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THE CONVENIENT SETTING FOR QUASIANALYTIC DENJOY–CARLEMAN DIFFERENTIABLE MAPPINGS

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Abstract. For quasianalytic Denjoy–Carleman differentiable function classes C^Q where the weight sequence $Q = (Q_k)$ is log-convex, stable under derivations, of moderate growth and also an $\mathcal{L}$-intersection (see (1.6)), we prove the following: The category of C^Q-mappings is cartesian closed in the sense that $C^Q(E, C^Q(F,G)) \cong C^Q(E \times F, G)$ for convenient vector spaces. Applications to manifolds of mappings are given: The group of C^Q-diffeomorphisms is a regular C^Q-Lie group but not better.

Classes of Denjoy-Carleman differentiable functions are in general situated between real analytic functions and smooth functions. They are described by growth conditions on the derivatives. Quasianalytic classes are those where infinite Taylor expansion is an injective mapping.

That a class of mappings $\mathcal{S}$ admits a convenient setting means essentially that we can extend the class to mappings between admissible infinite dimensional spaces $E, F, \dots$ so that $\mathcal{S}(E,F)$ is again admissible and we have $\mathcal{S}(E \times F, G)$ canonically $\mathcal{S}$-diffeomorphic to $\mathcal{S}(E, \mathcal{S}(F,G))$ (the exponential law). Usually this comes hand in hand with (partly nonlinear) uniform boundedness theorems which are easy $\mathcal{S}$-detection principles.

For the C^∞ convenient setting one can test smoothness along smooth curves. For the real analytic (C^ω) convenient setting we have: A mapping is C^ω if and only if it is C^∞ and in addition C^ω along C^ω-curves (C^ω along just affine lines suffices). We shall use convenient calculus of C^∞ and C^ω mappings in this paper; see the book [15], or the three appendices in [17] for a short overview.

In [17] we succeeded to show that non-quasianalytic log-convex Denjoy-Carleman classes C^M of moderate growth (hence derivation closed) admit a convenient setting, where the underlying admissible locally convex vector spaces are the same as for smooth or for real analytic mappings. A mapping is C^M if and only if it is C^M along all C^M-curves. The method of proof there relies on the existence of C^M partitions of unity.

In this paper we succeed to prove that quasianalytic log-convex Denjoy-Carleman classes C^Q of moderate growth which are also $\mathcal{L}$-intersections (see (1.6)), admit a convenient setting. The method consists of representing C^Q as the intersection $\bigcap\{C^L : L \in \mathcal{L}(Q)\}$ of all larger non-quasianalytic log-convex classes C^L; this is the meaning of: Q is an $\mathcal{L}$-intersection. In (1.9) we construct countably many classes Q which satisfy all these requirements. Taking intersections of derivation closed classes C^L only, or only of classes C^L of moderate growth, is not sufficient for yielding the intended results. Thus we have to strengthen many results from [17] before we are able to prove the exponential law. A mapping is C^Q if and only if

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it is C^L along each C^L-curve for each $L \in \mathcal{L}(Q)$. It is an open problem (even in $\mathbb{R}^2$), whether a smooth mapping which is C^Q along each C^Q-curve (or affine line), is indeed C^Q. As replacement we show that a mapping is C^Q if it is C^Q along each C^Q mapping from a Banach ball (5.2). The real analytic case from [14] is not covered by this approach.

The initial motivation of both [17] and this paper was the desire to prove the following result which is due to Rellich [19] in the real analytic case. *Let $t \mapsto A(t)$ for $t \in \mathbb{R}$ be a curve of unbounded self-adjoint operators in a Hilbert space with common domain of definition and with compact resolvent. If $t \mapsto A(t)$ is of a certain quasianalytic Denjoy-Carleman class C^Q, then the eigenvalues and the eigenvectors of $A(t)$ may be parameterized C^Q in t also.* We manage to prove this with the help of the results in this paper and in [17]. Due to length this will be explained in another paper [18].

Generally, one can hope that the space $C^M(A,B)$ of all Denjoy-Carleman C^M-mappings between finite dimensional C^M-manifolds (with A compact for simplicity) is again a C^M-manifold, that composition is C^M, and that the group $\operatorname{Diff}^M(A)$ of all C^M-diffeomorphisms of A is a regular infinite dimensional C^M-Lie group, for each class C^M which admits a convenient setting. For the non-quasianalytic classes this was proved in [17]. For quasianalytic classes this is proved in this paper.

1. Weight Sequences and function spaces

1.1. **Denjoy–Carleman C^M-functions in finite dimensions.** We mainly follow [17] and [25] (see also the references therein). We use $\mathbb{N} = \mathbb{N}_{>0} \cup \{0\}$. For each multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we write $\alpha! = \alpha_1! \cdots \alpha_n!$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$.

Let $M = (M_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers. Let $U \subseteq \mathbb{R}^n$ be open. We denote by $C^M(U)$ the set of all $f \in C^\infty(U)$ such that, for all compact $K \subseteq U$, there exist positive constants C and ρ such that

$$|\partial^\alpha f(x)| \le C\,\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|} \quad \text{for all } \alpha \in \mathbb{N}^n \text{ and } x \in K.$$

The set $C^M(U)$ is a *Denjoy–Carleman class* of functions on U. If $M_k = 1$, for all k, then $C^M(U)$ coincides with the ring $C^\omega(U)$ of real analytic functions on U.

A sequence $M = (M_k)$ is *log-convex* if $k \mapsto \log(M_k)$ is convex, i.e.,

$$M_k^2 \le M_{k-1}\,M_{k+1} \quad \text{for all } k.$$

If $M = (M_k)$ is log-convex, then $k \mapsto (M_k/M_0)^{1/k}$ is increasing and

$$M_l\,M_k \le M_0\,M_{l+k} \quad \text{for all } l,k \in \mathbb{N}. \tag{1}$$

Furthermore, we have that $k \mapsto k!M_k$ is log-convex (since Euler's Γ-function is so), and we call this weaker condition *weakly log-convex*. If M is weakly log-convex then $C^M(U,\mathbb{R})$ is a ring, for all open subsets $U \subseteq \mathbb{R}^n$.

If M is log-convex then (see the proof of [17, 2.9]) we have

$$M_1^j\,M_k \ge M_j\,M_{\alpha_1} \cdots M_{\alpha_j} \quad \text{for all } \alpha_i \in \mathbb{N}_{>0} \text{ with } \alpha_1 + \cdots + \alpha_j = k. \tag{2}$$

This implies that the class of C^M-mappings is stable under composition ([20], see also [2, 4.7]; this also follows from (1.4)). If M is log-convex then the inverse function theorem for C^M holds ([12]; see also [2, 4.10]), and C^M is closed under solving ODEs (due to [13]).

Suppose that $M = (M_k)$ and $N = (N_k)$ satisfy $M_k \le C^k\,N_k$, for a constant C and all k. Then $C^M(U) \subseteq C^N(U)$. The converse is true if M is weakly log-convex: There exists $f \in C^M(\mathbb{R})$ such that $|f^{(k)}(0)| \ge k!\,M_k$ for all k (see [25, Theorem 1]).

If M is weakly log-convex then C^M is *stable under derivations* (alias *derivation closed*) if and only if

$$\sup_{k\in\mathbb{N}_{>0}}\Big(\frac{M_{k+1}}{M_k}\Big)^{\frac{1}{k}}<\infty. \tag{3}$$

A weakly log-convex sequence M is called of *moderate growth* if

$$\sup_{j,k\in\mathbb{N}_{>0}}\Big(\frac{M_{j+k}}{M_j\,M_k}\Big)^{\frac{1}{j+k}}<\infty. \tag{4}$$

Moderate growth implies derivation closed.

Definition. A sequence $M=(M_k)_{k=0,1,2,\dots}$ is called a *weight sequence* if it satisfies $M_0=1\le M_1$ and is log-convex. Consequently, it is increasing (i.e. $M_k\le M_{k+1}$).

A *DC-weight sequence* $M=(M_k)_{k=0,1,2,\dots}$ is a weight sequence which is also derivation closed (DC stands for Denjoy-Carleman and also for derivation closed). This was the notion investigated in [17].

1.2. **Theorem** (Denjoy–Carleman [6], [5])**.** *For a sequence M of positive numbers the following statements are equivalent.*

1. *C^M is quasianalytic, i.e., for open connected $U\subseteq\mathbb{R}^n$ and each $a\in U$, the Taylor series homomorphism centered at a from $C^M(U,\mathbb{R})$ into the space of formal power series is injective.*
2. *$\sum_{k=1}^\infty \frac{1}{m_k^{\flat(i)}}=\infty$ where $m_k^{\flat(i)}:=\inf\{(j!\,M_j)^{1/j}:j\ge k\}$ is the increasing minorant of $(k!\,M_k)^{1/k}$.*
3. *$\sum_{k=1}^\infty(\frac{1}{M_k^{\flat(lc)}})^{1/k}=\infty$ where $M_k^{\flat(lc)}$ is the log-convex minorant of $k!\,M_k$, given by $M_k^{\flat(lc)}:=\inf\{(j!\,M_j)^{\frac{l-k}{l-j}}(l!\,M_l)^{\frac{k-j}{l-j}}:j\le k\le l, j<l\}$.*
4. *$\sum_{k=0}^\infty\frac{M_k^{\flat(lc)}}{M_{k+1}^{\flat(lc)}}=\infty$.*

For contemporary proofs see for instance [11, 1.3.8] or [22, 19.11].

1.3. **Sequence spaces.** Let $M=(M_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers and $\rho>0$. We consider (where $\mathcal{F}$ stands for 'formal power series')

$$\mathcal{F}^M_\rho:=\Big\{(f_k)_{k\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}:\exists C>0\,\forall k\in\mathbb{N}:|f_k|\le C\,\rho^k\,k!\,M_k\Big\}\text{ and }\mathcal{F}^M:=\bigcup_{\rho>0}\mathcal{F}^M_\rho.$$

Note that, for $U\subseteq\mathbb{R}^n$ open, a function $f\in C^\infty(U,\mathbb{R})$ is in $C^M(U,\mathbb{R})$ if and only if for each compact $K\subset U$

$$(\sup\{|\partial^\alpha f(x)|:x\in K,|\alpha|=k\})_{k\in\mathbb{N}}\in\mathcal{F}^M.$$

Lemma. *We have*

$$\begin{aligned}\mathcal{F}^{M^1}\subseteq\mathcal{F}^{M^2}&\Leftrightarrow\exists\rho>0\,\forall k:M^1_k\le\rho^{k+1}\,M^2_k\\&\Leftrightarrow\exists C,\rho>0\,\forall k:M^1_k\le C\,\rho^k\,M^2_k.\end{aligned}$$

Proof. ($\Rightarrow$) Let $f_k:=k!M^1_k$. Then $f=(f_k)_{k\in\mathbb{N}}\in\mathcal{F}^{M^1}\subseteq\mathcal{F}^{M^2}$, so there exists a $\rho>0$ such that $k!M^1_k\le\rho^{k+1}k!M^2_k$ for all k.

($\Leftarrow$) Let $f=(f_k)_{k\in\mathbb{N}}\in\mathcal{F}^{M^1}$, i.e. there exists a $\sigma>0$ with $|f_k|\le\sigma^{k+1}k!M^1_k\le(\rho\sigma)^{k+1}k!M^2_k$ for all k and thus $f\in\mathcal{F}^{M^2}$. □

1.4. **Lemma.** *Let M and L be sequences of positive numbers. Then for the composition of formal power series we have*

$$\mathcal{F}^M\circ\mathcal{F}^L_{>0}\subseteq\mathcal{F}^{M\circ L}$$

where $(M\circ L)_k:=\max\{M_jL_{\alpha_1}\dots L_{\alpha_j}:\alpha_i\in\mathbb{N}_{>0},\alpha_1+\dots+\alpha_j=k\}$

Here $\mathcal{F}^L_{>0} := \{(g_k)_{k\in\mathbb{N}} \in \mathcal{F}^L : g_0 = 0\}$ is the space of formal power series in $\mathcal{F}^L$ with vanishing constant term.

Proof. Let $f \in \mathcal{F}^M$ and $g \in \mathcal{F}^L$. For $k > 0$ we have (inspired by [7])

$$\begin{aligned}
\frac{(f\circ g)_k}{k!} &= \sum_{j=1}^k \frac{f_j}{j!} \sum_{\substack{\alpha\in\mathbb{N}^j_{>0}\\ \alpha_1+\dots+\alpha_j=k}} \frac{g_{\alpha_1}}{\alpha_1!}\cdots\frac{g_{\alpha_j}}{\alpha_j!}\\
\frac{|(f\circ g)_k|}{k!(M\circ L)_k} &\le \sum_{j=1}^k \frac{|f_j|}{j!M_j} \sum_{\substack{\alpha\in\mathbb{N}^j_{>0}\\ \alpha_1+\dots+\alpha_j=k}} \frac{|g_{\alpha_1}|}{\alpha_1!L_{\alpha_1}}\cdots\frac{|g_{\alpha_j}|}{\alpha_j!L_{\alpha_j}}\\
&\le \sum_{j=1}^k \rho_f^j C_f \sum_{\substack{\alpha\in\mathbb{N}^j_{>0}\\ \alpha_1+\dots+\alpha_j=k}} \rho_g^k C_g^j \le \sum_{j=1}^k \rho_f^j C_f \binom{k-1}{j-1}\rho_g^k C_g^j\\
&= \rho_g^k \rho_f C_f C_g \sum_{j=1}^k (\rho_f C_g)^{j-1}\binom{k-1}{j-1} = \rho_g^k\rho_f C_f C_g(1+\rho_f C_g)^{k-1}\\
&= (\rho_g(1+\rho_f C_g))^k \frac{\rho_f C_f C_g}{1+\rho_f C_g} \qquad \square
\end{aligned}$$

1.5. **Notation for quasianalytic weight sequences.** Let M be a sequence of positive numbers. We may replace M by $k \mapsto C\,\rho^k\,M_k$ with $C, \rho > 0$ without changing $\mathcal{F}^M$. In particular, it is no loss of generality to assume that $M_1 > 1$ (put $C\rho > 1/M_1$) and $M_0 = 1$ (put $C := 1/M_0$). If M is log-convex then so is the modified sequence and if in addition $\rho \ge M_0/M_1$ then the modified sequence is monotone increasing. Furthermore M is quasianalytic if and only if the modified sequence is so, since $M_k^{\flat(lc)}$ is modified in the same way. We tried to make all conditions equivariant under this modification. Unfortunately, the next construction does not react nicely to this modification.

For a quasianalytic sequence $M = (M_k)$ let the sequence $\check M = (\check M_k)$ be defined by

$$\check M_k := M_k \prod_{j=1}^k \left(1 - \frac{1}{(j!\,M_j)^{1/j}}\right)^k, \quad \check M_0 = 1.$$

We have $\check M_k \le M_k$. Note that if we put $m_k := (k!M_k)^{1/k}$ (and $m_0 := 1$) and $\check m_k := (k!\check M_k)^{1/k}$ (where we assume $\check M_k \ge 0$) then

$$\check m_k = m_k \prod_{j=1}^k \left(1 - \frac{1}{m_j}\right)$$

or, recursively,

$$\check m_{k+1} = \check m_k \frac{m_{k+1}-1}{m_k} \text{ and } \check m_0 = 1,\ \check m_1 = m_1 - 1.$$

And conversely, if all $\check M_k > 0$ (this is the case if M is increasing and $M_1 > 1$) then

$$m_{k+1} = 1 + m_k\frac{\check m_{k+1}}{\check m_k} \text{ and } m_0 = 1,\ m_1 = \check m_1 + 1$$

i.e.

$$m_k = \check m_k\Big(1 + \sum_{j=1}^k \frac{1}{\check m_j}\Big). \tag{1}$$

For sequences M we define (recall from (1.1) that M is called weakly log-convex if $k \mapsto \log(k!\,M_k)$ is convex):

$$\begin{aligned}\mathcal{L}(M) &:= \{L \ge M : L \text{ non-quasianalytic, log-convex}\}\\ \mathcal{L}_w(M) &:= \{L \ge M : L \text{ non-quasianalytic, weakly log-convex}\} \supseteq \mathcal{L}(M)\end{aligned}$$

1.6. **Theorem.** *Let $Q = (Q_k)_{k=0,1,2,\dots}$ be a quasianalytic sequence of positive real numbers. Then we have:*

(1) *If the sequence $\check{Q} = (\check{Q}_k)$ is log-convex and positive then*
$$\mathcal{F}^Q = \bigcap_{L\in\mathcal{L}(Q)} \mathcal{F}^L.$$
(2) *If Q is weakly log-convex, then for each $L^1, L^2 \in \mathcal{L}_w(Q)$ there exists an $L \in \mathcal{L}_w(Q)$ with $L \le L^1, L^2$.*
(3) *If Q is weakly log-convex of moderate growth, then for each $L \in \mathcal{L}_w(Q)$ there exists an $L' \in \mathcal{L}_w(Q)$ such that $L'_{j+k} \le C^{j+k} L_j L_k$ for some positive constant C and all $j,k \in \mathbb{N}$.*

We could not obtain (2) for log-convex instead of weakly log-convex, in particular for $\mathcal{L}(Q)$ instead of $\mathcal{L}_w(Q)$.

