, \mathbb{R})$ for $t \in \mathbb{R}$. Since the bornology of E' is generated by $\mathcal{S} := \{ev_x : x \in E\}$, $\tilde{\ell} : \mathbb{R} \to E'$ is C^M, for $\mathrm{ev}_x \circ \tilde{\ell} = \ell(x) \in C^M(\mathbb{R}, \mathbb{R})$, by (3.5). $\square$

4.8. **Lemma.** *Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence. By $\lambda^M(\mathbb{R})$ we denote the c^∞-closure of the linear subspace generated by $\{\mathrm{ev}_t : t \in \mathbb{R}\}$ in $C^M(\mathbb{R}, \mathbb{R})'$ and let $\delta : \mathbb{R} \to \lambda^M(\mathbb{R})$ be given by $t \mapsto \mathrm{ev}_t$. Then $\lambda^M(\mathbb{R})$ is the free convenient vector space over C^M, i.e. for every convenient vector space G the C^M-curve δ induces a bornological isomorphism*

$$L(\lambda^M(\mathbb{R}), G) \cong C^M(\mathbb{R}, G).$$

We expect $\lambda^M(\mathbb{R})$ to be equal to $C^M(\mathbb{R}, \mathbb{R})'$ as it is the case for the analogous situation of smooth mappings, see [KM97a, 23.11], and of holomorphic mappings, see [Sie95] and [Sie97].

Proof. The proof goes along the same lines as in [KM97a, 23.6] and in [FK88, 5.1.1]. Note first that $\lambda^M(\mathbb{R})$ is a convenient vector space since it is c^∞-closed in the convenient vector space $C^M(\mathbb{R}, \mathbb{R})'$. Moreover, δ is C^M by (3.5), since $\mathrm{ev}_h \circ \delta = h$ for all $h \in C^M(\mathbb{R}, \mathbb{R})$, so $\delta^* : L(\lambda^M(\mathbb{R}), G) \to C^M(\mathbb{R}, G)$ is a well-defined linear mapping. This mapping is injective, since each bounded linear mapping $\lambda^M(\mathbb{R}) \to G$ is uniquely determined on $\delta(\mathbb{R}) = \{\mathrm{ev}_t : t \in \mathbb{R}\}$. Let now $f \in C^M(\mathbb{R}, G)$. Then $\ell \circ f \in C^M(\mathbb{R}, \mathbb{R})$ for every $\ell \in G^*$ and hence $\tilde{f} : C^M(\mathbb{R}, \mathbb{R})' \to \prod_{G^*} \mathbb{R}$ given by $\tilde{f}(\varphi) = (\varphi(\ell \circ f))_{\ell \in G^*}$ is a well-defined bounded linear map. Since it maps ev_t to $\tilde{f}(\mathrm{ev}_t) = \delta(f(t))$, where $\delta : G \to \prod_{G^*} \mathbb{R}$ denotes the bornological embedding given by $x \mapsto (\ell(x))_{\ell \in G^*}$, it induces a bounded linear mapping $\tilde{f} : \lambda^M(\mathbb{R}) \to G$ satisfying $\tilde{f} \circ \delta = f$. Thus δ^* is a linear bijection. That it is a bornological isomorphism (i.e. δ^* and its inverse are both bounded) follows from the uniform boundedness principles (4.1) and (4.2). $\square$

4.9. **Corollary.** *Let $M = (M_k)$ and $N = (N_k)$ be non-quasianalytic DC-weight sequences. We have the following isomorphisms of linear spaces*

(1) $C^\infty(\mathbb{R}, C^M(\mathbb{R}, \mathbb{R})) \cong C^M(\mathbb{R}, C^\infty(\mathbb{R}, \mathbb{R}))$
(2) $C^\omega(\mathbb{R}, C^M(\mathbb{R}, \mathbb{R})) \cong C^M(\mathbb{R}, C^\omega(\mathbb{R}, \mathbb{R}))$
(3) $C^N(\mathbb{R}, C^M(\mathbb{R}, \mathbb{R})) \cong C^M(\mathbb{R}, C^N(\mathbb{R}, \mathbb{R}))$

Proof. For $\alpha \in \{\infty, \omega, N\}$ we get

$$\begin{aligned} C^M(\mathbb{R}, C^\alpha(\mathbb{R}, \mathbb{R})) &\cong L(\lambda^M(\mathbb{R}), C^\alpha(\mathbb{R}, \mathbb{R})) && \text{by (4.8)} \\ &\cong C^\alpha(\mathbb{R}, L(\lambda^M(\mathbb{R}), \mathbb{R})) && \text{by (4.7), [KM97a, 3.13.4, 5.3, 11.15]} \\ &\cong C^\alpha(\mathbb{R}, C^M(\mathbb{R}, \mathbb{R})) && \text{by (4.8).} \quad \square \end{aligned}$$

4.10. **Theorem.** *(Canonical isomorphisms) Let $M = (M_k)$ and $N = (N_k)$ be non-quasianalytic DC-weight sequences. Let E, F be convenient vector spaces and let W_i be c^∞-open subsets in such. We have the following natural bornological isomorphisms:*

(1) $C^M(W_1, C^N(W_2, F)) \cong C^N(W_2, C^M(W_1, F))$,
(2) $C^M(W_1, C^\infty(W_2, F)) \cong C^\infty(W_2, C^M(W_1, F))$.
(3) $C^M(W_1, C^\omega(W_2, F)) \cong C^\omega(W_2, C^M(W_1, F))$.
(4) $C^M(W_1, L(E, F)) \cong L(E, C^M(W_1, F))$.
(5) $C^M(W_1, \ell^\infty(X, F)) \cong \ell^\infty(X, C^M(W_1, F))$.
(6) $C^M(W_1, \mathcal{L}\mathrm{ip}^k(X, F)) \cong \mathcal{L}\mathrm{ip}^k(X, C^M(W_1, F))$.

In (5) *the space X is an ℓ^∞-space, i.e. a set together with a bornology induced by a family of real valued functions on X, cf.* [FK88, 1.2.4]. *In* (6) *the space X is a $\mathcal{L}\mathrm{ip}^k$-space, cf.* [FK88, 1.4.1]. *The spaces $\ell^\infty(X, F)$ and $\mathcal{L}\mathrm{ip}^k(W, F)$ are defined in* [FK88, 3.6.1 and 4.4.1].

Proof. All isomorphisms, as well as their inverse mappings, are given by the flip of coordinates: $f \mapsto \tilde{f}$, where $\tilde{f}(x)(y) := f(y)(x)$. Furthermore, all occurring function spaces are convenient and satisfy the uniform $\mathcal{S}$-boundedness theorem, where $\mathcal{S}$ is the set of point evaluations, by (4.1), [KM97a, 11.11, 11.14, 11.12], and by [FK88, 3.6.1, 4.4.2, 3.6.6, and 4.4.7].

That $\tilde{f}$ has values in the corresponding spaces follows from the equation $\tilde{f}(x) = ev_x \circ f$. One only has to check that $\tilde{f}$ itself is of the corresponding class, since it follows that $f \mapsto \tilde{f}$ is bounded. This is a consequence of the uniform boundedness principle, since

$$(\mathrm{ev}_x \circ (\ \tilde{}\ ))(f) = \mathrm{ev}_x(\tilde{f}) = \tilde{f}(x) = \mathrm{ev}_x \circ f = (\mathrm{ev}_x)_*(f).$$

That $\tilde{f}$ is of the appropriate class in (1) and in (2) follows by composing with the appropriate curves $c_1 : \mathbb{R} \to W_1$, $c_2 : \mathbb{R} \to W_2$ and $\lambda \in F^*$ and thereby reducing the statement to the special case in (4.9).

That $\tilde{f}$ is of the appropriate class in (3) follows by composing with $c_1 \in C^M(\mathbb{R}, W_1)$ and $C^{\beta_2}(c_2, \lambda) : C^\omega(W_2, F) \to C^{\beta_2}(\mathbb{R}, \mathbb{R})$ for all $\lambda \in F^*$ and $c_2 \in C^{\beta_2}(\mathbb{R}, W_2)$, where β_2 is in $\{\infty, \omega\}$. Then $C^{\beta_2}(c_2, \lambda) \circ \tilde{f} \circ c_1 = (C^M(c_1, \lambda) \circ f \circ c_2)^\sim : \mathbb{R} \to C^{\beta_2}(\mathbb{R}, \mathbb{R})$ is C^M by (4.9), since $C^M(c_1, \lambda) \circ f \circ c_2 : \mathbb{R} \to W_2 \to C^M(W_1, F) \to C^M(\mathbb{R}, \mathbb{R})$ is C^{β_2}.

That $\tilde{f}$ is of the appropriate class in (4) follows, since $L(E, F)$ is the c^∞-closed subspace of $C^M(E, F)$ formed by the linear C^M-mappings.

That $\tilde{f}$ is of the appropriate class in (5) or (6) follows from (4), using the free convenient vector spaces $\ell^1(X)$ or $\lambda^k(X)$ over the ℓ^∞-space X or the the $\mathcal{L}\mathrm{ip}^k$-space X, see [FK88, 5.1.24 or 5.2.3], satisfying $\ell^\infty(X, F) \cong L(\ell^1(X), F)$ or satisfying $\mathcal{L}\mathrm{ip}^k(X, F) \cong L(\lambda^k(X), F)$. Existence of these free convenient vector spaces can be proved in a similar way as in (4.8). □

5. Exponential law

5.1. **Difference quotients.** For the following see [FK88, 1.3]. For a subset $K \subseteq \mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, a linear space E, and $f : K \to E$ let:

$$\mathbb{R}^{\langle k \rangle} = \big\{(x_0, \dots, x_k) \in \mathbb{R}^{k+1} : x_i \neq x_j \text{ for } i \neq j\big\}$$
$$K^\alpha = \big\{(x^1, \dots, x^n) \in \mathbb{R}^{\alpha_1+1} \times \dots \times \mathbb{R}^{\alpha_n+1} : (x^1_{i_1}, \dots, x^n_{i_n}) \in K \text{ for } 0 \le i_j \le \alpha_j\big\}$$
$$K^{\langle \alpha \rangle} = K^\alpha \cap (\mathbb{R}^{\langle \alpha_1 \rangle} \times \dots \times \mathbb{R}^{\langle \alpha_n \rangle})$$

$$\beta_i(x) = k! \prod_{\substack{0\le j\le k\\ j\ne i}} \frac{1}{x_i - x_j} \text{ for } x = (x_0,\dots,x_k) \in \mathbb{R}^{\langle k\rangle}$$

$$\delta^\alpha f(x^1,\dots,x^n) = \sum_{i_1=0}^{\alpha_1} \cdots \sum_{i_n=0}^{\alpha_n} \beta_{i_1}(x^1)\dots\beta_{i_n}(x^n) f(x^1_{i_1},\dots,x^n_{i_n})$$

Note that $\delta^0 f = f$ and $\delta^\alpha = \delta_n^{\alpha_n} \circ \ldots \circ \delta_1^{\alpha_1}$ where

$$\delta_i^k g(x^1,\dots,x^n) = \delta^k(g(x^1,\dots,x^{i-1},\quad, x^{i+1},\dots,x^n))(x^i).$$

Lemma. *Let E be a convenient vector space. Let $U \subseteq \mathbb{R}^n$ be open. For $f : U \to E$ the following conditions are equivalent:*

(1) *$f : U \to E$ is C^M.*

(2) *For every compact convex set K in U and every $\ell \in E^*$ there exists $\rho > 0$ such that*
$$\Big\{ \frac{\delta^\alpha(\ell\circ f)(x)}{\rho^{|\alpha|}\, |\alpha|!\, M_{|\alpha|}} : \alpha \in \mathbb{N}^n, x \in K^{\langle\alpha\rangle} \Big\}$$
is bounded in $\mathbb{R}$.

Furthermore, the norm on the space $C^M_\rho(K,\mathbb{R})$ from (2.8) *(for convex K) is also given by*

$$\|f\|_{\rho,K} := \sup\Big\{ \frac{|\delta^\alpha f(x)|}{\rho^{|\alpha|}|\alpha|! M_{|\alpha|}} : \alpha \in \mathbb{N}^n, x \in K^{\langle\alpha\rangle} \Big\}.$$

Proof. By composing with bounded linear functionals we may assume that $E = \mathbb{R}$.

(1) $\implies$ (2) If f is C^M then for each compact convex set K in U there exists $\rho > 0$ such that

$$\Big\{ \tfrac{\partial^\alpha f(x)}{\rho^{|\alpha|}\, |\alpha|! M_{|\alpha|}} : \alpha \in \mathbb{N}^n, x \in K \Big\}$$

is bounded in $\mathbb{R}$.

For a differentiable function $g : \mathbb{R} \to \mathbb{R}$ and $t_0 < \cdots < t_j$ there exist s_i with $t_i < s_i < t_{i+1}$ such that

$$\delta^j g(t_0,\dots,t_j) = \delta^{j-1} g'(s_0,\dots,s_{j-1}).$$

This follows by Rolle's theorem, see [KM97a, 12.4]. Recursion, for $g = \partial^\alpha f$, shows that $\delta^\alpha f(x^0,\dots,x^n) = \partial^\alpha f(s)$ for some $s \in K$.

(2) $\implies$ (1) f is C^∞ by [FK88, 1.3.29] since each difference quotient $\delta^\alpha f$ is bounded on bounded sets.

For $g \in C^\infty(\mathbb{R},\mathbb{R})$, using (see [FK88, 1.3.6])

$$g(t_j) = \sum_{i=0}^{j} \tfrac{1}{i!} \prod_{l=0}^{i-1} (t_j - t_l)\; \delta^j g(t_0,\dots,t_j),$$

induction on j and differentiability of g shows that

$$\delta^j g'(t_0,\dots,t_j) = \tfrac{1}{j+1} \sum_{i=0}^{j} \delta^{j+1} g(t_0,\dots,t_j,t_i),$$

where $\delta^{j+1} g(t_0,\dots,t_j,t_i) := \lim_{t\to t_i} \delta^{j+1} g(t_0,\dots,t_j,t)$. If the right hand side divided by $\rho^{|\alpha|}\, |\alpha|!\, M_{|\alpha|}$ is bounded, then also $\delta^j g'/(\rho^{|\alpha|}\, |\alpha|!\, M_{|\alpha|})$ is bounded.

By recursion, applied to $g = \delta^\beta \partial^{\alpha-\beta} f$, we conclude that $f \in C^M$. □

5.2. **Lemma.** *Let E be a convenient vector space such that there exists a Baire vector space topology on the dual E^* for which the point evaluations ev_x are continuous for all $x \in E$. For a mapping $f : \mathbb{R}^n \to E$ the following are equivalent:*

(1) *$\ell \circ f$ is C^M for all $\ell \in E^*$.*

(2) *For every convex compact $K \subseteq \mathbb{R}^n$ there exists $\rho > 0$ such that*

$$\left\{\frac{\partial^\alpha f(x)}{\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|}} : \alpha \in \mathbb{N}^n, x \in K\right\} \text{ is bounded in } E.$$

(3) *For every convex compact $K \subseteq \mathbb{R}^n$ there exists $\rho > 0$ such that*

$$\left\{\frac{\delta^\alpha f(x)}{\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|}} : \alpha \in \mathbb{N}^n, x \in K^{\langle\alpha\rangle}\right\} \text{ is bounded in } E.$$

Proof. (2) $\implies$ (1) is obvious.

(1) $\implies$ (2) Let K be compact convex in $\mathbb{R}^n$. We consider the sets

$$A_{\rho,C} := \left\{\ell \in E^* : \frac{|\partial^\alpha(\ell\circ f)(x)|}{\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|}} \le C \text{ for all } \alpha \in \mathbb{N}^n, x \in K\right\}$$

which are closed subsets in E^* for the Baire topology. We have $\bigcup_{\rho,C} A_{\rho,C} = E^*$. By the Baire property there exists ρ and C such that the interior U of $A_{\rho,C}$ is non-empty. If $\ell_0 \in U$ then for all $\ell \in E^*$ there is an $\epsilon > 0$ such that $\epsilon\ell \in U - \ell_0$ and hence for all $x \in K$ and all α we have

$$|\partial^\alpha(\ell\circ f)(x)| \le \tfrac{1}{\epsilon}\left(|\partial^\alpha((\epsilon\ell+\ell_0)\circ f)(x)| + |\partial^\alpha(\ell_0\circ f)(x)|\right) \le \tfrac{2C}{\epsilon}\,\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|}.$$

So the set

$$\left\{\frac{\partial^\alpha f(x)}{\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|}} : \alpha \in \mathbb{N}^n, x \in K\right\}$$

is weakly bounded in E and hence bounded.

(3) $\implies$ (1) follows by lemma (5.1). (1) $\implies$ (3) follows as above for the difference quotients instead of the partial differentials. □

5.3. **Theorem.** *(Cartesian closedness) Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence of moderate growth* (2.5.1). *Then the category of C^M-mappings between convenient real vector spaces is cartesian closed. More precisely, for convenient vector spaces E, F and G and c^∞-open sets $U \subseteq E$ and $W \subseteq F$ a mapping $f : U \times W \to G$ is C^M if and only if $f^\vee : U \to C^M(W,G)$ is C^M.*

Proof. We first show the result for $U = \mathbb{R}$, $W = \mathbb{R}$, $G = \mathbb{R}$.

If $f \in C^M(\mathbb{R}^2,\mathbb{R})$ then clearly for any $x \in \mathbb{R}$ the function $f^\vee(x) = f(x,\quad) \in C^M(\mathbb{R},\mathbb{R})$. To show that $f^\vee : \mathbb{R} \to C^M(\mathbb{R},\mathbb{R})$ is C^M it suffices to check (5.1.2) for all $\ell \in C^M(\mathbb{R},\mathbb{R})^*$. Such an ℓ factors over $\varinjlim_\rho C^M_\rho(L)$ for some compact $L \subset \mathbb{R}$. Let $K \subset \mathbb{R}$ be compact. Since f is C^M there exists $C > 0$ and $\rho > 0$ by lemma (5.1) such that

$$\frac{|\delta^\alpha f(x,y)|}{\rho^{|\alpha|}|\alpha|!M_{|\alpha|}} \le C \quad \text{ for } \alpha \in \mathbb{N}^2, (x,y) \in (K\times L)^{\langle\alpha\rangle}.$$

Since M is of moderate growth (2.5.1) we have $M_{j+k} \le \sigma^{j+k}M_jM_k$ for some $\sigma > 0$. Let $\alpha = (\alpha_1,\alpha_2) \in \mathbb{N}^2$. Then:

$$\begin{aligned}
&\left\|\frac{\delta^{\alpha_1} f^\vee(x)}{\rho_1^{\alpha_1}\,\alpha_1!\,M_{\alpha_1}}\right\|_{\rho_2,L} = \sup\left\{\frac{|\delta_2^{\alpha_2}\delta_1^{\alpha_1} f(x,y)|}{\rho_1^{\alpha_1}\,\alpha_1!\,M_{\alpha_1}\,\rho_2^{\alpha_2}\,\alpha_2!\,M_{\alpha_2}} : \alpha_2 \in \mathbb{N}, y \in L^{\langle\alpha_2\rangle}\right\}\\
&\le \sup\left\{\frac{|\delta_2^{\alpha_2}\delta_1^{\alpha_1} f(x,y)|}{\rho_1^{\alpha_1}\,\rho_2^{\alpha_2}\,\frac{\alpha_1!\,\alpha_2!}{(\alpha_1+\alpha_2)!}\,(\alpha_1+\alpha_2)!\,\sigma^{-\alpha_1-\alpha_2}M_{\alpha_1+\alpha_2}} : \alpha_2 \in \mathbb{N}, y \in L^{\langle\alpha_2\rangle}\right\}\\
&\le \sup\left\{\frac{|\delta^\alpha f(x,y)|}{\rho_1^{\alpha_1}\,\rho_2^{\alpha_2}\,\sigma^{-|\alpha|}2^{-|\alpha|}\,|\alpha|!\,M_{|\alpha|}} : \alpha_2 \in \mathbb{N}, y \in L^{\langle\alpha_2\rangle}\right\}\\
&\le \sup\left\{\frac{|\delta^\alpha f(x,y)|}{\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|}} : \alpha_2 \in \mathbb{N}, y \in L^{\langle\alpha_2\rangle}\right\} \le C \text{ for } \alpha_1 \in \mathbb{N}, x \in K^{\langle\alpha_1\rangle}
\end{aligned}$$

for $\rho_1 = \rho_2 = 2\sigma\rho$. So $f^\vee : K \to C^M_{\rho_2}(L, \mathbb{R})$ is C^M. Thus $\ell \circ f^\vee$ is C^M.

Conversely, let $f^\vee : \mathbb{R} \to C^M(\mathbb{R}, \mathbb{R})$ be C^M. Then $f^\vee : \mathbb{R} \to \varinjlim_{\rho_2} C^M_{\rho_2}(L, \mathbb{R})$ is C^M for all compact subsets $L \subset \mathbb{R}$. The dual space $(\varinjlim_{\rho_2} C^M_{\rho_2}(L, \mathbb{R}))^*$ can be equipped with the Baire topology of the countable limit $\varprojlim_{\rho_2} C^M_{\rho_2}(L, \mathbb{R})^*$ of Banach spaces.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "f^\vee"] & C^M(\mathbb{R}, \mathbb{R}) \arrow[r] & \varinjlim_{\rho_2} C^M_{\rho_2}(L, \mathbb{R}) \\
K \arrow[u, hook] \arrow[rr, "f^\vee"] & & C^M_{\rho_2}(L, \mathbb{R}) \arrow[u]
\end{tikzcd}$$

Thus the mapping $f^\vee : \mathbb{R} \to \varinjlim_{\rho_2} C^M_{\rho_2}(L, \mathbb{R})$ is strongly C^M by (5.2). Since the inductive limit $\varinjlim_{\rho_2} C^M_{\rho_2}(L, \mathbb{R})$ is countable and regular ([Flo71, 7.4 and 7.5] or [KM97a, 52.37]), for each compact $K \subset \mathbb{R}$ there exists $\rho_1 > 0$ such that the bounded set

$$\Big\{ \frac{\partial^{\alpha_1} f^\vee(x)}{\rho_1^{\alpha_1}\, \alpha_1!\, M_{\alpha_1}} : \alpha_1 \in \mathbb{N}, x \in K \Big\}$$

is contained and bounded in $C^M_{\rho_2}(L, \mathbb{R})$ for some $\rho_2 > 0$. Thus for $\alpha_1 \in \mathbb{N}$ and $x \in K$ we have (using (2.1.3))

$$\begin{aligned}
\infty > C := \sup_{\substack{\alpha_1 \in \mathbb{N}\\ y \in K}} \Big\| \frac{\delta^{\alpha_1} f^\vee(y)}{\rho_1^{\alpha_1}\, \alpha_1!\, M_{\alpha_1}} \Big\|_{\rho_2, L} &\ge \Big\| \frac{\delta^{\alpha_1} f^\vee(x)}{\rho_1^{\alpha_1}\, \alpha_1!\, M_{\alpha_1}} \Big\|_{\rho_2, L} \\
&= \sup \Big\{ \frac{|\delta_2^{\alpha_2} \delta_1^{\alpha_1} f(x,y)|}{\rho_1^{\alpha_1}\, \alpha_1!\, M_{\alpha_1}\, \rho_2^{\alpha_2}\, \alpha_2!\, M_{\alpha_2}} : \alpha_2 \in \mathbb{N}, y \in L^{\langle \alpha_2 \rangle} \Big\} \\
&\ge \sup \Big\{ \frac{|\delta_2^{\alpha_2} \delta_1^{\alpha_1} f(x,y)|}{\rho_1^{\alpha_1}\, \rho_2^{\alpha_2}\, \frac{\alpha_1!\,\alpha_2!}{(\alpha_1+\alpha_2)!}\, (\alpha_1+\alpha_2)! M_{\alpha_1+\alpha_2}} : \alpha_2 \in \mathbb{N}, y \in L^{\langle \alpha_2 \rangle} \Big\} \\
&\ge \sup \Big\{ \frac{|\delta^{\alpha} f(x,y)|}{\rho^{|\alpha|}\, |\alpha|!\, M_{|\alpha|}} : \alpha_2 \in \mathbb{N}, y \in L^{\langle \alpha_2 \rangle} \Big\}
\end{aligned}$$

where $\rho = \max(\rho_1, \rho_2)$. Thus f is C^M.

Now we consider the general case. Given a C^M-mapping $f : U \times W \to G$ we have to show that $f^\vee : U \to C^M(W, G)$ is C^M. Any continuous linear functional on $C^M(W, G)$ factors over some step mapping $C^M(c_2, \ell) : C^M(W, G) \to C^M(\mathbb{R}, \mathbb{R})$ of the cone in (3.1) where c_2 is a C^M-curve in W and $\ell \in G^*$. So we have to check that $C^M(c_2, \ell) \circ f^\vee \circ c_1 : \mathbb{R} \to C^M(\mathbb{R}, \mathbb{R})$ is C^M for every C^M-curve c_1 in U. Since $(\ell \circ f \circ (c_1 \times c_2))^\vee = C^M(c_2, \ell) \circ f^\vee \circ c_1$ this follows from the special case proved above.

If $f^\vee : U \to C^M(W, G)$ is C^M then $(\ell \circ f \circ (c_1 \times c_2))^\vee = C^M(c_2, \ell) \circ f^\vee \circ c_1$ is C^M for all C^M-curves $c_1 : \mathbb{R} \to U$, $c_2 : \mathbb{R} \to W$ and $\ell \in G^*$. By the special case, f is then C^M. $\square$

5.4. **Example: Cartesian closedness is wrong in general.** Let M be a DC-weight sequence which is strongly non-quasianalytic but not of moderate growth. For example, $M_k = 2^{k^2}$ satisfies this by (2.7). Then by (2.4) there exists $f : \mathbb{R}^2 \to \mathbb{R}$ of class C^M with $\partial^\alpha f(0,0) = |\alpha|!\, M_{|\alpha|}$. *We claim that* $f^\vee : \mathbb{R} \to C^M(\mathbb{R}, \mathbb{R})$ *is not* C^M.

Since M is not of moderate growth there exist $j_n \nearrow \infty$ and $k_n > 0$ such that

$$\Big(\frac{M_{k_n + j_n}}{M_{k_n} M_{j_n}} \Big)^{\frac{1}{k_n + j_n}} \ge n.$$

Consider the linear functional $\ell : C^M(\mathbb{R},\mathbb{R}) \to \mathbb{R}$ given by

$$\ell(g) = \sum_n \frac{g^{(j_n)}(0)}{j_n!\, M_{j_n}\, n^{j_n}}.$$

This functional is continuous since

$$\left|\sum_n \frac{g^{(j_n)}(0)}{j_n!\, M_{j_n}\, n^{j_n}}\right| \le \sum_n \frac{g^{(j_n)}(0)}{j_n!\, \rho^{j_n}\, M_{j_n}} \frac{\rho^{j_n}}{n^{j_n}} \le C(\rho)\, \|g\|_{\rho,[-1,1]} < \infty$$

for suitable ρ where

$$C(\rho) := \sum_n \rho^{j_n} \frac{1}{n^{j_n}} < \infty$$

for all ρ. But $\ell \circ f^\vee$ is not C^M since

$$\begin{aligned}
&\|\ell \circ f^\vee\|_{\rho_1,[-1,1]} \ge \sup_k \frac{1}{\rho_1^k\, k!\, M_k} \sum_n \frac{f^{(j_n,k)}(0,0)}{j_n!\, M_{j_n} n^{j_n}} \\
&\ge \sup_n \frac{1}{\rho_1^{k_n}\, k_n!\, M_{k_n}} \frac{f^{(j_n,k_n)}(0,0)}{j_n!\, M_{j_n}\, n^{j_n}} \\
&\ge \sup_n \frac{(j_n+k_n)!\, M_{j_n+k_n}}{\rho_1^{k_n}\, k_n!\, j_n!\, M_{k_n}\, M_{j_n}\, n^{j_n}} \ge \sup_n \frac{n^{j_n+k_n}}{\rho_1^{k_n}\, n^{j_n}} = \infty
\end{aligned}$$

for all $\rho_1 > 0$.

