UNIFORM EXTENSION OF DEFINABLE $C^{m,\omega}$-WHITNEY JETS

ADAM PARUSIŃSKI AND ARMIN RAINER

Abstract. We show that definable Whitney jets of class $C^{m,\omega}$, where m is a nonnegative integer and ω is a modulus of continuity, are the restrictions of definable $C^{m,\omega}$-functions; "definable" refers to an arbitrary given o-minimal expansion of the real field. This is true in a uniform way: any definable bounded family of Whitney jets of class $C^{m,\omega}$ extends to a definable bounded family of $C^{m,\omega}$-functions. We also discuss a uniform C^m-version and how the extension depends on the modulus of continuity.

1. Introduction

Let an o-minimal expansion of the real field be fixed. Throughout the paper, a set $X \subseteq \mathbb{R}^n$ is called *definable* if it is definable in this fixed o-minimal structure. A map $\varphi : X \to \mathbb{R}^m$ is definable if its graph $\Gamma(\varphi) := \{(x, \varphi(x)) : x \in X\}$ is a definable subset of $\mathbb{R}^n \times \mathbb{R}^m$. We assume familiarity with the basics of o-minimal structures; cf. [15] and [14].

Due to Kurdyka and Pawłucki [7, 8] and Thamrongthanyalak [12] we have the definable C^m Whitney extension theorem:

Theorem 1.1. *Let $0 \le m \le p$ be integers. Let $E \subseteq \mathbb{R}^n$ be a definable closed set. Any definable Whitney jet of class C^m on E extends to a definable C^m-function on $\mathbb{R}^n$ which is of class C^p outside E.*

In this paper, we prove a $C^{m,\omega}$-version of this result.

Theorem 1.2. *Let $0 \le m \le p$ be integers. Let ω be a modulus of continuity. Let $E \subseteq \mathbb{R}^n$ be a definable closed set. Any definable Whitney jet of class $C^{m,\omega}$ on E extends to a definable $C^{m,\omega}$-function on $\mathbb{R}^n$ which is of class C^p outside E.*

By a *modulus of continuity* we always mean a positive, continuous, increasing, and concave function $\omega : (0,\infty) \to (0,\infty)$ such that $\omega(t) \to 0$ as $t \to 0$. The class $C^{m,\omega}$ consists of C^m-functions that are globally bounded together with its partial derivatives up to order m and whose partial derivatives of order m satisfy a global ω-Hölder condition. See Section 3.

We use Theorem 1.2 in our paper [9] to show that a definable function $f : E \to \mathbb{R}$ on a definable closed set $E \subseteq \mathbb{R}^n$ that has a $C^{1,\omega}$-extension to $\mathbb{R}^n$ also has a definable $C^{1,\omega}$-extension. (In [9] we assume that ω is definable, but not in the present paper.) In fact, this application was one of our main motivations for proving Theorem 1.2.

Furthermore, we will show that the definable extension of Whitney jets of class $C^{m,\omega}$ can be done in a bounded way:

Date: October 9, 2023.

2020 *Mathematics Subject Classification.* 03C64, 14P10, 26B35, 26E25, 32B20, 46E15.

Key words and phrases. O-minimal structures, Whitney extension theorem, $C^{m,\omega}$-extension of Whitney jets, uniform boundedness of the extension.

Supported by FWF-Project P 32905-N and Oberwolfach Research Fellows (OWRF) ID 2244p.

Theorem 1.3. *Let $0 \le m \le p$ be integers. Let ω be a modulus of continuity. Let $(E_a)_{a \in A}$ be a definable family of closed subsets of $\mathbb{R}^n$. For any definable bounded family $(F_a)_{a \in A}$ of Whitney jets of class $C^{m,\omega}$ on $(E_a)_{a \in A}$ there exists a definable bounded family $(f_a)_{a \in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(F_a)_{a \in A}$ such that f_a is of class C^p outside E_a for all $a \in A$.*

Clearly, boundedness is understood with respect to the natural norms; see Section 3 for precise definitions. Note that Theorem 1.2 follows as a special case from Theorem 1.3. And already the case that $(F_a)_{a \in A}$ is a definable bounded family of Whitney jets of class $C^{m,\omega}$ on a *fixed set* $E = E_a$, for all $a \in A$, is very interesting. However, the method of proof (by induction on dimension) necessitates to consider the general case that the families of Whitney jets are defined on definable families of sets $(E_a)_{a \in A}$.