Definition. A quasianalytic sequence Q of positive real numbers is called $\mathcal{L}$-*intersectable* or an $\mathcal{L}$-*intersection* if $\mathcal{F}^Q = \bigcap_{L\in\mathcal{L}(Q)} \mathcal{F}^L$ holds.

Note that we may replace any non-quasianalytic weight sequence L for which $k \mapsto (\frac{Q_k}{L_k})^{1/k}$ is bounded, by an $\tilde{L} \in \mathcal{L}(Q)$ with $\mathcal{F}^{\tilde{L}} = \mathcal{F}^L$: Choose $\rho \ge 1/L_1$ (see (1.5)) and $\rho \ge \sup\{(\frac{Q_k}{L_k})^{1/k} : k \in \mathbb{N}\}$ then $\tilde{L}_k := \rho^k L_k \ge Q_k$.

Proof. (1) The proof is partly adapted from [3].

Let $q_k = (k!\,Q_k)^{1/k}$ and $q_0 = 1$, similarly $\check{q}_k = (k!\,\check{Q}_k)^{1/k}$, $l_k = (k!L_k)^{1/k}$, etc.. Then $\check{q}$ is increasing since $\check{Q}_0 = 1$, and $\check{Q}$ and the Gamma function are log-convex.

Clearly $\mathcal{F}^Q \subseteq \bigcap_{L\in\mathcal{L}(Q)} \mathcal{F}^L$. To show the converse inclusion, let $f \notin \mathcal{F}^Q$ and $g_k := |f_k|^{1/k}$. Then
$$\overline{\lim}\, \frac{g_k}{q_k} = \infty.$$
Choose $a_j, b_j > 0$ with $a_j \nearrow \infty$, $b_j \searrow 0$, and $\sum \frac{1}{a_j b_j} < \infty$. There exist strictly increasing k_j such that $\frac{g_{k_j}}{q_{k_j}} \ge a_j$. Since $\frac{q_k}{\check{q}_k}$ is increasing by (1.5.1) we get $b_j \frac{g_{k_j}}{\check{q}_{k_j}} = b_j \frac{g_{k_j}}{q_{k_j}} \frac{q_{k_j}}{\check{q}_{k_j}} \ge a_j b_j \frac{q_{k_1}}{\check{q}_{k_1}} \to \infty$. Passing to a subsequence we may assume that $k_0 > 0$ and $1 < \beta_j := b_j \frac{g_{k_j}}{\check{q}_{k_j}} \nearrow \infty$. Passing to a subsequence again we may also get

$$\beta_{j+1} \ge (\beta_j)^{k_j}\,. \tag{4}$$

Define a piecewise affine function ϕ by
$$\phi(k) := \begin{cases} 0 & \text{if } k = 0,\\ k_j \log\beta_j & \text{if } k = k_j,\\ c_j + d_j k & \text{for the minimal } j \text{ with } k \le k_j,\end{cases}$$
where c_j and d_j are chosen such that ϕ is well defined and $\phi(k_{j-1}) = c_j + d_j k_{j-1}$, i.e., for $j \ge 1$,

$$\begin{aligned} c_j + d_j k_j &= k_j \log\beta_j,\\ c_j + d_j k_{j-1} &= k_{j-1}\log\beta_{j-1}, \qquad \text{and}\\ c_0 &= 0,\end{aligned} \tag{5}$$

$$d_0 = \log \beta_0.$$

This implies first that $c_j \le 0$ and then

$$\log \beta_j \le d_j = \frac{k_j \log \beta_j - k_{j-1} \log \beta_{j-1}}{k_j - k_{j-1}} \le \frac{k_j}{k_j - k_{j-1}} \log \beta_j \tag{6}$$
$$\overset{(4)}{\le} \frac{\log \beta_{j+1}}{k_j - k_{j-1}} \le \log \beta_{j+1}.$$

Thus $j \mapsto d_j$ is increasing. It follows that ϕ is convex. The fact that all $c_j \le 0$ implies that $\phi(k)/k$ is increasing.

Now let

$$L_k := e^{\phi(k)} \cdot \check{Q}_k.$$

Then $L = (L_k)$ is log-convex and satisfies $L_0 = 1$ by construction and $f \notin \mathcal{F}^L$, since we have $\frac{l_{k_j}}{g_{k_j}} = \frac{\check{q}_{k_j} \beta_j}{g_{k_j}} = b_j \to 0$ and so $\overline{\lim}\, \frac{g_k}{l_k} = \infty$.

Let us check that L is not quasianalytic. By (6) and since $(\check{q}_k)$ is increasing, we have, for $k_{j-1} \le k < k_j$,

$$\frac{L_k}{(k+1) L_{k+1}} = \frac{e^{\phi(k)-\phi(k+1)}\, \check{Q}_k}{(k+1)\, \check{Q}_{k+1}} = \frac{e^{\phi(k)-\phi(k+1)}\, \check{q}_k^k}{\check{q}_{k+1}^{k+1}} = e^{-d_j} \frac{\check{q}_k^k}{\check{q}_{k+1}^{k+1}}$$
$$\le \frac{1}{\beta_j\, \check{q}_k} = \frac{\check{q}_{k_j}}{b_j g_{k_j}} \frac{1}{\check{q}_k}.$$

Thus, by (1.5.1),

$$\sum_{k=k_{j-1}}^{k_j - 1} \frac{L_k}{(k+1) L_{k+1}} \le \frac{\check{q}_{k_j}}{b_j g_{k_j}} \sum_{k=k_{j-1}}^{k_j - 1} \frac{1}{\check{q}_k} \le \frac{q_{k_j}}{b_j g_{k_j}} \le \frac{1}{a_j b_j},$$

which shows that L is not quasianalytic and $C_1 := \sum_{k=1}^\infty \frac{1}{l_k} < \infty$ by (1.2).

Next we claim that $\mathcal{F}^Q \subseteq \mathcal{F}^L$. Since $\frac{l_k}{\check{q}_k} = \frac{(k! L_k)^{1/k}}{(k! \check{Q}_k)^{1/k}} = e^{\phi(k)/k}$ is increasing, we have

$$\infty > \frac{\check{q}_1}{l_1} + C_1 > \frac{\check{q}_1}{l_1} + \sum_{j=1}^k \frac{1}{l_j} = \frac{\check{q}_1}{l_1} + \sum_{j=1}^k \frac{\check{q}_j}{l_j} \frac{1}{\check{q}_j} \ge \frac{\check{q}_k}{l_k} \Big(1 + \sum_{j=1}^k \frac{1}{\check{q}_j} \Big) = \frac{q_k}{l_k},$$

which proves $\mathcal{F}^Q \subseteq \mathcal{F}^L$. Finally we may replace L by some $L \in \mathcal{L}(Q)$ without changing $\mathcal{F}^L$ by the remark before the proof. Thus (1) is proved.

(2) Assume without loss that $L_0^1 = L_0^2 = 1$. Let $k! L_k$ be the log-convex minorant of $k! \bar{L}_k$ where $\bar{L}_k := \min\{L_k^1, L_k^2\}$. Since $L^1, L^2 \ge \bar{L} \ge Q$ and $k! Q_k$ is log-convex we have $L^1, L^2 \ge L \ge Q$. Since L^1, L^2 are not quasianalytic and are weakly log-convex (hence $k \mapsto (k! L_k^j)^{1/k}$ is increasing), we get that $k \mapsto (k! \bar{L}_k)^{1/k}$ is increasing and

$$\sum_k \frac{1}{(k!\, \bar{L}_k)^{1/k}} \le \sum_k \frac{1}{(k!\, L_k^1)^{1/k}} + \sum_k \frac{1}{(k!\, L_k^2)^{1/k}} < \infty.$$

By (1.2, 2⇒1) we get that $\bar{L}$ is not quasianalytic. By (1.2, 1⇒3) we get $\sum_k \frac{1}{(k! L_k)^{1/k}} < \infty$ since $\bar{L}^{\flat(lc)} = L$, i.e. L is not quasianalytic.

(3) Let $\tilde{Q}_k := k! Q_k$, $\tilde{L}_k := k! L_k$, and so on. Since Q is of moderate growth we have

$$C_{\tilde{Q}} := \sup_{k,j} \left(\frac{\tilde{Q}_{k+j}}{\tilde{Q}_k \tilde{Q}_j} \right)^{1/(k+j)} \le 2 \sup_{k,j} \left(\frac{Q_{k+j}}{Q_k Q_j} \right)^{1/(k+j)} < \infty.$$

Let $L \in \mathcal{L}_w(Q)$; without loss we assume that $L_0 = 1$. We put

$$\tilde{L}_k' := C_{\tilde{Q}}^k \min\{\tilde{L}_j \tilde{L}_{k-j} : j = 0, \dots, k\} = C_{\tilde{Q}}^k \min\{\tilde{L}_j \tilde{L}_{k-j} : 0 \le j \le k/2\}.$$

Then

$$\sup_{k,j}\left(\frac{L'_{k+j}}{L_k L_j}\right)^{1/(k+j)} \le \sup_{k,j}\left(\frac{\tilde L'_{k+j}}{\tilde L_k \tilde L_j}\right)^{1/(k+j)} \le C_{\tilde Q} < \infty.$$

Since $\tilde L$ is log-convex we have $\tilde L_k^2 \le \tilde L_j \tilde L_{2k-j}$ and $\tilde L_k \tilde L_{k+1} \le \tilde L_j \tilde L_{2k+1-j}$ for $j = 0,\dots,k$; therefore $\tilde L'_{2k} = C_{\tilde Q}^{2k}\tilde L_k^2$ and $\tilde L'_{2k+1} = C_{\tilde Q}^{2k+1}\tilde L_k \tilde L_{k+1}$. It is easy to check that $\tilde L'$ is log-convex. To see that L' is not quasianalytic we will use that $(\tilde L'_k)^{1/k}$ is increasing since $\tilde L'$ is log-convex. So it suffices to compute the sum of the even indices only.

$$\sum_k \frac{1}{\tilde L'_{2k}{}^{1/(2k)}} = \frac{1}{C_{\tilde Q}}\sum_k \frac{1}{\tilde L_k{}^{1/k}} < \infty.$$

It remains to show that $L' \ge Q$. Since $L \in \mathcal{L}_w(Q)$ we have $Q \le L$ and for $j = \lfloor k/2 \rfloor$,

$$\frac{Q_k}{L'_k} = \frac{\tilde Q_k}{\tilde L'_k} = \frac{\tilde Q_k}{C_{\tilde Q}^k \tilde L_j \tilde L_{k-j}} \le \frac{\tilde Q_k}{\frac{\tilde Q_k}{\tilde Q_j \tilde Q_{k-j}}\tilde L_j \tilde L_{k-j}} \le \frac{\tilde Q_j}{\tilde L_j}\frac{\tilde Q_{k-j}}{\tilde L_{k-j}} \le 1. \quad \square$$

1.7. **Corollary.** *Let Q be a quasianalytic weight sequence. Then*

$$\mathcal{F}^Q = \bigcap_{L \in \mathcal{L}_w(Q)} \mathcal{F}^L.$$

Proof. Without loss we may assume that the sequence $\check q_k$ is increasing. Namely, by definition this is the case if and only if $q_k \le q_{k+1} - 1$. Since $Q_0 = 1$ and (Q_k) is log-convex, $Q_k^{1/k}$ is increasing and thus $q_{k+1} - q_k \ge Q_k^{\frac{1}{k}}((k+1)!^{\frac{1}{k+1}} - k!^{\frac{1}{k}}) \ge Q_1 \frac{1}{e} \ge \frac{1}{e}$. If we set $\tilde Q_k := e^k Q_k$, then $\tilde Q = (\tilde Q_k)$ is a quasianalytic weight sequence with $\tilde Q_1 > 1$, $\mathcal{F}^{\tilde Q} = \mathcal{F}^Q$, and $\check{\tilde q}_k$ is increasing.

Now a little adaptation of the proof of (1.6.1) shows the corollary: Define here

$$l_k := \beta_j \check q_k \qquad \text{for the minimal } j \text{ with } k \le k_j.$$

Then $\frac{l_{k_j}}{g_{k_j}} = \frac{\beta_j \check q_{k_j}}{g_{k_j}} = b_j \to 0$ and so $\overline{\lim}\, \frac{g_k}{l_k} = \infty$. We have

$$\sum_{k=k_{j-1}+1}^{k_j} \frac{1}{l_k} = \sum_{k=k_{j-1}+1}^{k_j} \frac{1}{\beta_j \check q_k} = \frac{\check q_{k_j}}{b_j g_{k_j}} \sum_{k=k_{j-1}+1}^{k_j} \frac{1}{\check q_k} \le \frac{q_{k_j}}{b_j g_{k_j}} \le \frac{1}{a_j b_j}$$

and thus $\sum_{k=1}^\infty \frac{1}{l_k} < \infty$. As l_k is increasing, the Denjoy–Carleman theorem (1.2) implies that $L_k = \frac{l_k^k}{k!}$ is non-quasianalytic. Since $\frac{l_k}{\check q_k} = \beta_j$ is increasing, we find (as in the proof of (1.6.1)) that $C := \max\{L_0/L_1, \sup_k \frac{q_k}{l_k}\} < \infty$. Replacing L_k by $C^k L_k$ we may assume that $Q \le L$. Let the sequence $k!\underline{L}_k$ be the log-convex minorant of $k! L_k$. Since Q_k is (weakly) log-convex, we have $Q \le \underline{L}$. By (1.2) and the fact that L is non-quasianalytic, $\underline{L}$ is non-quasianalytic as well. Thus $\underline{L} \in \mathcal{L}_w(Q)$ and still $f \notin \mathcal{F}^{\underline{L}}$. $\square$

Corollary (1.7) implies that for the sequence $\omega = (1)_k$ describing real analytic functions we have $\mathcal{F}^\omega = \bigcap_{L \in \mathcal{L}_w(\omega)} \mathcal{F}^L$. Note that $\mathcal{L}_w(\omega)$ consists of all weakly log-convex non-quasianalytic $L \ge 1$. This is slightly stronger than a result by T. Bang, who shows that $\mathcal{F}^\omega = \bigcap \mathcal{F}^L$ where L runs through all non-quasianalytic sequences with $l_k = (k!\, L_k)^{1/k}$ increasing, see [1], [3].

This result becomes wrong if we replace weakly log-convex by log-convex:

1.8. **The intersection of all $\mathcal{F}^L$, where L is any non-quasianalytic weight sequence.** Put

$$Q_k := \frac{(k\log(k+e))^k}{k!}, \quad Q_0 := 1.$$

Then $Q = (Q_k)$ is a quasianalytic weight sequence of moderate growth with $Q_1 > 1$. We claim that Q is $\mathcal{L}$-intersectable, i.e., $\mathcal{F}^Q = \bigcap_{L\in\mathcal{L}(Q)} \mathcal{F}^L$. We could check that $\check{Q}$ is log-convex. This can be done, but is quite cumbersome. A simpler argument is the following. We consider $\check{q}'_k := k$, $\check{q}'_0 := 1$. Then $\check{Q}'_k = k^k/k!$ is log-convex. Since $C_1 \log k \le \sum_{j=1}^k \frac{1}{j} \le C_2 \log k$, we have by (1.5.1)

$$C_3 k\log(k+e) \le q'_k \le C_4 k \log(k+e)$$

for suitable constants C_i. Hence $\mathcal{F}^Q = \mathcal{F}^{Q'}$. By theorem (1.6.1) we have

$$\mathcal{F}^Q = \mathcal{F}^{Q'} = \bigcap_{L\in\mathcal{L}(Q')} \mathcal{F}^L = \bigcap_{L\in\mathcal{L}(Q)} \mathcal{F}^L$$

since $\mathcal{L}(Q)$ and $\mathcal{L}(Q')$ contain only sequences which are "equivalent mod (ρ^k)". The claim is proved.