5.5. **Theorem.** *Let M be a non-quasianalytic DC-weight sequence which is of moderate growth. Let E, F, etc., be convenient vector spaces and let U and V be c^∞-open subsets of such.*
(1) *The exponential law holds:*

$$C^M(U, C^M(V,G)) \cong C^M(U \times V, G)$$

is a linear C^M-diffeomorphism of convenient vector spaces.
The following canonical mappings are C^M.

(2) $\mathrm{ev} : C^M(U,F) \times U \to F, \quad \mathrm{ev}(f,x) = f(x)$

(3) $\mathrm{ins} : E \to C^M(F, E \times F), \quad \mathrm{ins}(x)(y) = (x,y)$

(4) $(\ \ )^\wedge : C^M(U, C^M(V,G)) \to C^M(U \times V, G)$

(5) $(\ \ )^\vee : C^M(U \times V, G) \to C^M(U, C^M(V,G))$

(6) $\mathrm{comp} : C^M(F,G) \times C^M(U,F) \to C^M(U,G)$

(7) $C^M(\ \ ,\ \ ) : C^M(F,F_1) \times C^M(E_1,E) \to C^M(C^M(E,F), C^M(E_1,F_1))$
$\qquad (f,g) \mapsto (h \mapsto f \circ h \circ g)$

(8) $\prod : \prod C^M(E_i,F_i) \to C^M(\prod E_i, \prod F_i)$

Proof. (2) The mapping associated to ev via cartesian closedness is the identity on $C^M(U,F)$, which is C^M, thus ev is also C^M.

(3) The mapping associated to ins via cartesian closedness is the identity on $E \times F$, hence ins is C^M.

(4) The mapping associated to $(\ \ )^\wedge$ via cartesian closedness is the C^M-composition of evaluations $\mathrm{ev} \circ (\mathrm{ev} \times \mathrm{Id}) : (f; x, y) \mapsto f(x)(y)$.

(5) We apply cartesian closedness twice to get the associated mapping $(f; x; y) \mapsto f(x,y)$, which is just a C^M evaluation mapping.

(6) The mapping associated to comp via cartesian closedness is $(f, g; x) \mapsto f(g(x))$, which is the C^M-mapping $\mathrm{ev} \circ (\mathrm{Id} \times \mathrm{ev})$.

(7) The mapping associated to the one in question by applying cartesian closedness twice is $(f, g; h, x) \mapsto g(h(f(x)))$, which is the C^M-mapping $\operatorname{ev} \circ (\operatorname{Id} \times \operatorname{ev}) \circ (\operatorname{Id} \times \operatorname{Id} \times \operatorname{ev})$.

(8) Up to a flip of factors the mapping associated via cartesian closedness is the product of the evaluation mappings $C^M(E_i, F_i) \times E_i \to F_i$.

(1) follows from (4) and (5). $\square$

6. Manifolds of C^M-mappings

6.1. **C^M-manifolds.** Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence of moderate growth. A C^M-manifold is a smooth manifold such that all chart changings are C^M-mappings. Likewise for C^M-bundles and C^M Lie groups.

Note that any finite dimensional (always assumed paracompact) C^∞-manifold admits a C^∞-diffeomorphic real analytic structure thus also a C^M-structure. Maybe, any finite dimensional C^M-manifold admits a C^M-diffeomorphic real analytic structure.

6.2. **Spaces of C^M-sections.** Let $E \to B$ be a C^M vector bundle (possibly infinite dimensional). The space $C^M(B \leftarrow E)$ of all C^M sections is a convenient vector space with the structure induced by

$$C^M(B \leftarrow E) \to \prod_\alpha C^M(u_\alpha(U_\alpha), V)$$
$$s \mapsto \operatorname{pr}_2 \circ \psi_\alpha \circ s \circ u_\alpha^{-1}$$

where $B \supseteq U_\alpha \xrightarrow{u_\alpha} u_\alpha(U_\alpha) \subset W$ is a C^M-atlas for B which we assume to be modelled on a convenient vector space W, and where $\psi_\alpha : E|_{U_\alpha} \to U_\alpha \times V$ form a vector bundle atlas over charts U_α of B.

Lemma. *For a C^M vector bundle $E \to B$ a curve $c : \mathbb{R} \to C^M(B \leftarrow E)$ is C^M if and only if $c^\wedge : \mathbb{R} \times B \to E$ is C^M.*

Proof. By the description of the structure on $C^M(B \leftarrow E)$ we may assume that B is c^∞-open in a convenient vector space W and that $E = B \times V$. Then $C^M(B \leftarrow B \times V) \cong C^M(B, V)$. Then the statement follows from the exponential law (5.3). $\square$

An immediate consequence is the following: If $U \subset E$ is an open neighborhood of $s(B)$ for a section s, $F \to B$ is another vector bundle and if $f : U \to F$ is a fiber respecting C^M mapping, then $f_* : C^M(B \leftarrow U) \to C^M(B \leftarrow F)$ is C^M on the open neighborhood $C^M(B \leftarrow U)$ of s in $C^M(B \leftarrow E)$. We have $(d(f_*)(s)v)_x = d(f|_{U \cap E_x})(s(x))(v(x))$.

6.3. **Theorem.** *Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence of moderate growth. Let A and B be finite dimensional C^M manifolds with A compact. Then the space $C^M(A, B)$ of all C^M-mappings $A \to B$ is a C^M-manifold modelled on convenient vector spaces $C^M(A \leftarrow f^*TB)$ of C^M sections of pullback bundles along $f : A \to B$. Moreover, a curve $c : \mathbb{R} \to C^M(A, B)$ is C^M if and only if $c^\wedge : \mathbb{R} \times A \to B$ is C^M.*

Proof. Choose a C^M Riemannian metric on B which exists since we have C^M partitions of unity. C^M-vector fields have C^M-flows by [Kom80]; applying this to the geodesic spray we get the C^M exponential mapping $\exp : TB \supseteq U \to B$ of this Riemannian metric, defined on a suitable open neighborhood of the zero section. We may assume that U is chosen in such a way that $(\pi_B, \exp) : U \to B \times B$ is a C^M diffeomorphism onto an open neighborhood V of the diagonal, by the C^M inverse function theorem due to [Kom79].

For $f \in C^M(A,B)$ we consider the pullback vector bundle

$$\begin{tikzcd} A \times_B TB \arrow[r, equal] & f^*TB \arrow[r, "\pi_B^* f"] \arrow[d, "f^*\pi_B"'] & TB \arrow[d, "\pi_B"] \\ & A \arrow[r, "f"] & B \end{tikzcd}$$

Then $C^M(A \leftarrow f^*TB)$ is canonically isomorphic to the space $C^M(A,TB)_f := \{h \in C^M(A,TB) : \pi_B \circ h = f\}$ via $s \mapsto (\pi_B^* f) \circ s$ and $(\mathrm{Id}_A, h) \leftarrow\!\shortmid h$. Now let

$$U_f := \{g \in C^M(A,B) : (f(x),\ g(x)) \in V \text{ for all } x \in A\},$$
$$u_f : U_f \to C^M(A \leftarrow f^*TB),$$
$$u_f(g)(x) = (x, \exp_{f(x)}^{-1}(g(x))) = (x, ((\pi_B, \exp)^{-1} \circ (f,g))(x)).$$

Then u_f is a bijective mapping from U_f onto the set $\{s \in C^M(A \leftarrow f^*TB) : s(A) \subseteq f^*U = (\pi_B^* f)^{-1}(U)\}$, whose inverse is given by $u_f^{-1}(s) = \exp \circ (\pi_B^* f) \circ s$, where we view $U \to B$ as fiber bundle. The push forward u_f is C^M since it maps C^M-curves to C^M-curves by lemma (6.2). The set $u_f(U_f)$ is open in $C^M(A \leftarrow f^*TB)$ for the topology described above in (6.2).

Now we consider the atlas $(U_f, u_f)_{f \in C^M(A,B)}$ for $C^M(A,B)$. Its chart change mappings are given for $s \in u_g(U_f \cap U_g) \subseteq C^M(A \leftarrow g^*TB)$ by

$$\begin{aligned}(u_f \circ u_g^{-1})(s) &= (\mathrm{Id}_A, (\pi_B, \exp)^{-1} \circ (f, \exp \circ (\pi_B^* g) \circ s)) \\ &= (\tau_f^{-1} \circ \tau_g)_*(s),\end{aligned}$$

where $\tau_g(x, Y_{g(x)}) := (x, \exp_{g(x)}(Y_{g(x)}))$ is a C^M diffeomorphism $\tau_g : g^*TB \supseteq g^*U \to (g \times \mathrm{Id}_B)^{-1}(V) \subseteq A \times B$ which is fiber respecting over A. The chart change $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ is defined on an open subset and it is also C^M since it respects C^M-curves.

Finally for the topology on $C^M(A,B)$ we take the identification topology from this atlas (with the c^∞-topologies on the modeling spaces), which is obviously finer than the compact-open topology and thus Hausdorff.

The equation $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ shows that the C^M structure does not depend on the choice of the C^M Riemannian metric on B.

The statement on C^M-curves follows from lemma (6.2). □

6.4. **Corollary.** *Let A_1, A_2 and B be finite dimensional C^M manifolds with A_1 and A_2 compact. Then composition*

$$C^M(A_2, B) \times C^M(A_1, A_2) \to C^M(A_1, B), \quad (f,g) \mapsto f \circ g$$

is C^M. However, if $N = (N_k)$ is another non-quasianalytic DC-weight sequence of moderate growth with $(N_k/M_k)^{1/k} \searrow 0$ then composition is **not** *C^N.*

Proof. Composition maps C^M-curves to C^M-curves, so it is C^M.

Let $A_1 = A_2 = S^1$ and $B = \mathbb{R}$. Then by (2.1.5) there exists $f \in C^M(S^1, \mathbb{R}) \setminus C^N(S^1, \mathbb{R})$. We consider $f : \mathbb{R} \to \mathbb{R}$ periodic. The universal covering space of $C^M(S^1, S^1)$ consists of all $2\pi\mathbb{Z}$-equivariant mappings in $C^M(\mathbb{R}, \mathbb{R})$, namely the space of all $g + \mathrm{Id}_\mathbb{R}$ for 2π-periodic $g \in C^M$. Thus $C^M(S^1, S^1)$ is a real analytic manifold and $t \mapsto (x \mapsto x + t)$ induces a real analytic curve c in $C^M(S^1, S^1)$. But $f_* \circ c$ is not C^N since:

$$\frac{(\partial_t^k|_{t=0}(f_* \circ c)(t))(x)}{k!\rho^k N_k} = \frac{\partial_t^k|_{t=0} f(x+t)}{k!\rho^k N_k} = \frac{f^{(k)}(x)}{k!\rho^k N_k}$$

which is unbounded for x in a suitable compact set and for all $\rho > 0$ since $f \notin C^N$. $\square$

6.5. **Theorem.** *Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence of moderate growth. Let A be a compact ($\implies$ finite dimensional) C^M manifold. Then the group $\operatorname{Diff}^M(A)$ of all C^M-diffeomorphisms of A is an open subset of the C^M manifold $C^M(A, A)$. Moreover, it is a C^M-regular C^M Lie group: Inversion and composition are C^M. Its Lie algebra consists of all C^M-vector fields on A, with the negative of the usual bracket as Lie bracket. The exponential mapping is C^M. It is not surjective onto any neighborhood of Id_A.*

Following [KM97b], see also [KM97a, 38.4], a C^M-Lie group G with Lie algebra $\mathfrak{g} = T_eG$ is called C^M-regular if the following holds:

- For each C^M-curve $X \in C^M(\mathbb{R}, \mathfrak{g})$ there exists a C^M-curve $g \in C^M(\mathbb{R}, G)$ whose right logarithmic derivative is X, i.e.,
$$\begin{cases} g(0) &= e \\ \partial_t g(t) &= T_e(\mu^{g(t)})X(t) = X(t).g(t) \end{cases}$$
The curve g is uniquely determined by its initial value $g(0)$, if it exists.
- Put $\operatorname{evol}^r_G(X) = g(1)$ where g is the unique solution required above. Then $\operatorname{evol}^r_G : C^M(\mathbb{R}, \mathfrak{g}) \to G$ is required to be C^M also.

Proof. The group $\operatorname{Diff}^M(A)$ is open in $C^M(A, A)$ since it is open in the coarser C^1 compact open topology, see [KM97a, 43.1]. So $\operatorname{Diff}^M(A)$ is a C^M-manifold and composition is C^M by (6.3) and (6.4). To show that inversion is C^M let c be a C^M-curve in $\operatorname{Diff}^M(A)$. By (6.3) the map $c^\wedge : \mathbb{R} \times A \to A$ is C^M and $(\operatorname{inv} \circ c)^\wedge : \mathbb{R} \times A \to A$ satisfies the finite dimensional implicit equation $c^\wedge(t, (\operatorname{inv} \circ c)^\wedge(t, x)) = x$ for $x \in A$. By the finite dimensional C^M implicit function theorem [Kom79] the mapping $(\operatorname{inv} \circ c)^\wedge$ is locally C^M and thus C^M. By (6.3) again, $\operatorname{inv} \circ c$ is a C^M-curve in $\operatorname{Diff}^M(A)$. So $\operatorname{inv} : \operatorname{Diff}^M(A) \to \operatorname{Diff}^M(A)$ is C^M. The Lie algebra of $\operatorname{Diff}^M(A)$ is the convenient vector space of all C^M-vector fields on A, with the negative of the usual Lie bracket (compare with the proof of [KM97a, 43.1]).

To show that $\operatorname{Diff}^M(A)$ is a C^M-regular Lie group, we choose a C^M-curve in the space of C^M-curves in the Lie algebra of all C^M vector fields on A, $c : \mathbb{R} \to C^M(\mathbb{R}, C^M(A \leftarrow TA))$. By lemma (6.2) c corresponds to a $\mathbb{R}^2$-time-dependent C^M vector field $c^{\wedge\wedge} : \mathbb{R}^2 \times A \to TA$. Since C^M-vector fields have C^M-flows and since A is compact, $\operatorname{evol}^r(c^\wedge(s))(t) = \mathrm{Fl}_t^{c^\wedge(s)}$ is C^M in all variables by [Kom80]. Thus $\operatorname{Diff}^M(A)$ is a C^M-regular C^M Lie group.

The exponential mapping is evol^r applied to constant curves in the Lie algebra, i.e., it consists of flows of autonomous C^M vector fields. That the exponential map is not surjective onto any C^M-neighborhood of the identity follows from [KM97a, 43.5] for $A = S^1$. This example can be embedded into any compact manifold, see [Gra88]. $\square$

7. Appendix. Calculus beyond Banach spaces

The traditional differential calculus works well for finite dimensional vector spaces and for Banach spaces. For more general locally convex spaces we sketch here the convenient approach as explained in [FK88] and [KM97a]. The main difficulty is that composition of linear mappings stops to be jointly continuous at the level of Banach spaces, for any compatible topology. We use the notation of [KM97a] and this is the main reference for the whole appendix. We list results in the order in which one can prove them, without proofs for which we refer to [KM97a]. This should explain how to use these results.

7.1. **The c^∞-topology.** Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all derivatives exist and are continuous - this is a concept without problems. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth functions. It can be shown that the set $C^\infty(\mathbb{R}, E)$ does not depend on the locally convex topology of E, only on its associated bornology (system of bounded sets).

The final topologies with respect to the following sets of mappings into E coincide:

(1) $C^\infty(\mathbb{R}, E)$.
(2) The set of all Lipschitz curves (so that $\{\frac{c(t)-c(s)}{t-s} : t \neq s\}$ is bounded in E).
(3) The set of injections $E_B \to E$ where B runs through all bounded absolutely convex subsets in E, and where E_B is the linear span of B equipped with the Minkowski functional $\|x\|_B := \inf\{\lambda > 0 : x \in \lambda B\}$.
(4) The set of all Mackey-convergent sequences $x_n \to x$ (there exists a sequence $0 < \lambda_n \nearrow \infty$ with $\lambda_n(x_n - x)$ bounded).

This topology is called the c^∞-topology on E and we write $c^\infty E$ for the resulting topological space. In general (on the space $\mathcal{D}$ of test functions for example) it is finer than the given locally convex topology, it is not a vector space topology, since addition is no longer jointly continuous. The finest among all locally convex topologies on E which are coarser than $c^\infty E$ is the bornologification of the given locally convex topology. If E is a Fréchet space, then $c^\infty E = E$.

7.2. **Convenient vector spaces.** A locally convex vector space E is said to be a *convenient vector space* if one of the following equivalent conditions is satisfied (called c^∞-completeness):

(1) For any $c \in C^\infty(\mathbb{R}, E)$ the (Riemann-) integral $\int_0^1 c(t)dt$ exists in E.
(2) Any Lipschitz curve in E is locally Riemann integrable.
(3) A curve $c : \mathbb{R} \to E$ is smooth if and only if $\lambda \circ c$ is smooth for all $\lambda \in E^*$, where E^* is the dual consisting of all continuous linear functionals on E. Equivalently, we may use the dual E' consisting of all bounded linear functionals.
(4) Any Mackey-Cauchy-sequence (i. e. $t_{nm}(x_n - x_m) \to 0$ for some $t_{nm} \to \infty$ in $\mathbb{R}$) converges in E. This is visibly a mild completeness requirement.
(5) If B is bounded closed absolutely convex, then E_B is a Banach space.
(6) If $f : \mathbb{R} \to E$ is scalarwise $\mathcal{L}\mathrm{ip}^k$, then f is $\mathcal{L}\mathrm{ip}^k$, for $k > 1$.
(7) If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is differentiable at 0.
(8) If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is C^∞.

Here a mapping $f : \mathbb{R} \to E$ is called $\mathcal{L}\mathrm{ip}^k$ if all derivatives up to order k exist and are Lipschitz, locally on $\mathbb{R}$. That f is scalarwise C^∞ means $\lambda \circ f$ is C^∞ for all continuous linear functionals on E.

7.3. **Smooth mappings.** Let E, F, and G be convenient vector spaces, and let $U \subset E$ be c^∞-open. A mapping $f : U \to F$ is called *smooth* or C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, U)$. *The main properties of smooth calculus are the following.*

(1) *For mappings on Fréchet spaces this notion of smoothness coincides with all other reasonable definitions. Even on $\mathbb{R}^2$ this is non-trivial.*
(2) *Multilinear mappings are smooth if and only if they are bounded.*
(3) *If $f : E \supseteq U \to F$ is smooth then the derivative $df : U \times E \to F$ is smooth, and also $df : U \to L(E, F)$ is smooth where $L(E, F)$ denotes the space of all bounded linear mappings with the topology of uniform convergence on bounded subsets.*
(4) *The chain rule holds.*

(5) *The space $C^\infty(U,F)$ is again a convenient vector space where the structure is given by the obvious injection*

$$C^\infty(U,F) \xrightarrow{C^\infty(c,\ell)} \prod_{c\in C^\infty(\mathbb{R},U),\ell\in F^*} C^\infty(\mathbb{R},\mathbb{R}), \quad f\mapsto (\ell\circ f\circ c)_{c,\ell},$$

where $C^\infty(\mathbb{R},\mathbb{R})$ carries the topology of compact convergence in each derivative separately.

(6) *The exponential law holds: For c^∞-open $V\subset F$,*

$$C^\infty(U, C^\infty(V,G)) \cong C^\infty(U\times V, G)$$

is a linear diffeomorphism of convenient vector spaces. Note that this is the main assumption of variational calculus.

(7) *A linear mapping $f: E\to C^\infty(V,G)$ is smooth (bounded) if and only if $E \xrightarrow{f} C^\infty(V,G) \xrightarrow{\mathrm{ev}_v} G$ is smooth for each $v\in V$. This is called the smooth uniform boundedness theorem* [KM97a, 5.26].

(8) *The following canonical mappings are smooth.*

$$\mathrm{ev}: C^\infty(E,F)\times E\to F,\quad \mathrm{ev}(f,x)=f(x)$$
$$\mathrm{ins}: E\to C^\infty(F,E\times F),\quad \mathrm{ins}(x)(y)=(x,y)$$
$$(\quad)^\wedge : C^\infty(E,C^\infty(F,G))\to C^\infty(E\times F,G)$$
$$(\quad)^\vee : C^\infty(E\times F,G)\to C^\infty(E,C^\infty(F,G))$$
$$\mathrm{comp}: C^\infty(F,G)\times C^\infty(E,F)\to C^\infty(E,G)$$
$$C^\infty(\quad,\quad): C^\infty(F,F_1)\times C^\infty(E_1,E)\to C^\infty(C^\infty(E,F),C^\infty(E_1,F_1))$$
$$(f,g)\mapsto (h\mapsto f\circ h\circ g)$$
$$\prod : \prod C^\infty(E_i,F_i)\to C^\infty(\prod E_i,\prod F_i)$$

7.4. **Remarks.** Note that the conclusion of (7.3.6) is the starting point of the classical calculus of variations, where a smooth curve in a space of functions was assumed to be just a smooth function in one variable more. It is also the source of the name convenient calculus. This and some other obvious properties already determines the convenient calculus.

There are, however, smooth mappings which are not continuous. This is unavoidable and not so horrible as it might appear at first sight. For example the evaluation $E\times E^*\to\mathbb{R}$ is jointly continuous if and only if E is normable, but it is always smooth. Clearly smooth mappings are continuous for the c^∞-topology.

8. Calculus of holomorphic mappings

8.1. **Holomorphic curves.** Let E be a complex locally convex vector space whose underlying real space is convenient – this will be called convenient in the sequel. Let $\mathbb{D}\subset\mathbb{C}$ be the open unit disk and let us denote by $\mathcal{H}(\mathbb{D},E)$ the space of all mappings $c:\mathbb{D}\to E$ such that $\lambda\circ c:\mathbb{D}\to\mathbb{C}$ is holomorphic for each continuous complex-linear functional λ on E. Its elements will be called the holomorphic curves.

If E and F are convenient complex vector spaces (or c^∞-open sets therein), a mapping $f:E\to F$ is called *holomorphic* if $f\circ c$ is a holomorphic curve in F for each holomorphic curve c in E. Obviously f is holomorphic if and only if $\lambda\circ f: E\to\mathbb{C}$ is holomorphic for each complex linear continuous (equivalently: bounded) functional λ on F. Let $\mathcal{H}(E,F)$ denote the space of all holomorphic mappings from E to F.

8.2. **Lemma** (Hartog's theorem)**.** *Let E_k for $k=1,2$ and F be complex convenient vector spaces and let $U_k\subset E_k$ be c^∞-open. A mapping $f: U_1\times U_2\to F$ is holomorphic if and only if it is separately holomorphic (i. e. $f(\quad,y)$ and $f(x,\quad)$ are holomorphic for all $x\in U_1$ and $y\in U_2$).*

This implies also that in finite dimensions we have recovered the usual definition.

8.3. **Lemma.** *If $f : E \supset U \to F$ is holomorphic then $df : U \times E \to F$ exists, is holomorphic and $\mathbb{C}$-linear in the second variable.*

A multilinear mapping is holomorphic if and only if it is bounded.

8.4. **Lemma.** *If E and F are Banach spaces and U is open in E, then for a mapping $f : U \to F$ the following conditions are equivalent:*

(1) *f is holomorphic.*
(2) *f is locally a convergent series of homogeneous continuous polynomials.*
(3) *f is $\mathbb{C}$-differentiable in the sense of Fréchet.*

8.5. **Lemma.** *Let E and F be convenient vector spaces. A mapping $f : E \to F$ is holomorphic if and only if it is smooth and its derivative in each point is $\mathbb{C}$-linear.*

An immediate consequence of this result is that $\mathcal{H}(E, F)$ is a closed linear subspace of $C^\infty(E_\mathbb{R}, F_\mathbb{R})$ and so it is a convenient vector space if F is one, by (7.3.5). The chain rule follows from (7.3.4).

8.6. **Theorem.** *The category of convenient complex vector spaces and holomorphic mappings between them is cartesian closed, i. e.*

$$\mathcal{H}(E \times F, G) \cong \mathcal{H}(E, \mathcal{H}(F, G)).$$

An immediate consequence of this is again that all canonical structural mappings as in (7.3.8) are holomorphic.

9. Calculus of real analytic mappings

9.1. We now sketch the cartesian closed setting to real analytic mappings in infinite dimension following the lines of the Frölicher–Kriegl calculus, as it is presented in [KM97a]. Surprisingly enough one has to deviate from the most obvious notion of real analytic curves in order to get a meaningful theory, but again convenient vector spaces turn out to be the right kind of spaces.

9.2. **Real analytic curves.** Let E be a real convenient vector space with continuous dual E^*. A curve $c : \mathbb{R} \to E$ is called *real analytic* if $\lambda \circ c : \mathbb{R} \to \mathbb{R}$ is real analytic for each $\lambda \in E^*$. It turns out that the set of these curves depends only on the bornology of E. Thus we may use the dual E' consisting of all bounded linear functionals in the definition.

In contrast a curve is called *strongly real analytic* if it is locally given by power series which converge in the topology of E. They can be extended to germs of holomorphic curves along $\mathbb{R}$ in the complexification $E_\mathbb{C}$ of E. If the dual E^* of E admits a Baire topology which is compatible with the duality, then each real analytic curve in E is in fact topologically real analytic for the bornological topology on E.

9.3. **Real analytic mappings.** Let E and F be convenient vector spaces. Let U be a c^∞-open set in E. A mapping $f : U \to F$ is called *real analytic* if and only if it is smooth (maps smooth curves to smooth curves) and maps real analytic curves to real analytic curves.

Let $C^\omega(U, F)$ denote the space of all real analytic mappings. We equip the space $C^\omega(U, \mathbb{R})$ of all real analytic functions with the initial topology with respect to the families of mappings

$$C^\omega(U, \mathbb{R}) \xrightarrow{c^*} C^\omega(\mathbb{R}, \mathbb{R}), \text{ for all } c \in C^\omega(\mathbb{R}, U)$$
$$C^\omega(U, \mathbb{R}) \xrightarrow{c^*} C^\infty(\mathbb{R}, \mathbb{R}), \text{ for all } c \in C^\infty(\mathbb{R}, U),$$

where $C^\infty(\mathbb{R}, \mathbb{R})$ carries the topology of compact convergence in each derivative separately, and where $C^\omega(\mathbb{R}, \mathbb{R})$ is equipped with the final locally convex topology with respect to the embeddings (restriction mappings) of all spaces of holomorphic mappings from a neighborhood V of $\mathbb{R}$ in $\mathbb{C}$ mapping $\mathbb{R}$ to $\mathbb{R}$, and each of these spaces carries the topology of compact convergence.

Furthermore we equip the space $C^\omega(U,F)$ with the initial topology with respect to the family of mappings

$$C^\omega(U,F) \xrightarrow{\lambda_*} C^\omega(U,\mathbb{R}), \text{ for all } \lambda \in F^*.$$

It turns out that this is again a convenient space.

9.4. **Theorem.** *In the setting of* (9.3) *a mapping $f : U \to F$ is real analytic if and only if it is smooth and is real analytic along each affine line in E.*

9.5. **Lemma.** *The space $L(E,F)$ of all bounded linear mappings is a closed linear subspace of $C^\omega(E,F)$. A mapping $f : U \to L(E,F)$ is real analytic if and only if $\mathrm{ev}_x \circ f : U \to F$ is real analytic for each point $x \in E$.*

9.6. **Theorem.** *The category of convenient spaces and real analytic mappings is cartesian closed. So the equation*

$$C^\omega(U, C^\omega(V,F)) \cong C^\omega(U \times V, F)$$

is valid for all c^∞-open sets U in E and V in F, where E, F, and G are convenient vector spaces.

This implies again that all structure mappings as in (7.3.8) are real analytic. Furthermore the differential operator

$$d : C^\omega(U,F) \to C^\omega(U, L(E,F))$$

exists, is unique and real analytic. Multilinear mappings are real analytic if and only if they are bounded.