The construction of the extension in Theorem 1.3 depends on ω only in a weak sense. We may, for instance, let the modulus of continuity ω_a depend as well on $a \in A$ if we impose that there is a constant $C > 0$ such that $C^{-1} \le \omega_a(1) \le C$ for all $a \in A$. This will be discussed in detail in Section 5.2 in which we present a more general version of Theorem 1.3. As a consequence, we deduce from Theorem 1.3 a uniform version of the C^m-result 1.1 on compact sets:

Theorem 1.4. *Let $0 \le m \le p$ be integers. Let $(E_a)_{a \in A}$ be a definable family of compact subsets of $\mathbb{R}^n$. For any definable bounded family $(F_a)_{a \in A}$ of Whitney jets of class C^m on $(E_a)_{a \in A}$ there exists a definable bounded family $(f_a)_{a \in A}$ of C^m-extensions to $\mathbb{R}^n$ of $(F_a)_{a \in A}$ such that f_a is of class C^p outside E_a for all $a \in A$.*

Theorem 1.4 is proved in Section 5.3. Furthermore, we deduce a local version of Theorem 1.3 in Section 5.1 and apply Theorem 1.3 in Section 5.4 to get a definable version of a correspondence, due to Shvartsman [11], between Whitney jets of class $C^{m,\omega}$ and certain Lipschitz maps.

The proof of Theorem 1.3 (which builds upon the one of Theorem 1.1 devised in [7, 8, 12] and also Pawłucki [10] and is very different from Whitney's classical method [16]) rests on two main cornerstones:

(1) *Two versions of Gromov's inequality* [5]*;* one classical, the other incorporating the modulus of continuity. These are inequalities for the derivatives of a definable function. Since the constants that appear in them are universal, it is not difficult to get them uniform for definable families of functions. See Corollary 2.18 and Proposition 2.19.
(2) *Uniform Λ_p-stratification of definable families of sets.* Roughly speaking, definable families of sets admit a stratification into a finite number of cells that are defined by functions satisfying certain estimates (for their derivatives up to order p). The constants in these estimates and the number of cells are independent of the parameter of the family. See Theorem 2.16. This is essential for the uniform extension theorem 1.3. We think that it is also of independent interest.

It is a natural question if there exists even a continuous and/or linear extension operator for definable Whitney jets of class $C^{m,\omega}$ (or C^m) on a definable closed set $E \subseteq \mathbb{R}^n$. This remains an open problem. The theorem of Bartle and Graves [2] (see also [3, Theorem 1.6]) is not applicable since the normed spaces of definable jets and functions (defined in Section 3) are not complete.

Note that Azagra, Le Gruyer, and Mudarra [1] give an explicit formula for the extension of Whitney jets of class $C^{1,1}$ with an optimal control of the norms; for definable input this explicit formula yields a definable $C^{1,1}$-extension. See also the discussion in [9, Sections 4.2–4].

Let us point out that Pawłucki [10] presents a continuous linear extension operator for (not necessarily definable) Whitney jets on definable closed sets which preserves (up to a multiplicative constant) the modulus of continuity. This extension operator is a finite composite of operators that preserve definability on the one hand or are defined by integration with respect to a parameter (more precisely, convolution) on the other hand; in general, the latter leads out of the original o-minimal structure.

While [10] was a important source of inspiration for handling the modulus of continuity, the main difficulty (besides getting everything uniformly bounded) was to replace the convolution operators by definable operations which at the same time allow for preserving the modulus of continuity.

The paper is organized as follows. In Section 2 the main geometric tools are prepared: Gromov's inequality and uniform Λ_p-stratification. We present in Section 3 background on definable bounded families of Whitney jets of class $C^{m,\omega}$, most notably, how they behave under pullback along a definable family of Λ_p-regular maps. The proof of Theorem 1.3 is carried out in Section 4. In the final Section 5 we give the mentioned applications, discuss dependence on the modulus of continuity, and prove Theorem 1.4.