Let L be any non-quasianalytic weight sequence. Consider

$$\alpha_k := \frac{(k!\,L_k)^{\frac{1}{k}}}{k} = \frac{l_k}{k}.$$

Since L is log-convex and $L_0 = 1$, we find that $L_k^{1/k}$ is increasing. Thus, for $s \le k$ we find

$$\frac{\alpha_s}{\alpha_k} = \frac{k}{s} \cdot \frac{s!^{1/s}}{k!^{1/k}} \cdot \frac{L_s^{1/s}}{L_k^{1/k}} \le 2e$$

(using Stirling's formula for instance). Since L is not quasianalytic, we have $\sum_{k=1}^\infty \frac{1}{k\alpha_k} < \infty$. But

$$\sum_{\sqrt{k}\le s\le k} \frac{1}{s\alpha_s} \ge \frac{1}{2e}\cdot\frac{1}{\alpha_k} \sum_{\sqrt{k}\le s\le k} \frac{1}{s} \sim \frac{1}{2e}\cdot\frac{1}{\alpha_k}\cdot\frac{\log k}{2}.$$

The sum on the left tends to 0 as $k \to \infty$. So $\frac{\log k}{\alpha_k} = \frac{k\log k}{l_k}$ is bounded. Thus $\mathcal{F}^Q \subseteq \mathcal{F}^L$.

So we have proved the following theorem (which is intimately related to [21, Thm. C]).

Theorem. *Put $Q_k = (k\log(k+e))^k/k!$, $Q_0 = 1$. Then Q is $\mathcal{L}$-intersectable. In fact,*

$$\mathcal{F}^Q = \bigcap\{\mathcal{F}^L : L \text{ non-quasianalytic weight sequence}\}. \quad \square$$

Remark. Log-convexity of $\check{Q}$ is only sufficient for Q being an $\mathcal{L}$-intersection, see (1.6.1): Using Stirling's formula we see that $\mathcal{F}^Q = \mathcal{F}^{Q''}$ for $Q_k = (k\log(k+e))^k/k!$ and $Q''_k = (\log(k+e))^k$. Also $\mathcal{L}(Q)$ and $\mathcal{L}(Q'')$ contain only sequences which are "equivalent mod (ρ^k)" and (1.6.1) holds for Q, thus also for Q''. But $\check{Q}''$ is not log-convex.

1.9. **A class of examples.** Let $\log^n$ denote the n-fold composition of log defined recursively by

$$\log^1 := \log,$$
$$\log^n := \log\circ\log^{n-1}, \quad (n \ge 2).$$

For $0 < \delta \le 1$, $n \in \mathbb{N}_{>0}$, we recursively define sequences $q^{\delta,n} = (q_k^{\delta,n})_{k\ge\kappa_n}$ by

$$q_k^{1,1} := k\log k,$$

$$q_k^{\delta,n} := q_k^{1,n-1} \cdot (\log^n(k))^\delta, \quad (n \geq 2),$$

where κ_n is the smallest integer greater than $e \uparrow\uparrow n$, i.e.,

$$\kappa_n := \lceil e \uparrow\uparrow n \rceil, \quad e \uparrow\uparrow n := \underbrace{e^{e^{\cdot^{\cdot^{e}}}}}_{n \text{ times}}.$$

Let $Q^{\delta,n} := (Q_k^{\delta,n})_{k\in\mathbb{N}}$ with

$$Q_0^{\delta,n} := 1,$$
$$Q_k^{\delta,n} := \frac{1}{(k-1+\kappa_n)!}(q_{k-1+\kappa_n}^{\delta,n})^{k-1+\kappa_n}, \quad (k \geq 1),$$

and consider

$$\mathcal{Q} := \{Q^{1,1}\} \cup \{Q^{\delta,n} : 0 < \delta \leq 1, n \in \mathbb{N}_{>1}\}.$$

It is easy to check inductively that each $Q \in \mathcal{Q}$ is a quasianalytic weight sequence of moderate growth with $Q_1 > 1$. Namely, $(\log^n(k))^{\delta k}$ is increasing, log-convex, and has moderate growth. Quasianalyticity follows from Cauchy's condensation criterion or the integral test. By construction, $\mathcal{Q} \ni Q \mapsto \mathcal{F}^Q$ is injective.

Let us consider

$$\hat{q}_k^{1,n} := q_k^{1,n-1}\left(1 + \sum_{j=\kappa_n}^{k} \frac{1}{q_j^{1,n-1}}\right).$$

Since $\frac{d}{dx}\log^n(x) = \frac{1}{x\log(x)\cdots\log^{n-1}(x)}$, we have (by comparison with the corresponding integral)

$$C_1 \log^n(k) \leq \sum_{j=\kappa_n}^{k} \frac{1}{q_j^{1,n-1}} \leq C_2 \log^n(k)$$

and thus

$$C_3 q_k^{1,n} \leq \hat{q}_k^{1,n} \leq C_4 q_k^{1,n} \tag{1}$$

for suitable constants C_i. Hence $\mathcal{F}^{Q^{1,n}} = \mathcal{F}^{\hat{Q}^{1,n}}$. Since $Q^{1,n-1}$ is log-convex, theorem (1.6.1) implies

$$\mathcal{F}^{Q^{1,n}} = \mathcal{F}^{\hat{Q}^{1,n}} = \bigcap_{L\in\mathcal{L}(\hat{Q}^{1,n})} \mathcal{F}^L = \bigcap_{L\in\mathcal{L}(Q^{1,n})} \mathcal{F}^L$$

since $\mathcal{L}(\hat{Q}^{1,n})$ and $\mathcal{L}(Q^{1,n})$ contain only sequences which are "equivalent mod (ρ^k)".

Hence we have proved (the case $n = 1$ follows from (1.8)):

Theorem. *Each $Q^{1,n}$ ($n \in \mathbb{N}_{>0}$) is a quasianalytic weight sequence of moderate growth which is an $\mathcal{L}$-intersection, i.e.,*

$$\mathcal{F}^{Q^{1,n}} = \bigcap_{L\in\mathcal{L}(Q^{1,n})} \mathcal{F}^L. \quad \square$$

Conjecture. *This is true for each $Q \in \mathcal{Q}$.*

Remark. Let $\check{Q}$ be any quasianalytic log-convex sequence of positive numbers. Then the corresponding sequence Q (determined by (1.5.1)) is quasianalytic and $\mathcal{L}$-intersectable. However, the mapping $\check{Q} \mapsto \mathcal{F}^Q$ is not injective. For instance, the image of $(C\rho^k\check{Q}_k)_k$ is the same for all positive C and ρ (which follows from (1.5.1)). Here is a more striking example:

Let $Q^{\delta,n} \in \mathcal{Q}$ and let $P^{\delta,n} = (P_k^{\delta,n})_k$ be defined by

$$P_k^{\delta,n} := \frac{1}{(k-1+\kappa_n)!}(p_{k-1+\kappa_n}^{\delta,n})^{k-1+\kappa_n}, \quad P_0^{\delta,n} := 1,$$

where

$$p_k^{\delta,n} := q_k^{\delta,n} \left(1 + \sum_{j=\kappa_n}^{k} \frac{1}{q_j^{\delta,n}}\right), \qquad \text{for } 0 < \delta < 1,$$

$$p_k^{1,n} := \hat{q}_k^{1,n+1} = q_k^{1,n} \left(1 + \sum_{j=\kappa_{n+1}}^{k} \frac{1}{q_j^{1,n}}\right).$$

We claim that $\mathcal{F}^{P^{1,n-1}} = \mathcal{F}^{P^{\delta,n}} = \mathcal{F}^{P^{\epsilon,n}}$ *for all* $0 < \delta, \epsilon < 1$. For: Since

$$\frac{d}{dx} \frac{(\log^n(x))^{1-\delta}}{1-\delta} = \frac{1}{x \log(x) \cdots \log^{n-1}(x)(\log^n(x))^\delta},$$

we have

$$C_1 \frac{(\log^n(k))^{1-\delta}}{1-\delta} \le \sum_{j=\kappa_n}^{k} \frac{1}{q_j^{\delta,n}} \le C_2 \frac{(\log^n(k))^{1-\delta}}{1-\delta},$$

and thus

$$\frac{p_k^{\delta,n}}{p_k^{\epsilon,n}} = \frac{(\log^n(k))^\delta}{(\log^n(k))^\epsilon} \frac{\left(1 + \sum_{j=\kappa_n}^k \frac{1}{q_j^{\delta,n}}\right)}{\left(1 + \sum_{j=\kappa_n}^k \frac{1}{q_j^{\epsilon,n}}\right)} \le C_3 \frac{(\log^n(k))^\delta}{(\log^n(k))^\epsilon} \frac{(\log^n(k))^{1-\delta}}{(\log^n(k))^{1-\epsilon}} = C_3$$

and similarly

$$\frac{p_k^{\delta,n}}{p_k^{\epsilon,n}} \ge C_4$$

for suitable constants C_i. By lemma (1.3) we have $\mathcal{F}^{P^{\delta,n}} = \mathcal{F}^{P^{\epsilon,n}}$ for all $0 < \delta, \epsilon < 1$. The same reasoning with $\delta = 0$ proves that $\mathcal{F}^{P^{1,n-1}} = \mathcal{F}^{P^{\epsilon,n}}$.

1.10. **Definition of function spaces.** Let $M = (M_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers, E and F be Banach spaces, $U \subseteq E$ open, $K \subseteq U$ compact, and $\rho > 0$. We consider the non-Hausdorff Banach space

$$\begin{aligned} C^M_{K,\rho}(U,F) &:= \Big\{ f \in C^\infty(U,F) : (\sup_{x\in K} \|f^{(k)}(x)\|_{L^k(E,F)})_k \in \mathcal{F}^M_\rho \Big\} \\ &= \Big\{ f \in C^\infty(U,F) : \|f\|_{K,\rho} < \infty \Big\}, \quad \text{where} \\ \|f\|_{K,\rho} &:= \sup\Big\{ \frac{\|f^{(k)}(x)\|_{L^k(E,F)}}{k!\, M_k\, \rho^k} : x \in K, k \in \mathbb{N} \Big\}, \end{aligned}$$

the inductive limit

$$C^M_K(U,F) := \varinjlim_{\rho>0} C^M_{K,\rho}(U,F),$$

and the projective limit

$$C^M_b(U,F) := \varprojlim_{K\subseteq U} C^M_K(U,F), \text{ where } K \text{ runs through all compact subsets of } U.$$

Here $f^{(k)}(x)$ denotes the k^{th}-order Fréchet derivative of f at x.

Note that instead of $\|f^{(k)}(x)\|_{L^k(E,F)}$ we could equivalently use $\sup\{\|d_v^k f(x)\|_F : \|v\|_E \le 1\}$ by [15, 7.13.1]. For $E = \mathbb{R}^n$ and $F = \mathbb{R}$ this is the same space as in (1.1).

For convenient vector spaces E and F, and c^∞-open $U \subseteq E$ we define:

$$C^M_b(U,F) := \Big\{ f \in C^\infty(U,F) : \forall B\ \forall \text{ compact } K \subseteq U \cap E_B\ \exists \rho > 0 :$$

$$\Big\{\frac{f^{(k)}(x)(v_1,\dots,v_k)}{k!\,\rho^k\,M_k} : k\in\mathbb{N}, x\in K, \|v_i\|_B\le 1\Big\} \text{ is bounded in } F\Big\}$$
$$= \Big\{f\in C^\infty(U,F) : \forall B\ \forall \text{ compact } K\subseteq U\cap E_B\ \exists\rho>0 :$$
$$\Big\{\frac{d_v^k f(x)}{k!\,\rho^k\,M_k} : k\in\mathbb{N}, x\in K, \|v\|_B\le 1\Big\} \text{ is bounded in } F\Big\}.$$

Here B runs through all closed absolutely convex bounded subsets and E_B is the vector space generated by B with the Minkowski functional $\|v\|_B=\inf\{\lambda\ge 0 : v\in \lambda B\}$ as complete norm.

Now we define the spaces of main interest in this paper: First we put

$$C^M(\mathbb{R},U) := \{c:\mathbb{R}\to U\ :\ \ell\circ c\in C^M(\mathbb{R},\mathbb{R})\ \forall \ell\in E^*\}.$$

In general, for L log-convex non-quasianalytic we put

$$\begin{aligned} C^L(U,F) &:= \{f : f\circ c\in C^L(\mathbb{R},F)\ \forall c\in C^L(\mathbb{R},U)\}\\ &= \{f : \ell\circ f\circ c\in C^L(\mathbb{R},\mathbb{R})\ \forall c\in C^L(\mathbb{R},U), \forall \ell\in F^*\}\end{aligned}$$

supplied with the initial locally convex structure induced by all linear mappings $C^L(c,\ell) : f\mapsto \ell\circ f\circ c\in C^L(\mathbb{R},\mathbb{R})$, which is a convenient vector space as c^∞-closed subspace in the product. Note that in particular the family $\ell_* : C^L(U,F)\to C^L(U,\mathbb{R})$ with $\ell\in F^*$ is initial, whereas this is not the case for C^L replaced by C^L_b as example (1.11) for $\{\mathrm{inj}_k\circ g^\vee(k) : k\in\mathbb{N}\}\subseteq C^L(\mathbb{R},\mathbb{R}^\mathbb{N})$ shows, where inj_k denotes the inclusion of the k-th factor in $\mathbb{R}^\mathbb{N}$.
For Q a quasianalytic $\mathcal{L}$-intersection we define the space

$$C^Q(U,F) := \bigcap_{L\in\mathcal{L}(Q)} C^L(U,F)$$

supplied with the initial locally convex structure. By theorem (1.6.1) this definition coincides with the classical notion of C^Q if E and F are finite dimensional.

Lemma. *For Q a quasianalytic $\mathcal{L}$-intersection, the composite of C^Q-mappings is again C^Q, and bounded linear mappings are C^Q.*

Proof. This is true for C^L (see [17, 3.1 and 3.11.1]) for every $L\in\mathcal{L}(Q)$ since each such L is log-convex. □

1.11. **Example.** By [25, Theorem 1], for each weakly log-convex sequence M there exists $f\in C^M(\mathbb{R},\mathbb{R})$ such that $|f^{(k)}(0)|\ge k!\,M_k$ for all $k\in\mathbb{N}$. *Then $g:\mathbb{R}^2\to\mathbb{R}$ given by $g(s,t)=f(st)$ is C^M, whereas there is no reasonable topology on $C^M(\mathbb{R},\mathbb{R})$ such that the associated mapping $g^\vee:\mathbb{R}\to C^M(\mathbb{R},\mathbb{R})$ is C^M_b.* For a topology on $C^M(\mathbb{R},\mathbb{R})$ to be reasonable we require only that all evaluations $\mathrm{ev}_t : C^M(\mathbb{R},\mathbb{R})\to\mathbb{R}$ are bounded linear functionals.

Proof. The mapping g is obviously C^M. If $g^\vee$ were C^M_b, for $s=0$ there existed ρ such that

$$\Big\{\frac{(g^\vee)^{(k)}(0)}{k!\,\rho^k\,M_k} : k\in\mathbb{N}\Big\}$$

was bounded in $C^M(\mathbb{R},\mathbb{R})$. We apply the bounded linear functional ev_t for $t=2\rho$ and then get

$$\frac{(g^\vee)^{(k)}(0)(2\rho)}{k!\,\rho^k\,M_k} = \frac{(2\rho)^k f^{(k)}(0)}{k!\,\rho^k\,M_k}\ge 2^k,$$

a contradiction. □

This example shows that for C^M_b one cannot expect cartesian closedness. Using cartesian closedness (3.3) and (2.3) this also shows (for $F = C^M(\mathbb{R},\mathbb{R})$ and $U = \mathbb{R} = E$) that

$$C^M_b(U,F) \supsetneq \bigcap_{B,V} C^M_b(U\cap E_B, F_V)$$

where F_V is the completion of $F/p_V^{-1}(0)$ with respect to the seminorm p_V induced by the absolutely convex closed 0-neighborhood V.

If we compose $g^\vee$ with the restriction map $(\mathrm{incl}_\mathbb{N})^* : C^M(\mathbb{R},\mathbb{R}) \to \mathbb{R}^\mathbb{N} := \prod_{t\in\mathbb{N}}\mathbb{R}$ then we get a C^M-curve, since the continuous linear functionals on $\mathbb{R}^\mathbb{N}$ are linear combinations of coordinate projections ev_t with $t \in \mathbb{N}$. However, this curve cannot be C^M_b as the argument above for $t > \rho$ shows.