9.7. **Theorem** (Real analytic uniform boundedness principle)**.** *A linear mapping $f : E \to C^\omega(V,G)$ is real analytic (bounded) if and only if $E \xrightarrow{f} C^\omega(V,G) \xrightarrow{\mathrm{ev}_v} G$ is real analytic (bounded) for each $v \in V$.*

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Submitted

THE CONVENIENT SETTING FOR QUASIANALYTIC DENJOY–CARLEMAN DIFFERENTIABLE MAPPINGS

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Abstract. For quasianalytic Denjoy–Carleman differentiable function classes C^Q where the weight sequence $Q = (Q_k)$ is log-convex, stable under derivations, of moderate growth and also an $\mathcal{L}$-intersection (see (1.6)), we prove the following: The category of C^Q-mappings is cartesian closed in the sense that $C^Q(E, C^Q(F, G)) \cong C^Q(E \times F, G)$ for convenient vector spaces. Applications to manifolds of mappings are given: The group of C^Q-diffeomorphisms is a regular C^Q-Lie group but not better.

Classes of Denjoy-Carleman differentiable functions are in general situated between real analytic functions and smooth functions. They are described by growth conditions on the derivatives. Quasianalytic classes are those where infinite Taylor expansion is an injective mapping.

That a class of mappings $\mathcal{S}$ admits a convenient setting means essentially that we can extend the class to mappings between admissible infinite dimensional spaces $E, F, \dots$ so that $\mathcal{S}(E, F)$ is again admissible and we have $\mathcal{S}(E \times F, G)$ canonically $\mathcal{S}$-diffeomorphic to $\mathcal{S}(E, \mathcal{S}(F, G))$ (the exponential law). Usually this comes hand in hand with (partly nonlinear) uniform boundedness theorems which are easy $\mathcal{S}$-detection principles.

For the C^∞ convenient setting one can test smoothness along smooth curves. For the real analytic (C^ω) convenient setting we have: A mapping is C^ω if and only if it is C^∞ and in addition C^ω along C^ω-curves (C^ω along just affine lines suffices). We shall use convenient calculus of C^∞ and C^ω mappings in this paper; see the book [KM97a], or the three appendices in [KMR09a] for a short overview.

In [KMR09a] we succeeded to show that non-quasianalytic log-convex Denjoy-Carleman classes C^M of moderate growth (hence derivation closed) admit a convenient setting, where the underlying admissible locally convex vector spaces are the same as for smooth or for real analytic mappings. A mapping is C^M if and only if it is C^M along all C^M-curves. The method of proof there relies on the existence of C^M partitions of unity.

In this paper we succeed to prove that quasianalytic log-convex Denjoy-Carleman classes C^Q of moderate growth which are also $\mathcal{L}$-intersections (see (1.6)), admit a convenient setting. The method consists of representing C^Q as the intersection $\bigcap\{C^L : L \in \mathcal{L}(Q)\}$ of all larger non-quasianalytic log-convex classes C^L; this is the meaning of: Q is an $\mathcal{L}$-intersection. In (1.9) we construct countably many classes Q which satisfy all these requirements. Taking intersections of derivation closed classes C^L only, or only of classes C^L of moderate growth, is not sufficient for yielding the intended results. Thus we have to strengthen many results from [KMR09a] before we are able to prove the exponential law. A mapping is C^Q if and only if it is C^L along each C^L-curve for each $L \in \mathcal{L}(Q)$. It is an open problem (even in $\mathbb{R}^2$), whether a smooth mapping which is C^Q along each C^Q-curve (or

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affine line), is indeed C^Q. As replacement we show that a mapping is C^Q if it is C^Q along each C^Q mapping from a Banach ball (5.2). The real analytic case from [KM90] is not covered by this approach.

The initial motivation of both [KMR09a] and this paper was the desire to prove the following result which is due to Rellich [Rel42] in the real analytic case. *Let $t \mapsto A(t)$ for $t \in \mathbb{R}$ be a curve of unbounded self-adjoint operators in a Hilbert space with common domain of definition and with compact resolvent. If $t \mapsto A(t)$ is of a certain quasianalytic Denjoy-Carleman class C^Q, then the eigenvalues and the eigenvectors of $A(t)$ may be parameterized C^Q in t also.* We manage to prove this with the help of the results in this paper and in [KMR09a]. Due to length this will be explained in another paper [KMR09b].

Generally, one can hope that the space $C^M(A,B)$ of all Denjoy-Carleman C^M-mappings between finite dimensional C^M-manifolds (with A compact for simplicity) is again a C^M-manifold, that composition is C^M, and that the group $\mathrm{Diff}^M(A)$ of all C^M-diffeomorphisms of A is a regular infinite dimensional C^M-Lie group, for each class C^M which admits a convenient setting. For the non-quasianalytic classes this was proved in [KMR09a]. For quasianalytic classes this is proved in this paper.

1. Weight Sequences and function spaces

1.1. **Denjoy–Carleman C^M-functions in finite dimensions.** We mainly follow [KMR09a] and [Thi08] (see also the references therein). We use $\mathbb{N} = \mathbb{N}_{>0} \cup \{0\}$. For each multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we write $\alpha! = \alpha_1! \cdots \alpha_n!$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$.

Let $M = (M_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers. Let $U \subseteq \mathbb{R}^n$ be open. We denote by $C^M(U)$ the set of all $f \in C^\infty(U)$ such that, for all compact $K \subseteq U$, there exist positive constants C and ρ such that

$$|\partial^\alpha f(x)| \le C\, \rho^{|\alpha|}\, |\alpha|!\, M_{|\alpha|} \quad \text{for all } \alpha \in \mathbb{N}^n \text{ and } x \in K.$$

The set $C^M(U)$ is a *Denjoy–Carleman class* of functions on U. If $M_k = 1$, for all k, then $C^M(U)$ coincides with the ring $C^\omega(U)$ of real analytic functions on U.

A sequence $M = (M_k)$ is *log-convex* if $k \mapsto \log(M_k)$ is convex, i.e.,

$$M_k^2 \le M_{k-1}\, M_{k+1} \quad \text{for all } k.$$

If $M = (M_k)$ is log-convex, then $k \mapsto (M_k/M_0)^{1/k}$ is increasing and

$$M_l\, M_k \le M_0\, M_{l+k} \quad \text{for all } l, k \in \mathbb{N}. \tag{1}$$

Furthermore, we have that $k \mapsto k! M_k$ is log-convex (since Euler's Γ-function is so), and we call this weaker condition *weakly log-convex.* If M is weakly log-convex then $C^M(U, \mathbb{R})$ is a ring, for all open subsets $U \subseteq \mathbb{R}^n$.

If M is log-convex then (see the proof of [KMR09a, 2.9]) we have

$$M_1^j\, M_k \ge M_j\, M_{\alpha_1} \cdots M_{\alpha_j} \quad \text{for all } \alpha_i \in \mathbb{N}_{>0} \text{ with } \alpha_1 + \cdots + \alpha_j = k. \tag{2}$$

This implies that the class of C^M-mappings is stable under composition ([Rou63], see also [BM04, 4.7]; this also follows from (1.4)). If M is log-convex then the inverse function theorem for C^M holds ([Kom79]; see also [BM04, 4.10]), and C^M is closed under solving ODEs (due to [Kom80]).

Suppose that $M = (M_k)$ and $N = (N_k)$ satisfy $M_k \le C^k\, N_k$, for a constant C and all k. Then $C^M(U) \subseteq C^N(U)$. The converse is true if M is weakly log-convex: There exists $f \in C^M(\mathbb{R})$ such that $|f^{(k)}(0)| \ge k!\, M_k$ for all k (see [Thi08, Theorem 1]).

If M is weakly log-convex then C^M is *stable under derivations* (alias *derivation closed*) if and only if

$$\sup_{k\in\mathbb{N}_{>0}}\Big(\frac{M_{k+1}}{M_k}\Big)^{\frac{1}{k}}<\infty. \tag{3}$$

A weakly log-convex sequence M is called of *moderate growth* if

$$\sup_{j,k\in\mathbb{N}_{>0}}\Big(\frac{M_{j+k}}{M_j\,M_k}\Big)^{\frac{1}{j+k}}<\infty. \tag{4}$$

Moderate growth implies derivation closed.

Definition. A sequence $M=(M_k)_{k=0,1,2,\dots}$ is called a *weight sequence* if it satisfies $M_0=1\le M_1$ and is log-convex. Consequently, it is increasing (i.e. $M_k\le M_{k+1}$).

A *DC-weight sequence* $M=(M_k)_{k=0,1,2,\dots}$ is a weight sequence which is also derivation closed (DC stands for Denjoy-Carleman and also for derivation closed). This was the notion investigated in [KMR09a].

1.2. **Theorem** (Denjoy–Carleman [Den21], [Car26])**.** *For a sequence M of positive numbers the following statements are equivalent.*

(1) *C^M is quasianalytic, i.e., for open connected $U\subseteq\mathbb{R}^n$ and each $a\in U$, the Taylor series homomorphism centered at a from $C^M(U,\mathbb{R})$ into the space of formal power series is injective.*
(2) *$\sum_{k=1}^\infty\frac{1}{m_k^{\flat(i)}}=\infty$ where $m_k^{\flat(i)}:=\inf\{(j!\,M_j)^{1/j}:j\ge k\}$ is the increasing minorant of $(k!\,M_k)^{1/k}$.*
(3) *$\sum_{k=1}^\infty(\frac{1}{M_k^{\flat(lc)}})^{1/k}=\infty$ where $M_k^{\flat(lc)}$ is the log-convex minorant of $k!\,M_k$, given by $M_k^{\flat(lc)}:=\inf\{(j!\,M_j)^{\frac{l-k}{l-j}}(l!\,M_l)^{\frac{k-j}{l-j}}:j\le k\le l, j<l\}$.*
(4) *$\sum_{k=0}^\infty\frac{M_k^{\flat(lc)}}{M_{k+1}^{\flat(lc)}}=\infty$.*

For contemporary proofs see for instance [Hör83, 1.3.8] or [Rud87, 19.11].

1.3. **Sequence spaces.** Let $M=(M_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers and $\rho>0$. We consider (where $\mathcal{F}$ stands for 'formal power series')

$$\mathcal{F}^M_\rho:=\Big\{(f_k)_{k\in\mathbb{N}}\in\mathbb{R}^\mathbb{N}:\exists C>0\,\forall k\in\mathbb{N}:|f_k|\le C\,\rho^k\;k!\;M_k\Big\}\text{ and }\mathcal{F}^M:=\bigcup_{\rho>0}\mathcal{F}^M_\rho.$$

Note that, for $U\subseteq\mathbb{R}^n$ open, a function $f\in C^\infty(U,\mathbb{R})$ is in $C^M(U,\mathbb{R})$ if and only if for each compact $K\subset U$

$$(\sup\{|\partial^\alpha f(x)|:x\in K,|\alpha|=k\})_{k\in\mathbb{N}}\in\mathcal{F}^M.$$

Lemma. *We have*

$$\begin{aligned}\mathcal{F}^{M^1}\subseteq\mathcal{F}^{M^2}&\Leftrightarrow\exists\rho>0\;\forall k:M^1_k\le\rho^{k+1}\,M^2_k\\&\Leftrightarrow\exists C,\rho>0\;\forall k:M^1_k\le C\,\rho^k\,M^2_k.\end{aligned}$$

Proof. ($\Rightarrow$) Let $f_k:=k!M^1_k$. Then $f=(f_k)_{k\in\mathbb{N}}\in\mathcal{F}^{M^1}\subseteq\mathcal{F}^{M^2}$, so there exists a $\rho>0$ such that $k!M^1_k\le\rho^{k+1}k!M^2_k$ for all k.

($\Leftarrow$) Let $f=(f_k)_{k\in\mathbb{N}}\in\mathcal{F}^{M^1}$, i.e. there exists a $\sigma>0$ with $|f_k|\le\sigma^{k+1}k!M^1_k\le(\rho\sigma)^{k+1}k!M^2_k$ for all k and thus $f\in\mathcal{F}^{M^2}$. □

1.4. **Lemma.** *Let M and L be sequences of positive numbers. Then for the composition of formal power series we have*

$$\mathcal{F}^M\circ\mathcal{F}^L_{>0}\subseteq\mathcal{F}^{M\circ L}$$

where $(M\circ L)_k:=\max\{M_jL_{\alpha_1}\dots L_{\alpha_j}:\alpha_i\in\mathbb{N}_{>0},\alpha_1+\dots+\alpha_j=k\}$

Here $\mathcal{F}^L_{>0} := \{(g_k)_{k\in\mathbb{N}} \in \mathcal{F}^L : g_0 = 0\}$ is the space of formal power series in $\mathcal{F}^L$ with vanishing constant term.

Proof. Let $f \in \mathcal{F}^M$ and $g \in \mathcal{F}^L$. For $k > 0$ we have (inspired by [FdB55])

$$\begin{aligned}
\frac{(f\circ g)_k}{k!} &= \sum_{j=1}^k \frac{f_j}{j!} \sum_{\substack{\alpha\in\mathbb{N}^j_{>0}\\ \alpha_1+\dots+\alpha_j=k}} \frac{g_{\alpha_1}}{\alpha_1!}\cdots\frac{g_{\alpha_j}}{\alpha_j!}\\
\frac{|(f\circ g)_k|}{k!(M\circ L)_k} &\le \sum_{j=1}^k \frac{|f_j|}{j!M_j} \sum_{\substack{\alpha\in\mathbb{N}^j_{>0}\\ \alpha_1+\dots+\alpha_j=k}} \frac{|g_{\alpha_1}|}{\alpha_1!L_{\alpha_1}}\cdots\frac{|g_{\alpha_j}|}{\alpha_j!L_{\alpha_j}}\\
&\le \sum_{j=1}^k \rho_f^j C_f \sum_{\substack{\alpha\in\mathbb{N}^j_{>0}\\ \alpha_1+\dots+\alpha_j=k}} \rho_g^k C_g^j \le \sum_{j=1}^k \rho_f^j C_f \binom{k-1}{j-1}\rho_g^k C_g^j\\
&= \rho_g^k \rho_f C_f C_g \sum_{j=1}^k (\rho_f C_g)^{j-1}\binom{k-1}{j-1} = \rho_g^k\rho_f C_f C_g(1+\rho_f C_g)^{k-1}\\
&= (\rho_g(1+\rho_f C_g))^k \frac{\rho_f C_f C_g}{1+\rho_f C_g} \qquad \square
\end{aligned}$$

1.5. **Notation for quasianalytic weight sequences.** Let M be a sequence of positive numbers. We may replace M by $k \mapsto C\,\rho^k\,M_k$ with $C,\rho > 0$ without changing $\mathcal{F}^M$. In particular, it is no loss of generality to assume that $M_1 > 1$ (put $C\rho > 1/M_1$) and $M_0 = 1$ (put $C := 1/M_0$). If M is log-convex then so is the modified sequence and if in addition $\rho \ge M_0/M_1$ then the modified sequence is monotone increasing. Furthermore M is quasianalytic if and only if the modified sequence is so, since $M_k^{\flat(lc)}$ is modified in the same way. We tried to make all conditions equivariant under this modification. Unfortunately, the next construction does not react nicely to this modification.

For a quasianalytic sequence $M = (M_k)$ let the sequence $\check M = (\check M_k)$ be defined by

$$\check M_k := M_k \prod_{j=1}^k \left(1 - \frac{1}{(j!\,M_j)^{1/j}}\right)^k, \quad \check M_0 = 1.$$

We have $\check M_k \le M_k$. Note that if we put $m_k := (k!M_k)^{1/k}$ (and $m_0 := 1$) and $\check m_k := (k!\check M_k)^{1/k}$ (where we assume $\check M_k \ge 0$) then

$$\check m_k = m_k \prod_{j=1}^k \left(1 - \frac{1}{m_j}\right)$$

or, recursively,

$$\check m_{k+1} = \check m_k \frac{m_{k+1}-1}{m_k} \text{ and } \check m_0 = 1,\ \check m_1 = m_1 - 1.$$

And conversely, if all $\check M_k > 0$ (this is the case if M is increasing and $M_1 > 1$) then

$$m_{k+1} = 1 + m_k \frac{\check m_{k+1}}{\check m_k} \text{ and } m_0 = 1,\ m_1 = \check m_1 + 1$$

i.e.

$$m_k = \check m_k\Big(1 + \sum_{j=1}^k \frac{1}{\check m_j}\Big). \tag{1}$$

For sequences M we define (recall from (1.1) that M is called weakly log-convex if $k \mapsto \log(k!\, M_k)$ is convex):

$$\begin{aligned}
\mathcal{L}(M) &:= \{L \geq M : L \text{ non-quasianalytic, log-convex}\}\\
\mathcal{L}_w(M) &:= \{L \geq M : L \text{ non-quasianalytic, weakly log-convex}\} \supseteq \mathcal{L}(M)
\end{aligned}$$

1.6. **Theorem.** *Let $Q = (Q_k)_{k=0,1,2,\ldots}$ be a quasianalytic sequence of positive real numbers. Then we have:*

(1) *If the sequence $\check{Q} = (\check{Q}_k)$ is log-convex and positive then*
$$\mathcal{F}^Q = \bigcap_{L \in \mathcal{L}(Q)} \mathcal{F}^L.$$
(2) *If Q is weakly log-convex, then for each $L^1, L^2 \in \mathcal{L}_w(Q)$ there exists an $L \in \mathcal{L}_w(Q)$ with $L \leq L^1, L^2$.*
(3) *If Q is weakly log-convex of moderate growth, then for each $L \in \mathcal{L}_w(Q)$ there exists an $L' \in \mathcal{L}_w(Q)$ such that $L'_{j+k} \leq C^{j+k} L_j L_k$ for some positive constant C and all $j, k \in \mathbb{N}$.*

We could not obtain (2) for log-convex instead of weakly log-convex, in particular for $\mathcal{L}(Q)$ instead of $\mathcal{L}_w(Q)$.

Definition. A quasianalytic sequence Q of positive real numbers is called $\mathcal{L}$-*intersectable* or an $\mathcal{L}$-*intersection* if $\mathcal{F}^Q = \bigcap_{L \in \mathcal{L}(Q)} \mathcal{F}^L$ holds.

Note that we may replace any non-quasianalytic weight sequence L for which $k \mapsto (\frac{Q_k}{L_k})^{1/k}$ is bounded, by an $\tilde{L} \in \mathcal{L}(Q)$ with $\mathcal{F}^{\tilde{L}} = \mathcal{F}^L$: Choose $\rho \geq 1/L_1$ (see (1.5)) and $\rho \geq \sup\{(\frac{Q_k}{L_k})^{1/k} : k \in \mathbb{N}\}$ then $\tilde{L}_k := \rho^k L_k \geq Q_k$.

Proof. (1) The proof is partly adapted from [Bom65].

Let $q_k = (k!\, Q_k)^{1/k}$ and $q_0 = 1$, similarly $\check{q}_k = (k!\, \check{Q}_k)^{1/k}$, $l_k = (k! L_k)^{1/k}$, etc.. Then $\check{q}$ is increasing since $\check{Q}_0 = 1$, and $\check{Q}$ and the Gamma function are log-convex.

Clearly $\mathcal{F}^Q \subseteq \bigcap_{L \in \mathcal{L}(Q)} \mathcal{F}^L$. To show the converse inclusion, let $f \notin \mathcal{F}^Q$ and $g_k := |f_k|^{1/k}$. Then
$$\overline{\lim}\, \frac{g_k}{q_k} = \infty.$$
Choose $a_j, b_j > 0$ with $a_j \nearrow \infty$, $b_j \searrow 0$, and $\sum \frac{1}{a_j b_j} < \infty$. There exist strictly increasing k_j such that $\frac{g_{k_j}}{q_{k_j}} \geq a_j$. Since $\frac{q_k}{\check{q}_k}$ is increasing by (1.5.1) we get $b_j \frac{g_{k_j}}{\check{q}_{k_j}} = b_j \frac{g_{k_j}}{q_{k_j}} \frac{q_{k_j}}{\check{q}_{k_j}} \geq a_j b_j \frac{q_{k_1}}{\check{q}_{k_1}} \to \infty$. Passing to a subsequence we may assume that $k_0 > 0$ and $1 < \beta_j := b_j \frac{g_{k_j}}{\check{q}_{k_j}} \nearrow \infty$. Passing to a subsequence again we may also get

$$\beta_{j+1} \geq (\beta_j)^{k_j}\,. \tag{4}$$

Define a piecewise affine function ϕ by
$$\phi(k) := \begin{cases} 0 & \text{if } k = 0,\\ k_j \log \beta_j & \text{if } k = k_j,\\ c_j + d_j k & \text{for the minimal } j \text{ with } k \leq k_j, \end{cases}$$
where c_j and d_j are chosen such that ϕ is well defined and $\phi(k_{j-1}) = c_j + d_j k_{j-1}$, i.e., for $j \geq 1$,
$$\begin{aligned}
c_j + d_j k_j &= k_j \log \beta_j,\\
c_j + d_j k_{j-1} &= k_{j-1} \log \beta_{j-1}, \qquad \text{and}\\
c_0 &= 0,
\end{aligned} \tag{5}$$

$$d_0 = \log \beta_0.$$

This implies first that $c_j \le 0$ and then

$$\text{(6)} \qquad \log \beta_j \le d_j = \frac{k_j \log \beta_j - k_{j-1} \log \beta_{j-1}}{k_j - k_{j-1}} \le \frac{k_j}{k_j - k_{j-1}} \log \beta_j$$
$$\overset{(4)}{\le} \frac{\log \beta_{j+1}}{k_j - k_{j-1}} \le \log \beta_{j+1}.$$

Thus $j \mapsto d_j$ is increasing. It follows that ϕ is convex. The fact that all $c_j \le 0$ implies that $\phi(k)/k$ is increasing.

Now let

$$L_k := e^{\phi(k)} \cdot \check{Q}_k.$$

Then $L = (L_k)$ is log-convex and satisfies $L_0 = 1$ by construction and $f \notin \mathcal{F}^L$, since we have $\frac{l_{k_j}}{g_{k_j}} = \frac{\check{q}_{k_j} \beta_j}{g_{k_j}} = b_j \to 0$ and so $\overline{\lim}\, \frac{g_k}{l_k} = \infty$.

Let us check that L is not quasianalytic. By (6) and since $(\check{q}_k)$ is increasing, we have, for $k_{j-1} \le k < k_j$,

$$\frac{L_k}{(k+1)L_{k+1}} = \frac{e^{\phi(k)-\phi(k+1)}\, \check{Q}_k}{(k+1)\, \check{Q}_{k+1}} = \frac{e^{\phi(k)-\phi(k+1)}\, \check{q}_k^k}{\check{q}_{k+1}^{k+1}} = e^{-d_j} \frac{\check{q}_k^k}{\check{q}_{k+1}^{k+1}}$$
$$\le \frac{1}{\beta_j\, \check{q}_k} = \frac{\check{q}_{k_j}}{b_j g_{k_j}} \frac{1}{\check{q}_k}.$$

Thus, by (1.5.1),

$$\sum_{k=k_{j-1}}^{k_j - 1} \frac{L_k}{(k+1)L_{k+1}} \le \frac{\check{q}_{k_j}}{b_j g_{k_j}} \sum_{k=k_{j-1}}^{k_j-1} \frac{1}{\check{q}_k} \le \frac{q_{k_j}}{b_j g_{k_j}} \le \frac{1}{a_j b_j},$$

which shows that L is not quasianalytic and $C_1 := \sum_{k=1}^\infty \frac{1}{l_k} < \infty$ by (1.2).

Next we claim that $\mathcal{F}^Q \subseteq \mathcal{F}^L$. Since $\frac{l_k}{\check{q}_k} = \frac{(k!L_k)^{1/k}}{(k!\check{Q}_k)^{1/k}} = e^{\phi(k)/k}$ is increasing, we have

$$\infty > \frac{\check{q}_1}{l_1} + C_1 > \frac{\check{q}_1}{l_1} + \sum_{j=1}^k \frac{1}{l_j} = \frac{\check{q}_1}{l_1} + \sum_{j=1}^k \frac{\check{q}_j}{l_j} \frac{1}{\check{q}_j} \ge \frac{\check{q}_k}{l_k}\Big(1 + \sum_{j=1}^k \frac{1}{\check{q}_j}\Big) = \frac{q_k}{l_k},$$

which proves $\mathcal{F}^Q \subseteq \mathcal{F}^L$. Finally we may replace L by some $L \in \mathcal{L}(Q)$ without changing $\mathcal{F}^L$ by the remark before the proof. Thus (1) is proved.

(2) Assume without loss that $L_0^1 = L_0^2 = 1$. Let $k!L_k$ be the log-convex minorant of $k!\bar{L}_k$ where $\bar{L}_k := \min\{L_k^1, L_k^2\}$. Since $L^1, L^2 \ge \bar{L} \ge Q$ and $k!Q_k$ is log-convex we have $L^1, L^2 \ge L \ge Q$. Since L^1, L^2 are not quasianalytic and are weakly log-convex (hence $k \mapsto (k!L_k^j)^{1/k}$ is increasing), we get that $k \mapsto (k!\bar{L}_k)^{1/k}$ is increasing and

$$\sum_k \frac{1}{(k!\,\bar{L}_k)^{1/k}} \le \sum_k \frac{1}{(k!\,L_k^1)^{1/k}} + \sum_k \frac{1}{(k!\,L_k^2)^{1/k}} < \infty.$$

By (1.2, 2⇒1) we get that $\bar{L}$ is not quasianalytic. By (1.2, 1⇒3) we get $\sum_k \frac{1}{(k!L_k)^{1/k}} < \infty$ since $\bar{L}^{\flat(lc)} = L$, i.e. L is not quasianalytic.

(3) Let $\tilde{Q}_k := k!Q_k$, $\tilde{L}_k := k!L_k$, and so on. Since Q is of moderate growth we have

$$C_{\tilde{Q}} := \sup_{k,j} \left(\frac{\tilde{Q}_{k+j}}{\tilde{Q}_k \tilde{Q}_j}\right)^{1/(k+j)} \le 2 \sup_{k,j} \left(\frac{Q_{k+j}}{Q_k Q_j}\right)^{1/(k+j)} < \infty.$$

Let $L \in \mathcal{L}_w(Q)$; without loss we assume that $L_0 = 1$. We put

$$\tilde{L}'_k := C_{\tilde{Q}}^k \min\{\tilde{L}_j \tilde{L}_{k-j} : j = 0, \dots, k\} = C_{\tilde{Q}}^k \min\{\tilde{L}_j \tilde{L}_{k-j} : 0 \le j \le k/2\}.$$

Then

$$\sup_{k,j}\left(\frac{L'_{k+j}}{L_k L_j}\right)^{1/(k+j)} \le \sup_{k,j}\left(\frac{\tilde L'_{k+j}}{\tilde L_k \tilde L_j}\right)^{1/(k+j)} \le C_{\tilde Q} < \infty.$$

Since $\tilde L$ is log-convex we have $\tilde L_k^2 \le \tilde L_j \tilde L_{2k-j}$ and $\tilde L_k \tilde L_{k+1} \le \tilde L_j \tilde L_{2k+1-j}$ for $j = 0,\dots,k$; therefore $\tilde L'_{2k} = C_{\tilde Q}^{2k}\tilde L_k^2$ and $\tilde L'_{2k+1} = C_{\tilde Q}^{2k+1}\tilde L_k \tilde L_{k+1}$. It is easy to check that $\tilde L'$ is log-convex. To see that L' is not quasianalytic we will use that $(\tilde L'_k)^{1/k}$ is increasing since $\tilde L'$ is log-convex. So it suffices to compute the sum of the even indices only.

$$\sum_k \frac{1}{{\tilde L'_{2k}}^{1/(2k)}} = \frac{1}{C_{\tilde Q}}\sum_k \frac{1}{{\tilde L_k}^{1/k}} < \infty.$$

It remains to show that $L' \ge Q$. Since $L \in \mathcal{L}_w(Q)$ we have $Q \le L$ and for $j = \lfloor k/2 \rfloor$,

$$\frac{Q_k}{L'_k} = \frac{\tilde Q_k}{\tilde L'_k} = \frac{\tilde Q_k}{C_{\tilde Q}^k \tilde L_j \tilde L_{k-j}} \le \frac{\tilde Q_k}{\frac{\tilde Q_k}{\tilde Q_j \tilde Q_{k-j}}\tilde L_j \tilde L_{k-j}} \le \frac{\tilde Q_j}{\tilde L_j}\frac{\tilde Q_{k-j}}{\tilde L_{k-j}} \le 1. \quad \square$$

1.7. **Corollary.** *Let Q be a quasianalytic weight sequence. Then*

$$\mathcal{F}^Q = \bigcap_{L \in \mathcal{L}_w(Q)} \mathcal{F}^L.$$

Proof. Without loss we may assume that the sequence $\check q_k$ is increasing. Namely, by definition this is the case if and only if $q_k \le q_{k+1} - 1$. Since $Q_0 = 1$ and (Q_k) is log-convex, $Q_k^{1/k}$ is increasing and thus $q_{k+1} - q_k \ge Q_k^{\frac{1}{k}}\big((k+1)!^{\frac{1}{k+1}} - k!^{\frac{1}{k}}\big) \ge Q_1 \frac{1}{e} \ge \frac{1}{e}$. If we set $\tilde Q_k := e^k Q_k$, then $\tilde Q = (\tilde Q_k)$ is a quasianalytic weight sequence with $\tilde Q_1 > 1$, $\mathcal{F}^{\tilde Q} = \mathcal{F}^Q$, and $\check{\tilde q}_k$ is increasing.