Notation. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of nonnegative integers. We denote by $d(x, S) := \inf_{y\in S} |x - y|$ the Euclidean distance in $\mathbb{R}^n$ of a point x to a subset S of $\mathbb{R}^n$, with the convention $d(x, \emptyset) := +\infty$. The open Euclidean ball with center $x \in \mathbb{R}^n$ and radius $r > 0$ is denoted by $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$. The closure of a set S is denoted by $\overline{S}$ and the frontier of S by $\partial S := \overline{S} \setminus S$; we will not use the topological boundary $\overline{S} \cap \overline{\mathbb{R}^n \setminus S}$. If S is a subset of $\mathbb{R}^k$, we write $S \times 0$ for the set $\{(u, w) \in \mathbb{R}^k \times \mathbb{R}^\ell : u \in \mathbb{R}^k, w = 0\}$. The graph of a map φ is denoted by $\Gamma(\varphi)$. For real valued nonnegative functions f, g we write $f \lesssim g$ if $f \le Cg$ for some universal constant $C > 0$. In particular, it should be always understood that C is independent of $a \in A$, i.e., the parameter which we consistently use in parameterized families of sets and maps. We write $f \approx g$ if $f \lesssim g$ and $g \lesssim f$. We use standard multi-index notation and in this context $(i) \in \mathbb{N}^n$ is the multi-index $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the i-th entry.

2. Uniform Λ_p-stratifications

The existence of uniform Λ_p-stratifications (Theorem 2.16) is based on an inequality of Gromov [5], of which we need two versions, and on uniform L-regular decomposition due to Kurdyka and Parusiński [6].

2.1. **Definable families of sets and maps.** Let A be a definable subset of $\mathbb{R}^N$. A family $(E_a)_{a\in A}$ of definable sets $E_a \subseteq \mathbb{R}^n$ is called a *definable family* if the associated set

$$E := \bigcup_{a\in A} \{a\} \times E_a \tag{2.1}$$

is a definable subset of $\mathbb{R}^N \times \mathbb{R}^n$. Conversely, any definable subset $E \subseteq \mathbb{R}^N \times \mathbb{R}^n$ defines a definable family $(E_a)_{a\in A}$ by setting $A := \{a \in \mathbb{R}^N : \exists x \in \mathbb{R}^n : (a, x) \in E\}$ and $E_a := \{x \in \mathbb{R}^n : (a, x) \in E\}$, $a \in A$. If we allow $E_a = \emptyset$, we may just take $A = \mathbb{R}^N$.

A family $(E'_a)_{a\in A}$ of subsets $E'_a \subseteq E_a$ is said to be a *definable subfamily* of $(E_a)_{a\in A}$ if the associated set E' (defined in analogy to (2.1)) is a definable subset of E.

A family $(\varphi_a)_{a\in A}$ of definable maps $\varphi_a : E_a \to \mathbb{R}^m$ is called a *definable family* if the map $\varphi : E \to \mathbb{R}^m$, where E is the associated set (2.1) and

$$\varphi(a, u) := \varphi_a(u), \quad u \in E_a, \tag{2.2}$$

is definable. This is consistent with the first paragraph, since

$$\begin{aligned}\Gamma(\varphi) &= \{(a, u, \varphi(a, u)) \in \mathbb{R}^N \times \mathbb{R}^n \times \mathbb{R}^m : (a, u) \in E\}\\ &= \bigcup_{a\in A} \{(a, u, \varphi_a(u)) \in \mathbb{R}^N \times \mathbb{R}^n \times \mathbb{R}^m : u \in E_a\}\\ &= \bigcup_{a\in A} \{a\} \times \{(u, \varphi_a(u)) \in \mathbb{R}^n \times \mathbb{R}^m : u \in E_a\}\\ &= \bigcup_{a\in A} \{a\} \times \Gamma(\varphi_a).\end{aligned}$$

2.2. **Gromov's inequality.** We need two versions of an inequality due to Gromov [5]. We start with a C^m-version.

Lemma 2.1 ([7, Lemma 2])**.** *Let $m \ge 1$. Let $f : I \to \mathbb{R}$ be a C^{m+1}-function, where $I = [t_0 - r, t_0 + r] \subseteq \mathbb{R}$, $r > 0$, is an interval. Suppose that, for all $j = 2, \dots, m+1$, we have either $f^{(j)} \ge 0$ on I or $f^{(j)} \le 0$ on I. Then*

$$|f^{(m)}(t_0)| \le 2^{\binom{m+2}{2}-2} \frac{\sup_{t\in I} |f(t)|}{r^m}.$$

We combine Lemma 2.1 with the following lemma in order to get a $C^{m,\omega}$-version in Lemma 2.3.