2. Working up to cartesian closedness: More on non-quasianalytic functions

In [17] we developed convenient calculus for C^M where M was log-convex, increasing, derivation closed, and of moderate growth for the exponential law. In this paper we describe quasianalytic mappings as intersections of non-quasianalytic classes C^L, but we cannot assume that L is derivation closed. Thus we need stronger versions of many results of [17] for non-quasianalytic L which are **not** derivation closed, and sometimes even **not** log-convex. This section collects an almost minimal set of results which allow to prove cartesian closedness for certain quasianalytic function classes.

2.1. **Lemma** (cf. [17, 3.3]). *Let $M = (M_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers and let E be a convenient vector space such that there exists a Baire vector space topology on the dual E^* for which the point evaluations ev_x are continuous for all $x \in E$. Then a curve $c : \mathbb{R} \to E$ is C^M if and only if c is C^M_b.*

Proof. Let K be compact in $\mathbb{R}$ and c be a C^M-curve. We consider the sets

$$A_{\rho,C} := \Big\{\ell \in E^* : \frac{|\ell(c^{(k)}(x))|}{\rho^k\, k!\, M_k} \le C \text{ for all } k \in \mathbb{N}, x \in K\Big\}$$

which are closed subsets in E^* for the given Baire topology. We have $\bigcup_{\rho,C} A_{\rho,C} = E^*$. By the Baire property there exists ρ and C such that the interior U of $A_{\rho,C}$ is non-empty. If $\ell_0 \in U$ then for each $\ell \in E^*$ there is a $\delta > 0$ such that $\delta\ell \in U - \ell_0$ and hence for all $x \in K$ and all k we have

$$|(\ell\circ c)^{(k)}(x)| \le \tfrac{1}{\delta}\Big(|((\delta\,\ell+\ell_0)\circ c)^{(k)}(x)| + |(\ell_0\circ c)^{(k)}(x)|\Big) \le \tfrac{2C}{\delta}\,\rho^k\, k!\, M_k.$$

So the set

$$\Big\{\frac{c^{(k)}(x)}{\rho^k\, k!\, M_k} : k \in \mathbb{N}, x \in K\Big\}$$

is weakly bounded in E and hence bounded. □

2.2. **Lemma** (cf. [17, 3.4]). *Let $M = (M_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers and let E be a Banach space. For a smooth curve $c : \mathbb{R} \to E$ the following are equivalent.*

(1) *c is $C^M = C^M_b$.*
(2) *For each sequence (r_k) with $r_k\,\rho^k \to 0$ for all $\rho > 0$, and each compact set K in $\mathbb{R}$, the set $\{\frac{1}{k!M_k}\, c^{(k)}(a)\, r_k : a \in K, k \in \mathbb{N}\}$ is bounded in E.*
(3) *For each sequence (r_k) satisfying $r_k > 0$, $r_k r_\ell \ge r_{k+\ell}$, and $r_k\,\rho^k \to 0$ for all $\rho > 0$, and each compact set K in $\mathbb{R}$, there exists an $\delta > 0$ such that $\{\frac{1}{k!M_k}\, c^{(k)}(a)\, r_k\, \delta^k : a \in K, k \in \mathbb{N}\}$ is bounded in E.*

Proof. (1) $\implies$ (2) For K, there exists $\rho > 0$ such that

$$\Big\|\frac{c^{(k)}(a)}{k!\,M_k} r_k\Big\|_E = \Big\|\frac{c^{(k)}(a)}{k!\,\rho^k\,M_k}\Big\|_E \cdot |r_k \rho^k|$$

is bounded uniformly in $k \in \mathbb{N}$ and $a \in K$ by (2.1).

(2) $\implies$ (3) Use $\delta = 1$.

(3) $\implies$ (1) Let $a_k := \sup_{a\in K} \|\frac{1}{k!\,M_k}\, c^{(k)}(a)\|_E$. Using (4$\Rightarrow$1) in [15, 9.2] these are the coefficients of a power series with positive radius of convergence. Thus a_k/ρ^k is bounded for some $\rho > 0$. $\square$

2.3. **Lemma** (cf. [17, 3.5]). *Let $M = (M_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers. Let E be a convenient vector space, and let $\mathcal{S}$ be a family of bounded linear functionals on E which together detect bounded sets (i.e., $B \subseteq E$ is bounded if and only if $\ell(B)$ is bounded for all $\ell \in \mathcal{S}$). Then a curve $c : \mathbb{R} \to E$ is C^M if and only if $\ell \circ c : \mathbb{R} \to \mathbb{R}$ is C^M for all $\ell \in \mathcal{S}$.*

Proof. For smooth curves this follows from [15, 2.1, 2.11]. By (2.2), for $\ell \in \mathcal{S}$, the function $\ell \circ c$ is C^M if and only if:

(1) For each sequence (r_k) with $r_k\, t^k \to 0$ for all $t > 0$, and each compact set K in $\mathbb{R}$, the set $\{\frac{1}{k!M_k}\,(\ell \circ c)^{(k)}(a)\, r_k : a \in K, k \in \mathbb{N}\}$ is bounded.

By (1) the curve c is C^M if and only if the set $\{\frac{1}{k!M_k}\, c^{(k)}(a)\, r_k : a \in K, k \in \mathbb{N}\}$ is bounded in E. By (1) again this is in turn equivalent to $\ell \circ c \in C^M$ for all $\ell \in \mathcal{S}$, since $\mathcal{S}$ detects bounded sets. $\square$

2.4. **Corollary.** *Let $M = (M_k)_{k\in\mathbb{N}}$ be a non-quasianalytic weight sequence or an $\mathcal{L}$-intersectable quasianalytic weight sequence. Let U be c^∞-open in a convenient vector space E, and let $\mathcal{S} = \{\ell : F \to F_\ell\}$ be a family of bounded linear mappings between convenient vector spaces which together detect bounded sets. Then a mapping $f : U \to F$ is C^M if and only if $\ell \circ f$ is C^M for all $\ell \in \mathcal{S}$.*

In particular, a mapping $f : U \to L(G, H)$ is C^M if and only if $\mathrm{ev}_v \circ f : U \to H$ is C^M for each $v \in G$, where G and H are convenient vector spaces.

This result is not valid for C^M_b instead of C^M, by a variant of (1.11): Replace $C^M(\mathbb{R}, \mathbb{R})$ by $\mathbb{R}^{\mathbb{N}}$.

Proof. First, let M be non-quasianalytic. By composing with curves we may reduce to $U = E = \mathbb{R}$. By composing each $\ell \in \mathcal{S}$ with all bounded linear functionals on F_ℓ we get a family of bounded linear functionals on F to which we can apply (2.3). For quasianalytic M the result follows by definition. The case $F = L(G, H)$ follows since the ev_v together detect bounded sets, by the uniform boundedness principle [15, 5.18]. $\square$

2.5. C^L**-curve lemma** (cf. [17, 3.6]). A sequence x_n in a locally convex space E is said to be *Mackey convergent* to x, if there exists some $\lambda_n \nearrow \infty$ such that $\lambda_n(x_n - x)$ is bounded. If we fix $\lambda = (\lambda_n)$ we say that x_n is λ-converging.

Lemma. *Let L be a non-quasianalytic weight sequence. Then there exist sequences $\lambda_k \to 0$, $t_k \to t_\infty$, $s_k > 0$ in $\mathbb{R}$ with the following property: For $1/\lambda = (1/\lambda_n)$-converging sequences x_n and v_n in a convenient vector space E there exists a strong uniform C^L-curve $c : \mathbb{R} \to E$ with $c(t_k + t) = x_k + t.v_k$ for $|t| \le s_k$.*

Proof. Since C^L is not quasianalytic we have $\sum_k 1/(k!L_k)^{1/k} < \infty$ by (1.2). We choose another non-quasianalytic weight sequence $\bar{L} = (\bar{L}_k)$ with $(L_k/\bar{L}_k)^{1/k} \to \infty$.

By [17, 2.3] there is a $C^{\bar L}$-function $\phi : \mathbb{R} \to [0,1]$ which is 0 on $\{t : |t| \ge \frac{1}{2}\}$ and which is 1 on $\{t : |t| \le \frac{1}{3}\}$, i.e. there exist $\bar C, \rho > 0$ such that

$$|\phi^{(k)}(t)| \le \bar C\,\rho^k\,k!\,\bar L_k \qquad \text{for all } t \in \mathbb{R} \text{ and } k \in \mathbb{N}.$$

For x, v in an absolutely convex bounded set $B \subseteq E$ and $0 < T \le 1$ the curve $c : t \mapsto \phi(t/T)\cdot(x + t\,v)$ satisfies (cf. [4, Lemma 2]):

$$\begin{aligned} c^{(k)}(t) &= T^{-k}\phi^{(k)}(\tfrac{t}{T}).(x+t.v) + k\,T^{1-k}\,\phi^{(k-1)}(\tfrac{t}{T}).v \\ &\in T^{-k}\bar C\,\rho^k\,k!\,\bar L_k(1+\tfrac{T}{2}).B + k\,T^{1-k}\,\bar C\,\rho^{k-1}\,(k-1)!\,\bar L_{k-1}.B \\ &\subseteq T^{-k}\bar C\,\rho^k\,k!\,\bar L_k(1+\tfrac{T}{2}).B + T\,T^{-k}\,\bar C\tfrac{1}{\rho}\,\rho^k\,k!\,\bar L_k.B \\ &\subseteq \bar C(\tfrac{3}{2}+\tfrac{1}{\rho})\,T^{-k}\,\rho^k\,k!\,\bar L_k.B \end{aligned}$$

So there are $\rho, C := \bar C(\frac{3}{2}+\frac{1}{\rho}) > 0$ which do not depend on x, v and T such that $c^{(k)}(t) \in C\,T^{-k}\,\rho^k\,k!\,\bar L_k.B$ for all k and t.

Let $0 < T_j \le 1$ with $\sum_j T_j < \infty$ and $t_k := 2\sum_{j<k} T_j + T_k$. We choose the λ_j such that $0 < \lambda_j/T_j^k \le L_k/\bar L_k$ (note that $T_j^k\,L_k/\bar L_k \to \infty$ for $k \to \infty$) for all j and k, and that $\lambda_j/T_j^k \to 0$ for $j \to \infty$ and each k.

Without loss we may assume that $x_n \to 0$. By assumption there exists a closed bounded absolutely convex subset B in E such that $x_n, v_n \in \lambda_n \cdot B$. We consider $c_j : t \mapsto \phi\big((t-t_j)/T_j\big)\cdot\big(x_j + (t-t_j)\,v_j\big)$ and $c := \sum_j c_j$. The c_j have disjoint support $\subseteq [t_j - T_j, t_j + T_j]$, hence c is C^∞ on $\mathbb{R}\setminus\{t_\infty\}$ with

$$c^{(k)}(t) \in C\,T_j^{-k}\,\rho^k k!\bar L_k\,\lambda_j\cdot B \qquad \text{for } |t-t_j| \le T_j.$$

Then

$$\|c^{(k)}(t)\|_B \le C\,\rho^k\,k!\bar L_k\,\frac{\lambda_j}{T_j^k} \le C\rho^k k!\bar L_k\,\frac{L_k}{\bar L_k} = C\,\rho^k\,k!L_k$$

for $t \ne t_\infty$. Hence $c : \mathbb{R} \to E_B$ is smooth at t_∞ as well, and is strongly C^L by the following lemma. □

2.6. **Lemma** (cf. [17, 3.7]). *Let $c : \mathbb{R}\setminus\{0\} \to E$ be strongly C^L in the sense that c is smooth and for all bounded $K \subset \mathbb{R}\setminus\{0\}$ there exists $\rho > 0$ such that*

$$\left\{\frac{c^{(k)}(x)}{\rho^k\,k!\,L_k} : k \in \mathbb{N}, x \in K\right\} \text{ is bounded in } E.$$

Then c has a unique extension to a strongly C^L-curve on $\mathbb{R}$.

Proof. The curve c has a unique extension to a smooth curve by [15, 2.9]. The strong C^L condition extends by continuity. □

2.7. **Theorem** (cf. [17, 3.9]). *Let $L = (L_k)$ be a non-quasianalytic weight sequence. Let $U \subseteq E$ be c^∞-open in a convenient vector space, let F be a Banach space and $f : U \to F$ a mapping. Furthermore, let $\overline{L} \le L$ be another non-quasianalytic weight sequence. Then the following statements are equivalent:*

1. *f is C^L, i.e. $f\circ c$ is C^L for all C^L-curves c.*
2. *$f|_{U\cap E_B} : E_B \supseteq U \cap E_B \to F$ is C^L for each closed bounded absolutely convex B in E.*
3. *$f\circ c$ is C^L for all $C_b^{\overline{L}}$-curves c.*
4. *$f \in C_b^L(U, F)$.*

Proof. (1) $\implies$ (2) is clear, since $E_B \to E$ is continuous and linear, hence all C^L-curves c into the Banach space E_B are also C^L into E and hence $f\circ c$ is C^L by assumption.

(2) $\implies$ (3) is clear, since $C_b^{\overline{L}} \subseteq C^L$.

(3) $\implies$ (4) Without loss let $E = E_B$ be a Banach space. For each $v \in E$ and $x \in U$ the iterated directional derivative $d_v^k f(x)$ exists since f is C^L along affine lines. To show that f is smooth it suffices to check that $d_{v_n}^k f(x_n)$ is bounded for each $k \in \mathbb{N}$ and each Mackey convergent sequences x_n and $v_n \to 0$, by [15, 5.20]. For contradiction let us assume that there exist k and sequences x_n and v_n with $\|d_{v_n}^k f(x_n)\| \to \infty$. By passing to a subsequence we may assume that x_n and v_n are $(1/\lambda_n)$-converging for the λ_n from (2.5) for the weight sequence $\overline{L}$. Hence there exists a $C_b^{\overline{L}}$-curve c in E and with $c(t+t_n) = x_n + t.v_n$ for t near 0 for each n separately, and for t_n from (2.5). But then $\|(f\circ c)^{(k)}(t_n)\| = \|d_{v_n}^k f(x_n)\| \to \infty$, a contradiction. So f is smooth.

Assume for contradiction that the boundedness condition in (4) does not hold: There exists a compact set $K \subseteq U$ such that for each $n \in \mathbb{N}$ there are $k_n \in \mathbb{N}$, $x_n \in K$, and v_n with $\|v_n\| = 1$ such that

$$\|d_{v_n}^{k_n} f(x_n)\| > k_n!\, L_{k_n}\, \left(\frac{1}{\lambda_n^2}\right)^{k_n+1},$$

where we used $C = \rho := 1/\lambda_n^2$ with the λ_n from (2.5) for the weight sequence $\overline{L}$. By passing to a subsequence (again denoted n) we may assume that the x_n are $1/\lambda$-converging, thus there exists a $C_b^{\overline{L}}$-curve $c : \mathbb{R} \to E$ with $c(t_n + t) = x_n + t.\lambda_n.v_n$ for t near 0 by (2.5). Since

$$(f\circ c)^{(k)}(t_n) = \lambda_n^k d_{v_n}^k f(x_n),$$

we get

$$\left(\frac{\|(f\circ c)^{(k_n)}(t_n)\|}{k_n! L_{k_n}}\right)^{\frac{1}{k_n+1}} = \left(\lambda_n^{k_n}\frac{\|d_{v_n}^{k_n} f(x_n)\|}{k_n! L_{k_n}}\right)^{\frac{1}{k_n+1}} > \frac{1}{\lambda_n^{\frac{k_n+2}{k_n+1}}} \to \infty,$$

a contradiction to $f\circ c \in C^L$.

(4) $\implies$ (1) We have to show that $f\circ c$ is C^L for each C^L-curve $c : \mathbb{R} \to E$. By (2.2.3) it suffices to show that for each sequence (r_k) satisfying $r_k > 0$, $r_k r_\ell \ge r_{k+\ell}$, and $r_k\, t^k \to 0$ for all $t > 0$, and each compact interval I in $\mathbb{R}$, there exists an $\epsilon > 0$ such that $\{\frac{1}{k! L_k}\,(f\circ c)^{(k)}(a)\, r_k\, \epsilon^k : a \in I, k \in \mathbb{N}\}$ is bounded.