Now a little adaptation of the proof of (1.6.1) shows the corollary: Define here

$$l_k := \beta_j \check q_k \qquad \text{for the minimal } j \text{ with } k \le k_j.$$

Then $\frac{l_{k_j}}{g_{k_j}} = \frac{\beta_j \check q_{k_j}}{g_{k_j}} = b_j \to 0$ and so $\overline{\lim}\, \frac{g_k}{l_k} = \infty$. We have

$$\sum_{k=k_{j-1}+1}^{k_j} \frac{1}{l_k} = \sum_{k=k_{j-1}+1}^{k_j} \frac{1}{\beta_j \check q_k} = \frac{\check q_{k_j}}{b_j g_{k_j}} \sum_{k=k_{j-1}+1}^{k_j} \frac{1}{\check q_k} \le \frac{q_{k_j}}{b_j g_{k_j}} \le \frac{1}{a_j b_j}$$

and thus $\sum_{k=1}^\infty \frac{1}{l_k} < \infty$. As l_k is increasing, the Denjoy–Carleman theorem (1.2) implies that $L_k = \frac{l_k^k}{k!}$ is non-quasianalytic. Since $\frac{l_k}{\check q_k} = \beta_j$ is increasing, we find (as in the proof of (1.6.1)) that $C := \max\{L_0/L_1, \sup_k \frac{q_k}{l_k}\} < \infty$. Replacing L_k by $C^k L_k$ we may assume that $Q \le L$. Let the sequence $k!\underline{L}_k$ be the log-convex minorant of $k! L_k$. Since Q_k is (weakly) log-convex, we have $Q \le \underline{L}$. By (1.2) and the fact that L is non-quasianalytic, $\underline{L}$ is non-quasianalytic as well. Thus $\underline{L} \in \mathcal{L}_w(Q)$ and still $f \notin \mathcal{F}^{\underline{L}}$. $\square$

Corollary (1.7) implies that for the sequence $\omega = (1)_k$ describing real analytic functions we have $\mathcal{F}^\omega = \bigcap_{L \in \mathcal{L}_w(\omega)} \mathcal{F}^L$. Note that $\mathcal{L}_w(\omega)$ consists of all weakly log-convex non-quasianalytic $L \ge 1$. This is slightly stronger than a result by T. Bang, who shows that $\mathcal{F}^\omega = \bigcap \mathcal{F}^L$ where L runs through all non-quasianalytic sequences with $l_k = (k!\, L_k)^{1/k}$ increasing, see [Ban46], [Bom65].

This result becomes wrong if we replace weakly log-convex by log-convex:

1.8. **The intersection of all $\mathcal{F}^L$, where L is any non-quasianalytic weight sequence.** Put

$$Q_k := \frac{(k\log(k+e))^k}{k!}, \quad Q_0 := 1.$$

Then $Q = (Q_k)$ is a quasianalytic weight sequence of moderate growth with $Q_1 > 1$. We claim that Q is $\mathcal{L}$-intersectable, i.e., $\mathcal{F}^Q = \bigcap_{L\in\mathcal{L}(Q)} \mathcal{F}^L$. We could check that $\check{Q}$ is log-convex. This can be done, but is quite cumbersome. A simpler argument is the following. We consider $\check{q}'_k := k$, $\check{q}'_0 := 1$. Then $\check{Q}'_k = k^k/k!$ is log-convex. Since $C_1 \log k \le \sum_{j=1}^k \frac{1}{j} \le C_2 \log k$, we have by (1.5.1)

$$C_3 k\log(k+e) \le q'_k \le C_4 k \log(k+e)$$

for suitable constants C_i. Hence $\mathcal{F}^Q = \mathcal{F}^{Q'}$. By theorem (1.6.1) we have

$$\mathcal{F}^Q = \mathcal{F}^{Q'} = \bigcap_{L\in\mathcal{L}(Q')} \mathcal{F}^L = \bigcap_{L\in\mathcal{L}(Q)} \mathcal{F}^L$$

since $\mathcal{L}(Q)$ and $\mathcal{L}(Q')$ contain only sequences which are "equivalent mod (ρ^k)". The claim is proved.

Let L be any non-quasianalytic weight sequence. Consider

$$\alpha_k := \frac{(k!\,L_k)^{\frac{1}{k}}}{k} = \frac{l_k}{k}.$$

Since L is log-convex and $L_0 = 1$, we find that $L_k^{1/k}$ is increasing. Thus, for $s \le k$ we find

$$\frac{\alpha_s}{\alpha_k} = \frac{k}{s}\cdot\frac{s!^{1/s}}{k!^{1/k}}\cdot\frac{L_s^{1/s}}{L_k^{1/k}} \le 2e$$

(using Stirling's formula for instance). Since L is not quasianalytic, we have $\sum_{k=1}^\infty \frac{1}{k\alpha_k} < \infty$. But

$$\sum_{\sqrt{k}\le s\le k} \frac{1}{s\alpha_s} \ge \frac{1}{2e}\cdot\frac{1}{\alpha_k}\sum_{\sqrt{k}\le s\le k}\frac{1}{s} \sim \frac{1}{2e}\cdot\frac{1}{\alpha_k}\cdot\frac{\log k}{2}.$$

The sum on the left tends to 0 as $k\to\infty$. So $\frac{\log k}{\alpha_k} = \frac{k\log k}{l_k}$ is bounded. Thus $\mathcal{F}^Q \subseteq \mathcal{F}^L$.

So we have proved the following theorem (which is intimately related to [Rud62, Thm. C]).

Theorem. *Put $Q_k = (k\log(k+e))^k/k!$, $Q_0 = 1$. Then Q is $\mathcal{L}$-intersectable. In fact,*

$$\mathcal{F}^Q = \bigcap\{\mathcal{F}^L : L \text{ non-quasianalytic weight sequence}\}. \quad \square$$

Remark. Log-convexity of $\check{Q}$ is only sufficient for Q being an $\mathcal{L}$-intersection, see (1.6.1): Using Stirling's formula we see that $\mathcal{F}^Q = \mathcal{F}^{Q''}$ for $Q_k = (k\log(k+e))^k/k!$ and $Q''_k = (\log(k+e))^k$. Also $\mathcal{L}(Q)$ and $\mathcal{L}(Q'')$ contain only sequences which are "equivalent mod (ρ^k)" and (1.6.1) holds for Q, thus also for Q''. But $\check{Q}''$ is not log-convex.

1.9. **A class of examples.** Let $\log^n$ denote the n-fold composition of log defined recursively by

$$\log^1 := \log,$$
$$\log^n := \log\circ\log^{n-1}, \quad (n\ge 2).$$

For $0 < \delta \le 1$, $n \in \mathbb{N}_{>0}$, we recursively define sequences $q^{\delta,n} = (q_k^{\delta,n})_{k\ge\kappa_n}$ by

$$q_k^{1,1} := k\log k,$$

$$q_k^{\delta,n} := q_k^{1,n-1} \cdot (\log^n(k))^\delta, \quad (n \geq 2),$$

where κ_n is the smallest integer greater than $e \uparrow\uparrow n$, i.e.,

$$\kappa_n := \lceil e \uparrow\uparrow n \rceil, \quad e \uparrow\uparrow n := \underbrace{e^{e^{\cdot^{\cdot^{e}}}}}_{n \text{ times}}.$$

Let $Q^{\delta,n} := (Q_k^{\delta,n})_{k\in\mathbb{N}}$ with

$$Q_0^{\delta,n} := 1,$$
$$Q_k^{\delta,n} := \frac{1}{(k-1+\kappa_n)!}(q_{k-1+\kappa_n}^{\delta,n})^{k-1+\kappa_n}, \quad (k \geq 1),$$

and consider

$$\mathcal{Q} := \{Q^{1,1}\} \cup \{Q^{\delta,n} : 0 < \delta \leq 1, n \in \mathbb{N}_{>1}\}.$$

It is easy to check inductively that each $Q \in \mathcal{Q}$ is a quasianalytic weight sequence of moderate growth with $Q_1 > 1$. Namely, $(\log^n(k))^{\delta k}$ is increasing, log-convex, and has moderate growth. Quasianalyticity follows from Cauchy's condensation criterion or the integral test. By construction, $\mathcal{Q} \ni Q \mapsto \mathcal{F}^Q$ is injective.

Let us consider

$$\hat{q}_k^{1,n} := q_k^{1,n-1}\left(1 + \sum_{j=\kappa_n}^{k} \frac{1}{q_j^{1,n-1}}\right).$$

Since $\frac{d}{dx}\log^n(x) = \frac{1}{x\log(x)\cdots\log^{n-1}(x)}$, we have (by comparison with the corresponding integral)

$$C_1 \log^n(k) \leq \sum_{j=\kappa_n}^{k} \frac{1}{q_j^{1,n-1}} \leq C_2 \log^n(k)$$

and thus

$$C_3 q_k^{1,n} \leq \hat{q}_k^{1,n} \leq C_4 q_k^{1,n} \tag{1}$$

for suitable constants C_i. Hence $\mathcal{F}^{Q^{1,n}} = \mathcal{F}^{\hat{Q}^{1,n}}$. Since $Q^{1,n-1}$ is log-convex, theorem (1.6.1) implies

$$\mathcal{F}^{Q^{1,n}} = \mathcal{F}^{\hat{Q}^{1,n}} = \bigcap_{L\in\mathcal{L}(\hat{Q}^{1,n})} \mathcal{F}^L = \bigcap_{L\in\mathcal{L}(Q^{1,n})} \mathcal{F}^L$$

since $\mathcal{L}(\hat{Q}^{1,n})$ and $\mathcal{L}(Q^{1,n})$ contain only sequences which are "equivalent mod (ρ^k)".

Hence we have proved (the case $n = 1$ follows from (1.8)):

Theorem. *Each $Q^{1,n}$ ($n \in \mathbb{N}_{>0}$) is a quasianalytic weight sequence of moderate growth which is an $\mathcal{L}$-intersection, i.e.,*

$$\mathcal{F}^{Q^{1,n}} = \bigcap_{L\in\mathcal{L}(Q^{1,n})} \mathcal{F}^L. \quad \square$$

Conjecture. *This is true for each $Q \in \mathcal{Q}$.*

Remark. Let $\check{Q}$ be any quasianalytic log-convex sequence of positive numbers. Then the corresponding sequence Q (determined by (1.5.1)) is quasianalytic and $\mathcal{L}$-intersectable. However, the mapping $\check{Q} \mapsto \mathcal{F}^Q$ is not injective. For instance, the image of $(C\rho^k\check{Q}_k)_k$ is the same for all positive C and ρ (which follows from (1.5.1)). Here is a more striking example:

Let $Q^{\delta,n} \in \mathcal{Q}$ and let $P^{\delta,n} = (P_k^{\delta,n})_k$ be defined by

$$P_k^{\delta,n} := \frac{1}{(k-1+\kappa_n)!}(p_{k-1+\kappa_n}^{\delta,n})^{k-1+\kappa_n}, \quad P_0^{\delta,n} := 1,$$

where

$$p_k^{\delta,n} := q_k^{\delta,n}\left(1+\sum_{j=\kappa_n}^{k}\frac{1}{q_j^{\delta,n}}\right),\qquad \text{for } 0<\delta<1,$$

$$p_k^{1,n} := \hat q_k^{1,n+1} = q_k^{1,n}\left(1+\sum_{j=\kappa_{n+1}}^{k}\frac{1}{q_j^{1,n}}\right).$$

We claim that $\mathcal{F}^{P^{1,n-1}}=\mathcal{F}^{P^{\delta,n}}=\mathcal{F}^{P^{\epsilon,n}}$ *for all* $0<\delta,\epsilon<1$. For: Since

$$\frac{d}{dx}\frac{(\log^n(x))^{1-\delta}}{1-\delta}=\frac{1}{x\log(x)\cdots\log^{n-1}(x)(\log^n(x))^{\delta}},$$

we have

$$C_1\frac{(\log^n(k))^{1-\delta}}{1-\delta}\le\sum_{j=\kappa_n}^{k}\frac{1}{q_j^{\delta,n}}\le C_2\frac{(\log^n(k))^{1-\delta}}{1-\delta},$$

and thus

$$\frac{p_k^{\delta,n}}{p_k^{\epsilon,n}}=\frac{(\log^n(k))^{\delta}}{(\log^n(k))^{\epsilon}}\frac{\left(1+\sum_{j=\kappa_n}^{k}\frac{1}{q_j^{\delta,n}}\right)}{\left(1+\sum_{j=\kappa_n}^{k}\frac{1}{q_j^{\epsilon,n}}\right)}\le C_3\frac{(\log^n(k))^{\delta}}{(\log^n(k))^{\epsilon}}\frac{(\log^n(k))^{1-\delta}}{(\log^n(k))^{1-\epsilon}}=C_3$$

and similarly

$$\frac{p_k^{\delta,n}}{p_k^{\epsilon,n}}\ge C_4$$

for suitable constants C_i. By lemma (1.3) we have $\mathcal{F}^{P^{\delta,n}}=\mathcal{F}^{P^{\epsilon,n}}$ for all $0<\delta,\epsilon<1$. The same reasoning with $\delta=0$ proves that $\mathcal{F}^{P^{1,n-1}}=\mathcal{F}^{P^{\epsilon,n}}$.

1.10. **Definition of function spaces.** Let $M=(M_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers, E and F be Banach spaces, $U\subseteq E$ open, $K\subseteq U$ compact, and $\rho>0$. We consider the non-Hausdorff Banach space

$$\begin{aligned}C^M_{K,\rho}(U,F) &:= \Big\{f\in C^\infty(U,F):(\sup_{x\in K}\|f^{(k)}(x)\|_{L^k(E,F)})_k\in\mathcal{F}^M_\rho\Big\}\\ &=\Big\{f\in C^\infty(U,F):\|f\|_{K,\rho}<\infty\Big\},\quad\text{where}\\ \|f\|_{K,\rho} &:= \sup\Big\{\frac{\|f^{(k)}(x)\|_{L^k(E,F)}}{k!\,M_k\,\rho^k}:x\in K,k\in\mathbb{N}\Big\},\end{aligned}$$

the inductive limit

$$C^M_K(U,F):=\varinjlim_{\rho>0}C^M_{K,\rho}(U,F),$$

and the projective limit

$$C^M_b(U,F):=\varprojlim_{K\subseteq U}C^M_K(U,F),\ \text{where } K \text{ runs through all compact subsets of } U.$$

Here $f^{(k)}(x)$ denotes the k^{th}-order Fréchet derivative of f at x.

Note that instead of $\|f^{(k)}(x)\|_{L^k(E,F)}$ we could equivalently use $\sup\{\|d_v^k f(x)\|_F : \|v\|_E\le 1\}$ by [KM97a, 7.13.1]. For $E=\mathbb{R}^n$ and $F=\mathbb{R}$ this is the same space as in (1.1).

For convenient vector spaces E and F, and c^∞-open $U\subseteq E$ we define:

$$C^M_b(U,F):=\Big\{f\in C^\infty(U,F):\forall B\ \forall\text{ compact } K\subseteq U\cap E_B\ \exists\rho>0:$$

$$\Big\{\frac{f^{(k)}(x)(v_1,\dots,v_k)}{k!\,\rho^k\,M_k} : k\in\mathbb{N}, x\in K, \|v_i\|_B\le 1\Big\} \text{ is bounded in } F\Big\}$$

$$= \Big\{f\in C^\infty(U,F) : \forall B\ \forall \text{ compact } K\subseteq U\cap E_B\ \exists\rho>0 :$$

$$\Big\{\frac{d_v^k f(x)}{k!\,\rho^k\,M_k} : k\in\mathbb{N}, x\in K, \|v\|_B\le 1\Big\} \text{ is bounded in } F\Big\}.$$

Here B runs through all closed absolutely convex bounded subsets and E_B is the vector space generated by B with the Minkowski functional $\|v\|_B = \inf\{\lambda\ge 0 : v\in \lambda B\}$ as complete norm.

Now we define the spaces of main interest in this paper: First we put

$$C^M(\mathbb{R},U) := \{c:\mathbb{R}\to U\ :\ \ell\circ c\in C^M(\mathbb{R},\mathbb{R})\ \forall\ell\in E^*\}.$$

In general, for L log-convex non-quasianalytic we put

$$\begin{aligned} C^L(U,F) &:= \{f : f\circ c\in C^L(\mathbb{R},F)\ \forall c\in C^L(\mathbb{R},U)\} \\ &= \{f : \ell\circ f\circ c\in C^L(\mathbb{R},\mathbb{R})\ \forall c\in C^L(\mathbb{R},U), \forall\ell\in F^*\} \end{aligned}$$

supplied with the initial locally convex structure induced by all linear mappings $C^L(c,\ell) : f\mapsto \ell\circ f\circ c\in C^L(\mathbb{R},\mathbb{R})$, which is a convenient vector space as c^∞-closed subspace in the product. Note that in particular the family $\ell_* : C^L(U,F)\to C^L(U,\mathbb{R})$ with $\ell\in F^*$ is initial, whereas this is not the case for C^L replaced by C^L_b as example (1.11) for $\{\mathrm{inj}_k\circ g^\vee(k) : k\in\mathbb{N}\}\subseteq C^L(\mathbb{R},\mathbb{R}^\mathbb{N})$ shows, where inj_k denotes the inclusion of the k-th factor in $\mathbb{R}^\mathbb{N}$.
For Q a quasianalytic $\mathcal{L}$-intersection we define the space

$$C^Q(U,F) := \bigcap_{L\in\mathcal{L}(Q)} C^L(U,F)$$

supplied with the initial locally convex structure. By theorem (1.6.1) this definition coincides with the classical notion of C^Q if E and F are finite dimensional.

Lemma. *For Q a quasianalytic $\mathcal{L}$-intersection, the composite of C^Q-mappings is again C^Q, and bounded linear mappings are C^Q.*

Proof. This is true for C^L (see [KMR09a, 3.1 and 3.11.1]) for every $L\in\mathcal{L}(Q)$ since each such L is log-convex. □

1.11. **Example.** By [Thi08, Theorem 1], for each weakly log-convex sequence M there exists $f\in C^M(\mathbb{R},\mathbb{R})$ such that $|f^{(k)}(0)|\ge k!\,M_k$ for all $k\in\mathbb{N}$. *Then $g : \mathbb{R}^2\to\mathbb{R}$ given by $g(s,t) = f(st)$ is C^M, whereas there is no reasonable topology on $C^M(\mathbb{R},\mathbb{R})$ such that the associated mapping $g^\vee : \mathbb{R}\to C^M(\mathbb{R},\mathbb{R})$ is C^M_b.* For a topology on $C^M(\mathbb{R},\mathbb{R})$ to be reasonable we require only that all evaluations $\mathrm{ev}_t : C^M(\mathbb{R},\mathbb{R})\to\mathbb{R}$ are bounded linear functionals.

Proof. The mapping g is obviously C^M. If $g^\vee$ were C^M_b, for $s=0$ there existed ρ such that

$$\Big\{\frac{(g^\vee)^{(k)}(0)}{k!\,\rho^k\,M_k} : k\in\mathbb{N}\Big\}$$

was bounded in $C^M(\mathbb{R},\mathbb{R})$. We apply the bounded linear functional ev_t for $t=2\rho$ and then get

$$\frac{(g^\vee)^{(k)}(0)(2\rho)}{k!\,\rho^k\,M_k} = \frac{(2\rho)^k f^{(k)}(0)}{k!\,\rho^k\,M_k}\ge 2^k,$$

a contradiction. □

This example shows that for C_b^M one cannot expect cartesian closedness. Using cartesian closedness (3.3) and (2.3) this also shows (for $F = C^M(\mathbb{R},\mathbb{R})$ and $U = \mathbb{R} = E$) that

$$C_b^M(U,F) \supsetneq \bigcap_{B,V} C_b^M(U\cap E_B, F_V)$$

where F_V is the completion of $F/p_V^{-1}(0)$ with respect to the seminorm p_V induced by the absolutely convex closed 0-neighbourhood V.

If we compose $g^\vee$ with the restriction map $(\mathrm{incl}_{\mathbb{N}})^* : C^M(\mathbb{R},\mathbb{R}) \to \mathbb{R}^{\mathbb{N}} := \prod_{t\in\mathbb{N}}\mathbb{R}$ then we get a C^M-curve, since the continuous linear functionals on $\mathbb{R}^{\mathbb{N}}$ are linear combinations of coordinate projections ev_t with $t\in\mathbb{N}$. However, this curve cannot be C_b^M as the argument above for $t>\rho$ shows.

2. Working up to cartesian closedness: More on non-quasianalytic functions

In [KMR09a] we developed convenient calculus for C^M where M was log-convex, increasing, derivation closed, and of moderate growth for the exponential law. In this paper we describe quasianalytic mappings as intersections of non-quasianalytic classes C^L, but we cannot assume that L is derivation closed. Thus we need stronger versions of many results of [KMR09a] for non-quasianalytic L which are **not** derivation closed, and sometimes even **not** log-convex. This section collects an almost minimal set of results which allow to prove cartesian closedness for certain quasianalytic function classes.

2.1. **Lemma** (cf. [KMR09a, 3.3])**.** *Let $M=(M_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers and let E be a convenient vector space such that there exists a Baire vector space topology on the dual E^* for which the point evaluations ev_x are continuous for all $x\in E$. Then a curve $c:\mathbb{R}\to E$ is C^M if and only if c is C_b^M.*

Proof. Let K be compact in $\mathbb{R}$ and c be a C^M-curve. We consider the sets

$$A_{\rho,C} := \Big\{\ell\in E^* : \frac{|\ell(c^{(k)}(x))|}{\rho^k\,k!\,M_k} \le C \text{ for all } k\in\mathbb{N}, x\in K\Big\}$$

which are closed subsets in E^* for the given Baire topology. We have $\bigcup_{\rho,C} A_{\rho,C} = E^*$. By the Baire property there exists ρ and C such that the interior U of $A_{\rho,C}$ is non-empty. If $\ell_0\in U$ then for each $\ell\in E^*$ there is a $\delta>0$ such that $\delta\ell\in U-\ell_0$ and hence for all $x\in K$ and all k we have

$$|(\ell\circ c)^{(k)}(x)| \le \tfrac{1}{\delta}\Big(|((\delta\,\ell+\ell_0)\circ c)^{(k)}(x)| + |(\ell_0\circ c)^{(k)}(x)|\Big) \le \tfrac{2C}{\delta}\,\rho^k\,k!\,M_k.$$

So the set

$$\Big\{\frac{c^{(k)}(x)}{\rho^k\,k!\,M_k} : k\in\mathbb{N}, x\in K\Big\}$$

is weakly bounded in E and hence bounded. □

2.2. **Lemma** (cf. [KMR09a, 3.4])**.** *Let $M=(M_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers and let E be a Banach space. For a smooth curve $c:\mathbb{R}\to E$ the following are equivalent.*

(1) *c is $C^M = C_b^M$.*
(2) *For each sequence (r_k) with $r_k\,\rho^k\to 0$ for all $\rho>0$, and each compact set K in $\mathbb{R}$, the set $\{\frac{1}{k!M_k}\,c^{(k)}(a)\,r_k : a\in K, k\in\mathbb{N}\}$ is bounded in E.*
(3) *For each sequence (r_k) satisfying $r_k>0$, $r_k r_\ell \ge r_{k+\ell}$, and $r_k\,\rho^k\to 0$ for all $\rho>0$, and each compact set K in $\mathbb{R}$, there exists an $\delta>0$ such that $\{\frac{1}{k!M_k}\,c^{(k)}(a)\,r_k\,\delta^k : a\in K, k\in\mathbb{N}\}$ is bounded in E.*

Proof. (1) $\implies$ (2) For K, there exists $\rho > 0$ such that

$$\Big\| \frac{c^{(k)}(a)}{k!\, M_k} r_k \Big\|_E = \Big\| \frac{c^{(k)}(a)}{k!\, \rho^k\, M_k} \Big\|_E \cdot |r_k \rho^k|$$

is bounded uniformly in $k \in \mathbb{N}$ and $a \in K$ by (2.1).

(2) $\implies$ (3) Use $\delta = 1$.

(3) $\implies$ (1) Let $a_k := \sup_{a\in K} \|\frac{1}{k!\, M_k}\, c^{(k)}(a)\|_E$. Using (4$\Rightarrow$1) in [KM97a, 9.2] these are the coefficients of a power series with positive radius of convergence. Thus a_k/ρ^k is bounded for some $\rho > 0$. □

2.3. **Lemma** (cf. [KMR09a, 3.5])**.** *Let $M = (M_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers. Let E be a convenient vector space, and let $\mathcal{S}$ be a family of bounded linear functionals on E which together detect bounded sets (i.e., $B \subseteq E$ is bounded if and only if $\ell(B)$ is bounded for all $\ell \in \mathcal{S}$). Then a curve $c : \mathbb{R} \to E$ is C^M if and only if $\ell \circ c : \mathbb{R} \to \mathbb{R}$ is C^M for all $\ell \in \mathcal{S}$.*

Proof. For smooth curves this follows from [KM97a, 2.1, 2.11]. By (2.2), for $\ell \in \mathcal{S}$, the function $\ell \circ c$ is C^M if and only if:

(1) For each sequence (r_k) with $r_k\, t^k \to 0$ for all $t > 0$, and each compact set K in $\mathbb{R}$, the set $\{\frac{1}{k!M_k}\, (\ell \circ c)^{(k)}(a)\, r_k : a \in K, k \in \mathbb{N}\}$ is bounded.

By (1) the curve c is C^M if and only if the set $\{\frac{1}{k!M_k}\, c^{(k)}(a)\, r_k : a \in K, k \in \mathbb{N}\}$ is bounded in E. By (1) again this is in turn equivalent to $\ell \circ c \in C^M$ for all $\ell \in \mathcal{S}$, since $\mathcal{S}$ detects bounded sets. □

2.4. **Corollary.** *Let $M = (M_k)_{k\in\mathbb{N}}$ be a non-quasianalytic weight sequence or an $\mathcal{L}$-intersectable quasianalytic weight sequence. Let U be c^∞-open in a convenient vector space E, and let $\mathcal{S} = \{\ell : F \to F_\ell\}$ be a family of bounded linear mappings between convenient vector spaces which together detect bounded sets. Then a mapping $f : U \to F$ is C^M if and only if $\ell \circ f$ is C^M for all $\ell \in \mathcal{S}$.*

In particular, a mapping $f : U \to L(G,H)$ is C^M if and only if $\mathrm{ev}_v \circ f : U \to H$ is C^M for each $v \in G$, where G and H are convenient vector spaces.

This result is not valid for C^M_b instead of C^M, by a variant of (1.11): Replace $C^M(\mathbb{R},\mathbb{R})$ by $\mathbb{R}^{\mathbb{N}}$.

Proof. First, let M be non-quasianalytic. By composing with curves we may reduce to $U = E = \mathbb{R}$. By composing each $\ell \in \mathcal{S}$ with all bounded linear functionals on F_ℓ we get a family of bounded linear functionals on F to which we can apply (2.3). For quasianalytic M the result follows by definition. The case $F = L(G,H)$ follows since the ev_v together detect bounded sets, by the uniform boundedness principle [KM97a, 5.18]. □

2.5. C^L**-curve lemma** (cf. [KMR09a, 3.6])**.** A sequence x_n in a locally convex space E is said to be *Mackey convergent* to x, if there exists some $\lambda_n \nearrow \infty$ such that $\lambda_n(x_n - x)$ is bounded. If we fix $\lambda = (\lambda_n)$ we say that x_n is λ-converging.