Lemma 2.2. *Let $f : I \to \mathbb{R}$ be a C^2-function, where $I = [t_0 - r, t_0 + r] \subseteq \mathbb{R}$, $r > 0$, is an interval, and let ω be a modulus of continuity for f. Suppose that $f'' \ge 0$ on I or $f'' \le 0$ on I. Then*

$$|f'(t_0)| \le \frac{\omega(r)}{r}.$$

Proof. We may assume that $t_0 = 0$. Suppose that $f'' \ge 0$ on I. Then f is convex and, for $0 < s < r$,

$$\frac{f(s) - f(0)}{s} \le \frac{f(r) - f(0)}{r} \le \frac{\omega(r)}{r}.$$

Letting $s \to 0$, we find that $f'(0) \le \omega(r)/r$. The same reasoning applied to $f(-t)$ shows that also $-f'(0) \le \omega(r)/r$ so that the assertion is proved.

The case $f'' \le 0$ follows from the previous one by considering $-f$. □

Lemma 2.3. *Let $m \ge 1$. Let $f : I \to \mathbb{R}$ be a C^{m+1}-function, where $I = [t_0 - r, t_0 + r] \subseteq \mathbb{R}$, $r > 0$, is an interval, and let ω be a modulus of continuity for f. Suppose that, for all $j = 2, \dots, m+1$, we have either $f^{(j)} \ge 0$ on I or $f^{(j)} \le 0$ on I. Then*

$$|f^{(m)}(t_0)| \le 2^{\binom{m+1}{2}+m-2} \frac{\omega(r)}{r^m}.$$

Proof. If $m = 1$, then the statement is immediate from Lemma 2.2. If $m \ge 2$, then, by Lemma 2.1 applied to f' and in turn Lemma 2.2, we have

$$|f^{(m)}(t_0)| \le \frac{2^{\binom{m+1}{2}-2}}{(\frac{r}{2})^{m-1}} \sup_{|t-t_0|\le\frac{r}{2}} |f'(t)| \le \frac{2^{\binom{m+1}{2}-2}}{(\frac{r}{2})^{m-1}} \cdot \frac{\omega(\frac{r}{2})}{\frac{r}{2}} \le 2^{\binom{m+1}{2}+m-2}\,\frac{\omega(r)}{r^m}$$

as claimed, since ω is increasing. □

2.3. Uniform bounds for definable families of functions.

Proposition 2.4. *Let $(U_a)_{a\in A}$ be a definable family of open sets $U_a \subseteq \mathbb{R}^k$ and let $U \subseteq \mathbb{R}^N \times \mathbb{R}^k$ be the associated definable set (cf. (2.1)). Let $(\varphi_a)_{a\in A}$ be a definable family of functions $\varphi_a : U_a \to \mathbb{R}$ and let $\varphi : U \to \mathbb{R}$ be the associated definable function (cf. (2.2)). Let $\alpha \in \mathbb{N}^k$ with $|\alpha| = p$. There exists a definable subset $Z \subseteq U$ such that, for all $a \in A$, Z_a is closed in U_a, $\dim Z_a < k$, φ_a is C^p on $U_a \setminus Z_a$, and, for each open ball $B = B(u, r)$, $r > 0$, contained in $U_a \setminus Z_a$, we have*

$$|\partial^\alpha \varphi_a(u)| \le C(k,p) \sup_{v\in B} |\varphi_a(v)|\, r^{-|\alpha|}. \tag{2.3}$$

Proof. Consider the definable set

$$\begin{aligned} X &:= \{(b,v) \in U : (a,u) \mapsto \varphi(a,u) \text{ is not } C^p \text{ at } (b,v) \text{ in } u\} \\ &= \{(b,v) \in U : \varphi_b \text{ is not } C^p \text{ at } v\} \end{aligned}$$

and note that

$$X_a = \{u \in U_a : \varphi_a \text{ is not } C^p \text{ at } u\}$$

is closed in U_a and, by o-minimality, $\dim X_a < k$.