By (2.2.2) applied to $r_k 2^k$ instead of r_k, for each $\ell \in E^*$, each sequence (r_k) with $r_k\, t^k \to 0$ for all $t > 0$, and each compact interval I in $\mathbb{R}$ the set $\{\frac{1}{k! L_k}\,(\ell\circ c)^{(k)}(a)\, r_k\, 2^k : a \in I, k \in \mathbb{N}\}$ is bounded in $\mathbb{R}$. Thus $\{\frac{1}{k! L_k}\, c^{(k)}(a)\, r_k\, 2^k : a \in I, k \in \mathbb{N}\}$ is contained in some closed absolutely convex $B \subseteq E$. Consequently, $c^{(k)} : I \to E_B$ is smooth and hence $K_k := \{\frac{1}{k! L_k}\, c^{(k)}(a)\, r_k\, 2^k : a \in I\}$ is compact in E_B for each k. Then each sequence (x_n) in the set

$$K := \left\{\frac{1}{k! L_k}\, c^{(k)}(a)\, r_k : a \in I, k \in \mathbb{N}\right\} = \bigcup_{k\in\mathbb{N}} \frac{1}{2^k} K_k$$

has a cluster point in $K \cup \{0\}$: either there is a subsequence in one K_k, or $2^{k_n} x_{k_n} \in K_{k_n} \subseteq B$ for $k_n \to \infty$, hence $x_{k_n} \to 0$ in E_B. So $K \cup \{0\}$ is compact.

By Faà di Bruno ([7] for the 1-dimensional version, $k \ge 1$)

$$\frac{(f\circ c)^{(k)}(a)}{k!} = \sum_{j\ge 1} \sum_{\substack{\alpha\in\mathbb{N}_{>0}^j \\ \alpha_1+\dots+\alpha_j = k}} \frac{1}{j!} d^j f(c(a))\Big(\frac{c^{(\alpha_1)}(a)}{\alpha_1!}, \dots, \frac{c^{(\alpha_j)}(a)}{\alpha_j!}\Big)$$

and (1.1.2) for $a \in I$ and $k \in \mathbb{N}_{>0}$ we have

$$\left\|\frac{1}{k! L_k}\,(f\circ c)^{(k)}(a)\, r_k\right\| \le$$

$$\le \sum_{j\ge 1} L_1^j \sum_{\substack{\alpha\in\mathbb{N}_{>0}^j\\ \alpha_1+\dots+\alpha_j=k}} \frac{\|d^j f(c(a))\|_{L^j(E_B,F)}}{j!L_j} \prod_{i=1}^j \frac{\|c^{(\alpha_i)}(a)\|_B\, r_{\alpha_i}}{\alpha_i! L_{\alpha_i}}$$

$$\le \sum_{j\ge 1} L_1^j \binom{k-1}{j-1} C\,\rho^j\, \frac{1}{2^k} = L_1\rho(1+L_1\,\rho)^{k-1} C\,\frac{1}{2^k}.$$

So $\left\{ \frac{1}{k!L_k}\,(f\circ c)^{(k)}(a)\left(\frac{2}{1+L_1\,\rho}\right)^k r_k : a\in I, k\in\mathbb{N}\right\}$ is bounded as required. $\square$

2.8. **Corollary.** *Let $L=(L_k)$ be a non-quasianalytic weight sequence. Let $U\subseteq E$ be c^∞-open in a convenient vector space, let F be a convenient vector space and $f:U\to F$ a mapping. Furthermore, let $\overline{L}\le L$ be a non-quasianalytic weight sequence. Then the following statements are equivalent:*

1. *f is C^L.*
2. *$f|_{U\cap E_B} : E_B \supseteq U\cap E_B \to F$ is C^L for each closed bounded absolutely convex B in E.*
3. *$f\circ c$ is C^L for all $C_b^{\overline{L}}$-curves c.*
4. *$\pi_V\circ f \in C_b^L(U,F_V)$ for each absolutely convex 0-neighborhood $V\subseteq F$, where $\pi_V : F\to F_V$ denotes the natural mapping.*

Proof. Each of the statements holds for f if and only if it holds for $\pi_V\circ f$ for each absolutely convex 0-neighborhood $V\subseteq F$. So the corollary follows from (2.7). $\square$

2.9. **Theorem** (Uniform boundedness principle for C^M, cf. [17, 4.1])**.** *Let $M=(M_k)$ be a non-quasianalytic weight sequence or an $\mathcal{L}$-intersectable quasianalytic weight sequence. Let E, F, G be convenient vector spaces and let $U\subseteq F$ be c^∞-open. A linear mapping $T:E\to C^M(U,G)$ is bounded if and only if $\operatorname{ev}_x\circ T : E\to G$ is bounded for every $x\in U$.*

Proof. Let first M be non-quasianalytic. For $x\in U$ and $\ell\in G^*$ the linear mapping $\ell\circ\operatorname{ev}_x = C^M(x,\ell) : C^M(U,G)\to\mathbb{R}$ is continuous, thus ev_x is bounded. Therefore, if T is bounded then so is $\operatorname{ev}_x\circ T$.

Conversely, suppose that $\operatorname{ev}_x\circ T$ is bounded for all $x\in U$. For each closed absolutely convex bounded $B\subseteq E$ we consider the Banach space E_B. For each $\ell\in G^*$, each C^M-curve $c:\mathbb{R}\to U$, each $t\in\mathbb{R}$, and each compact $K\subset\mathbb{R}$ the composite given by the following diagram is bounded.

$$\begin{tikzcd}
E \arrow[r, "T"] & C^M(U,G) \arrow[rr, "\operatorname{ev}_{c(t)}"] \arrow[d, "C^M(c,\ell)"] & & G \arrow[d, "\ell"] \\
E_B \arrow[u] \arrow[r] & C^M(\mathbb{R},\mathbb{R}) \arrow[r] & \varinjlim_\rho C^M_\rho(K,\mathbb{R}) \arrow[r, "\operatorname{ev}_t"] & \mathbb{R}
\end{tikzcd}$$

By [15, 5.24, 5.25] the map T is bounded. In more detail: Since $\varinjlim_\rho C^M_\rho(K,\mathbb{R})$ is webbed, the closed graph theorem [15, 52.10] yields that the mapping $E_B\to \varinjlim_\rho C^M_\rho(K,\mathbb{R})$ is continuous. Thus T is bounded.

For quasianalytic M the result follows since the structure of a convenient vector space on $C^M(U,G)$ is the initial one with respect to all inclusions $C^M(U,G)\to C^L(U,G)$ for all $L\in\mathcal{L}(M)$. $\square$

As a consequence we can show that the equivalences of (2.7) and (2.8) are not only valid for single functions f but also for the bornology of $C^M(U,F)$:

2.10. **Corollary** (cf. [17, 4.6]). *Let $L = (L_k)$ be a non-quasianalytic weight sequence. Let E and F be Banach spaces and let $U \subseteq E$ be open. Then*

$$C^L(U,F) = C^L_b(U,F) := \varprojlim_K \varinjlim_\rho C^L_{K,\rho}(U,F)$$

as vector spaces with bornology. Here K runs through all compact subsets of U ordered by inclusion and ρ runs through the positive real numbers.

Proof. The second equality is by definition (1.10). The first equality, as vector spaces, is by (2.7). By (1.10) the space $C^L(U,F)$ is convenient.

The identity from right to left is continuous since $C^L(U,F)$ carries the initial structure with respect to the mappings

$$C^L(c|_I,\ell) : C^L(U,F) \to C^L(\mathbb{R},\mathbb{R}) = \varprojlim_{I\subseteq\mathbb{R}} \varinjlim_{\rho>0} C^L_{I,\rho}(\mathbb{R},\mathbb{R}) \to \varinjlim_{\rho>0} C^L_{I,\rho}(\mathbb{R},\mathbb{R}),$$

where c runs through the $C^L \overset{(2.1)}{=\!=} C^L_b$-curves, $\ell \in F^*$ and I runs through the compact intervals in $\mathbb{R}$, and for $K := c(I)$ and $\rho' := (1+\rho\,\|c\|_{I,\sigma})\cdot\sigma$, where $\sigma > 0$ is chosen such that $\|c\|_{I,\sigma} < \infty$, the mapping $C^L(c|_I,\ell) : C^L_{K,\rho}(U,F) \to C^L_{I,\rho'}(\mathbb{R},\mathbb{R}) \to \varinjlim_{\rho'>0} C^L_{I,\rho'}(\mathbb{R},\mathbb{R})$ is continuous by (1.4). These arguments are collected in the diagram:

$$\begin{tikzcd}
\varprojlim_I C^L_I(\mathbb{R},\mathbb{R}) \arrow[r, equal] \arrow[d] & C^L(\mathbb{R},\mathbb{R}) & C^L(U,F) \arrow[l, "C^L(c,\ell)"'] & C^L_b(U,F) \arrow[l, dotted] \arrow[r, equal] & \varprojlim_K C^L_K(U,F) \arrow[d] \\
C^L_I(\mathbb{R},\mathbb{R}) \arrow[r, equal] & \varinjlim_\rho C^L_{I,\rho}(\mathbb{R},\mathbb{R}) & & \varinjlim_\rho C^L_{K,\rho}(U,F) \arrow[ll, dotted] \arrow[r, equal] & C^L_K(U,F) \\
 & C^L_{I,\rho'}(\mathbb{R},\mathbb{R}) \arrow[u] & & C^L_{K,\rho}(U,F) \arrow[ll, "C^L(c|_I,\ell)"'] \arrow[u] &
\end{tikzcd}$$

The identity from left to right is bounded since the countable (take $\rho \in \mathbb{N}$) inductive limit $\varinjlim_\rho$ of the (non-Hausdorff) Banach spaces $C^L_{K,\rho}(U,F)$ is webbed and hence satisfies the $\mathcal{S}$-boundedness principle [15, 5.24] where $\mathcal{S} = \{\mathrm{ev}_x : x \in U\}$, and by [15, 5.25] the same is true for $C^L_b(U,F)$. □

2.11. **Corollary** (cf. [17, 4.4]). *Let $L = (L_k)$ be a non-quasianalytic weight sequence. Let E and F be convenient vector spaces and let $U \subseteq E$ be c^∞-open. Then*

$$C^L(U,F) = \varprojlim_{c\in C^L} C^L(\mathbb{R},F) = \varprojlim_{B\subseteq E} C^L(U\cap E_B,F) = \varprojlim_{s\in C^L_b} C^L(\mathbb{R},F)$$

as vector spaces with bornology, where c runs through all C^L-curves in U, B runs through all bounded closed absolutely convex subsets of E, and s runs through all C^L_b-curves in U.

Proof. The first and third inverse limit is formed with $g^* : C^L(\mathbb{R},F) \to C^L(\mathbb{R},F)$ for $g \in C^L(\mathbb{R},\mathbb{R})$ as connecting mappings. Each element $(f_c)_c$ determines a unique function $f : U \to F$ given by $f(x) := (f\circ\mathrm{const}_x)(0)$ with $f\circ c = f_c$ for all such curves c, and $f \in C^L$ if and only if $f_c \in C^L$ for all such c, by (2.8). The second inverse limit is formed with $\mathrm{incl}^* : C^L(U\cap E_B,F) \to C^L(U\cap E_{B'},F)$ for $B' \subseteq B$ as connecting mappings. Each element $(f_B)_B$ determines a unique function $f : U \to F$ given by $f(x) := f_{[-1,1]x}(x)$ with $f|_{E_B} = f_B$ for all B, and $f \in C^L$ if and only if $f_B \in C^L$ for all such B, by (2.8). Thus all equalities hold as vector spaces.

The first identity is continuous from left to right, since the family of $\ell_*: C^L(\mathbb{R},F) \to C^L(\mathbb{R},\mathbb{R})$ with $\ell \in F^*$ is initial and $C^L(c,\ell) = \ell_* \circ c^* : C^L(U,F) \to C^L(\mathbb{R},\mathbb{R})$ is continuous and linear by definition.

Continuity for the second one from left to right is obvious, since C^L-curves in $U \cap E_B$ are C^L into $U \subseteq E$.

In order to show the continuity of the last identity from left to right choose a C^L_b-curve s in U, an $\ell \in F^*$ and a compact interval $I \subseteq \mathbb{R}$. Then there exists a bounded absolutely convex closed $B \subseteq E$ such that $s|_I$ is $C^L_b = C^L$ into $U \cap E_B$, hence $C^L(s|_I, \ell) : C^L(U,F) \to C^L(I,\mathbb{R})$ factors by (1.4) as continuous linear mapping $(s|_I)^* : C^L_b(U \cap E_B, \mathbb{R}) \to C^L(I,\mathbb{R})$ over $C^L(U,F) \to C^L(U \cap E_B, F) \to C^L(U \cap E_B, \mathbb{R}) \xlongequal{(2.10)} C^L_b(U \cap E_B, \mathbb{R})$. Since the structure of $C^L(\mathbb{R},F)$ is initial with respect to $\mathrm{incl}^* \circ \ell_* : C^L(\mathbb{R},F) \to C^L(I,\mathbb{R})$ the identity $\varprojlim_{B \subseteq E} C^L(U \cap E_B, F) \to \varprojlim_{s \in C^L_b} C^L(\mathbb{R},F)$ is continuous.

Conversely, the identity $\varprojlim_{s\in C^L_b} C^L(\mathbb{R},F) \to C^L(U,F)$ is bounded, since $C^L(\mathbb{R},F)$ is convenient and hence also the inverse limit $\varprojlim_{s\in C^L_b} C^L(\mathbb{R},F)$ and $C^L(U,F)$ satisfies the uniform boundedness theorem (2.9) with respect to the point-evaluations ev_x and they factor over $(\mathrm{const}_x)^* : C^L(U,F) \to C^L(\mathbb{R},F)$. $\square$

3. The exponential law for certain quasianalytic function classes

We start with some preparations. Let $Q = (Q_k)$ be an $\mathcal{L}$-intersectable quasianalytic weight sequence. Let E and F be convenient vector spaces and let $U \subseteq E$ be c^∞-open.

3.1. **Lemma.** *For Banach spaces E and F we have*

$$C^Q(U,F) = C^Q_b(U,F) = \bigcap_{N \in \mathcal{L}_w(Q)} C^N_b(U,F)$$

as vector spaces.

Proof. Since Q is $\mathcal{L}$-intersectable we have $\mathcal{F}^Q = \bigcap_{L\in\mathcal{L}(Q)} \mathcal{F}^L$. Hence

$$\begin{aligned}
C^Q_b(U,F) &= \{f \in C^\infty(U,F) : \forall K : (\sup_{x\in K} \|f^{(k)}(x)\|_{L^k(E,F)})_k \in \mathcal{F}^Q = \bigcap_{L\in\mathcal{L}(Q)} \mathcal{F}^L\} \\
&= \{f \in C^\infty(U,F) : \forall K\ \forall L \in \mathcal{L}(Q) : (\sup_{x\in K} \|f^{(k)}(x)\|)_k \in \mathcal{F}^L\} \\
&= \{f \in C^\infty(U,F) : \forall L \in \mathcal{L}(Q)\ \forall K : (\sup_{x\in K} \|f^{(k)}(x)\|)_k \in \mathcal{F}^L\} \\
&= \bigcap_{L\in\mathcal{L}(Q)} C^L_b(U,F) \xlongequal{(2.7)} \bigcap_{L\in\mathcal{L}(Q)} C^L(U,F) = C^Q(U,F).
\end{aligned}$$

$$C^Q_b(U,F) \xlongequal{(1.6.1)} \bigcap_{L\in\mathcal{L}(Q)} C^L_b(U,F) \supseteq \bigcap_{L\in\mathcal{L}_w(Q)} C^L_b(U,F) \supseteq C^Q_b(U,F). \quad \square$$

3.2. **Lemma.** *For log-convex non-quasianalytic L^1, L^2 and weakly log-convex non-quasianalytic N with $N_{k+n} \le C^{k+n} L^1_k L^2_n$ for some positive constant C and all $k,n \in \mathbb{N}$, for Banach-spaces E_1 and E_2, and for $f \in C^N_b(U_1 \times U_2, \mathbb{R})$ we have $f^\vee \in C^{L^1}(U_1, C^{L^2}_b(U_2,\mathbb{R}))$.*

Proof. Since f is C^N_b, by definition, for all compact $K_i \subseteq U_i$ there exists a $\rho > 0$ such that for all $k,j \in \mathbb{N}$, $x_i \in K_i$ and $\|v_1\| = \dots = \|v_j\| = 1 = \|w_1\| = \dots = \|w_k\|$ we have

$$|\partial_2^k \partial_1^j f(x_1,x_2)(v_1,\dots,v_j,w_1,\dots,w_k)| \le \rho^{k+j+1}(k+j)!N_{k+j}$$

$$\leq \rho^{k+j+1}2^{k+j}k!j!C^{k+j}L^1_jL^2_k = \rho(2C\rho)^j j!L^1_j \cdot (2C\rho)^k k!L^2_k.$$

In particular $(\partial_1^j f)^\vee(K_1)(oE_1^k)$ is contained and bounded in $C_b^{L^2}(U_2,\mathbb{R})$, where oE_1 denotes the unit ball in E_1, since $d^k((\partial_1^j f)^\vee(x_1))(x_2) = \partial_2^k\partial_1^j f(x_1,x_2)$.