Lemma. *Let L be a non-quasianalytic weight sequence. Then there exist sequences $\lambda_k \to 0$, $t_k \to t_\infty$, $s_k > 0$ in $\mathbb{R}$ with the following property: For $1/\lambda = (1/\lambda_n)$-converging sequences x_n and v_n in a convenient vector space E there exists a strong uniform C^L-curve $c : \mathbb{R} \to E$ with $c(t_k + t) = x_k + t.v_k$ for $|t| \le s_k$.*

Proof. Since C^L is not quasianalytic we have $\sum_k 1/(k!L_k)^{1/k} < \infty$ by (1.2). We choose another non-quasianalytic weight sequence $\bar{L} = (\bar{L}_k)$ with $(L_k/\bar{L}_k)^{1/k} \to \infty$.

By [KMR09a, 2.3] there is a $C^{\bar{L}}$-function $\phi:\mathbb{R}\to[0,1]$ which is 0 on $\{t:|t|\ge\frac12\}$ and which is 1 on $\{t:|t|\le\frac13\}$, i.e. there exist $\bar{C},\rho>0$ such that

$$|\phi^{(k)}(t)|\le\bar{C}\,\rho^k\,k!\,\bar{L}_k\qquad\text{for all } t\in\mathbb{R}\text{ and }k\in\mathbb{N}.$$

For x,v in an absolutely convex bounded set $B\subseteq E$ and $0<T\le 1$ the curve $c:t\mapsto\phi(t/T)\cdot(x+t\,v)$ satisfies (cf. [Bom67, Lemma 2]):

$$\begin{aligned}c^{(k)}(t)&=T^{-k}\phi^{(k)}(\tfrac{t}{T}).(x+t.v)+k\,T^{1-k}\,\phi^{(k-1)}(\tfrac{t}{T}).v\\&\in T^{-k}\bar{C}\,\rho^k\,k!\,\bar{L}_k(1+\tfrac{T}{2}).B+k\,T^{1-k}\,\bar{C}\,\rho^{k-1}\,(k-1)!\,\bar{L}_{k-1}.B\\&\subseteq T^{-k}\bar{C}\,\rho^k\,k!\,\bar{L}_k(1+\tfrac{T}{2}).B+T\,T^{-k}\,\bar{C}\tfrac{1}{\rho}\,\rho^k\,k!\,\bar{L}_k.B\\&\subseteq\bar{C}(\tfrac32+\tfrac1\rho)\,T^{-k}\,\rho^k\,k!\,\bar{L}_k.B\end{aligned}$$

So there are $\rho, C:=\bar{C}(\frac32+\frac1\rho)>0$ which do not depend on x,v and T such that $c^{(k)}(t)\in C\,T^{-k}\,\rho^k\,k!\,\bar{L}_k.B$ for all k and t.

Let $0<T_j\le 1$ with $\sum_j T_j<\infty$ and $t_k:=2\sum_{j<k}T_j+T_k$. We choose the λ_j such that $0<\lambda_j/T_j^k\le L_k/\bar{L}_k$ (note that $T_j^k\,L_k/\bar{L}_k\to\infty$ for $k\to\infty$) for all j and k, and that $\lambda_j/T_j^k\to 0$ for $j\to\infty$ and each k.

Without loss we may assume that $x_n\to 0$. By assumption there exists a closed bounded absolutely convex subset B in E such that $x_n,v_n\in\lambda_n\cdot B$. We consider $c_j:t\mapsto\phi\big((t-t_j)/T_j\big)\cdot\big(x_j+(t-t_j)\,v_j\big)$ and $c:=\sum_j c_j$. The c_j have disjoint support $\subseteq[t_j-T_j,t_j+T_j]$, hence c is C^∞ on $\mathbb{R}\setminus\{t_\infty\}$ with

$$c^{(k)}(t)\in C\,T_j^{-k}\,\rho^k k!\bar{L}_k\,\lambda_j\cdot B\qquad\text{for } |t-t_j|\le T_j.$$

Then

$$\|c^{(k)}(t)\|_B\le C\,\rho^k\,k!\bar{L}_k\,\frac{\lambda_j}{T_j^k}\le C\rho^k k!\bar{L}_k\,\frac{L_k}{\bar{L}_k}=C\,\rho^k\,k!L_k$$

for $t\ne t_\infty$. Hence $c:\mathbb{R}\to E_B$ is smooth at t_∞ as well, and is strongly C^L by the following lemma. □

2.6. **Lemma** (cf. [KMR09a, 3.7]). *Let $c:\mathbb{R}\setminus\{0\}\to E$ be strongly C^L in the sense that c is smooth and for all bounded $K\subset\mathbb{R}\setminus\{0\}$ there exists $\rho>0$ such that*

$$\left\{\frac{c^{(k)}(x)}{\rho^k\,k!\,L_k}:k\in\mathbb{N},x\in K\right\}\ \textit{is bounded in } E.$$

Then c has a unique extension to a strongly C^L-curve on $\mathbb{R}$.

Proof. The curve c has a unique extension to a smooth curve by [KM97a, 2.9]. The strong C^L condition extends by continuity. □

2.7. **Theorem** (cf. [KMR09a, 3.9]). *Let $L=(L_k)$ be a non-quasianalytic weight sequence. Let $U\subseteq E$ be c^∞-open in a convenient vector space, let F be a Banach space and $f:U\to F$ a mapping. Furthermore, let $\overline{L}\le L$ be another non-quasianalytic weight sequence. Then the following statements are equivalent:*

(1) *f is C^L, i.e. $f\circ c$ is C^L for all C^L-curves c.*
(2) *$f|_{U\cap E_B}:E_B\supseteq U\cap E_B\to F$ is C^L for each closed bounded absolutely convex B in E.*
(3) *$f\circ c$ is C^L for all $C_b^{\overline{L}}$-curves c.*
(4) *$f\in C_b^L(U,F)$.*

Proof. (1) $\Longrightarrow$ (2) is clear, since $E_B\to E$ is continuous and linear, hence all C^L-curves c into the Banach space E_B are also C^L into E and hence $f\circ c$ is C^L by assumption.

(2) $\Longrightarrow$ (3) is clear, since $C_b^{\overline{L}}\subseteq C^L$.

(3) $\implies$ (4) Without loss let $E = E_B$ be a Banach space. For each $v \in E$ and $x \in U$ the iterated directional derivative $d_v^k f(x)$ exists since f is C^L along affine lines. To show that f is smooth it suffices to check that $d_{v_n}^k f(x_n)$ is bounded for each $k \in \mathbb{N}$ and each Mackey convergent sequences x_n and $v_n \to 0$, by [KM97a, 5.20]. For contradiction let us assume that there exist k and sequences x_n and v_n with $\|d_{v_n}^k f(x_n)\| \to \infty$. By passing to a subsequence we may assume that x_n and v_n are $(1/\lambda_n)$-converging for the λ_n from (2.5) for the weight sequence $\overline{L}$. Hence there exists a $C_b^{\overline{L}}$-curve c in E and with $c(t+t_n) = x_n + t.v_n$ for t near 0 for each n separately, and for t_n from (2.5). But then $\|(f\circ c)^{(k)}(t_n)\| = \|d_{v_n}^k f(x_n)\| \to \infty$, a contradiction. So f is smooth.

Assume for contradiction that the boundedness condition in (4) does not hold: There exists a compact set $K \subseteq U$ such that for each $n \in \mathbb{N}$ there are $k_n \in \mathbb{N}$, $x_n \in K$, and v_n with $\|v_n\| = 1$ such that

$$\|d_{v_n}^{k_n} f(x_n)\| > k_n!\, L_{k_n} \left(\frac{1}{\lambda_n^2}\right)^{k_n+1},$$

where we used $C = \rho := 1/\lambda_n^2$ with the λ_n from (2.5) for the weight sequence $\overline{L}$. By passing to a subsequence (again denoted n) we may assume that the x_n are $1/\lambda$-converging, thus there exists a $C_b^{\overline{L}}$-curve $c : \mathbb{R} \to E$ with $c(t_n + t) = x_n + t.\lambda_n.v_n$ for t near 0 by (2.5). Since

$$(f\circ c)^{(k)}(t_n) = \lambda_n^k d_{v_n}^k f(x_n),$$

we get

$$\left(\frac{\|(f\circ c)^{(k_n)}(t_n)\|}{k_n! L_{k_n}}\right)^{\frac{1}{k_n+1}} = \left(\lambda_n^{k_n} \frac{\|d_{v_n}^{k_n} f(x_n)\|}{k_n! L_{k_n}}\right)^{\frac{1}{k_n+1}} > \frac{1}{\lambda_n^{\frac{k_n+2}{k_n+1}}} \to \infty,$$

a contradiction to $f \circ c \in C^L$.

(4) $\implies$ (1) We have to show that $f\circ c$ is C^L for each C^L-curve $c : \mathbb{R} \to E$. By (2.2.3) it suffices to show that for each sequence (r_k) satisfying $r_k > 0$, $r_k r_\ell \ge r_{k+\ell}$, and $r_k\, t^k \to 0$ for all $t > 0$, and each compact interval I in $\mathbb{R}$, there exists an $\epsilon > 0$ such that $\{\frac{1}{k! L_k}\,(f\circ c)^{(k)}(a)\, r_k\, \epsilon^k : a \in I, k \in \mathbb{N}\}$ is bounded.

By (2.2.2) applied to $r_k 2^k$ instead of r_k, for each $\ell \in E^*$, each sequence (r_k) with $r_k\, t^k \to 0$ for all $t > 0$, and each compact interval I in $\mathbb{R}$ the set $\{\frac{1}{k! L_k}\,(\ell\circ c)^{(k)}(a)\, r_k\, 2^k : a \in I, k \in \mathbb{N}\}$ is bounded in $\mathbb{R}$. Thus $\{\frac{1}{k! L_k}\, c^{(k)}(a)\, r_k\, 2^k : a \in I, k \in \mathbb{N}\}$ is contained in some closed absolutely convex $B \subseteq E$. Consequently, $c^{(k)} : I \to E_B$ is smooth and hence $K_k := \{\frac{1}{k! L_k}\, c^{(k)}(a)\, r_k\, 2^k : a \in I\}$ is compact in E_B for each k. Then each sequence (x_n) in the set

$$K := \left\{\frac{1}{k! L_k}\, c^{(k)}(a)\, r_k : a \in I, k \in \mathbb{N}\right\} = \bigcup_{k\in\mathbb{N}} \frac{1}{2^k} K_k$$

has a cluster point in $K \cup \{0\}$: either there is a subsequence in one K_k, or $2^{k_n} x_{k_n} \in K_{k_n} \subseteq B$ for $k_n \to \infty$, hence $x_{k_n} \to 0$ in E_B. So $K \cup \{0\}$ is compact.

By Faà di Bruno ([FdB55] for the 1-dimensional version, $k \ge 1$)

$$\frac{(f\circ c)^{(k)}(a)}{k!} = \sum_{j\ge 1} \sum_{\substack{\alpha\in\mathbb{N}_{>0}^j\\ \alpha_1+\dots+\alpha_j = k}} \frac{1}{j!} d^j f(c(a))\Big(\frac{c^{(\alpha_1)}(a)}{\alpha_1!}, \dots, \frac{c^{(\alpha_j)}(a)}{\alpha_j!}\Big)$$

and (1.1.2) for $a \in I$ and $k \in \mathbb{N}_{>0}$ we have

$$\left\|\frac{1}{k! L_k}\,(f\circ c)^{(k)}(a)\, r_k\right\| \le$$

$$\le \sum_{j\ge 1} L_1^j \sum_{\substack{\alpha\in\mathbb{N}_{>0}^j\\ \alpha_1+\dots+\alpha_j=k}} \frac{\|d^j f(c(a))\|_{L^j(E_B,F)}}{j!L_j} \prod_{i=1}^j \frac{\|c^{(\alpha_i)}(a)\|_B\, r_{\alpha_i}}{\alpha_i! L_{\alpha_i}}$$
$$\le \sum_{j\ge 1} L_1^j \binom{k-1}{j-1} C\,\rho^j\,\frac{1}{2^k} = L_1\rho(1+L_1\,\rho)^{k-1} C\,\frac{1}{2^k}.$$

So $\left\{\frac{1}{k!L_k}\,(f\circ c)^{(k)}(a)\left(\frac{2}{1+L_1\,\rho}\right)^k r_k : a\in I, k\in\mathbb{N}\right\}$ is bounded as required. $\square$

2.8. **Corollary.** *Let $L=(L_k)$ be a non-quasianalytic weight sequence. Let $U\subseteq E$ be c^∞-open in a convenient vector space, let F be a convenient vector space and $f:U\to F$ a mapping. Furthermore, let $\overline{L}\le L$ be a non-quasianalytic weight sequence. Then the following statements are equivalent:*

(1) *f is C^L.*
(2) *$f|_{U\cap E_B} : E_B\supseteq U\cap E_B\to F$ is C^L for each closed bounded absolutely convex B in E.*
(3) *$f\circ c$ is C^L for all $C_b^{\overline{L}}$-curves c.*
(4) *$\pi_V\circ f\in C_b^L(U,\mathbb{R})$ for each absolutely convex 0-neighborhood $V\subseteq F$, where $\pi_V: F\to F_V$ denotes the natural mapping.*

Proof. Each of the statements holds for f if and only if it holds for $\pi_V\circ f$ for each absolutely convex 0-neighborhood $V\subseteq F$. So the corollary follows from (2.7). $\square$

2.9. **Theorem** (Uniform boundedness principle for C^M, cf. [KMR09a, 4.1])**.** *Let $M=(M_k)$ be a non-quasianalytic weight sequence or an $\mathcal{L}$-intersectable quasianalytic weight sequence. Let E, F, G be convenient vector spaces and let $U\subseteq F$ be c^∞-open. A linear mapping $T: E\to C^M(U,G)$ is bounded if and only if* $\operatorname{ev}_x\circ T: E\to G$ *is bounded for every $x\in U$.*

Proof. Let first M be non-quasianalytic. For $x\in U$ and $\ell\in G^*$ the linear mapping $\ell\circ\operatorname{ev}_x = C^M(x,\ell): C^M(U,G)\to\mathbb{R}$ is continuous, thus ev_x is bounded. Therefore, if T is bounded then so is $\operatorname{ev}_x\circ T$.

Conversely, suppose that $\operatorname{ev}_x\circ T$ is bounded for all $x\in U$. For each closed absolutely convex bounded $B\subseteq E$ we consider the Banach space E_B. For each $\ell\in G^*$, each C^M-curve $c:\mathbb{R}\to U$, each $t\in\mathbb{R}$, and each compact $K\subset\mathbb{R}$ the composite given by the following diagram is bounded.

$$\begin{tikzcd}
E \arrow[r, "T"] & C^M(U,G) \arrow[rr, "\operatorname{ev}_{c(t)}"] \arrow[d, "C^M(c,\ell)"] & & G \arrow[d, "\ell"] \\
E_B \arrow[u] \arrow[r] & C^M(\mathbb{R},\mathbb{R}) \arrow[r] & \varinjlim_\rho C^M_\rho(K,\mathbb{R}) \arrow[r, "\operatorname{ev}_t"] & \mathbb{R}
\end{tikzcd}$$

By [KM97a, 5.24, 5.25] the map T is bounded. In more detail: Since $\varinjlim_\rho C^M_\rho(K,\mathbb{R})$ is webbed, the closed graph theorem [KM97a, 52.10] yields that the mapping $E_B\to\varinjlim_\rho C^M_\rho(K,\mathbb{R})$ is continuous. Thus T is bounded.

For quasianalytic M the result follows since the structure of a convenient vector space on $C^M(U,G)$ is the initial one with respect to all inclusions $C^M(U,G)\to C^L(U,G)$ for all $L\in\mathcal{L}(M)$. $\square$

As a consequence we can show that the equivalences of (2.7) and (2.8) are not only valid for single functions f but also for the bornology of $C^M(U,F)$:

2.10. **Corollary** (cf. [KMR09a, 4.6]). *Let $L = (L_k)$ be a non-quasianalytic weight sequence. Let E and F be Banach spaces and let $U \subseteq E$ be open. Then*

$$C^L(U, F) = C^L_b(U, F) := \varprojlim_K \varinjlim_\rho C^L_{K,\rho}(U, F)$$

as vector spaces with bornology. Here K runs through all compact subsets of U ordered by inclusion and ρ runs through the positive real numbers.

Proof. The second equality is by definition (1.10). The first equality, as vector spaces, is by (2.7). By (1.10) the space $C^L(U, F)$ is convenient.

The identity from right to left is continuous since $C^L(U, F)$ carries the initial structure with respect to the mappings

$$C^L(c|_I, \ell) : C^L(U, F) \to C^L(\mathbb{R}, \mathbb{R}) = \varprojlim_{I \subseteq \mathbb{R}} \varinjlim_{\rho > 0} C^L_{I,\rho}(\mathbb{R}, \mathbb{R}) \to \varinjlim_{\rho > 0} C^L_{I,\rho}(\mathbb{R}, \mathbb{R}),$$

where c runs through the $C^L \overset{(2.1)}{=\!=} C^L_b$-curves, $\ell \in F^*$ and I runs through the compact intervals in $\mathbb{R}$, and for $K := c(I)$ and $\rho' := (1 + \rho \, \|c\|_{I,\sigma}) \cdot \sigma$, where $\sigma > 0$ is chosen such that $\|c\|_{I,\sigma} < \infty$, the mapping $C^L(c|_I, \ell) : C^L_{K,\rho}(U, F) \to C^L_{I,\rho'}(\mathbb{R}, \mathbb{R}) \to \varinjlim_{\rho' > 0} C^L_{I,\rho'}(\mathbb{R}, \mathbb{R})$ is continuous by (1.4). These arguments are collected in the diagram:

$$\begin{tikzcd}
\varprojlim_I C^L_I(\mathbb{R},\mathbb{R}) \arrow[r, equal] \arrow[d] & C^L(\mathbb{R},\mathbb{R}) & C^L(U,F) \arrow[l, "C^L(c,\ell)"'] & C^L_b(U,F) \arrow[l, dotted] \arrow[r, equal] & \varprojlim_K C^L_K(U,F) \arrow[d] \\
C^L_I(\mathbb{R},\mathbb{R}) \arrow[r, equal] & \varinjlim_\rho C^L_{I,\rho}(\mathbb{R},\mathbb{R}) & & \varinjlim_\rho C^L_{K,\rho}(U,F) \arrow[ll, dotted] \arrow[r, equal] & C^L_K(U,F) \\
& C^L_{I,\rho'}(\mathbb{R},\mathbb{R}) \arrow[u] & & C^L_{K,\rho}(U,F) \arrow[ll, "C^L(c|_I,\ell)"'] \arrow[u] &
\end{tikzcd}$$

The identity from left to right is bounded since the countable (take $\rho \in \mathbb{N}$) inductive limit $\varinjlim_\rho$ of the (non-Hausdorff) Banach spaces $C^L_{K,\rho}(U, F)$ is webbed and hence satisfies the $\mathcal{S}$-boundedness principle [KM97a, 5.24] where $\mathcal{S} = \{\mathrm{ev}_x : x \in U\}$, and by [KM97a, 5.25] the same is true for $C^L_b(U, F)$. $\square$

2.11. **Corollary** (cf. [KMR09a, 4.4]). *Let $L = (L_k)$ be a non-quasianalytic weight sequence. Let E and F be convenient vector spaces and let $U \subseteq E$ be c^∞-open. Then*

$$C^L(U, F) = \varprojlim_{c \in C^L} C^L(\mathbb{R}, F) = \varprojlim_{B \subseteq E} C^L(U \cap E_B, F) = \varprojlim_{s \in C^L_b} C^L(\mathbb{R}, F)$$

as vector spaces with bornology, where c runs through all C^L-curves in U, B runs through all bounded closed absolutely convex subsets of E, and s runs through all C^L_b-curves in U.

Proof. The first and third inverse limit is formed with $g^* : C^L(\mathbb{R}, F) \to C^L(\mathbb{R}, F)$ for $g \in C^L(\mathbb{R}, \mathbb{R})$ as connecting mappings. Each element $(f_c)_c$ determines a unique function $f : U \to F$ given by $f(x) := (f \circ \mathrm{const}_x)(0)$ with $f \circ c = f_c$ for all such curves c, and $f \in C^L$ if and only if $f_c \in C^L$ for all such c, by (2.8). The second inverse limit is formed with $\mathrm{incl}^* : C^L(U \cap E_B, F) \to C^L(U \cap E_{B'}, F)$ for $B' \subseteq B$ as connecting mappings. Each element $(f_B)_B$ determines a unique function $f : U \to F$ given by $f(x) := f_{[-1,1]x}(x)$ with $f|_{E_B} = f_B$ for all B, and $f \in C^L$ if and only if $f_B \in C^L$ for all such B, by (2.8). Thus all equalities hold as vector spaces.

The first identity is continuous from left to right, since the family of $\ell_* : C^L(\mathbb{R},F) \to C^L(\mathbb{R},\mathbb{R})$ with $\ell \in F^*$ is initial and $C^L(c,\ell) = \ell_* \circ c^* : C^L(U,F) \to C^L(\mathbb{R},\mathbb{R})$ is continuous and linear by definition.

Continuity for the second one from left to right is obvious, since C^L-curves in $U \cap E_B$ are C^L into $U \subseteq E$.

In order to show the continuity of the last identity from left to right choose a C^L_b-curve s in U, an $\ell \in F^*$ and a compact interval $I \subseteq \mathbb{R}$. Then there exists a bounded absolutely convex closed $B \subseteq E$ such that $s|_I$ is $C^L_b = C^L$ into $U \cap E_B$, hence $C^L(s|_I,\ell) : C^L(U,F) \to C^L(I,\mathbb{R})$ factors by (1.4) as continuous linear mapping $(s|_I)^* : C^L_b(U \cap E_B,\mathbb{R}) \to C^L(I,\mathbb{R})$ over $C^L(U,F) \to C^L(U \cap E_B,F) \to C^L(U \cap E_B,\mathbb{R}) \overset{(2.10)}{=\!=} C^L_b(U \cap E_B,\mathbb{R})$. Since the structure of $C^L(\mathbb{R},F)$ is initial with respect to $\mathrm{incl}^* \circ \ell_* : C^L(\mathbb{R},F) \to C^L(I,\mathbb{R})$ the identity $\varprojlim_{B\subseteq E} C^L(U \cap E_B,F) \to \varprojlim_{s\in C^L_b} C^L(\mathbb{R},F)$ is continuous.

Conversely, the identity $\varprojlim_{s\in C^L_b} C^L(\mathbb{R},F) \to C^L(U,F)$ is bounded, since $C^L(\mathbb{R},F)$ is convenient and hence also the inverse limit $\varprojlim_{s\in C^L_b} C^L(\mathbb{R},F)$ and $C^L(U,F)$ satisfies the uniform boundedness theorem (2.9) with respect to the point-evaluations ev_x and they factor over $(\mathrm{const}_x)^* : C^L(U,F) \to C^L(\mathbb{R},F)$. $\square$

3. The exponential law for certain quasianalytic function classes

We start with some preparations. Let $Q = (Q_k)$ be an $\mathcal{L}$-intersectable quasianalytic weight sequence. Let E and F be convenient vector spaces and let $U \subseteq E$ be c^∞-open.

3.1. Lemma. *For Banach spaces E and F we have*
$$C^Q(U,F) = C^Q_b(U,F) = \bigcap_{N\in\mathcal{L}_w(Q)} C^N_b(U,F)$$
as vector spaces.

Proof. Since Q is $\mathcal{L}$-intersectable we have $\mathcal{F}^Q = \bigcap_{L\in\mathcal{L}(Q)} \mathcal{F}^L$. Hence
$$\begin{aligned}
C^Q_b(U,F) &= \{f \in C^\infty(U,F) : \forall K : (\sup_{x\in K} \|f^{(k)}(x)\|_{L^k(E,F)})_k \in \mathcal{F}^Q = \bigcap_{L\in\mathcal{L}(Q)} \mathcal{F}^L\}\\
&= \{f \in C^\infty(U,F) : \forall K\ \forall L \in \mathcal{L}(Q) : (\sup_{x\in K} \|f^{(k)}(x)\|)_k \in \mathcal{F}^L\}\\
&= \{f \in C^\infty(U,F) : \forall L \in \mathcal{L}(Q)\ \forall K : (\sup_{x\in K} \|f^{(k)}(x)\|)_k \in \mathcal{F}^L\}\\
&= \bigcap_{L\in\mathcal{L}(Q)} C^L_b(U,F) \overset{(2.7)}{=\!=} \bigcap_{L\in\mathcal{L}(Q)} C^L(U,F) = C^Q(U,F).
\end{aligned}$$
$$C^Q_b(U,F) \overset{(1.6.1)}{=\!=} \bigcap_{L\in\mathcal{L}(Q)} C^L_b(U,F) \supseteq \bigcap_{L\in\mathcal{L}_w(Q)} C^L_b(U,F) \supseteq C^Q_b(U,F). \quad \square$$

3.2. Lemma. *For log-convex non-quasianalytic L^1, L^2 and weakly log-convex non-quasianalytic N with $N_{k+n} \le C^{k+n} L^1_k L^2_k$ for some positive constant C and all $k,n \in \mathbb{N}$, for Banach-spaces E_1 and E_2, and for $f \in C^N_b(U_1 \times U_2,\mathbb{R})$ we have $f^\vee \in C^{L^1}(U_1, C^{L^2}_b(U_2,\mathbb{R}))$.*

Proof. Since f is C^N_b, by definition, for all compact $K_i \subseteq U_i$ there exists a $\rho > 0$ such that for all $k,j \in \mathbb{N}$, $x_i \in K_i$ and $\|v_1\| = \dots = \|v_j\| = 1 = \|w_1\| = \dots = \|w_k\|$ we have
$$|\partial_2^k \partial_1^j f(x_1,x_2)(v_1,\dots,v_j,w_1,\dots,w_k)| \le \rho^{k+j+1}(k+j)!N_{k+j}$$

$$\le \rho^{k+j+1} 2^{k+j} k! j! C^{k+j} L^1_j L^2_k = \rho (2C\rho)^j j! L^1_j \cdot (2C\rho)^k k! L^2_k.$$

In particular $(\partial_1^j f)^\vee (K_1)(oE_1^k)$ is contained and bounded in $C_b^{L^2}(U_2,\mathbb{R})$, where oE_1 denotes the unit ball in E_1, since $d^k((\partial_1^j f)^\vee(x_1))(x_2) = \partial_2^k \partial_1^j f(x_1,x_2)$.

Claim. *If $f \in C_b^N$ then $f^\vee : U_1 \to C_b^{L^2}(U_2,\mathbb{R})$ is C^∞ with $d^j f^\vee = (\partial_1^j f)^\vee$.*

Since $C_b^{L^2}(U_2,\mathbb{R})$ is a convenient vector space, by [KM97a, 5.20] it is enough to show that the iterated unidirectional derivatives $d_v^j f^\vee(x)$ exist, equal $\partial_1^j f(x, \quad)(v^j)$, and are separately bounded for x, resp. v, in compact subsets. For $j = 1$ and fixed x, v, and y consider the smooth curve $c : t \mapsto f(x+tv, y)$. By the fundamental theorem

$$\begin{aligned}\frac{f^\vee(x+tv) - f^\vee(x)}{t}(y) - (\partial_1 f)^\vee(x)(y)(v) &= \frac{c(t)-c(0)}{t} - c'(0)\\ &= t\int_0^1 s \int_0^1 c''(tsr)\,dr\,ds\\ &= t\int_0^1 s\int_0^1 \partial_1^2 f(x+tsrv, y)(v,v)\,dr\,ds.\end{aligned}$$

Since $(\partial_1^2 f)^\vee(K_1)(oE_1^2)$ is bounded in $C_b^{L^2}(U_2,\mathbb{R})$ for each compact subset $K_1 \subseteq U_1$ this expression is Mackey convergent to 0 in $C_b^{L^2}(U_2,\mathbb{R})$, for $t \to 0$. Thus $d_v f^\vee(x)$ exists and equals $\partial_1 f(x, \quad)(v)$.