The operator ∂^α is a linear combination of directional derivatives d_v^p for a finite collection V of suitably chosen unit directions v in $\mathbb{R}^k$. Let $\varphi_1, \ldots, \varphi_s$ be an enumeration of all functions $d_v^j\varphi : U \setminus X \to \mathbb{R}$, $j = 2, \ldots, p+1$, $v \in V$ (where d_v^j acts only in the u-variable: $d_v^j\varphi(a,u) = \partial_t^j|_{t=0}\, \varphi(a, u+tv)$). For $i = 1, \ldots, s$, set

$$\begin{aligned} Y_i &:= \bigcup_{a\in A} \{(a,u) \in U \setminus X : \exists\epsilon > 0\, \forall v \in B(u,\epsilon) : \varphi_i(a,v) = 0\}, \\ Z_i &:= \{(a,u) \in U \setminus X : \varphi_i(a,u) = 0\} \setminus Y_i, \end{aligned}$$

and

$$Z := X \cup \bigcup_{i=1}^{s} Z_i.$$

Then Z is a definable subset of U and, for all $a \in A$, Z_a is closed in U_a and $\dim Z_a < k$.

Now (2.3) follows easily by applying Lemma 2.1 to $t \mapsto \varphi(a, u+tv)$. □

Corollary 2.5. *Let $(U_a)_{a\in A}$ be a definable family of open sets $U_a \subseteq \mathbb{R}^k$ and let $(\varphi_a)_{a\in A}$ be a definable family of C^1-functions $\varphi_a : U_a \to \mathbb{R}$. Suppose that there is a constant $M > 0$ such that*

$$|\partial_j\varphi_a(u)| \le M, \quad a \in A,\, u \in U_a,\, j = 1, \ldots, k.$$

Let p be a positive integer. There exists a definable family $(Z_a)_{a\in A}$ of closed definable sets $Z_a \subseteq U_a$ of dimension $\dim Z_a < k$ such that, for all $a \in A$, φ_a is of class C^p on $U_a \setminus Z_a$ and

$$|\partial^\gamma \varphi_a(u)| \le C(k,p)\, M\, d(u, Z_a \cup \partial U_a)^{1-|\gamma|}, \quad u \in U_a \setminus Z_a,\, 1 \le |\gamma| \le p.$$

Proof. Apply Proposition 2.4 to $\partial_j \varphi_a$. □

Remark 2.6. We may assume that Z_a is not empty so that $d(u, Z_a \cup \partial U_a)$ is always finite. We will tacitly make this assumption in all subsequent results of this type.

Proposition 2.7. *Let $(U_a)_{a\in A}$ be a definable family of open sets $U_a \subseteq \mathbb{R}^k$ and let $(\varphi_a)_{a\in A}$ be a definable family of continuous functions $\varphi_a : U_a \to \mathbb{R}$. Let p be a positive integer. Then there exists a definable family $(Z_a)_{a\in A}$ of closed subsets $Z_a \subseteq U_a$ of dimension $\dim Z_a < k$ such that, for all $a \in A$, φ_a is C^p on $U_a \setminus Z_a$ and*

$$|\partial^\gamma \varphi_a(x)| \le C(k,p)\, \frac{\omega(d(x, Z_a \cup \partial U_a))}{d(x, Z_a \cup \partial U_a)^{|\gamma|}}, \quad x \in U_a \setminus Z_a,\ 1 \le |\gamma| \le p, \tag{2.4}$$

where ω is a modulus of continuity for φ_a.

Proof. Follow the proof of Proposition 2.4 and use Lemma 2.3. □

Remark 2.8. We want to emphasize that the construction of $(Z_a)_{a\in A}$ is independent of ω.

2.4. **Λ_p-regular mappings.** Let $U \subseteq \mathbb{R}^k$ be an open set. Let p be a positive integer. A C^p-mapping $\varphi : U \to \mathbb{R}^n$ is said to be *Λ_p-regular* if there exists a constant $C > 0$ such that

$$|\partial^\gamma \varphi(u)| \le C\, d(u, \partial U)^{1-|\gamma|}, \quad u \in U,\ 1 \le |\gamma| \le p. \tag{2.5}$$

Λ_p-regular maps behave nicely on quasiconvex sets. Let us first recall the definition of quasiconvexity.