Claim. *If $f \in C_b^N$ then $f^\vee : U_1 \to C_b^{L^2}(U_2,\mathbb{R})$ is C^∞ with $d^j f^\vee = (\partial_1^j f)^\vee$.*

Since $C_b^{L^2}(U_2,\mathbb{R})$ is a convenient vector space, by [15, 5.20] it is enough to show that the iterated unidirectional derivatives $d_v^j f^\vee(x)$ exist, equal $\partial_1^j f(x,\quad)(v^j)$, and are separately bounded for x, resp. v, in compact subsets. For $j=1$ and fixed x, v, and y consider the smooth curve $c : t \mapsto f(x+tv,y)$. By the fundamental theorem

$$\begin{aligned}\frac{f^\vee(x+tv)-f^\vee(x)}{t}(y) - (\partial_1 f)^\vee(x)(y)(v) &= \frac{c(t)-c(0)}{t} - c'(0)\\ &= t\int_0^1 s\int_0^1 c''(tsr)\,dr\,ds\\ &= t\int_0^1 s\int_0^1 \partial_1^2 f(x+tsrv,y)(v,v)\,dr\,ds.\end{aligned}$$

Since $(\partial_1^2 f)^\vee(K_1)(oE_1^2)$ is bounded in $C_b^{L^2}(U_2,\mathbb{R})$ for each compact subset $K_1 \subseteq U_1$ this expression is Mackey convergent to 0 in $C_b^{L^2}(U_2,\mathbb{R})$, for $t \to 0$. Thus $d_v f^\vee(x)$ exists and equals $\partial_1 f(x,\quad)(v)$.

Now we proceed by induction, applying the same arguments as before to $(d_v^j f^\vee)^\wedge : (x,y) \mapsto \partial_1^j f(x,y)(v^j)$ instead of f. Again $(\partial_1^2(d_v^j f^\vee)^\wedge)^\vee(K_1)(oE_1^2) = (\partial_1^{j+2} f)^\vee(K_1)(oE_1,oE_1,v,\dots,v)$ is bounded, and also the separated boundedness of $d_v^j f^\vee(x)$ follows. So the claim is proved.

It remains to show that $f^\vee : U_1 \to C_b^{L^2}(U_2,\mathbb{R}) := \varprojlim_K \varinjlim_\rho C_{K,\rho}^{L^2}(U_2,\mathbb{R})$ is C^{L^1}. By (2.4), it suffices to show that $f^\vee : U_1 \to \varinjlim_\rho C_{K_2,\rho}^{L^2}(U_2,\mathbb{R})$ is $C_b^{L^1} \subseteq C^{L^1}$ for all K_2, i.e., for all compact $K_2 \subset U_2$ and $K_1 \subset U_1$ there exists $\rho_1 > 0$ such that

$$\Big\{\frac{d^k f^\vee(K_1)(v_1,\dots,v_k)}{k!\rho_1^k L^1_k} : k\in\mathbb{N}, \|v_i\|\leq 1\Big\} \text{ is bounded in } \varinjlim_\rho C_{K_2,\rho}^{L^2}(U_2,\mathbb{R}),$$

or equivalently: For all compact $K_2 \subset U_2$ and $K_1 \subset U_1$ there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\Big\{\frac{\partial_2^l\partial_1^k f(K_1,K_2)(v_1,\dots,v_{k+l})}{l!k!\rho_2^l L^2_l \rho_1^k L^1_k} : k\in\mathbb{N}, l\in\mathbb{N}, \|v_i\|\leq 1\Big\} \text{ is bounded in } \mathbb{R}.$$

For $k_1 \in \mathbb{N}$, $x \in K_1$, $\rho_i := 2C\rho$, and $\|v_i\| \leq 1$ we get:

$$\begin{aligned}&\Big\|\frac{d^{k_1} f^\vee(x)(v_1,\dots,v_{k_1})}{\rho_1^{k_1}\,k_1!\,L^1_{k_1}}\Big\|_{K_2,\rho_2} =\\ &:= \sup\Big\{\frac{|\partial_2^{k_2}\partial_1^{k_1} f(x,y)(v_1,\dots;w_1,\dots)|}{\rho_1^{k_1}\,k_1!\,L^1_{k_1}\,\rho_2^{k_2}\,k_2!\,L^2_{k_2}} : k_2\in\mathbb{N}, y\in K_2, \|w_i\|\leq 1\Big\}\\ &\leq \sup\Big\{\frac{\frac{(k_1+k_2)!}{k_1!\,k_2!}C^{k_1+k_2}|\partial_2^{k_2}\partial_1^{k_1} f(x,y)(v_1,\dots;w_1,\dots)|}{\rho_1^{k_1}\,\rho_2^{k_2}\,(k_1+k_2)!\,N_{k_1+k_2}} : k_2\in\mathbb{N}, y\in K_2, \|w_i\|\leq 1\Big\}\\ &\leq \sup\Big\{\frac{(2C)^{k_1+k_2}|\partial^{(k_1,k_2)} f(x,y)(v_1,\dots;w_1,\dots)|}{\rho_1^{k_1}\,\rho_2^{k_2}\,(k_1+k_2)!\,N_{k_1+k_2}} : k_2\in\mathbb{N}, y\in K_2, \|w_i\|\leq 1\Big\}\\ &= \sup\Big\{\frac{|\partial^{(k_1,k_2)} f(x,y)(v_1,\dots;w_1,\dots)|}{\rho^{k_1+k_2}\,(k_1+k_2)!\,N_{k_1+k_2}} : k_2\in\mathbb{N}, y\in K_2 : \|w_i\|\leq 1\Big\} \leq \rho\end{aligned}$$

So $f^\vee$ is C^{L_1}. $\square$

3.3. **Theorem** (Cartesian closedness). *Let $Q = (Q_k)$ be an $\mathcal{L}$-intersectable quasianalytic weight sequence of moderate growth. Then the category of C^Q-mappings between convenient real vector spaces is cartesian closed. More precisely, for convenient vector spaces E_1, E_2 and F and c^∞-open sets $U_1 \subseteq E_1$ and $U_2 \subseteq E_2$ a mapping $f : U_1 \times U_2 \to F$ is C^Q if and only if $f^\vee : U_1 \to C^Q(U_2, F)$ is C^Q.*

Actually, we prove that the direction ($\Leftarrow$) holds without the assumption of moderate growth.

Proof. ($\Rightarrow$) Let $f : U_1 \times U_2 \to F$ be C^Q, i.e. C^L for all $L \in \mathcal{L}(Q)$. Since $(E_i)_{B_i} \to E_i$ is bounded and linear and since C^L is closed under composition we get that $\ell \circ f : (U_1 \cap (E_1)_{B_1}) \times (U_2 \cap (E_2)_{B_2}) \to \mathbb{R}$ is $C^L = C^L_b$ (by (2.7) since $(E_i)_{B_i}$ are Banach-spaces) for $\ell \in F^*$, arbitrary bounded closed $B_i \subseteq E_i$ and all $L \in \mathcal{L}(Q)$. Hence $\ell \circ f$ is C^L_b even for all $L \in \mathcal{L}_w(Q)$ by (3.1). For arbitrary $L^1, L^2 \in \mathcal{L}(Q)$, by (1.6.3) and (1.6.2), there exists an $N \in \mathcal{L}_w(Q)$ with $N_{k+n} \le C^{k+n} L^1_k L^2_n$ for some positive constant C and all $k, n \in \mathbb{N}$. Thus $\ell \circ f : (U_1 \cap (E_1)_{B_1}) \times (U_2 \cap (E_2)_{B_2}) \to \mathbb{R}$ is C^N_b. By (3.2), the function $(\ell \circ f)^\vee : U_1 \cap (E_1)_{B_1} \to C^{L^2}_b(U_2 \cap (E_2)_{B_2}, \mathbb{R})$ is C^{L^1}. Since the cone

$$C^Q(U_2, F) \to C^{L^2}(U_2, F) \xrightarrow{C^{L^2}(i_{B_2}, \ell)} C^{L^2}(U_2 \cap (E_2)_{B_2}, \mathbb{R}) = C^{L^2}_b(U_2 \cap (E_2)_{B_2}, \mathbb{R}),$$

with $L_2 \in \mathcal{L}(Q)$, $\ell \in F^*$, and bounded closed $B_2 \subseteq E_2$, generates the bornology by (2.11), and since obviously $f^\vee(x) = f(x, \quad) \in C^Q(U_2, F)$, we have that $f^\vee : U_1 \cap (E_1)_{B_1} \to C^Q(U_2, F)$ is C^{L^1}, by (2.4). From this we get by (2.8) that $f^\vee : U_1 \to C^Q(U_2, F)$ is C^{L^1} for all $L^1 \in \mathcal{L}(Q)$, i.e., $f^\vee : U_1 \to C^Q(U_2, F)$ is C^Q as required. The whole argument above is collected in the following diagram where $U^i_{B_i}$ stands for $U_i \cap E_{B_i}$:

$$\begin{array}{ccc}
U^1 \times U^2 & \xrightarrow{f \in C^Q} & F \\
\uparrow{\scriptstyle \text{incl}} & & \downarrow{\scriptstyle \ell} \\
U^1_{B_1} \times U^2_{B_2} & \xrightarrow[(3.1)]{f \in C^Q \subseteq C^N_b} & \mathbb{R}
\end{array}
\quad \Longrightarrow \quad
\begin{array}{ccccc}
U^1 & \xrightarrow[(2.8)]{f^\vee \in C^{L^1}} & C^Q(U^2, F) & \longrightarrow & C^{L^2}(U^2, F) \\
\uparrow{\scriptstyle \text{incl}_1} & \nearrow{\scriptstyle (2.3)} & & & \downarrow{\scriptstyle \ell_* \circ \text{incl}_2^* \; (2.11)} \\
U^1_{B_1} & \xrightarrow[(3.2)]{C^{L^1}} & C^{L^2}(U^2_{B_2}, \mathbb{R}) & = & C^{L^2}_b(U^2_{B_2}, \mathbb{R})
\end{array}$$

($\Leftarrow$) Let, conversely, $f^\vee : U_1 \to C^Q(U_2, F)$ be C^Q, i.e., C^L for all $L \in \mathcal{L}(Q)$. By the description of the structure of $C^Q(U, F)$ in (1.10) the mapping $f^\vee : U_1 \to C^L(U_2, F)$ is C^L. We now conclude that $f : U_1 \times U_2 \to F$ is C^L; this direction of cartesian closedness for C^L holds even if L is not of moderate growth, see [17, 5.3] and its proof. This is true for all $L \in \mathcal{L}(Q)$. Hence f is C^Q. $\square$

3.4. **Corollary.** *Let Q be an $\mathcal{L}$-intersectable quasianalytic weight sequence of moderate growth. Let E, F, etc., be convenient vector spaces and let U and V be c^∞-open subsets of such. Then we have:*

(1) *The exponential law holds:*

$$C^Q(U, C^Q(V, G)) \cong C^Q(U \times V, G)$$

is a linear C^Q-diffeomorphism of convenient vector spaces.

The following canonical mappings are C^Q.

(2) $\quad \mathrm{ev} : C^Q(U, F) \times U \to F, \quad \mathrm{ev}(f, x) = f(x)$

(3) $\quad \mathrm{ins} : E \to C^Q(F, E \times F), \quad \mathrm{ins}(x)(y) = (x, y)$

(4) $\quad (\quad)^\wedge : C^Q(U, C^Q(V, G)) \to C^Q(U \times V, G)$

(5) $\quad (\quad)^\vee : C^Q(U \times V, G) \to C^Q(U, C^Q(V, G))$

(6) $\quad \mathrm{comp} : C^Q(F, G) \times C^Q(U, F) \to C^Q(U, G)$

$$(7)\qquad C^Q(\quad,\quad) : C^Q(F,F_1)\times C^Q(E_1,E)\to C^Q\Big(C^Q(E,F),C^Q(E_1,F_1)\Big)$$
$$(f,g)\mapsto (h\mapsto f\circ h\circ g)$$

$$(8)\qquad \prod : \prod C^Q(E_i,F_i)\to C^Q\Big(\prod E_i,\prod F_i\Big)$$

Proof. This is a direct consequence of cartesian closedness (3.3). See [17, 5.5] or even [15, 3.13] for the detailed arguments. □

4. More on function spaces

In this section we collect results for function classes C^M where M is either a non-quasianalytic weight sequence or an $\mathcal{L}$-intersectable quasianalytic weight sequence. In order to treat both cases simultaneously, the proofs will often use non-quasianalytic weight sequences $L \ge M$. These are either M itself if M is non-quasianalytic or are in $\mathcal{L}(M)$ if M is $\mathcal{L}$-intersectable quasianalytic. In both cases we may assume without loss that L is increasing, by (1.5).

4.1. **Proposition.** *Let $M = (M_k)$ be a non-quasianalytic weight sequence or an $\mathcal{L}$-intersectable quasianalytic weight sequence. Then we have:*

(1) *Multilinear mappings between convenient vector spaces are C^M if and only if they are bounded.*
(2) *If $f : E \supseteq U \to F$ is C^M, then the derivative $df : U \to L(E,F)$ is $C^{M_{+1}}$, and also $(df)^\wedge : U\times E \to F$ is $C^{M_{+1}}$, where the space $L(E,F)$ of all bounded linear mappings is considered with the topology of uniform convergence on bounded sets.*
(3) *The chain rule holds.*

Proof. (1) If f is C^M then it is smooth by (2.8) and hence bounded by [15, 5.5]. Conversely, if f is multilinear and bounded then it is smooth, again by [15, 5.5]. Furthermore, $f\circ i_B$ is multilinear and continuous and all derivatives of high order vanish. Thus condition (2.8.4) is satisfied, so f is C^M.

(2) Since f is smooth, by [15, 3.18] the map $df : U \to L(E,F)$ exists and is smooth. Let $L \ge M_{+1}$ be a non-quasianalytic weight sequence and $c : \mathbb{R}\to U$ be a C^L-curve. We have to show that $t\mapsto df(c(t))\in L(E,F)$ is C^L. By the uniform boundedness principle [15, 5.18] and by (2.3) it suffices to show that the mapping $t\mapsto c(t)\mapsto \ell(df(c(t))(v))\in\mathbb{R}$ is C^L for each $\ell\in F^*$ and $v\in E$. We are reduced to show that $x\mapsto \ell(df(x)(v))$ satisfies the conditions of (2.7). By (2.7) applied to $\ell\circ f$, for each $L\ge M$, each closed bounded absolutely convex B in E, and each $x\in U\cap E_B$ there are $r>0$, $\rho>0$, and $C>0$ such that

$$\frac{1}{k!\,L_k}\|d^k(\ell\circ f\circ i_B)(a)\|_{L^k(E_B,\mathbb{R})}\le C\,\rho^k$$

for all $a\in U\cap E_B$ with $\|a-x\|_B\le r$ and all $k\in\mathbb{N}$. For $v\in E$ and those B containing v we then have:

$$\begin{aligned}
\|d^k(d(\ell\circ f)(\quad)(v))\circ i_B)(a)\|_{L^k(E_B,\mathbb{R})} &= \|d^{k+1}(\ell\circ f\circ i_B)(a)(v,\dots)\|_{L^k(E_B,\mathbb{R})}\\
&\le \|d^{k+1}(\ell\circ f\circ i_B)(a)\|_{L^{k+1}(E_B,\mathbb{R})}\|v\|_B \le C\,\rho^{k+1}\,(k+1)!\,L_{k+1}\\
&= C\rho\,((k+1)^{1/k}\rho)^k\,k!\,L_{k+1}\le C\rho\,(2\rho)^k\,k!\,(L_{+1})_k
\end{aligned}$$

By (4.2) below also $(df)^\wedge$ is $C^{L_{+1}}$.