Now we proceed by induction, applying the same arguments as before to $(d_v^j f^\vee)^\wedge : (x,y) \mapsto \partial_1^j f(x,y)(v^j)$ instead of f. Again $(\partial_1^2 (d_v^j f^\vee)^\wedge)^\vee (K_1)(oE_1^2) = (\partial_1^{j+2} f)^\vee (K_1)(oE_1, oE_1, v, \dots, v)$ is bounded, and also the separated boundedness of $d_v^j f^\vee(x)$ follows. So the claim is proved.

It remains to show that $f^\vee : U_1 \to C_b^{L^2}(U_2,\mathbb{R}) := \varprojlim_K \varinjlim_\rho C^{L^2}_{K,\rho}(U_2,\mathbb{R})$ is C^{L^1}. By (2.4), it suffices to show that $f^\vee : U_1 \to \varinjlim_\rho C^{L^2}_{K_2,\rho}(U_2,\mathbb{R})$ is $C_b^{L^1} \subseteq C^{L^1}$ for all K_2, i.e., for all compact $K_2 \subset U_2$ and $K_1 \subset U_1$ there exists $\rho_1 > 0$ such that

$$\left\{ \frac{d^k f^\vee(K_1)(v_1,\dots,v_k)}{k!\rho_1^k L^1_k} : k \in \mathbb{N}, \|v_i\| \le 1 \right\} \text{ is bounded in } \varinjlim_\rho C^{L^2}_{K_2,\rho}(U_2,\mathbb{R}),$$

or equivalently: For all compact $K_2 \subset U_2$ and $K_1 \subset U_1$ there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\left\{ \frac{\partial_2^l \partial_1^k f(K_1,K_2)(v_1,\dots,v_{k+l})}{l!k!\rho_2^l L^2_l \rho_1^k L^1_k} : k \in \mathbb{N}, l \in \mathbb{N}, \|v_i\| \le 1 \right\} \text{ is bounded in } \mathbb{R}.$$

For $k_1 \in \mathbb{N}$, $x \in K_1$, $\rho_i := 2C\rho$, and $\|v_i\| \le 1$ we get:

$$\begin{aligned}&\left\| \frac{d^{k_1} f^\vee(x)(v_1,\dots,v_{k_1})}{\rho_1^{k_1}\, k_1!\, L^1_{k_1}} \right\|_{K_2,\rho_2} =\\ &:= \sup\left\{ \frac{|\partial_2^{k_2}\partial_1^{k_1} f(x,y)(v_1,\dots;w_1,\dots)|}{\rho_1^{k_1}\, k_1!\, L^1_{k_1}\, \rho_2^{k_2}\, k_2!\, L^2_{k_2}} : k_2 \in \mathbb{N}, y \in K_2, \|w_i\| \le 1 \right\}\\ &\le \sup\left\{ \frac{\frac{(k_1+k_2)!}{k_1!\,k_2!} C^{k_1+k_2} |\partial_2^{k_2}\partial_1^{k_1} f(x,y)(v_1,\dots;w_1,\dots)|}{\rho_1^{k_1}\, \rho_2^{k_2}\, (k_1+k_2)!\, N_{k_1+k_2}} : k_2 \in \mathbb{N}, y \in K_2, \|w_i\| \le 1 \right\}\\ &\le \sup\left\{ \frac{(2C)^{k_1+k_2} |\partial^{(k_1,k_2)} f(x,y)(v_1,\dots;w_1,\dots)|}{\rho_1^{k_1}\, \rho_2^{k_2}\, (k_1+k_2)!\, N_{k_1+k_2}} : k_2 \in \mathbb{N}, y \in K_2, \|w_i\| \le 1 \right\}\\ &= \sup\left\{ \frac{|\partial^{(k_1,k_2)} f(x,y)(v_1,\dots;w_1,\dots)|}{\rho^{k_1+k_2}\, (k_1+k_2)!\, N_{k_1+k_2}} : k_2 \in \mathbb{N}, y \in K_2 : \|w_i\| \le 1 \right\} \le \rho\end{aligned}$$

So $f^\vee$ is C^{L_1}. $\square$

3.3. **Theorem** (Cartesian closedness). *Let $Q = (Q_k)$ be an $\mathcal{L}$-intersectable quasianalytic weight sequence of moderate growth. Then the category of C^Q-mappings between convenient real vector spaces is cartesian closed. More precisely, for convenient vector spaces E_1, E_2 and F and c^∞-open sets $U_1 \subseteq E_1$ and $U_2 \subseteq E_2$ a mapping $f : U_1 \times U_2 \to F$ is C^Q if and only if $f^\vee : U_1 \to C^Q(U_2, F)$ is C^Q.*

Actually, we prove that the direction ($\Leftarrow$) holds without the assumption of moderate growth.

Proof. ($\Rightarrow$) Let $f : U_1 \times U_2 \to F$ be C^Q, i.e. C^L for all $L \in \mathcal{L}(Q)$. Since $(E_i)_{B_i} \to E_i$ is bounded and linear and since C^L is closed under composition we get that $\ell \circ f : (U_1 \cap (E_1)_{B_1}) \times (U_2 \cap (E_2)_{B_2}) \to \mathbb{R}$ is $C^L = C^L_b$ (by (2.7) since $(E_i)_{B_i}$ are Banach-spaces) for $\ell \in F^*$, arbitrary bounded closed $B_i \subseteq E_i$ and all $L \in \mathcal{L}(Q)$. Hence $\ell \circ f$ is C^L_b even for all $L \in \mathcal{L}_w(Q)$ by (3.1). For arbitrary $L^1, L^2 \in \mathcal{L}(Q)$, by (1.6.3) and (1.6.2), there exists an $N \in \mathcal{L}_w(Q)$ with $N_{k+n} \le C^{k+n} L^1_k L^2_n$ for some positive constant C and all $k, n \in \mathbb{N}$. Thus $\ell \circ f : (U_1 \cap (E_1)_{B_1}) \times (U_2 \cap (E_2)_{B_2}) \to \mathbb{R}$ is C^N_b. By (3.2), the function $(\ell \circ f)^\vee : U_1 \cap (E_1)_{B_1} \to C^{L^2}_b(U_2 \cap (E_2)_{B_2}, \mathbb{R})$ is C^{L^1}. Since the cone

$$C^Q(U_2, F) \to C^{L^2}(U_2, F) \xrightarrow{C^{L^2}(i_{B_2}, \ell)} C^{L^2}(U_2 \cap (E_2)_{B_2}, \mathbb{R}) = C^{L^2}_b(U_2 \cap (E_2)_{B_2}, \mathbb{R}),$$

with $L_2 \in \mathcal{L}(Q)$, $\ell \in F^*$, and bounded closed $B_2 \subseteq E_2$, generates the bornology by (2.11), and since obviously $f^\vee(x) = f(x, \quad) \in C^Q(U_2, F)$, we have that $f^\vee : U_1 \cap (E_1)_{B_1} \to C^Q(U_2, F)$ is C^{L^1}, by (2.4). From this we get by (2.8) that $f^\vee : U_1 \to C^Q(U_2, F)$ is C^{L^1} for all $L^1 \in \mathcal{L}(Q)$, i.e., $f^\vee : U_1 \to C^Q(U_2, F)$ is C^Q as required. The whole argument above is collected in the following diagram where $U^i_{B_i}$ stands for $U_i \cap E_{B_i}$:

$$\begin{tikzcd}
U^1 \times U^2 \arrow[r, "f \in C^Q"] & F \arrow[d, "\ell"'] & & U^1 \arrow[r, "f^\vee \in C^{L^1}", "(2.8)"'] & C^Q(U^2, F) \arrow[r] & C^{L^2}(U^2, F) \arrow[d, "\ell_* \circ \mathrm{incl}_2^*"', "(2.11)"] \\
U^1_{B_1} \times U^2_{B_2} \arrow[u, "\mathrm{incl}"'] \arrow[r, "f \in C^Q \subseteq C^N_b", "(3.1)"'] & \mathbb{R} & \Longrightarrow & U^1_{B_1} \arrow[u, "\mathrm{incl}_1"'] \arrow[ur, "(2.3)"'] \arrow[r, "C^{L^1}", "(3.2)"'] & C^{L^2}(U^2_{B_2}, \mathbb{R}) \arrow[r, equal] & C^{L^2}_b(U^2_{B_2}, \mathbb{R})
\end{tikzcd}$$

($\Leftarrow$) Let, conversely, $f^\vee : U_1 \to C^Q(U_2, F)$ be C^Q, i.e., C^L for all $L \in \mathcal{L}(Q)$. By the description of the structure of $C^Q(U, F)$ in (1.10) the mapping $f^\vee : U_1 \to C^L(U_2, F)$ is C^L. We now conclude that $f : U_1 \times U_2 \to F$ is C^L; this direction of cartesian closedness for C^L holds even if L is not of moderate growth, see [KMR09a, 5.3] and its proof. This is true for all $L \in \mathcal{L}(Q)$. Hence f is C^Q. $\square$

3.4. **Corollary.** *Let Q be an $\mathcal{L}$-intersectable quasianalytic weight sequence of moderate growth. Let E, F, etc., be convenient vector spaces and let U and V be c^∞-open subsets of such. Then we have:*

(1) *The exponential law holds:*

$$C^Q(U, C^Q(V, G)) \cong C^Q(U \times V, G)$$

is a linear C^Q-diffeomorphism of convenient vector spaces.

The following canonical mappings are C^Q.

(2) $\quad \mathrm{ev} : C^Q(U, F) \times U \to F, \quad \mathrm{ev}(f, x) = f(x)$

(3) $\quad \mathrm{ins} : E \to C^Q(F, E \times F), \quad \mathrm{ins}(x)(y) = (x, y)$

(4) $\quad (\quad)^\wedge : C^Q(U, C^Q(V, G)) \to C^Q(U \times V, G)$

(5) $\quad (\quad)^\vee : C^Q(U \times V, G) \to C^Q(U, C^Q(V, G))$

(6) $\quad \mathrm{comp} : C^Q(F, G) \times C^Q(U, F) \to C^Q(U, G)$

$$(7)\qquad C^Q(\ \ ,\ \ ): C^Q(F,F_1)\times C^Q(E_1,E)\to C^Q\Big(C^Q(E,F),C^Q(E_1,F_1)\Big)$$
$$(f,g)\mapsto (h\mapsto f\circ h\circ g)$$
$$(8)\qquad \prod : \prod C^Q(E_i,F_i)\to C^Q\Big(\prod E_i,\prod F_i\Big)$$

Proof. This is a direct consequence of cartesian closedness (3.3). See [KMR09a, 5.5] or even [KM97a, 3.13] for the detailed arguments. □

4. More on function spaces

In this section we collect results for function classes C^M where M is either a non-quasianalytic weight sequence or an $\mathcal{L}$-intersectable quasianalytic weight sequence. In order to treat both cases simultaneously, the proofs will often use non-quasianalytic weight sequences $L \ge M$. These are either M itself if M is non-quasianalytic or are in $\mathcal{L}(M)$ if M is $\mathcal{L}$-intersectable quasianalytic. In both cases we may assume without loss that L is increasing, by (1.5).

4.1. **Proposition.** *Let $M = (M_k)$ be a non-quasianalytic weight sequence or an $\mathcal{L}$-intersectable quasianalytic weight sequence. Then we have:*

(1) *Multilinear mappings between convenient vector spaces are C^M if and only if they are bounded.*
(2) *If $f : E \supseteq U \to F$ is C^M, then the derivative $df : U \to L(E,F)$ is $C^{M_{+1}}$, and also $(df)^\wedge : U\times E \to F$ is $C^{M_{+1}}$, where the space $L(E,F)$ of all bounded linear mappings is considered with the topology of uniform convergence on bounded sets.*
(3) *The chain rule holds.*

Proof. (1) If f is C^M then it is smooth by (2.8) and hence bounded by [KM97a, 5.5]. Conversely, if f is multilinear and bounded then it is smooth, again by [KM97a, 5.5]. Furthermore, $f\circ i_B$ is multilinear and continuous and all derivatives of high order vanish. Thus condition (2.8.4) is satisfied, so f is C^M.

(2) Since f is smooth, by [KM97a, 3.18] the map $df : U \to L(E,F)$ exists and is smooth. Let $L \ge M_{+1}$ be a non-quasianalytic weight sequence and $c : \mathbb{R}\to U$ be a C^L-curve. We have to show that $t\mapsto df(c(t)) \in L(E,F)$ is C^L. By the uniform boundedness principle [KM97a, 5.18] and by (2.3) it suffices to show that the mapping $t\mapsto c(t)\mapsto \ell(df(c(t))(v))\in\mathbb{R}$ is C^L for each $\ell\in F^*$ and $v\in E$. We are reduced to show that $x\mapsto \ell(df(x)(v))$ satisfies the conditions of (2.7). By (2.7) applied to $\ell\circ f$, for each $L\ge M$, each closed bounded absolutely convex B in E, and each $x\in U\cap E_B$ there are $r>0$, $\rho>0$, and $C>0$ such that

$$\frac{1}{k!\,L_k}\|d^k(\ell\circ f\circ i_B)(a)\|_{L^k(E_B,\mathbb{R})}\le C\,\rho^k$$

for all $a\in U\cap E_B$ with $\|a-x\|_B\le r$ and all $k\in\mathbb{N}$. For $v\in E$ and those B containing v we then have:

$$\begin{aligned}\|d^k(d(\ell\circ f)(\ \ )(v))\circ i_B)(a)\|_{L^k(E_B,\mathbb{R})} &= \|d^{k+1}(\ell\circ f\circ i_B)(a)(v,\dots)\|_{L^k(E_B,\mathbb{R})}\\ &\le \|d^{k+1}(\ell\circ f\circ i_B)(a)\|_{L^{k+1}(E_B,\mathbb{R})}\|v\|_B \le C\,\rho^{k+1}\,(k+1)!\,L_{k+1}\\ &= C\rho\,((k+1)^{1/k}\rho)^k\,k!\,L_{k+1}\le C\rho\,(2\rho)^k\,k!\,(L_{+1})_k\end{aligned}$$

By (4.2) below also $(df)^\wedge$is $C^{L_{+1}}$.

(3) This is valid even for all smooth f by [KM97a, 3.18]. □

4.2. **Proposition.** *Let $M = (M_k)$ be a non-quasianalytic weight sequence or an $\mathcal{L}$-intersectable quasianalytic weight sequence.*

(1) *For convenient vector spaces E and F, on $L(E,F)$ the following bornologies coincide which are induced by:*
- *The topology of uniform convergence on bounded subsets of E.*
- *The topology of pointwise convergence.*
- *The embedding $L(E,F) \subset C^\infty(E,F)$.*
- *The embedding $L(E,F) \subset C^M(E,F)$.*

(2) *Let E, F, G be convenient vector spaces and let $U \subset E$ be c^∞-open. A mapping $f : U \times F \to G$ which is linear in the second variable is C^M if and only if $f^\vee : U \to L(F,G)$ is well defined and C^M.*

Analogous results hold for spaces of multilinear mappings.

Proof. (1) That the first three topologies on $L(E,F)$ have the same bounded sets has been shown in [KM97a, 5.3, 5.18]. The inclusion $C^M(E,F) \to C^\infty(E,F)$ is bounded by the uniform boundedness principle [KM97a, 5.18]. Conversely, the inclusion $L(E,F) \to C^M(E,F)$ is bounded by the uniform boundedness principle (2.9).

(2) The assertion for C^∞ is true by [KM97a, 3.12] since $L(E,F)$ is closed in $C^\infty(E,F)$.

If f is C^M let $L \geq M$ be a non-quasianalytic weight-sequence and let $c : \mathbb{R} \to U$ be a C^L-curve. We have to show that $f^\vee \circ c$ is C^L into $L(F,G)$. By the uniform boundedness principle [KM97a, 5.18] and (2.3) it suffices to show that $t \mapsto \ell\big(f^\vee(c(t))(v)\big) = \ell\big(f(c(t),v)\big) \in \mathbb{R}$ is C^L for each $\ell \in G^*$ and $v \in F$; this is obviously true.

Conversely, let $f^\vee : U \to L(F,G)$ be C^M and let $L \geq M$ be a non-quasianalytic weight-sequence. We claim that $f : U \times F \to G$ is C^L. By composing with $\ell \in G^*$ we may assume that $G = \mathbb{R}$. By induction we have

$$\begin{aligned} d^k f(x,w_0)\big((v_k,w_k),\dots,(v_1,w_1)\big) &= d^k(f^\vee)(x)(v_k,\dots,v_1)(w_0) + \\ &+ \sum_{i=1}^k d^{k-1}(f^\vee)(x)(v_k,\dots,\widehat{v_i},\dots,v_1)(w_i) \end{aligned}$$

We check condition (2.7.4) for f where $x \in K$ which is compact in U:

$$\begin{aligned} &\|d^k f(x,w_0)\|_{L^k(E_B \times F_{B'},\mathbb{R})} \leq \\ &\leq \|d^k(f^\vee)(x)(\dots)(w_0)\|_{L^k(E_B,\mathbb{R})} + \sum_{i=1}^k \|d^{k-1}(f^\vee)(x)\|_{L^{k-1}(E_B,L(F_{B'},\mathbb{R}))} \\ &\leq \|d^k(f^\vee)(x)\|_{L^k(E_B,L(F_{B'},\mathbb{R}))}\|w_0\|_{B'} + \sum_{i=1}^k \|d^{k-1}(f^\vee)(x)\|_{L^{k-1}(E_B,L(F_{B'},\mathbb{R}))} \\ &\leq C\,\rho^k\,k!\,L_k\|w_0\|_{B'} + \sum_{i=1}^k C\,\rho^{k-1}\,(k-1)!\,L_{k-1} = C\,\rho^k\,k!\,L_k\Big(\|w_0\|_{B'} + \tfrac{L_{k-1}}{\rho\,L_k}\Big) \end{aligned}$$

where we used (2.7.4) for $L(i_{B'},\mathbb{R}) \circ f^\vee : U \to L(F_{B'},\mathbb{R})$. Since L is increasing, f is C^L. □

4.3. **Theorem.** *Let $Q = (Q_k)$ be an $\mathcal{L}$-intersectable quasianalytic weight sequence. Let $U \subseteq E$ be c^∞-open in a convenient vector space, let F be another convenient vector space, and $f : U \to F$ a mapping. Then the following statements are equivalent:*

(1) *f is C^Q, i.e., for all $L \in \mathcal{L}(Q)$ we have $f \circ c$ is C^L for all C^L-curves c.*

(2) *$f|_{U \cap E_B} : E_B \supseteq U \cap E_B \to F$ is C^Q for each closed bounded absolutely convex B in E.*

(3) *For all $L \in \mathcal{L}(Q)$ the curve $f \circ c$ is C^L for all C^L_b-curves c.*

(4) *$\pi_V \circ f$ is C_b^Q for all absolutely convex 0-neighborhoods V in F and the associated mapping $\pi_V : F \to F_V$.*

Proof. This follows from (2.8) for $\overline{L} := L$ since $C^Q := \bigcap_{L\in\mathcal{L}(Q)} C^L$ and $C_b^Q = \bigcap_{L\in\mathcal{L}(Q)} C_b^L$. □

4.4. **Theorem** (cf. [KMR09a, 4.4]). *Let $Q = (Q_k)$ be an $\mathcal{L}$-intersectable quasianalytic weight sequence. Let E and F be convenient vector spaces and let $U \subseteq E$ be c^∞-open. Then*

$$C^Q(U,F) = \varprojlim_{L\in\mathcal{L}(Q), c\in C^L} C^L(\mathbb{R},F) = \varprojlim_{B\subseteq E} C^Q(U\cap E_B, F) = \varprojlim_{L\in\mathcal{L}(Q), s\in C_b^L} C^L(\mathbb{R},F)$$

as vector spaces with bornology, where c runs through all C^L-curves in U for $L \in \mathcal{L}(Q)$, B runs through all bounded closed absolutely convex subsets of E, and s runs through all C_b^L-curves in U for $L \in \mathcal{L}(Q)$.

Proof. This follows by applying $\varprojlim_{L\in\mathcal{L}(Q)}$ to (2.11). □

4.5. **Jet spaces.** Let E and F be Banach spaces and $A \subseteq E$ convex. We consider the linear space $C^\infty(A,F)$ consisting of all sequences $(f^k)_k \in \prod_{k\in\mathbb{N}} C(A, L^k(E,F))$ satisfying

$$f^k(y)(v) - f^k(x)(v) = \int_0^1 f^{k+1}(x + t(y-x))(y-x, v)\, dt$$

for all $k \in \mathbb{N}$, $x, y \in A$, and $v \in E^k$. If A is open we can identify this space with that of all smooth functions $A \to F$ by the passage to jets.

In addition, let $M = (M_k)$ be a weight sequence and (r_k) a sequence of positive real numbers. Then we consider the normed spaces

$$C^M_{(r_k)}(A,F) := \Big\{(f^k)_k \in C^\infty(A,F) : \|(f^k)\|_{(r_k)} < \infty\Big\}$$

where the norm is given by

$$\|(f^k)\|_{(r_k)} := \sup\Big\{\frac{\|f^k(a)(v_1,\dots,v_k)\|}{k!\, r_k\, M_k\, \|v_1\|\cdot\dots\cdot\|v_k\|} : k\in\mathbb{N}, a\in A, v_i \in E\Big\}.$$

If $(r_k) = (\rho^k)$ for some $\rho > 0$ we just write ρ instead of (r_k) as indices. The spaces $C^M_{(r_k)}(A,F)$ are Banach spaces, since they are closed in $\ell^\infty(\mathbb{N}, \ell^\infty(A, L^k(E,F)))$ via $(f^k)_k \mapsto (k \mapsto \frac{1}{k!\, r_k\, M_k} f^k)$.

If A is open, $C^\infty(A,F)$ and $C^M_\rho(A,F)$ coincide with the convenient spaces treated before.

4.6. **Theorem** (cf. [KMR09a, 4.6]). *Let $M = (M_k)$ be a non-quasianalytic weight sequence or an $\mathcal{L}$-intersectable quasianalytic weight sequence. Let E and F be Banach spaces and let $U \subseteq E$ be open and convex. Then the space $C^M(U,F) = C_b^M(U,F)$ can be described bornologically in the following equivalent ways, i.e. these constructions give the same vector space and the same bounded sets*

(1) $$\varprojlim_K \varinjlim_{\rho,W} C^M_\rho(W,F)$$

(2) $$\varprojlim_K \varinjlim_{\rho} C^M_\rho(K,F)$$

(3) $$\varprojlim_{K,(r_k)} C^M_{(r_k)}(K,F)$$

Moreover, all involved inductive limits are regular, i.e. the bounded sets of the inductive limits are contained and bounded in some step.

Here K runs through all compact convex subsets of U ordered by inclusion, W runs through the open subsets $K \subseteq W \subseteq U$ again ordered by inclusion, ρ runs through the positive real numbers, (r_k) runs through all sequences of positive real numbers for which $\rho^k/r_k \to 0$ for all $\rho > 0$.

Proof. This proof is almost identical with that of [KMR09a, 4.6]. The only change is to use (2.7) and (4.3) instead of [KMR09a, 3.9] to show that all these descriptions give $C^M(U,F)$ as vector space. □

4.7. **Lemma** (cf. [KMR09a, 4.7]). *Let M be a non-quasianalytic weight sequence. For any convenient vector space E the flip of variables induces an isomorphism $L(E, C^M(\mathbb{R},\mathbb{R})) \cong C^M(\mathbb{R}, E')$ as vector spaces.*

Proof. This proof is identical with that of [KMR09a, 4.7] but uses (2.9) instead of [KMR09a, 4.1] and (2.3) instead of [KMR09a, 3.5]. □

4.8. **Lemma** (cf. [KMR09a, 4.8]). *Let $M = (M_k)$ be a non-quasianalytic weight sequence. By $\lambda^M(\mathbb{R})$ we denote the c^∞-closure of the linear subspace generated by $\{\mathrm{ev}_t : t \in \mathbb{R}\}$ in $C^M(\mathbb{R},\mathbb{R})'$ and let $\delta : \mathbb{R} \to \lambda^M(\mathbb{R})$ be given by $t \mapsto \mathrm{ev}_t$. Then $\lambda^M(\mathbb{R})$ is the free convenient vector space over C^M, i.e. for every convenient vector space G the C^M-curve δ induces a bornological isomorphism*

$$\delta^* : L(\lambda^M(\mathbb{R}), G) \cong C^M(\mathbb{R}, G).$$

We expect $\lambda^M(\mathbb{R})$ to be equal to $C^M(\mathbb{R},\mathbb{R})'$ as it is the case for the analogous situation of smooth mappings, see [KM97a, 23.11], and of holomorphic mappings, see [Sie95] and [Sie97].

Proof. The proof goes along the same lines as in [KM97a, 23.6] and in [FK88, 5.1.1]. It is identical with that of [KMR09a, 4.8] but uses (2.3), (2.9), and (4.2) in that order. □

4.9. **Corollary** (cf. [KMR09a, 4.9]). *Let $L = (L_k)$ and $L' = (L'_k)$ be non-quasianalytic weight sequences. We have the following isomorphisms of linear spaces*

(1) $C^\infty(\mathbb{R}, C^L(\mathbb{R},\mathbb{R})) \cong C^L(\mathbb{R}, C^\infty(\mathbb{R},\mathbb{R}))$
(2) $C^\omega(\mathbb{R}, C^L(\mathbb{R},\mathbb{R})) \cong C^L(\mathbb{R}, C^\omega(\mathbb{R},\mathbb{R}))$
(3) $C^{L'}(\mathbb{R}, C^L(\mathbb{R},\mathbb{R})) \cong C^L(\mathbb{R}, C^{L'}(\mathbb{R},\mathbb{R}))$

Proof. This proof is that of [KMR09a, 4.9] with other refernces: For $\alpha \in \{\infty, \omega, L'\}$ we get

$$\begin{aligned} C^L(\mathbb{R}, C^\alpha(\mathbb{R},\mathbb{R})) &\cong L(\lambda^L(\mathbb{R}), C^\alpha(\mathbb{R},\mathbb{R})) \qquad \text{by (4.8)} \\ &\cong C^\alpha(\mathbb{R}, L(\lambda^L(\mathbb{R}),\mathbb{R})) \qquad \text{by (4.7), [KM97a, 3.13.4, 5.3, 11.15]} \\ &\cong C^\alpha(\mathbb{R}, C^L(\mathbb{R},\mathbb{R})) \qquad \text{by (4.8).} \quad \square \end{aligned}$$

4.10. **Theorem** (Canonical isomorphisms). *Let $M = (M_k)$ be a non-quasianalytic weight sequences or an $\mathcal{L}$-intersectable quasianalytic weight-sequences; likewise $M' = (M'_k)$. Let E, F be convenient vector spaces and let W_i be c^∞-open subsets in such. We have the following natural bornological isomorphisms:*

(1) $C^M(W_1, C^{M'}(W_2, F)) \cong C^{M'}(W_2, C^M(W_1, F))$,
(2) $C^M(W_1, C^\infty(W_2, F)) \cong C^\infty(W_2, C^M(W_1, F))$.
(3) $C^M(W_1, C^\omega(W_2, F)) \cong C^\omega(W_2, C^M(W_1, F))$.
(4) $C^M(W_1, L(E, F)) \cong L(E, C^M(W_1, F))$.

(5) $C^M(W_1, \ell^\infty(X,F)) \cong \ell^\infty(X, C^M(W_1,F))$.
(6) $C^M(W_1, \mathcal{L}\mathrm{ip}^k(X,F)) \cong \mathcal{L}\mathrm{ip}^k(X, C^M(W_1,F))$.

In (5) *the space X is an ℓ^∞-space, i.e. a set together with a bornology induced by a family of real valued functions on X, cf.* [FK88, 1.2.4]. *In* (6) *the space X is a $\mathcal{L}\mathrm{ip}^k$-space, cf.* [FK88, 1.4.1]. *The spaces $\ell^\infty(X,F)$ and $\mathcal{L}\mathrm{ip}^k(W,F)$ are defined in* [FK88, 3.6.1 and 4.4.1].