Definition 2.9 (Quasiconvex sets)**.** A set $E \subseteq \mathbb{R}^n$ is called *quasiconvex* if there is a constant $C > 0$ such any two points $x, y \in E$ can be joined in E by a rectifiable path of length at most $C\,|x-y|$.

Let $\varphi : U \to \mathbb{R}^n$ by Λ_p-regular. If $E \subseteq U$ is a quasiconvex subset, then $\varphi|_E$ is Lipschitz on E and extends continuously to a map $\overline{\varphi}$ on $\overline{E}$.

2.5. **Λ_p-cells.** Let p be a positive integer. We define recursively Λ_p-cells in $\mathbb{R}^n$:

A definable subset $S \subseteq \mathbb{R}^n$ is an *open Λ_p-cell in $\mathbb{R}^n$* if,

in case $n = 1$: S is an open interval in $\mathbb{R}$,

in case $n > 1$: S is of the form

$$S = (\psi_1, \psi_2, T) := \{(x', x_n) : x' \in T,\ \psi_1(x') < x_n < \psi_2(x')\},$$

where T is an open Λ_p-cell in $\mathbb{R}^{n-1}$ and each ψ_i, $i = 1, 2$, is either a Λ_p-regular definable function $T \to \mathbb{R}$ or identically $-\infty$ or $+\infty$, and $\psi_1 < \psi_2$ on T. (Here $x' = (x_1, \dots, x_{n-1})$.)

Note that S is quasiconvex. Moreover, if ψ_i is finite, then it is Lipschitz on T and has a continuous extension $\overline{\psi}_i$ to $\overline{T}$.

A definable subset S of $\mathbb{R}^n$ is a *k-dimensional Λ_p-cell in $\mathbb{R}^n$*, where $k = 0, \dots, n-1$, if

$$S = \{(u, w) : u \in T,\ w = \varphi(u)\} = \Gamma(\varphi),$$

where $u = (x_1, \dots, x_k)$, $w = (x_{k+1}, \dots, x_n)$, T is an open Λ_p-cell in $\mathbb{R}^k$, and $\varphi : T \to \mathbb{R}^{n-k}$ is a Λ_p-regular definable map.

Definition 2.10 (Λ_p-cell with constant C)**.** A Λ_p*-cell* S *in* $\mathbb{R}^n$ is an open or a k-dimensional Λ_p-cell in $\mathbb{R}^n$. We say that S is a Λ_p-cell in $\mathbb{R}^n$ *with constant* C if all the Λ_p-regular maps involved in the recursive definition of S satisfy (2.5) with the same constant C.

2.6. **Associated functions.** We associate with any open Λ_p-cell S in $\mathbb{R}^n$ a sequence of $2n+1$ definable functions $\rho_j : \overline{S} \to [0, \infty]$, $j = 0, 1, 2, \ldots, 2n$. We put $\rho_0 \equiv 1$ and define ρ_j for $j \ge 1$ as follows:

Case $n = 1$**:** If $S = (a_1, a_2)$ we set

$$\rho_1(x) := \begin{cases} x - a_1 & \text{if } a_1 \in \mathbb{R}, \\ +\infty & \text{if } a_1 = -\infty, \end{cases}$$

$$\rho_2(x) := \begin{cases} a_2 - x & \text{if } a_2 \in \mathbb{R}, \\ +\infty & \text{if } a_2 = +\infty. \end{cases}$$

Case $n > 1$**:** If $S = (\psi_1, \psi_2, T)$ and σ_j, $j = 1, \ldots, 2n-2$, are the functions associated with T, we set $\rho_j(x) = \sigma_j(x')$, for $j = 1, \ldots, 2n-2$, and

$$\rho_{2n-1}(x) := \begin{cases} x_n - \overline{\psi}_1(x') & \text{if } \psi_1 \text{ is finite}, \\ +\infty & \text{if } \psi_1 \equiv -\infty, \end{cases}$$

$$\rho_{2n}(x) := \begin{cases} \overline{\psi}_2(x') - x_n & \text{if } \psi_2 \text{ is finite}, \\ +\infty & \text{if } \psi_2 \equiv +\infty. \end{cases}$$

Remark 2.11. We add the function ρ_0 (which is not present in [7, 8, 12]) in order to handle the extension from unbounded Λ_p-cells (cf. proof of Theorem 1.3).