(3) This is valid even for all smooth f by [15, 3.18]. □

4.2. **Proposition.** *Let $M = (M_k)$ be a non-quasianalytic weight sequence or an $\mathcal{L}$-intersectable quasianalytic weight sequence.*

(1) *For convenient vector spaces E and F, on $L(E,F)$ the following bornologies coincide which are induced by:*
   - *The topology of uniform convergence on bounded subsets of E.*
   - *The topology of pointwise convergence.*
   - *The embedding $L(E,F) \subset C^\infty(E,F)$.*
   - *The embedding $L(E,F) \subset C^M(E,F)$.*

(2) *Let E, F, G be convenient vector spaces and let $U \subset E$ be c^∞-open. A mapping $f : U \times F \to G$ which is linear in the second variable is C^M if and only if $f^\vee : U \to L(F,G)$ is well defined and C^M.*

Analogous results hold for spaces of multilinear mappings.

Proof. (1) That the first three topologies on $L(E,F)$ have the same bounded sets has been shown in [15, 5.3, 5.18]. The inclusion $C^M(E,F) \to C^\infty(E,F)$ is bounded by the uniform boundedness principle [15, 5.18]. Conversely, the inclusion $L(E,F) \to C^M(E,F)$ is bounded by the uniform boundedness principle (2.9).

(2) The assertion for C^∞ is true by [15, 3.12] since $L(E,F)$ is closed in $C^\infty(E,F)$.

If f is C^M let $L \geq M$ be a non-quasianalytic weight-sequence and let $c : \mathbb{R} \to U$ be a C^L-curve. We have to show that $f^\vee \circ c$ is C^L into $L(F,G)$. By the uniform boundedness principle [15, 5.18] and (2.3) it suffices to show that $t \mapsto \ell\big(f^\vee(c(t))(v)\big) = \ell\big(f(c(t),v)\big) \in \mathbb{R}$ is C^L for each $\ell \in G^*$ and $v \in F$; this is obviously true.

Conversely, let $f^\vee : U \to L(F,G)$ be C^M and let $L \geq M$ be a non-quasianalytic weight-sequence. We claim that $f : U \times F \to G$ is C^L. By composing with $\ell \in G^*$ we may assume that $G = \mathbb{R}$. By induction we have

$$d^k f(x,w_0)\big((v_k,w_k),\dots,(v_1,w_1)\big) = d^k(f^\vee)(x)(v_k,\dots,v_1)(w_0) + \\ + \sum_{i=1}^k d^{k-1}(f^\vee)(x)(v_k,\dots,\widehat{v_i},\dots,v_1)(w_i)$$

We check condition (2.7.4) for f where $x \in K$ which is compact in U:

$$\begin{aligned}
&\|d^k f(x,w_0)\|_{L^k(E_B\times F_{B'},\mathbb{R})} \leq \\
&\leq \|d^k(f^\vee)(x)(\dots)(w_0)\|_{L^k(E_B,\mathbb{R})} + \sum_{i=1}^k \|d^{k-1}(f^\vee)(x)\|_{L^{k-1}(E_B,L(F_{B'},\mathbb{R}))} \\
&\leq \|d^k(f^\vee)(x)\|_{L^k(E_B,L(F_{B'},\mathbb{R}))}\|w_0\|_{B'} + \sum_{i=1}^k \|d^{k-1}(f^\vee)(x)\|_{L^{k-1}(E_B,L(F_{B'},\mathbb{R}))} \\
&\leq C\,\rho^k\,k!\,L_k\|w_0\|_{B'} + \sum_{i=1}^k C\,\rho^{k-1}\,(k-1)!\,L_{k-1} = C\,\rho^k\,k!\,L_k\Big(\|w_0\|_{B'} + \tfrac{L_{k-1}}{\rho\,L_k}\Big)
\end{aligned}$$

where we used (2.7.4) for $L(i_{B'},\mathbb{R}) \circ f^\vee : U \to L(F_{B'},\mathbb{R})$. Since L is increasing, f is C^L. □

4.3. **Theorem.** *Let $Q = (Q_k)$ be an $\mathcal{L}$-intersectable quasianalytic weight sequence. Let $U \subseteq E$ be c^∞-open in a convenient vector space, let F be another convenient vector space, and $f : U \to F$ a mapping. Then the following statements are equivalent:*

(1) *f is C^Q, i.e., for all $L \in \mathcal{L}(Q)$ we have $f \circ c$ is C^L for all C^L-curves c.*

(2) *$f|_{U\cap E_B} : E_B \supseteq U \cap E_B \to F$ is C^Q for each closed bounded absolutely convex B in E.*

(3) *For all $L \in \mathcal{L}(Q)$ the curve $f \circ c$ is C^L for all C^L_b-curves c.*

(4) *$\pi_V \circ f$ is C^Q_b for all absolutely convex 0-neighborhoods V in F and the associated mapping $\pi_V : F \to F_V$.*

Proof. This follows from (2.8) for $\overline{L} := L$ since $C^Q := \bigcap_{L\in\mathcal{L}(Q)} C^L$ and $C_b^Q = \bigcap_{L\in\mathcal{L}(Q)} C_b^L$. $\square$

4.4. **Theorem** (cf. [17, 4.4])**.** *Let $Q = (Q_k)$ be an $\mathcal{L}$-intersectable quasianalytic weight sequence. Let E and F be convenient vector spaces and let $U \subseteq E$ be c^∞-open. Then*

$$C^Q(U,F) = \varprojlim_{L\in\mathcal{L}(Q),c\in C^L} C^L(\mathbb{R},F) = \varprojlim_{B\subseteq E} C^Q(U\cap E_B,F) = \varprojlim_{L\in\mathcal{L}(Q),s\in C_b^L} C^L(\mathbb{R},F)$$

as vector spaces with bornology, where c runs through all C^L-curves in U for $L \in \mathcal{L}(Q)$, B runs through all bounded closed absolutely convex subsets of E, and s runs through all C_b^L-curves in U for $L \in \mathcal{L}(Q)$.

Proof. This follows by applying $\varprojlim_{L\in\mathcal{L}(Q)}$ to (2.11). $\square$

4.5. **Jet spaces.** Let E and F be Banach spaces and $A \subseteq E$ convex. We consider the linear space $C^\infty(A,F)$ consisting of all sequences $(f^k)_k \in \prod_{k\in\mathbb{N}} C(A, L^k(E,F))$ satisfying

$$f^k(y)(v) - f^k(x)(v) = \int_0^1 f^{k+1}(x+t(y-x))(y-x,v)\,dt$$

for all $k \in \mathbb{N}$, $x, y \in A$, and $v \in E^k$. If A is open we can identify this space with that of all smooth functions $A \to F$ by the passage to jets.

In addition, let $M = (M_k)$ be a weight sequence and (r_k) a sequence of positive real numbers. Then we consider the normed spaces

$$C^M_{(r_k)}(A,F) := \Big\{(f^k)_k \in C^\infty(A,F) : \|(f^k)\|_{(r_k)} < \infty\Big\}$$

where the norm is given by

$$\|(f^k)\|_{(r_k)} := \sup\Big\{\frac{\|f^k(a)(v_1,\dots,v_k)\|}{k!\,r_k\,M_k\,\|v_1\|\cdot\dots\cdot\|v_k\|} : k\in\mathbb{N}, a\in A, v_i\in E\Big\}.$$

If $(r_k) = (\rho^k)$ for some $\rho > 0$ we just write ρ instead of (r_k) as indices. The spaces $C^M_{(r_k)}(A,F)$ are Banach spaces, since they are closed in $\ell^\infty(\mathbb{N}, \ell^\infty(A, L^k(E,F)))$ via $(f^k)_k \mapsto (k \mapsto \frac{1}{k!\,r_k\,M_k} f^k)$.

If A is open, $C^\infty(A,F)$ and $C^M_\rho(A,F)$ coincide with the convenient spaces treated before.

4.6. **Theorem** (cf. [17, 4.6])**.** *Let $M = (M_k)$ be a non-quasianalytic weight sequence or an $\mathcal{L}$-intersectable quasianalytic weight sequence. Let E and F be Banach spaces and let $U \subseteq E$ be open and convex. Then the space $C^M(U,F) = C^M_b(U,F)$ can be described bornologically in the following equivalent ways, i.e. these constructions give the same vector space and the same bounded sets*

$$\varprojlim_{K} \varinjlim_{\rho,W} C^M_\rho(W,F) \tag{1}$$

$$\varprojlim_{K} \varinjlim_{\rho} C^M_\rho(K,F) \tag{2}$$

$$\varprojlim_{K,(r_k)} C^M_{(r_k)}(K,F) \tag{3}$$

Moreover, all involved inductive limits are regular, i.e. the bounded sets of the inductive limits are contained and bounded in some step.

Here K runs through all compact convex subsets of U ordered by inclusion, W runs through the open subsets $K \subseteq W \subseteq U$ again ordered by inclusion, ρ runs *through the positive real numbers, (r_k) runs through all sequences of positive real numbers for which $\rho^k/r_k \to 0$ for all $\rho > 0$.*

Proof. This proof is almost identical with that of [17, 4.6]. The only change is to use (2.7) and (4.3) instead of [17, 3.9] to show that all these descriptions give $C^M(U,F)$ as vector space. $\square$

4.7. **Lemma** (cf. [17, 4.7]). *Let M be a non-quasianalytic weight sequence. For any convenient vector space E the flip of variables induces an isomorphism $L(E, C^M(\mathbb{R},\mathbb{R})) \cong C^M(\mathbb{R}, E')$ as vector spaces.*

Proof. This proof is identical with that of [17, 4.7] but uses (2.9) instead of [17, 4.1] and (2.3) instead of [17, 3.5]. $\square$

4.8. **Lemma** (cf. [17, 4.8]). *Let $M = (M_k)$ be a non-quasianalytic weight sequence. By $\lambda^M(\mathbb{R})$ we denote the c^∞-closure of the linear subspace generated by $\{\mathrm{ev}_t : t \in \mathbb{R}\}$ in $C^M(\mathbb{R},\mathbb{R})'$ and let $\delta : \mathbb{R} \to \lambda^M(\mathbb{R})$ be given by $t \mapsto \mathrm{ev}_t$. Then $\lambda^M(\mathbb{R})$ is the free convenient vector space over C^M, i.e. for every convenient vector space G the C^M-curve δ induces a bornological isomorphism*

$$\delta^* : L(\lambda^M(\mathbb{R}), G) \cong C^M(\mathbb{R}, G).$$

We expect $\lambda^M(\mathbb{R})$ to be equal to $C^M(\mathbb{R},\mathbb{R})'$ as it is the case for the analogous situation of smooth mappings, see [15, 23.11], and of holomorphic mappings, see [23] and [24].

Proof. The proof goes along the same lines as in [15, 23.6] and in [8, 5.1.1]. It is identical with that of [17, 4.8] but uses (2.3), (2.9), and (4.2) in that order. $\square$

4.9. **Corollary** (cf. [17, 4.9]). *Let $L = (L_k)$ and $L' = (L'_k)$ be non-quasianalytic weight sequences. We have the following isomorphisms of linear spaces*

(1) $C^\infty(\mathbb{R}, C^L(\mathbb{R},\mathbb{R})) \cong C^L(\mathbb{R}, C^\infty(\mathbb{R},\mathbb{R}))$
(2) $C^\omega(\mathbb{R}, C^L(\mathbb{R},\mathbb{R})) \cong C^L(\mathbb{R}, C^\omega(\mathbb{R},\mathbb{R}))$
(3) $C^{L'}(\mathbb{R}, C^L(\mathbb{R},\mathbb{R})) \cong C^L(\mathbb{R}, C^{L'}(\mathbb{R},\mathbb{R}))$

Proof. This proof is that of [17, 4.9] with other references: For $\alpha \in \{\infty, \omega, L'\}$ we get

$$\begin{aligned} C^L(\mathbb{R}, C^\alpha(\mathbb{R},\mathbb{R})) &\cong L(\lambda^L(\mathbb{R}), C^\alpha(\mathbb{R},\mathbb{R})) \qquad \text{by (4.8)} \\ &\cong C^\alpha(\mathbb{R}, L(\lambda^L(\mathbb{R}),\mathbb{R})) \qquad \text{by (4.7), [15, 3.13.4, 5.3, 11.15]} \\ &\cong C^\alpha(\mathbb{R}, C^L(\mathbb{R},\mathbb{R})) \qquad \text{by (4.8).} \quad \square \end{aligned}$$

4.10. **Theorem** (Canonical isomorphisms). *Let $M = (M_k)$ be a non-quasianalytic weight sequences or an $\mathcal{L}$-intersectable quasianalytic weight-sequences; likewise $M' = (M'_k)$. Let E, F be convenient vector spaces and let W_i be c^∞-open subsets in such. We have the following natural bornological isomorphisms:*

(1) $C^M(W_1, C^{M'}(W_2, F)) \cong C^{M'}(W_2, C^M(W_1, F))$,
(2) $C^M(W_1, C^\infty(W_2, F)) \cong C^\infty(W_2, C^M(W_1, F))$.
(3) $C^M(W_1, C^\omega(W_2, F)) \cong C^\omega(W_2, C^M(W_1, F))$.
(4) $C^M(W_1, L(E, F)) \cong L(E, C^M(W_1, F))$.
(5) $C^M(W_1, \ell^\infty(X, F)) \cong \ell^\infty(X, C^M(W_1, F))$.
(6) $C^M(W_1, \mathcal{L}\mathrm{ip}^k(X, F)) \cong \mathcal{L}\mathrm{ip}^k(X, C^M(W_1, F))$.

In (5) the space X is an ℓ^∞-space, i.e. a set together with a bornology induced by a family of real valued functions on X, cf. [8, 1.2.4]. *In (6) the space X is a $\mathcal{L}\mathrm{ip}^k$-space, cf.* [8, 1.4.1]. *The spaces $\ell^\infty(X, F)$ and $\mathcal{L}\mathrm{ip}^k(W, F)$ are defined in* [8, 3.6.1 and 4.4.1].

Proof. This proof is very similar with that of [17, 4.8] but written differently. Let $\mathcal{C}^1$ and $\mathcal{C}^2$ denote any of the functions spaces mentioned above and X_1 and X_2 the corresponding domains. In order to show that the flip of coordinates $f \mapsto \tilde{f}$, $\mathcal{C}^1(X_1, \mathcal{C}^2(X_2, F)) \to \mathcal{C}^2(X_2, \mathcal{C}^1(X_1, F))$ is a well-defined bounded linear mapping we have to show:

- $\tilde{f}(x_2) \in \mathcal{C}^1(X_1, F)$, which is obvious, since $\tilde{f}(x_2) = \mathrm{ev}_{x_2} \circ f : X_1 \to \mathcal{C}^2(X_2, F) \to F$.
- $\tilde{f} \in \mathcal{C}^2(X_2, \mathcal{C}^1(X_1, F))$, which we will show below.
- $f \mapsto \tilde{f}$ is bounded and linear, which follows by applying the appropriate uniform boundedness theorem for $\mathcal{C}^2$ and $\mathcal{C}^1$ since $f \mapsto \mathrm{ev}_{x_1} \circ \mathrm{ev}_{x_2} \circ \tilde{f} = \mathrm{ev}_{x_2} \circ \mathrm{ev}_{x_1} \circ f$ is bounded and linear.