Proof. This proof is very similar with that of [KMR09a, 4.8] but written differently. Let $\mathcal{C}^1$ and $\mathcal{C}^2$ denote any of the functions spaces mentioned above and X_1 and X_2 the corresponding domains. In order to show that the flip of coordinates $f \mapsto \tilde{f}$, $\mathcal{C}^1(X_1, \mathcal{C}^2(X_2,F)) \to \mathcal{C}^2(X_2, \mathcal{C}^1(X_1,F))$ is a well-defined bounded linear mapping we have to show:

- $\tilde{f}(x_2) \in \mathcal{C}^1(X_1,F)$, which is obvious, since $\tilde{f}(x_2) = \mathrm{ev}_{x_2} \circ f : X_1 \to \mathcal{C}^2(X_2,F) \to F$.
- $\tilde{f} \in \mathcal{C}^2(X_2, \mathcal{C}^1(X_1,F))$, which we will show below.
- $f \mapsto \tilde{f}$ is bounded and linear, which follows by applying the appropriate uniform boundedness theorem for $\mathcal{C}^2$ and $\mathcal{C}^1$ since $f \mapsto \mathrm{ev}_{x_1} \circ \mathrm{ev}_{x_2} \circ \tilde{f} = \mathrm{ev}_{x_2} \circ \mathrm{ev}_{x_1} \circ f$ is bounded and linear.

All occurring function spaces are convenient and satisfy the uniform $\mathcal{S}$-boundedness theorem, where $\mathcal{S}$ is the set of point evaluations:

C^M by (1.10) and (2.9).
C^∞ by [KM97a, 2.14.3, 5.26],
C^ω by [KM97a, 11.11, 11.12],
L by [KM97a, 2.14.3, 5.18],
ℓ^∞ by [KM97a, 2.15, 5.24, 5.25] or [FK88, 3.6.1 and 3.6.6]
$\mathcal{L}\mathrm{ip}^k$ by [FK88, 4.4.2 and 4.4.7]

It remains to check that $\tilde{f}$ is of the appropriate class:

(1) follows by composing with the appropriate (non-quasianalytic) curves $c_1 : \mathbb{R} \to W_1$, $c_2 : \mathbb{R} \to W_2$ and $\lambda \in F^*$ and thereby reducing the statement to the special case in (4.9.3).
(2) as for (1) using (4.9.1).
(3) follows by composing with $c_2 \in C^{\beta_2}(\mathbb{R}, W_2)$, where β_2 is in $\{\infty, \omega\}$, and with $C^L(c_1,\lambda) : C^M(W_1,F) \to C^L(\mathbb{R},\mathbb{R})$ where $c_1 \in C^L(\mathbb{R}, W_1)$ with $L \geq M$ non-quasianalytic and $\lambda \in F^*$. Then $C^L(c_1,\lambda) \circ \tilde{f} \circ c_2 = (C^{\beta_2}(c_2,\lambda) \circ f \circ c_1)^\sim : \mathbb{R} \to C^L(\mathbb{R},\mathbb{R})$ is C^{β_2} by (4.9.1) and (4.9.2), since $C^{\beta_2}(c_2,\lambda) \circ f \circ c_1 : \mathbb{R} \to W_1 \to C^\omega(W_2,F) \to C^{\beta_2}(\mathbb{R},\mathbb{R})$ is C^L.
For the inverse, compose with c_1 and $C^{\beta_2}(c_2,\lambda) : C^\omega(W_2,F) \to C^{\beta_2}(\mathbb{R},\mathbb{R})$. Then $C^{\beta_2}(c_2,\lambda) \circ \tilde{f} \circ c_1 = (C^L(c_1,\lambda) \circ f \circ c_2)^\sim : \mathbb{R} \to C^{\beta_2}(\mathbb{R},\mathbb{R})$ is C^L by (4.9.1) and (4.9.2), since $C^L(c_1,\lambda) \circ f \circ c_2 : \mathbb{R} \to W_2 \to C^L(W_1,F) \to C^L(\mathbb{R},\mathbb{R})$ is C^{β_2}.
(4) since $L(E,F)$ is the c^∞-closed subspace of $C^M(E,F)$ formed by the linear C^M-mappings.
(5) follows from (4), using the free convenient vector spaces $\ell^1(X)$ over the ℓ^∞-space X, see [FK88, 5.1.24 or 5.2.3], satisfying $\ell^\infty(X,F) \cong L(\ell^1(X),F)$.
(6) follows from (4), using the free convenient vector spaces $\lambda^k(X)$ over the $\mathcal{L}\mathrm{ip}^k$-space X, satisfying $\mathcal{L}\mathrm{ip}^k(X,F) \cong L(\lambda^k(X),F)$. Existence of this free convenient vector space can be proved in a similar way as in (4.8). □

5. Manifolds of quasianalytic mappings

For manifolds of real analytic mappings [KM90] we could prove that composition and inversion (on groups of real analytic diffeomorphisms) are again C^ω by testing along C^∞-curves and C^ω-curves separately. Here this does not (yet) work. We have to test along C^L-curves for all L in $\mathcal{L}(Q)$, but for those L we do not have cartesian closedness in general. But it suffices to test along C^Q-mappings from open sets in Banach spaces, and this is a workable replacement.

5.1. **C^Q-manifolds.** Let $Q = (Q_k)$ be an $\mathcal{L}$-intersectable quasianalytic weight sequence of moderate growth. A C^Q-manifold is a smooth manifold such that all chart changings are C^Q-mappings. Likewise for C^Q-bundles and C^Q Lie groups.

Note that any finite dimensional (always assumed paracompact) C^∞-manifold admits a C^∞-diffeomorphic real analytic structure thus also a C^Q-structure. Maybe, any finite dimensional C^Q-manifold admits a C^Q-diffeomorphic real analytic structure. This would follow from:

Conjecture. *Let X be a finite dimensional real analytic manifold. Consider the space $C^Q(X, \mathbb{R})$ of all C^Q-functions on X, equipped with the (obvious) Whitney C^Q-topology. Then $C^\omega(X, \mathbb{R})$ is dense in $C^Q(X, \mathbb{R})$.*

This conjecture is the analogon of [Gra58, Proposition 9].

5.2. **Banach plots.** Let $Q = (Q_k)$ be an $\mathcal{L}$-intersectable quasianalytic weight sequence of moderate growth. Let X be a C^Q-manifold. By a *C^Q-plot in X* we mean a C^Q-mapping $c : D \to X$ where $D \subset E$ is the open unit ball in a Banach space E.

Lemma. *A mapping between C^Q-manifolds is C^Q if and only if it maps C^Q-plots to C^Q-plots.*

Proof. For a convenient vector space E the c^∞-topology is the final topology for all injections $E_B \to E$ where B runs through all closed absolutely convex bounded subsets of E. The c^∞-topology on a c^∞-open subset $U \subseteq E$ is final with respect to all injections $E_B \cap U \to U$. For a C^Q-manifold the topology is the final one for all C^Q-plots. Let $f : X \to Y$ be the mapping. If f respects C^Q-plots it is continuous and so we may assume that Y is c^∞-open in a convenient vector space F and then likewise for $X \subseteq E$. The (affine) plots induced by $X \cap E_B \subset X$ are C^Q. By definition f is C^Q if and only if it is C^L for all $L \in \mathcal{L}(Q)$ and this is the case if f is C^L on $X \cap E_B$ for all B by (2.8). □

5.3. **Spaces of C^Q-sections.** Let $p : E \to B$ be a C^Q vector bundle (possibly infinite dimensional). The space $C^Q(B \leftarrow E)$ of all C^Q-sections is a convenient vector space with the structure induced by

$$C^Q(B \leftarrow E) \to \prod_\alpha C^Q(u_\alpha(U_\alpha), V)$$
$$s \mapsto \operatorname{pr}_2 \circ \psi_\alpha \circ s \circ u_\alpha^{-1}$$

where $B \supseteq U_\alpha \xrightarrow{u_\alpha} u_\alpha(U_\alpha) \subseteq W$ is a C^Q-atlas for B which we assume to be modeled on a convenient vector space W, and where $\psi_\alpha : E|_{U_\alpha} \to U_\alpha \times V$ form a vector bundle atlas over charts U_α of B.

Lemma. *Let D be a unit ball in a Banach space. A mapping $c : D \to C^Q(B \leftarrow E)$ is a C^Q-plot if and only if $c^\wedge : D \times B \to E$ is C^Q.*

Proof. By the description of the structure on $C^Q(B \leftarrow E)$ we may assume that B is c^∞-open in a convenient vector space W and that $E = B \times V$. Then we have $C^Q(B \leftarrow B \times V) \cong C^Q(B, V)$. Thus the statement follows from the exponential law (3.3). □

Let $U \subseteq E$ be an open neighborhood of $s(B)$ for a section s and let $q : F \to B$ be another vector bundle. The set $C^Q(B \leftarrow U)$ of all C^Q-sections $s' : B \to E$ with $s'(B) \subset U$ is open in the convenient vector space $C^Q(B \leftarrow E)$ if B is compact.

An immediate consequence of the lemma is the following: If $U \subseteq E$ is an open neighborhood of $s(B)$ for a section s, $F \to B$ is another vector bundle and if $f : U \to F$ is a fiber respecting C^Q-mapping, then $f_* : C^Q(B \leftarrow U) \to C^Q(B \leftarrow F)$ is C^Q on the open neighborhood $C^Q(B \leftarrow U)$ of s in $C^Q(B \leftarrow E)$. We have $(d(f_*)(s)v)_x = d(f|_{U\cap E_x})(s(x))(v(x))$.

5.4. **Theorem.** *Let $Q = (Q_k)$ be an $\mathcal{L}$-intersectable quasianalytic weight sequence of moderate growth. Let A and B be finite dimensional C^Q-manifolds with A compact and B equipped with a C^Q Riemann metric. Then the space $C^Q(A, B)$ of all C^Q-mappings $A \to B$ is a C^Q-manifold modeled on convenient vector spaces $C^Q(A \leftarrow f^*TB)$ of C^Q-sections of pullback bundles along $f : A \to B$. Moreover, a mapping $c : D \to C^Q(A, B)$ is a C^Q-plot if and only if $c^\wedge : D \times A \to B$ is C^Q.*

If the C^Q-structure on B is induced by a real analytic structure then there exists a real analytic Riemann metric which in turn is C^Q.

Proof. C^Q-vector fields have C^Q-flows by [Kom80]; applying this to the geodesic spray we get the C^Q exponential mapping $\exp : TB \supseteq U \to B$ of the Riemann metric, defined on a suitable open neighborhood of the zero section. We may assume that U is chosen in such a way that $(\pi_B, \exp) : U \to B \times B$ is a C^Q-diffeomorphism onto an open neighborhood V of the diagonal, by the C^Q inverse function theorem due to [Kom79].

For $f \in C^Q(A, B)$ we consider the pullback vector bundle

$$\begin{tikzcd} A \times TB & A \times_B TB \arrow[l, hook'] \arrow[r, equal] & f^*TB \arrow[r, "\pi_B^* f"] \arrow[d, "f^*\pi_B"'] & TB \arrow[d, "\pi_B"] \\ & & A \arrow[r, "f"] & B \end{tikzcd}$$

Then the convenient space of sections $C^Q(A \leftarrow f^*TB)$ is canonically isomorphic to the space $C^Q(A, TB)_f := \{h \in C^Q(A, TB) : \pi_B \circ h = f\}$ via $s \mapsto (\pi_B^* f) \circ s$ and $(\mathrm{Id}_A, h) \leftarrow h$. Now let

$$U_f := \{g \in C^Q(A, B) : (f(x),\ g(x)) \in V \text{ for all } x \in A\},$$
$$u_f : U_f \to C^Q(A \leftarrow f^*TB),$$
$$u_f(g)(x) = (x, \exp^{-1}_{f(x)}(g(x))) = (x, ((\pi_B, \exp)^{-1} \circ (f, g))(x)).$$

Then $u_f : U_f \to \{s \in C^Q(A \leftarrow f^*TB) : s(A) \subseteq f^*U = (\pi_B^* f)^{-1}(U)\}$ is a bijection with inverse $u_f^{-1}(s) = \exp \circ (\pi_B^* f) \circ s$, where we view $U \to B$ as a fiber bundle. The set $u_f(U_f)$ is open in $C^Q(A \leftarrow f^*TB)$ for the topology described above in (5.3) since A is compact and the push forward u_f is C^Q since it respects C^Q-plots by lemma (5.3).

Now we consider the atlas $(U_f, u_f)_{f\in C^Q(A,B)}$ for $C^Q(A, B)$. Its chart change mappings are given for $s \in u_g(U_f \cap U_g) \subseteq C^Q(A \leftarrow g^*TB)$ by

$$\begin{aligned}(u_f \circ u_g^{-1})(s) &= (\mathrm{Id}_A, (\pi_B, \exp)^{-1} \circ (f, \exp \circ (\pi_B^* g) \circ s)) \\ &= (\tau_f^{-1} \circ \tau_g)_*(s),\end{aligned}$$

where $\tau_g(x, Y_{g(x)}) := (x, \exp_{g(x)}(Y_{g(x)}))$ is a C^Q-diffeomorphism $\tau_g : g^*TB \supseteq g^*U \to (g \times \mathrm{Id}_B)^{-1}(V) \subseteq A \times B$ which is fiber respecting over A. The chart change $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ is defined on an open subset and it is also C^Q since it respects C^Q-plots by lemma (5.3).

Finally for the topology on $C^Q(A, B)$ we take the identification topology from this atlas (with the c^∞-topologies on the modeling spaces), which is obviously finer than the compact-open topology and thus Hausdorff.

The equation $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ shows that the C^Q-structure does not depend on the choice of the C^Q Riemannian metric on B.

The statement on C^Q-plots follows from lemma (5.3). □

5.5. **Corollary.** *Let A_1, A_2 and B be finite dimensional C^Q-manifolds with A_1 and A_2 compact. Then composition*

$$C^Q(A_2, B) \times C^Q(A_1, A_2) \to C^Q(A_1, B), \quad (f, g) \mapsto f \circ g$$

is C^Q. However, if $N = (N_k)$ is another weight sequence ($\mathcal{L}$-intersectable quasianalytic) with $(N_k/Q_k)^{1/k} \searrow 0$ then composition is **not** *C^N.*

Proof. Composition maps C^Q-plots to C^Q-plots, so it is C^Q.

Let $A_1 = A_2 = S^1$ and $B = \mathbb{R}$. Then by [Thi08, Theorem 1] or [KMR09a, 2.1.5] there exists $f \in C^Q(S^1, \mathbb{R}) \setminus C^N(S^1, \mathbb{R})$. We consider f as a periodic function $\mathbb{R} \to \mathbb{R}$. The universal covering space of $C^Q(S^1, S^1)$ consists of all $2\pi\mathbb{Z}$-equivariant mappings in $C^Q(\mathbb{R}, \mathbb{R})$, namely the space of all $g + \mathrm{Id}_{\mathbb{R}}$ for 2π-periodic $g \in C^Q$. Thus $C^Q(S^1, S^1)$ is a real analytic manifold and $t \mapsto (x \mapsto x + t)$ induces a real analytic curve c in $C^Q(S^1, S^1)$. But $f_* \circ c$ is not C^N since:

$$\frac{(\partial_t^k|_{t=0}(f_* \circ c)(t))(x)}{k!\rho^k N_k} = \frac{\partial_t^k|_{t=0} f(x+t)}{k!\rho^k N_k} = \frac{f^{(k)}(x)}{k!\rho^k N_k}$$

which is unbounded in k for x in a suitable compact set and for all $\rho > 0$, since $f \notin C^N$. □

5.6. **Theorem.** *Let $Q = (Q_k)$ be a, $\mathcal{L}$-intersectable quasianalytic weight sequence of moderate growth. Let A be a compact ($\Rightarrow$ finite dimensional) C^Q-manifold. Then the group $\mathrm{Diff}^Q(A)$ of all C^Q-diffeomorphisms of A is an open subset of the C^Q-manifold $C^Q(A, A)$. Moreover, it is a C^Q-regular C^Q Lie group: Inversion and composition are C^Q. Its Lie algebra consists of all C^Q-vector fields on A, with the negative of the usual bracket as Lie bracket. The exponential mapping is C^Q. It is not surjective onto any neighborhood of Id_A.*

Following [KM97b], see also [KM97a, 38.4], a C^Q-Lie group G with Lie algebra $\mathfrak{g} = T_e G$ is called C^Q-regular if the following holds:

- For each C^Q-curve $X \in C^Q(\mathbb{R}, \mathfrak{g})$ there exists a C^Q-curve $g \in C^Q(\mathbb{R}, G)$ whose right logarithmic derivative is X, i.e.,

$$\begin{cases} g(0) &= e \\ \partial_t g(t) &= T_e(\mu^{g(t)})X(t) = X(t).g(t) \end{cases}$$

  The curve g is uniquely determined by its initial value $g(0)$, if it exists.
- Put $\mathrm{evol}^r_G(X) = g(1)$ where g is the unique solution required above. Then $\mathrm{evol}^r_G : C^Q(\mathbb{R}, \mathfrak{g}) \to G$ is required to be C^Q also.

Proof. The group $\mathrm{Diff}^Q(A)$ is open in $C^Q(A, A)$ since it is open in the coarser C^1 compact-open topology, see [KM97a, 43.1]. So $\mathrm{Diff}^Q(A)$ is a C^Q-manifold and composition is C^Q by (5.4) and (5.5). To show that inversion is C^Q let c be a C^Q-plot in $\mathrm{Diff}^Q(A)$. By (5.4) the map $c^\wedge : D \times A \to A$ is C^Q and $(\mathrm{inv} \circ c)^\wedge : D \times A \to A$ satisfies the Banach manifold implicit equation $c^\wedge(t, (\mathrm{inv} \circ c)^\wedge(t, x)) = x$ for $x \in A$. By the Banach C^Q implicit function theorem [Yam89] the mapping $(\mathrm{inv} \circ c)^\wedge$ is locally C^Q and thus C^Q. By (5.4) again, $\mathrm{inv} \circ c$ is a C^Q-plot in $\mathrm{Diff}^Q(A)$. So $\mathrm{inv} : \mathrm{Diff}^Q(A) \to \mathrm{Diff}^Q(A)$ is C^Q. The Lie algebra of $\mathrm{Diff}^Q(A)$ is the convenient vector space of all C^Q-vector fields on A, with the negative of the usual Lie bracket (compare with the proof of [KM97a, 43.1]).

To show that $\mathrm{Diff}^Q(A)$ is a C^Q-regular Lie group, we choose a C^Q-plot in the space of C^Q-curves in the Lie algebra of all C^Q vector fields on A, $c : D \to C^Q(\mathbb{R}, C^Q(A \leftarrow TA))$. By lemma (5.3) c corresponds to a $(D\times\mathbb{R})$-time-dependent C^Q vector field $c^{\wedge\wedge} : D \times \mathbb{R} \times A \to TA$. Since C^Q-vector fields have C^Q-flows and since A is compact, $\mathrm{evol}^r(c^\wedge(s))(t) = \mathrm{Fl}_t^{c^\wedge(s)}$ is C^Q in all variables by [Yam91]. Thus $\mathrm{Diff}^Q(A)$ is a C^Q-regular C^Q Lie group.

The exponential mapping is evol^r applied to constant curves in the Lie algebra, i.e., it consists of flows of autonomous C^Q vector fields. That the exponential map is not surjective onto any C^Q-neighborhood of the identity follows from [KM97a, 43.5] for $A = S^1$. This example can be embedded into any compact manifold, see [Gra88]. □

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Part 3

Perturbation theory for unbounded operators

Submitted

MANY PARAMETER HÖLDER PERTURBATION OF UNBOUNDED OPERATORS

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Abstract. If $u \mapsto A(u)$ is a $C^{0,\alpha}$-mapping, for $0 < \alpha \leq 1$, having as values unbounded self-adjoint operators with compact resolvents and common domain of definition, parametrized by u in an (even infinite dimensional) space, then any continuous (in u) arrangement of the eigenvalues of $A(u)$ is indeed $C^{0,\alpha}$ in u.

Theorem. *Let $U \subseteq E$ be a c^∞-open subset in a convenient vector space E, and $0 < \alpha \leq 1$. Let $u \mapsto A(u)$, for $u \in U$, be a $C^{0,\alpha}$-mapping with values unbounded self-adjoint operators in a Hilbert space H with common domain of definition and with compact resolvent. Then any (in u) continuous eigenvalue $\lambda(u)$ of $A(u)$ is $C^{0,\alpha}$ in u.*

Remarks and definitions. This paper is a complement to [KM03] and builds upon it. A function $f : \mathbb{R} \to \mathbb{R}$ is called $C^{0,\alpha}$ if $\frac{f(t)-f(s)}{|t-s|^\alpha}$ is locally bounded in $t \neq s$. For $\alpha = 1$ this is Lipschitz.

Due to [Bom67] a mapping $f : \mathbb{R}^n \to \mathbb{R}$ is $C^{0,\alpha}$ if and only if $f \circ c$ is $C^{0,\alpha}$ for each smooth (i.e. C^∞) curve c. [Fau89] has shown that this holds for even more general concepts of Hölder differentiable maps.

A convenient vector space (see [KM97]) is a locally convex vector space E satisfying the following equivalent conditions: Mackey Cauchy sequences converge; C^∞-curves in E are locally integrable in E; a curve $c : \mathbb{R} \to E$ is C^∞ (Lipschitz) if and only if $\ell \circ c$ is C^∞ (Lipschitz) for all continuous linear functionals ℓ. The c^∞-topology on E is the final topology with respect to all smooth curves (Lipschitz curves). Mappings f defined on open (or even c^∞-open) subsets of convenient vector spaces E are called $C^{0,\alpha}$ (Lipschitz) if $f \circ c$ is $C^{0,\alpha}$ (Lipschitz) for every smooth curve c. If E is a Banach space then a $C^{0,\alpha}$-mapping is locally Hölder-continuous of order α in the usual sense. This has been proved in [Fau91], which is not easily accessible, thus we include a proof in the lemma below. For the Lipschitz case see [FK88] and [KM97, 12.7].

That a mapping $t \mapsto A(t)$ defined on a c^∞-open subset U of a convenient vector space E is $C^{0,\alpha}$ with values in unbounded operators means the following: There is a dense subspace V of the Hilbert space H such that V is the domain of definition of each $A(t)$, and such that $A(t)^* = A(t)$. And furthermore, $t \mapsto \langle A(t)u, v\rangle$ is $C^{0,\alpha}$ for each $u \in V$ and $v \in H$ in the sense of the definition given above.

This implies that $t \mapsto A(t)u$ is of the same class $U \to H$ for each $u \in V$ by [KM97, 2.3], [FK88, 2.6.2], or[Fau91, 4.1.14]. This is true because $C^{0,\alpha}$ can be described by boundedness conditions only; and for these the uniform boundedness principle is valid.

Lemma ([Fau91]). *Let E and F be Banach spaces, U open in E. Then, a mapping $f : U \to F$ is $C^{0,\alpha}$ if and only if f is locally Hölder of order α, i.e., $\frac{\|f(x)-f(y)\|}{\|x-y\|^\alpha}$ is locally bounded.*

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Proof. If f is $C^{0,\alpha}$ but not locally Hölder near $z \in U$, there are $x_n \neq y_n$ in U with $\|x_n - z\| \le 1/4^n$ and $\|y_n - z\| \le 1/4^n$, such that $\|f(y_n) - f(x_n)\| \ge n.2^n.\|y_n - x_n\|^\alpha$. Now we apply the general curve lemma [KM97, 12.2] with $s_n := 2^n.\|y_n - x_n\|$ and $c_n(t) := x_n - z + t\frac{y_n - x_n}{2^n\|y_n - x_n\|}$ to get a smooth curve c with $c(t + t_n) - z = c_n(t)$ for $0 \le t \le s_n$. Then $\frac{1}{s_n^\alpha}\|(f \circ c)(t_n + s_n) - (f \circ c)(t_n)\| = \frac{1}{2^{n\alpha}.\|y_n - x_n\|^\alpha}\|f(y_n) - f(x_n)\| \ge n$. The converse is obvious. □

The theorem holds for $E = \mathbb{R}$. Let $t \mapsto A(t)$ be a $C^{0,\alpha}$-curve. Going through the proof of the resolvent lemma in [KM03] carefully, we find that $t \mapsto A(t)$ is a $C^{0,\alpha}$-mapping $U \to L(V, H)$, and thus the resolvent $(A(t) - z)^{-1}$ is $C^{0,\alpha}$ into $L(H, H)$ in t and z jointly.

For a continuous eigenvalue $t \mapsto \lambda(t)$ as in the theorem, let the eigenvalue $\lambda(s)$ of $A(s)$ have multiplicity N for s fixed. Choose a simple closed curve γ in the resolvent set of $A(s)$ enclosing only $\lambda(s)$ among all eigenvalues of $A(s)$. Since the global resolvent set $\{(t, z) \in \mathbb{R} \times \mathbb{C} : (A(t) - z) : V \to H \text{ is invertible}\}$ is open, no eigenvalue of $A(t)$ lies on γ, for t near s. Consider

$$t \mapsto -\frac{1}{2\pi i}\int_\gamma (A(t) - z)^{-1} dz =: P(t),$$

a $C^{0,\alpha}$-curve of projections (on the direct sum of all eigenspaces corresponding to eigenvalues in the interior of γ) with finite dimensional ranges and constant ranks. So for t near s, there are equally many eigenvalues (repeated with multiplicity) in the interior of γ. Let us order them by size, $\mu_1(t) \le \mu_2(t) \le \dots \le \mu_N(t)$, for all t. The image of $t \mapsto P(t)$, for t near s describes a finite dimensional $C^{0,\alpha}$ vector subbundle of $\mathbb{R} \times H \to \mathbb{R}$, since its rank is constant. The set $\{\mu_i(t) : 1 \le i \le N\}$ represents the eigenvalues of $P(t)A(t)|_{P(t)(H)}$. By the following result, it forms a $C^{0,\alpha}$-parametrization of the eigenvalues of $A(t)$ inside γ, for t near s.

The eigenvalue $\lambda(t)$ is a continuous (in t) choice among the $\mu_i(t)$, and it is $C^{0,\alpha}$ in t by the proposition below.

Result ([Wey12], see also [Bha97, III.2.6])**.** *Let A, B be $N \times N$ Hermitian matrices. Let $\mu_1(A) \le \mu_2(A) \le \dots \le \mu_N(A)$ and $\mu_1(B) \le \mu_2(B) \le \dots \le \mu_N(B)$ denote the eigenvalues of A and B, respectively. Then*

$$\max_j |\mu_j(A) - \mu_j(B)| \le \|A - B\|.$$

Here $\|.\|$ is the operator norm.

Proposition. *Let $0 < \alpha \le 1$. Let $U \ni u \mapsto A(u)$ be a $C^{0,\alpha}$-mapping of Hermitian $N \times N$ matrices. Let $u \mapsto \lambda_i(u)$, $i = 1, \dots, N$ be continuous mappings which together parametrize the eigenvalues of $A(u)$. Then each λ_i is $C^{0,\alpha}$.*

Proof. It suffices to check that λ_i is $C^{0,\alpha}$ along each smooth curve in U, so we may assume without loss that $U = \mathbb{R}$. We have to show that each continuous eigenvalue $t \mapsto \lambda(t)$ is a $C^{0,\alpha}$-function on each compact interval I in U. Let $\mu_1(u) \le \dots \le \mu_N(u)$ be the increasingly ordered arrangement of eigenvalues. Then each μ_i is a $C^{0,\alpha}$-function on I with a common Hölder constant C by the result above. Let $t < s$ be in I. Then there is an i_0 such that $\lambda(t) = \mu_{i_0}(t)$. Now let t_1 be the maximum of all $r \in [t, s]$ such that $\lambda(r) = \mu_{i_0}(r)$. If $t_1 < s$ then $\mu_{i_0}(t_1) = \mu_{i_1}(t_1)$ for some $i_1 \neq i_0$. Let t_2 be the maximum of all $r \in [t_1, s]$ such that $\lambda(r) = \mu_{i_1}(r)$. If $t_2 < s$ then $\mu_{i_1}(t_2) = \mu_{i_2}(t_2)$ for some $i_2 \notin \{i_0, i_1\}$. And so on until $s = t_k$ for some $k \le N$. Then we have (where $t_0 = t$)

$$\frac{|\lambda(s) - \lambda(t)|}{(s - t)^\alpha} \le \sum_{j=0}^{k-1} \frac{|\mu_{i_j}(t_{j+1}) - \mu_{i_j}(t_j)|}{(t_{j+1} - t_j)^\alpha} \cdot \left(\frac{t_{j+1} - t_j}{s - t}\right)^\alpha \le Ck \le CN. \quad \square$$

Proof of the theorem. For each smooth curve $c : \mathbb{R} \to U$ the curve $\mathbb{R} \ni t \mapsto A(c(t))$ is $C^{0,\alpha}$, and by the 1-parameter case the eigenvalue $\lambda(c(t))$ is $C^{0,\alpha}$. But then $u \mapsto \lambda(u)$ is $C^{0,\alpha}$. □

Remark. Let $u \mapsto A(u)$ be Lipschitz. Choose a fixed continuous ordering of the eigenvalues, e.g., by size. We claim that along a smooth or Lipschitz curve $c(t)$ in U, none of these can accelerate to ∞ or $-\infty$ in finite time. Thus we may denote them as $\dots \lambda_i(u) \le \lambda_{i+1}(u) \le \dots$, for all $u \in U$. Then each λ_i is Lipschitz.

The claim can be proved as follows: Let $t \mapsto A(t)$ be a Lipschitz curve. By reducing to the projection $P(t)A(t)|_{P(t)(H)}$, we may assume that $t \mapsto A(t)$ is a Lipschitz curve of $N \times N$ Hermitian matrices. So $A'(t)$ exists a.e. and is locally bounded. Let $t \mapsto \lambda(t)$ be a continuous eigenvalue. It follows that λ satisfies [KM03, (6)] a.e. and, as in the proof of [KM03, (7)], one shows that for each compact interval I there is a constant C such that $|\lambda'(t)| \le C + C|\lambda(t)|$ a.e. in I. Since $t \mapsto \lambda(t)$ is Lipschitz, in particular, absolutely continuous, Gronwall's lemma (e.g. [Die60, (10.5.1.3)]) implies that $|\lambda(s) - \lambda(t)| \le (1 + |\lambda(t)|)(e^{a|s-t|} - 1)$ for a constant a depending only on I.

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Submitted

DENJOY-CARLEMAN DIFFERENTIABLE PERTURBATION OF POLYNOMIALS AND UNBOUNDED OPERATORS

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Abstract. Let $t \mapsto A(t)$ for $t \in T$ be a C^M-mapping with values unbounded operators with compact resolvents and common domain of definition which are self-adjoint or normal. Here C^M stands for C^ω (real analytic), a quasianalytic or non-quasianalytic Denjoy-Carleman class, C^∞, or a Hölder continuity class $C^{0,\alpha}$. The parameter domain T is either $\mathbb{R}$ or $\mathbb{R}^n$ or an infinite dimensional convenient vector space. We prove and review results on C^M-dependence on t of the eigenvalues and eigenvectors of $A(t)$.

Theorem. *Let $t \mapsto A(t)$ for $t \in T$ be a parameterized family of unbounded operators in a Hilbert space H with common domain of definition and with compact resolvent.*

If $t \in T = \mathbb{R}$ and all $A(t)$ are self-adjoint then the following holds:

(A) *If $A(t)$ is real analytic in $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ may be parameterized real analytically in t.*
(B) *If $A(t)$ is quasianalytic of class C^Q in $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ may be parameterized C^Q in t.*
(C) *If $A(t)$ is non-quasianalytic of class C^L in $t \in \mathbb{R}$ and if no two unequal continuously parameterized eigenvalues meet of infinite order at any $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ can be parameterized C^L in t.*
(D) *If $A(t)$ is C^∞ in $t \in \mathbb{R}$ and if no two unequal continuously parameterized eigenvalues meet of infinite order at any $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ can be parameterized C^∞ in t.*
(E) *If $A(t)$ is C^∞ in $t \in \mathbb{R}$, then the eigenvalues of $A(t)$ may be parameterized twice differentiably in t.*
(F) *If $A(t)$ is $C^{1,\alpha}$ in $t \in \mathbb{R}$ for some $\alpha > 0$, then the eigenvalues of $A(t)$ may be parameterized in a C^1 way in t.*

If $t \in T = \mathbb{R}$ and all $A(t)$ are normal then the following holds:

(G) *If $A(t)$ is real analytic in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue λ of $A(t_0)$ there exists $N \in \mathbb{N}$ such that the eigenvalues near λ of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized real analytically in s near $s = 0$.*
(H) *If $A(t)$ is C^Q in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue λ of $A(t_0)$ there exists $N \in \mathbb{N}$ such that the eigenvalues near λ of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized C^Q in s near $s = 0$.*
(I) *If $A(t)$ is C^L in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue λ of $A(t_0)$ at which no two of the unequal continuously arranged eigenvalues (see* [Kat76, II.5.2]*) meet of infinite order, there exists $N \in \mathbb{N}$ such that the eigenvalues near λ of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized C^L in s near $s = 0$.*

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(J) *If $A(t)$ is C^∞ in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue λ of $A(t_0)$ at which no two of the unequal continuously arranged eigenvalues (see* [Kat76, II.5.2]*) meet of infinite order, there exists $N \in \mathbb{N}$ such that the eigenvalues near λ of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized C^∞ in s near $s = 0$.*

(K) *If $A(t)$ is C^∞ in $t \in \mathbb{R}$ and no two of the unequal continuously parameterized eigenvalues meet of infinite order at any $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ can be parameterized by absolutely continuous functions, locally in t.*

If $t \in T = \mathbb{R}^n$ and all $A(t)$ are normal then the following holds:

(L) *If $A(t)$ is C^ω or C^Q in $t \in \mathbb{R}^n$, then for each $t_0 \in \mathbb{R}^n$ and for each eigenvalue λ of $A(t_0)$, there exists a finite covering $\{\pi_k : U_k \to W\}$ of a neighborhood W of t_0, where each π_k is a composite of finitely many mappings each of which is either a local blow-up along a C^ω or C^Q submanifold or a local power substitution, such that the eigenvalues and the eigenvectors of $A(\pi_k(s))$ can be chosen C^ω or C^Q in s. If A is self-adjoint, then we do not need power substitutions.*

(M) *If $A(t)$ is C^ω or C^Q in $t \in \mathbb{R}^n$, then the eigenvalues and their eigenvectors of $A(t)$ can be parameterized by functions which are special functions of bounded variation (SBV), see* [DGA88] *or* [AFP00]*, locally in t.*

If $t \in T \subseteq E$, a c^∞-open subset in an infinite dimensional convenient vector space then the following holds:

(N) *For $0 < \alpha \leq 1$, if $A(t)$ is $C^{0,\alpha}$ (Hölder continuous of exponent α) in $t \in T$ and all $A(t)$ are self-adjoint, then the eigenvalues of $A(t)$ may be parameterized in a $C^{0,\alpha}$ way in t.*

(O) *For $0 < \alpha \leq 1$, if $A(t)$ is $C^{0,\alpha}$ (Hölder continuous of exponent α) in $t \in T$ and all $A(t)$ are normal, then we have: For each $t_0 \in T$ and each eigenvalue z_0 of $A(t_0)$ consider a simple closed C^1-curve γ in the resolvent set of $A(t_0)$ enclosing only z_0 among all eigenvalues of $A(t_0)$. Then for t near t_0 in the c^∞-topology on T, no eigenvalue of $A(t)$ lies on γ. Let $\lambda(t) = (\lambda_1(t), \ldots, \lambda_N(t))$ be the N-tuple of all eigenvalues (repeated according to their multiplicity) of $A(t)$ inside of γ. Then $t \mapsto \lambda(t)$ is $C^{0,\alpha}$ for t near t_0 with respect to the non-separating metric*

$$d(\lambda, \mu) = \min_{\sigma \in \mathcal{S}_N} \max_{1 \leq i \leq N} |\lambda_i - \mu_{\sigma(i)}|$$

on the space of N-tuples.

Part (A) is due to Rellich [Rel42] in 1942, see also [Bau72] and [Kat76, VII, 3.9]. Part (D) has been proved in [AKLM98, 7.8], see also [KM97, 50.16], in 1997, which contains also a different proof of (A). (E) and (F) have been proved in [KM03] in 2003. (G) was proved in [Rai09a, 7.1]; it can be proved as (H) with some obvious changes, but it is not a special case since C^ω does not correspond to a sequence which is an $\mathcal{L}$-intersection (see [KMR09b]). (J) and (K) were proved in [Rai09a, 7.1]. (N) was proved in [KMR09c].

The purpose of this paper is to prove the remaining parts (B), (C), (H), (I), (L), (M), and (O).

Definitions and remarks. Let $M = (M_k)_{k \in \mathbb{N} = \mathbb{N}_{\geq 0}}$ be an increasing sequence ($M_{k+1} \geq M_k$) of positive real numbers with $M_0 = 1$. Let $U \subseteq \mathbb{R}^n$ be open. We denote by $C^M(U)$ the set of all $f \in C^\infty(U)$ such that, for each compact $K \subseteq U$, there exist positive constants C and ρ such that

$$|\partial^\alpha f(x)| \leq C\, \rho^{|\alpha|}\, |\alpha|!\, M_{|\alpha|} \quad \text{for all } \alpha \in \mathbb{N}^n \text{ and } x \in K.$$

The set $C^M(U)$ is a *Denjoy–Carleman class* of functions on U. If $M_k = 1$, for all k, then $C^M(U)$ coincides with the ring $C^\omega(U)$ of real analytic functions on U. In general, $C^\omega(U) \subseteq C^M(U) \subseteq C^\infty(U)$.

Here $Q = (Q_k)_{k\in\mathbb{N}}$ is a sequence as above which is quasianalytic, log-convex, and which is also an $\mathcal{L}$-intersection, see [KMR09b] or [KMR09a] and references therein. Moreover, $L = (L_k)_{k\in\mathbb{N}}$ is a sequence as above which is non-quasianalytic and log-convex.

That $A(t)$ is a real analytic, C^M (where M is either Q or L), C^∞, or $C^{k,\alpha}$ family of unbounded operators means the following: There is a dense subspace V of the Hilbert space H such that V is the domain of definition of each $A(t)$, and such that $A(t)^* = A(t)$ in the self-adjoint case, or $A(t)$ has closed graph and $A(t)A(t)^* = A(t)^*A(t)$ wherever defined in the normal case. Moreover, we require that $t \mapsto \langle A(t)u, v\rangle$ is of the respective differentiability class for each $u \in V$ and $v \in H$. From now on we treat only $C^M = C^\omega$, C^M for $M = Q$, $M = L$, and $C^M = C^{0,\alpha}$.

This implies that $t \mapsto A(t)u$ is of the same class $C^M(E, H)$ (where E is either $\mathbb{R}$ or $\mathbb{R}^n$) or is in $C^{0,\alpha}(E, H)$ (if E is a convenient vector space) for each $u \in V$ by [KM97, 2.14.4, 10.3] for C^ω, by [KMR09a, 3.1, 3.3, 3.5] for $M = L$, by [KMR09b, 1.10, 2.1, 2.3] for $M = Q$, and by [KM97, 2.3], [FK88, 2.6.2] or [Fau91, 4.14.4] for $C^{0,\alpha}$ because $C^{0,\alpha}$ can be described by boundedness conditions only and for these the uniform boundedness principle is valid.

A sequence of functions λ_i is said to *parameterize the eigenvalues, if for each* $z \in \mathbb{C}$ *the cardinality* $|\{i : \lambda_i(t) = z\}|$ *equals the multiplicity of* z *as eigenvalue of* $A(t)$.

Let X be a C^ω or C^Q manifold. A *local blow-up* Φ over an open subset U of X means the composition $\Phi = \iota \circ \varphi$ of a blow-up $\varphi : U' \to U$ with center a C^ω or C^Q submanifold and of the inclusion $\iota : U \to X$. A *local power substitution* is a mapping $\Psi : V \to X$ of the form $\Psi = \iota \circ \psi$, where $\iota : W \to X$ is the inclusion of a coordinate chart W of X and $\psi : V \to W$ is given by

$$(y_1, \dots, y_q) = ((-1)^{\epsilon_1} x_1^{\gamma_1}, \dots, (-1)^{\epsilon_q} x_q^{\gamma_q}),$$

for some $\gamma = (\gamma_1, \dots, \gamma_q) \in (\mathbb{N}_{>0})^q$ and all $\epsilon = (\epsilon_1, \dots, \epsilon_q) \in \{0, 1\}^q$, where $y_1, \dots, y_q$ denote the coordinates of W (and $q = \dim X$).

This paper became possible only after some of the results of [KMR09a] and [KMR09b] were proved, in particular the uniform boundedness principles. The wish to prove the results of this paper was the main motivation for us to work on [KMR09a] and [KMR09b].

Applications. Let X be a compact C^Q manifold and let $t \mapsto g_t$ be a C^Q-curve of C^Q Riemannian metrics on X. Then we get the corresponding C^Q curve $t \mapsto \Delta(g_t)$ of Laplace-Beltrami operators on $L^2(X)$. By theorem (B) the eigenvalues and eigenvectors can be arranged C^Q. Question: Are the eigenfunctions then also C^Q?

Let Ω be a bounded region in $\mathbb{R}^n$ with C^Q boundary, and let $H(t) = -\Delta + V(t)$ be a C^Q-curve of Schrödinger operators with varying C^Q potential and Dirichlet boundary conditions. Then the eigenvalues and eigenvectors can be arranged C^Q. Question: Are the eigenvectors viewed as eigenfunctions then also in $C^Q(\Omega \times \mathbb{R})$?

Example. This is an elaboration of [AKLM98, 7.4] and [KM03, Example]. Let $S(2)$ be the vector space of all symmetric real (2×2)-matrices. We use the C^L-curve lemma [KMR09a, 3.6] or [KMR09b, 2.5]: *There exists a converging sequence of reals* t_n *with the following property: Let* $A_n, B_n \in S(2)$ *be any sequences which converge fast to 0, i.e., for each* $k \in \mathbb{N}$ *the sequences* $n^k A_n$ *and* $n^k B_n$ *are bounded in* $S(2)$. *Then there exists a curve* $A \in C^L(\mathbb{R}, S(2))$ *such that* $A(t_n + s) = A_n + sB_n$ *for* $|s| \leq \frac{1}{n^2}$, *for all* n.

We use it for

$$A_n := \frac{1}{2^{n^2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_n := \frac{1}{2^{n^2}\, s_n} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{where } s_n := 2^{n-n^2} \leq \frac{1}{n^2}.$$

The eigenvalues of $A_n + tB_n$ and their derivatives are

$$\lambda_n(t) = \pm \frac{1}{2^{n^2}} \sqrt{1 + (\tfrac{t}{s_n})^2}, \quad \lambda_n'(t) = \pm \frac{2^{n^2-2n} t}{\sqrt{1 + (\tfrac{t}{s_n})^2}}.$$

Then

$$\begin{aligned} \frac{\lambda'(t_n + s_n) - \lambda'(t_n)}{s_n^\alpha} &= \frac{\lambda_n'(s_n) - \lambda_n'(0)}{s_n^\alpha} = \pm \frac{2^{n^2-2n} s_n}{s_n^\alpha \sqrt{2}} \\ &= \pm \frac{2^{n(\alpha(n-1)-1)}}{\sqrt{2}} \to \infty \text{ for } \alpha > 0. \end{aligned}$$

So the condition (in (C), (D), (I), (J), and (K)) that no two unequal continuously parameterized eigenvalues meet of infinite order cannot be dropped. By [AKLM98, 2.1], we may always find a twice differentiable square root of a non-negative smooth function, so that the eigenvalues λ are functions which are twice differentiable but not $C^{1,\alpha}$ for any $\alpha > 0$.

Note that the normed eigenvectors cannot be chosen continuously in this example (see also example [Rel37, §2]). Namely, we have

$$A(t_n) = A_n = \frac{1}{2^{n^2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A(t_n + s_n) = A_n + s_n\, B_n = \frac{1}{2^{n^2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Resolvent Lemma. *Let C^M be any of C^ω, C^Q, C^L, C^∞, or $C^{0,\alpha}$, and let $A(t)$ be normal. If A is C^M then the resolvent $(t,z) \mapsto (A(t) - z)^{-1} \in L(H,H)$ is C^M on its natural domain, the global resolvent set*

$$\{(t,z) \in T \times \mathbb{C} : (A(t) - z) : V \to H \text{ is invertible}\}$$

which is open (and even connected).

Proof. By definition the function $t \mapsto \langle A(t)v, u\rangle$ is of class C^M for each $v \in V$ and $u \in H$. We may conclude that the mapping $t \mapsto A(t)v$ is of class C^M into H as follows: For $C^M = C^\infty$ we use [KM97, 2.14.4]. For $C^M = C^\omega$ we use in addition [KM97, 10.3]. For $C^M = C^Q$ or $C^M = C^L$ we use [KMR09b, 2.1] and/or [KMR09a, 3.3] where we replace $\mathbb{R}$ by $\mathbb{R}^n$. For $C^M = C^{0,\alpha}$ we use [KM97, 2.3], [FK88, 2.6.2], or [Fau91, 4.1.14] because $C^{0,\alpha}$ can be described by boundedness conditions only and for these the uniform boundedness principle is valid.

For each t consider the norm $\|u\|_t^2 := \|u\|^2 + \|A(t)u\|^2$ on V. Since $A(t)$ is closed, $(V, \|\ \ \|_t)$ is again a Hilbert space with inner product $\langle u,v\rangle_t := \langle u,v\rangle + \langle A(t)u, A(t)v\rangle$.

(1) *Claim (see* [AKLM98, in the proof of 7.8]*,* [KM97, in the proof of 50.16]*, or* [KM03, Claim 1]*). All these norms $\|\ \ \|_t$ on V are equivalent, locally uniformly in t. We then equip V with one of the equivalent Hilbert norms, say $\|\ \ \|_0$.*

We reduce this to $C^{0,\alpha}$. Namely, note first that $A(t) : (V, \|\ \ \|_s) \to H$ is bounded since the graph of $A(t)$ is closed in $H \times H$, contained in $V \times H$ and thus also closed in $(V, \|\ \ \|_s) \times H$. For fixed $u, v \in V$, the function $t \mapsto \langle u,v\rangle_t = \langle u,v\rangle + \langle A(t)u, A(t)v\rangle$ is $C^{0,\alpha}$ since $t \mapsto A(t)u$ is it. By the multilinear uniform boundedness principle ([KM97, 5.18] or [FK88, 3.7.4]) the mapping $t \mapsto \langle\ \ ,\ \ \rangle_t$ is $C^{0,\alpha}$ into the space of bounded sesquilinear forms on $(V, \|\ \ \|_s)$ for each fixed s. Thus the inverse image of $\langle\ \ ,\ \ \rangle_s + \frac{1}{2}(\text{unit ball})$ in $L(\overline{(V, \|\ \ \|_s)} \oplus (V, \|\ \ \|_s); \mathbb{C})$ is a c^∞-open neighborhood U of s in T. Thus $\sqrt{1/2}\|u\|_s \leq \|u\|_t \leq \sqrt{3/2}\|u\|_s$ for all $t \in U$, i.e. all Hilbert norms $\|\ \ \|_t$ are locally uniformly equivalent, and claim (1) follows.

By the linear uniform boundedness theorem we see that $t \mapsto A(t)$ is in $C^M(T, L(V,H))$ as follows (here it suffices to use a set of linear functionals which together recognize bounded sets instead of the whole dual): For $C^M = C^\infty$ we use [KM97, 1.7 and 2.14.3]. For $C^M = C^\omega$ we use in addition [KM97, 9.4]. For $C^M = C^Q$ or $C^M = C^L$ we use [KMR09b, 2.2 and 2.3] and/or [KMR09a, 3.5] where we replace $\mathbb{R}$ by $\mathbb{R}^n$. For $C^M = C^{0,\alpha}$ see above.

If for some $(t,z) \in T \times \mathbb{C}$ the bounded operator $A(t) - z : V \to H$ is invertible, then this is true locally with respect to the c^∞-topology on the product which is the product topology by [KM97, 4.16], and $(t,z) \mapsto (A(t)-z)^{-1} : H \to V$ is C^M, by the chain rule, since inversion is real analytic on the Banach space $L(V,H)$. □

Note that $(A(t)-z)^{-1} : H \to H$ is a compact operator for some (equivalently any) (t,z) if and only if the inclusion $i : V \to H$ is compact, since $i = (A(t)-z)^{-1} \circ (A(t)-z) : V \to H \to H$.

Polynomial proposition. *Let P be a curve of polynomials*

$$P(t)(x) = x^n - a_1(t)x^{n-1} + \cdots + (-1)^n a_n(t), \quad t \in \mathbb{R}.$$

- (a) *If P is hyperbolic (all roots real) and if the coefficient functions a_i are all C^Q then there exist C^Q functions λ_i which parameterize all roots.*
- (b) *If P is hyperbolic (all roots real), if the coefficient functions a_i are C^L and no two of the different roots meet of infinite order, then there exist C^L functions λ_i which parameterize all roots.*
- (c) *If the coefficient functions a_i are C^Q, then for each t_0 there exists $N \in \mathbb{N}$ such that the roots of $s \mapsto P(t_0 \pm s^N)$ can be parameterized C^Q in s for s near 0.*
- (d) *If the coefficient functions a_i are C^L and no two of the different roots meet of infinite order, then for each t_0 there exists $N \in \mathbb{N}$ such that the roots of $s \mapsto P(t_0 \pm s^N)$ can be parameterized C^L in s for s near 0.*

All C^Q or C^L solutions differ by permutations.

The proof of parts (a) and (b) is exactly as in [AKLM98] where the corresponding results were proven for C^∞ instead of C^L, and for C^ω instead of C^Q. For this we need only the following properties of C^Q and C^L:

- They allow for the implicit function theorem (for [AKLM98, 3.3]).
- They contain C^ω and are closed under composition (for [AKLM98, 3.4]).
- They are derivation closed (for [AKLM98, 3.7]).

Part (a) is also in [CC04, 7.6] which follows [AKLM98]. It also follows from the multidimensional version [Rai09b, 6.10] since blow-ups in dimension 1 are trivial. The proofs of parts (c) and (d) are exactly as in [Rai09a, 3.2] where the corresponding result was proven for C^ω instead of C^Q, and for C^∞ instead of C^L, if none of the different roots meet of infinite order. For these we need the properties of C^Q and C^L listed above.

Matrix proposition. *Let $A(t)$ for $t \in T$ be a family of $(N \times N)$-matrices.*

- (e) *If $T = \mathbb{R} \ni t \mapsto A(t)$ is a C^Q-curve of Hermitian matrices, then the eigenvalues and the eigenvectors can be chosen C^Q.*
- (f) *If $T = \mathbb{R} \ni t \mapsto A(t)$ is a C^L-curve of Hermitian matrices such that no two eigenvalues meet of infinite order, then the eigenvalues and the eigenvectors can be chosen C^L.*
- (g) *If $T = \mathbb{R} \ni t \mapsto A(t)$ is a C^L-curve of normal matrices such that no two eigenvalues meet of infinite order, then for each t_0 there exists $N_1 \in \mathbb{N}$ such that the eigenvalues and eigenvectors of $s \mapsto A(t_0 \pm s^{N_1})$ can be parameterized C^L in s for s near 0.*

(h) *Let $T \subseteq \mathbb{R}^n$ be open and let $T \ni t \mapsto A(t)$ be a C^ω or C^Q-mapping of normal matrices. Let $K \subseteq T$ be compact. Then there exist a neighborhood W of K, and a finite covering $\{\pi_k : U_k \to W\}$ of W, where each π_k is a composite of finitely many mappings each of which is either a local blow-up along a C^ω or C^Q submanifold or a local power substitution, such that the eigenvalues and the eigenvectors of $A(\pi_k(s))$ can be chosen C^ω or C^Q in s. Consequently, the eigenvalues and eigenvectors of $A(t)$ are locally special functions of bounded variation (SBV). If A is a family of Hermitian matrices, then we do not need power substitutions.*

The proof of the matrix proposition in case (e) and (f) is exactly as in [AKLM98, 7.6], using the polynomial proposition and properties of C^Q and C^L. Item (g) is exactly as in [Rai09a, 6.2], using the polynomial proposition and properties of C^L. Item (h) is proved in [Rai09b, 9.1 and 9.6], see also [KP08].

Proof of the theorem. We have to prove parts (B), (C), (H), (I), (L), (M), and (O). So let C^M be any of C^ω, C^Q, C^L, or $C^{0,\alpha}$, and let $A(t)$ be normal. Let z be an eigenvalue of $A(t_0)$ of multiplicity N. We choose a simple closed C^1 curve γ in the resolvent set of $A(t_0)$ for fixed t_0 enclosing only z among all eigenvalues of $A(t_0)$. Since the global resolvent set is open, see the resolvent lemma, no eigenvalue of $A(t)$ lies on γ, for t near t_0. By the resolvent lemma, $A : T \to L((V, \|\quad\|_0), H)$ is C^M, thus also

$$t \mapsto -\frac{1}{2\pi i}\int_\gamma (A(t)-z)^{-1}\,dz =: P(t,\gamma) = P(t)$$

is a C^M mapping. Each $P(t)$ is a projection, namely onto the direct sum of all eigenspaces corresponding to eigenvalues of $A(t)$ in the interior of γ, with finite rank. Thus the rank must be constant: It is easy to see that the (finite) rank cannot fall locally, and it cannot increase, since the distance in $L(H,H)$ of $P(t)$ to the subset of operators of rank $\le N = \operatorname{rank}(P(t_0))$ is continuous in t and is either 0 or 1.

So for t in a neighborhood U of t_0 there are equally many eigenvalues in the interior of γ, and we may call them $\lambda_i(t)$ for $1 \le i \le N$ (repeated with multiplicity).

Now we consider the family of N-dimensional complex vector spaces $t \mapsto P(t)(H) \subseteq H$, for $t \in U$. They form a C^M Hermitian vector subbundle over U of $U \times H \to U$: For given t, choose $v_1, \dots v_N \in H$ such that the $P(t)v_i$ are linearly independent and thus span $P(t)H$. This remains true locally in t. Now we use the Gram Schmidt orthonormalization procedure (which is C^ω) for the $P(t)v_i$ to obtain a local orthonormal C^M frame of the bundle.

Now $A(t)$ maps $P(t)H$ to itself; in a C^M local frame it is given by a normal $(N \times N)$-matrix parameterized C^M by $t \in U$.

Now all local assertions of the theorem follow:

(B) Use the matrix proposition, part (e).
(C) Use the matrix proposition, part (f).
(H) Use the matrix proposition, part (h), and note that in dimension 1 blowups are trivial.
(I) Use the matrix proposition, part (g).
(L,M) Use the matrix proposition, part (h), for $\mathbb{R}^n$.
(O) We use the following
Result ([BDM83], [Bha97, VII.4.1]). *Let A, B be normal $(N \times N)$-matrices and let $\lambda_i(A)$ and $\lambda_i(B)$ for $i = 1, \dots, N$ denote the respective eigenvalues. Then*

$$\min_{\sigma \in \mathcal{S}_N} \max_j |\lambda_j(A) - \lambda_{\sigma(j)}(B)| \le C\|A - B\|$$

for a universal constant C with $1 < C < 3$. Here $\| \quad \|$ is the operator norm.

Finally, it remains to extend the local choices to global ones for the cases (B) and (C) only. There $t \mapsto A(t)$ is C^Q or C^L, respectively, which imply both C^∞, and no two different eigenvalues meet of infinite order. So we may apply [AKLM98, 7.8] (in fact we need only the end of the proof) to conclude that the eigenvalues can be chosen C^∞ on $T = \mathbb{R}$, uniquely up to a global permutation. By the local result above they are then C^Q or C^L. The same proof then gives us, for each eigenvalue $\lambda_i : T \to \mathbb{R}$ with generic multiplicity N, a unique N-dimensional smooth vector subbundle of $\mathbb{R} \times H$ whose fiber over t consists of eigenvectors for the eigenvalue $\lambda_i(t)$. In fact this vector bundle is C^Q or C^L by the local result above, namely the matrix proposition, part (e) or (f), respectively. □

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