All occurring function spaces are convenient and satisfy the uniform $\mathcal{S}$-boundedness theorem, where $\mathcal{S}$ is the set of point evaluations:

C^M by (1.10) and (2.9).
C^∞ by [15, 2.14.3, 5.26]
C^ω by [15, 11.11, 11.12],
L by [15, 2.14.3, 5.18]
ℓ^∞ by [15, 2.15, 5.24, 5.25] or [8, 3.6.1 and 3.6.6]
$\mathcal{L}\mathrm{ip}^k$ by [8, 4.4.2 and 4.4.7]

It remains to check that $\tilde{f}$ is of the appropriate class:

(1) follows by composing with the appropriate (non-quasianalytic) curves $c_1 : \mathbb{R} \to W_1$, $c_2 : \mathbb{R} \to W_2$ and $\lambda \in F^*$ and thereby reducing the statement to the special case in (4.9.3).
(2) as for (1) using (4.9.1).
(3) follows by composing with $c_2 \in C^{\beta_2}(\mathbb{R}, W_2)$, where β_2 is in $\{\infty, \omega\}$, and with $C^L(c_1, \lambda) : C^M(W_1, F) \to C^L(\mathbb{R}, \mathbb{R})$ where $c_1 \in C^L(\mathbb{R}, W_1)$ with $L \geq M$ non-quasianalytic and $\lambda \in F^*$. Then $C^L(c_1, \lambda) \circ \tilde{f} \circ c_2 = (C^{\beta_2}(c_2, \lambda) \circ f \circ c_1)^\sim : \mathbb{R} \to C^L(\mathbb{R}, \mathbb{R})$ is C^{β_2} by (4.9.1) and (4.9.2), since $C^{\beta_2}(c_2, \lambda) \circ f \circ c_1 : \mathbb{R} \to W_1 \to C^\omega(W_2, F) \to C^{\beta_2}(\mathbb{R}, \mathbb{R})$ is C^L.
For the inverse, compose with c_1 and $C^{\beta_2}(c_2, \lambda) : C^\omega(W_2, F) \to C^{\beta_2}(\mathbb{R}, \mathbb{R})$. Then $C^{\beta_2}(c_2, \lambda) \circ \tilde{f} \circ c_1 = (C^L(c_1, \lambda) \circ f \circ c_2)^\sim : \mathbb{R} \to C^{\beta_2}(\mathbb{R}, \mathbb{R})$ is C^L by (4.9.1) and (4.9.2), since $C^L(c_1, \lambda) \circ f \circ c_2 : \mathbb{R} \to W_2 \to C^L(W_1, F) \to C^L(\mathbb{R}, \mathbb{R})$ is C^{β_2}.
(4) since $L(E, F)$ is the c^∞-closed subspace of $C^M(E, F)$ formed by the linear C^M-mappings.
(5) follows from (4), using the free convenient vector spaces $\ell^1(X)$ over the ℓ^∞-space X, see [8, 5.1.24 or 5.2.3], satisfying $\ell^\infty(X, F) \cong L(\ell^1(X), F)$.
(6) follows from (4), using the free convenient vector spaces $\lambda^k(X)$ over the $\mathcal{L}\mathrm{ip}^k$-space X, satisfying $\mathcal{L}\mathrm{ip}^k(X, F) \cong L(\lambda^k(X), F)$. Existence of this free convenient vector space can be proved in a similar way as in (4.8). $\square$

5. Manifolds of quasianalytic mappings

For manifolds of real analytic mappings [14] we could prove that composition and inversion (on groups of real analytic diffeomorphisms) are again C^ω by testing along C^∞-curves and C^ω-curves separately. Here this does not (yet) work. We have to test along C^L-curves for all L in $\mathcal{L}(Q)$, but for those L we do not have cartesian closedness in general. But it suffices to test along C^Q-mappings from open sets in Banach spaces, and this is a workable replacement.

5.1. **C^Q-manifolds.** Let $Q = (Q_k)$ be an $\mathcal{L}$-intersectable quasianalytic weight sequence of moderate growth. A C^Q-manifold is a smooth manifold such that all chart changings are C^Q-mappings. Likewise for C^Q-bundles and C^Q Lie groups.

Note that any finite dimensional (always assumed paracompact) C^∞-manifold admits a C^∞-diffeomorphic real analytic structure thus also a C^Q-structure. Maybe, any finite dimensional C^Q-manifold admits a C^Q-diffeomorphic real analytic structure. This would follow from:

Conjecture. *Let X be a finite dimensional real analytic manifold. Consider the space $C^Q(X,\mathbb{R})$ of all C^Q-functions on X, equipped with the (obvious) Whitney C^Q-topology. Then $C^\omega(X,\mathbb{R})$ is dense in $C^Q(X,\mathbb{R})$.*

This conjecture is the analog of [10, Proposition 9].

5.2. **Banach plots.** Let $Q = (Q_k)$ be an $\mathcal{L}$-intersectable quasianalytic weight sequence of moderate growth. Let X be a C^Q-manifold. By a *C^Q-plot in X* we mean a C^Q-mapping $c : D \to X$ where $D \subset E$ is the open unit ball in a Banach space E.

Lemma. *A mapping between C^Q-manifolds is C^Q if and only if it maps C^Q-plots to C^Q-plots.*

Proof. For a convenient vector space E the c^∞-topology is the final topology for all injections $E_B \to E$ where B runs through all closed absolutely convex bounded subsets of E. The c^∞-topology on a c^∞-open subset $U \subseteq E$ is final with respect to all injections $E_B \cap U \to U$. For a C^Q-manifold the topology is the final one for all C^Q-plots. Let $f : X \to Y$ be the mapping. If f respects C^Q-plots it is continuous and so we may assume that Y is c^∞-open in a convenient vector space F and then likewise for $X \subseteq E$. The (affine) plots induced by $X \cap E_B \subset X$ are C^Q. By definition f is C^Q if and only if it is C^L for all $L \in \mathcal{L}(Q)$ and this is the case if f is C^L on $X \cap E_B$ for all B by (2.8). □

5.3. **Spaces of C^Q-sections.** Let $p : E \to B$ be a C^Q vector bundle (possibly infinite dimensional). The space $C^Q(B \leftarrow E)$ of all C^Q-sections is a convenient vector space with the structure induced by

$$C^Q(B \leftarrow E) \to \prod_\alpha C^Q(u_\alpha(U_\alpha), V)$$

$$s \mapsto \mathrm{pr}_2 \circ \psi_\alpha \circ s \circ u_\alpha^{-1}$$

where $B \supseteq U_\alpha \xrightarrow{u_\alpha} u_\alpha(U_\alpha) \subseteq W$ is a C^Q-atlas for B which we assume to be modeled on a convenient vector space W, and where $\psi_\alpha : E|_{U_\alpha} \to U_\alpha \times V$ form a vector bundle atlas over charts U_α of B.

Lemma. *Let D be a unit ball in a Banach space. A mapping $c : D \to C^Q(B \leftarrow E)$ is a C^Q-plot if and only if $c^\wedge : D \times B \to E$ is C^Q.*

Proof. By the description of the structure on $C^Q(B \leftarrow E)$ we may assume that B is c^∞-open in a convenient vector space W and that $E = B \times V$. Then we have $C^Q(B \leftarrow B \times V) \cong C^Q(B, V)$. Thus the statement follows from the exponential law (3.3). □

Let $U \subseteq E$ be an open neighborhood of $s(B)$ for a section s and let $q : F \to B$ be another vector bundle. The set $C^Q(B \leftarrow U)$ of all C^Q-sections $s' : B \to E$ with $s'(B) \subset U$ is open in the convenient vector space $C^Q(B \leftarrow E)$ if B is compact. An immediate consequence of the lemma is the following: If $U \subseteq E$ is an open neighborhood of $s(B)$ for a section s, $F \to B$ is another vector bundle and if $f : U \to F$ is a fiber respecting C^Q-mapping, then $f_* : C^Q(B \leftarrow U) \to C^Q(B \leftarrow F)$

is C^Q on the open neighborhood $C^Q(B \leftarrow U)$ of s in $C^Q(B \leftarrow E)$. We have $(d(f_*)(s)v)_x = d(f|_{U\cap E_x})(s(x))(v(x))$.

5.4. **Theorem.** *Let $Q = (Q_k)$ be an $\mathcal{L}$-intersectable quasianalytic weight sequence of moderate growth. Let A and B be finite dimensional C^Q-manifolds with A compact and B equipped with a C^Q Riemann metric. Then the space $C^Q(A,B)$ of all C^Q-mappings $A \to B$ is a C^Q-manifold modeled on convenient vector spaces $C^Q(A \leftarrow f^*TB)$ of C^Q-sections of pullback bundles along $f : A \to B$. Moreover, a mapping $c : D \to C^Q(A,B)$ is a C^Q-plot if and only if $c^\wedge : D \times A \to B$ is C^Q.*

If the C^Q-structure on B is induced by a real analytic structure then there exists a real analytic Riemann metric which in turn is C^Q.

Proof. C^Q-vector fields have C^Q-flows by [13]; applying this to the geodesic spray we get the C^Q exponential mapping $\exp : TB \supseteq U \to B$ of the Riemann metric, defined on a suitable open neighborhood of the zero section. We may assume that U is chosen in such a way that $(\pi_B, \exp) : U \to B \times B$ is a C^Q-diffeomorphism onto an open neighborhood V of the diagonal, by the C^Q inverse function theorem due to [12].

For $f \in C^Q(A,B)$ we consider the pullback vector bundle

$$\begin{tikzcd} A \times TB & A \times_B TB \arrow[l, hook'] \arrow[r, equal] & f^*TB \arrow[r, "\pi_B^* f"] \arrow[d, "f^*\pi_B"'] & TB \arrow[d, "\pi_B"] \\ & & A \arrow[r, "f"] & B \end{tikzcd}$$

Then the convenient space of sections $C^Q(A \leftarrow f^*TB)$ is canonically isomorphic to the space $C^Q(A,TB)_f := \{h \in C^Q(A,TB) : \pi_B \circ h = f\}$ via $s \mapsto (\pi_B^* f) \circ s$ and $(\mathrm{Id}_A, h) \hookleftarrow h$. Now let

$$\begin{aligned} U_f &:= \{g \in C^Q(A,B) : (f(x),\ g(x)) \in V \text{ for all } x \in A\}, \\ u_f &: U_f \to C^Q(A \leftarrow f^*TB), \\ u_f(g)(x) &= (x, \exp^{-1}_{f(x)}(g(x))) = (x, ((\pi_B, \exp)^{-1} \circ (f,g))(x)). \end{aligned}$$

Then $u_f : U_f \to \{s \in C^Q(A \leftarrow f^*TB) : s(A) \subseteq f^*U = (\pi_B^* f)^{-1}(U)\}$ is a bijection with inverse $u_f^{-1}(s) = \exp \circ (\pi_B^* f) \circ s$, where we view $U \to B$ as a fiber bundle. The set $u_f(U_f)$ is open in $C^Q(A \leftarrow f^*TB)$ for the topology described above in (5.3) since A is compact and the push forward u_f is C^Q since it respects C^Q-plots by lemma (5.3).

Now we consider the atlas $(U_f, u_f)_{f \in C^Q(A,B)}$ for $C^Q(A,B)$. Its chart change mappings are given for $s \in u_g(U_f \cap U_g) \subseteq C^Q(A \leftarrow g^*TB)$ by

$$\begin{aligned} (u_f \circ u_g^{-1})(s) &= (\mathrm{Id}_A, (\pi_B, \exp)^{-1} \circ (f, \exp \circ (\pi_B^* g) \circ s)) \\ &= (\tau_f^{-1} \circ \tau_g)_*(s), \end{aligned}$$

where $\tau_g(x, Y_{g(x)}) := (x, \exp_{g(x)}(Y_{g(x)}))$ is a C^Q-diffeomorphism $\tau_g : g^*TB \supseteq g^*U \to (g \times \mathrm{Id}_B)^{-1}(V) \subseteq A \times B$ which is fiber respecting over A. The chart change $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ is defined on an open subset and it is also C^Q since it respects C^Q-plots by lemma (5.3).

Finally for the topology on $C^Q(A,B)$ we take the identification topology from this atlas (with the c^∞-topologies on the modeling spaces), which is obviously finer than the compact-open topology and thus Hausdorff.

The equation $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ shows that the C^Q-structure does not depend on the choice of the C^Q Riemannian metric on B.

The statement on C^Q-plots follows from lemma (5.3). $\square$

5.5. **Corollary.** *Let A_1, A_2 and B be finite dimensional C^Q-manifolds with A_1 and A_2 compact. Then composition*

$$C^Q(A_2, B) \times C^Q(A_1, A_2) \to C^Q(A_1, B), \quad (f, g) \mapsto f \circ g$$

is C^Q. However, if $N = (N_k)$ is another weight sequence ($\mathcal{L}$-intersectable quasianalytic) with $(N_k/Q_k)^{1/k} \searrow 0$ then composition is **not** *C^N.*

Proof. Composition maps C^Q-plots to C^Q-plots, so it is C^Q.

Let $A_1 = A_2 = S^1$ and $B = \mathbb{R}$. Then by [25, Theorem 1] or [17, 2.1.5] there exists $f \in C^Q(S^1, \mathbb{R}) \setminus C^N(S^1, \mathbb{R})$. We consider f as a periodic function $\mathbb{R} \to \mathbb{R}$. The universal covering space of $C^Q(S^1, S^1)$ consists of all $2\pi\mathbb{Z}$-equivariant mappings in $C^Q(\mathbb{R}, \mathbb{R})$, namely the space of all $g + \mathrm{Id}_{\mathbb{R}}$ for 2π-periodic $g \in C^Q$. Thus $C^Q(S^1, S^1)$ is a real analytic manifold and $t \mapsto (x \mapsto x + t)$ induces a real analytic curve c in $C^Q(S^1, S^1)$. But $f_* \circ c$ is not C^N since:

$$\frac{(\partial_t^k|_{t=0}(f_* \circ c)(t))(x)}{k!\rho^k N_k} = \frac{\partial_t^k|_{t=0} f(x+t)}{k!\rho^k N_k} = \frac{f^{(k)}(x)}{k!\rho^k N_k}$$

which is unbounded in k for x in a suitable compact set and for all $\rho > 0$, since $f \notin C^N$. □

5.6. **Theorem.** *Let $Q = (Q_k)$ be an $\mathcal{L}$-intersectable quasianalytic weight sequence of moderate growth. Let A be a compact ($\Rightarrow$ finite dimensional) C^Q-manifold. Then the group $\mathrm{Diff}^Q(A)$ of all C^Q-diffeomorphisms of A is an open subset of the C^Q-manifold $C^Q(A, A)$. Moreover, it is a C^Q-regular C^Q Lie group: Inversion and composition are C^Q. Its Lie algebra consists of all C^Q-vector fields on A, with the negative of the usual bracket as Lie bracket. The exponential mapping is C^Q. It is not surjective onto any neighborhood of Id_A.*

Following [16], see also [15, 38.4], a C^Q-Lie group G with Lie algebra $\mathfrak{g} = T_eG$ is called C^Q-regular if the following holds:

- For each C^Q-curve $X \in C^Q(\mathbb{R}, \mathfrak{g})$ there exists a C^Q-curve $g \in C^Q(\mathbb{R}, G)$ whose right logarithmic derivative is X, i.e.,

$$\begin{cases} g(0) &= e \\ \partial_t g(t) &= T_e(\mu^{g(t)})X(t) = X(t).g(t) \end{cases}$$

  The curve g is uniquely determined by its initial value $g(0)$, if it exists.
- Put $\mathrm{evol}_G^r(X) = g(1)$ where g is the unique solution required above. Then $\mathrm{evol}_G^r : C^Q(\mathbb{R}, \mathfrak{g}) \to G$ is required to be C^Q also.

Proof. The group $\mathrm{Diff}^Q(A)$ is open in $C^Q(A, A)$ since it is open in the coarser C^1 compact-open topology, see [15, 43.1]. So $\mathrm{Diff}^Q(A)$ is a C^Q-manifold and composition is C^Q by (5.4) and (5.5). To show that inversion is C^Q let c be a C^Q-plot in $\mathrm{Diff}^Q(A)$. By (5.4) the map $c^\wedge : D \times A \to A$ is C^Q and $(\mathrm{inv} \circ c)^\wedge : D \times A \to A$ satisfies the Banach manifold implicit equation $c^\wedge(t, (\mathrm{inv} \circ c)^\wedge(t, x)) = x$ for $x \in A$. By the Banach C^Q implicit function theorem [26] the mapping $(\mathrm{inv} \circ c)^\wedge$ is locally C^Q and thus C^Q. By (5.4) again, $\mathrm{inv} \circ c$ is a C^Q-plot in $\mathrm{Diff}^Q(A)$. So $\mathrm{inv} : \mathrm{Diff}^Q(A) \to \mathrm{Diff}^Q(A)$ is C^Q. The Lie algebra of $\mathrm{Diff}^Q(A)$ is the convenient vector space of all C^Q-vector fields on A, with the negative of the usual Lie bracket (compare with the proof of [15, 43.1]).

To show that $\mathrm{Diff}^Q(A)$ is a C^Q-regular Lie group, we choose a C^Q-plot in the space of C^Q-curves in the Lie algebra of all C^Q vector fields on A, $c : D \to C^Q(\mathbb{R}, C^Q(A \leftarrow TA))$. By lemma (5.3) c corresponds to a $(D \times \mathbb{R})$-time-dependent C^Q vector field $c^{\wedge\wedge} : D \times \mathbb{R} \times A \to TA$. Since C^Q-vector fields have C^Q-flows and since A is compact, $\operatorname{evol}^r(c^\wedge(s))(t) = \operatorname{Fl}_t^{c^\wedge(s)}$ is C^Q in all variables by [27]. Thus $\operatorname{Diff}^Q(A)$ is a C^Q-regular C^Q Lie group.

The exponential mapping is evol^r applied to constant curves in the Lie algebra, i.e., it consists of flows of autonomous C^Q vector fields. That the exponential map is not surjective onto any C^Q-neighborhood of the identity follows from [15, 43.5] for $A = S^1$. This example can be embedded into any compact manifold, see [9]. □

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