Each of the functions ρ_j, that is finite, is Λ_p-regular on S and Lipschitz on $\overline{S}$, cf. [7, Lemma 4]. There is a positive constant $C > 0$ such that

$$\frac{1}{C} \min_{j \ge 1} \rho_j(x) \le d(x, \partial S) \le \min_{j \ge 1} \rho_j(x), \quad x \in \overline{S}, \tag{2.6}$$

where $d(x, \emptyset) = +\infty$ by convention, since the faces of S are graphs of Lipschitz maps; cf. [7, Lemma 3]. Consequently,

$$\frac{1}{C} \min_{j \ge 0} \rho_j(x) \le \min\{1, d(x, \partial S)\} \le \min_{j \ge 0} \rho_j(x), \quad x \in \overline{S}. \tag{2.7}$$

Furthermore, if ρ_j for $j \ge 1$ is finite, then there exists a positive constant $C > 0$ such that

$$|\partial^\gamma (1/\rho_j)(x)| \le C\, d(x, \partial S)^{-|\gamma|-1}, \quad x \in S,\ |\gamma| \le p, \tag{2.8}$$

cf. [7, Lemma 5] and (2.10). It follows that for all finite ρ_j, $j \ge 0$, we have

$$|\partial^\gamma (1/\rho_j)(x)| \le C\, \min\{1, d(x, \partial S)\}^{-|\gamma|-1}, \quad x \in S,\ |\gamma| \le p. \tag{2.9}$$

Remark 2.12. The constants C in (2.6)–(2.9) only depend on the constants of the Λ_p-regular maps involved in the definition of S.

For later reference, we recall that for a non-vanishing C^p-function r we have, for $1 \le |\gamma| \le p$.

$$\partial^\gamma (1/r) = \sum_{j=1}^{|\gamma|} \Big(\sum_{\substack{\delta_1 + \cdots + \delta_j = \gamma \\ \delta_1 \ne 0, \ldots, \delta_j \ne 0}} a^\gamma_{\delta_1 \ldots \delta_j} \partial^{\delta_1} r \cdots \partial^{\delta_j} r \Big) r^{-j-1}, \tag{2.10}$$

where $a^\gamma_{\delta_1\dots\delta_j}$ are integers that only depend on γ and $\delta_1,\dots,\delta_j$.

2.7. **Λ_p-stratification of definable sets.** Recall that a definable C^p*-stratification* of a definable set $E \subseteq \mathbb{R}^n$ is a finite decomposition $\mathscr{S}$ of E into definable C^p-submanifolds of $\mathbb{R}^n$, called strata, such that, for each stratum $S \in \mathscr{S}$, the frontier $(\overline{S}\setminus S)\cap E$ in E is the union of some strata of dimension $< \dim S$. A stratification is called *compatible* with a collection of finitely many definable subsets of E if each subset is a union of strata.

A definable C^p-stratification $\mathscr{S}$ of E is called a Λ_p*-stratification* if each stratum $S \in \mathscr{S}$ is a Λ_p-cell in $\mathbb{R}^n$ in some linear coordinate system.

Theorem 2.13 ([7, Proposition 4] and [8, Theorem 3])**.** *Let $E \subseteq \mathbb{R}^n$ be a definable set and let $E_1,\dots,E_\ell$ be definable subsets of E. Then there exists a Λ_p-stratification $\mathscr{S}$ of E that is compatible with $E_1,\dots,E_\ell$.*

2.8. **Uniform Λ_p-stratifications of definable families of sets.** We prove a uniform version of Theorem 2.13. Let us first recall a result on uniform L-regular decompositions.

Theorem 2.14 ([6, Proposition 1.4])**.** *Let $E^k \subseteq \mathbb{R}^N \times \mathbb{R}^n$, $k \in K$, be a finite collection of definable sets. Then there exist finitely many disjoint definable sets $B^i \subseteq \mathbb{R}^N \times \mathbb{R}^n$, $i \in I$, and linear orthogonal mappings $\varphi^i : \mathbb{R}^n \to \mathbb{R}^n$, $i \in I$, such that:*

1. *For every $a \in \mathbb{R}^N$, each $\varphi^i(B^i_a)$ is a standard L-regular cell in $\mathbb{R}^n$ with constant $C = C(n)$.*
2. *For every $a \in \mathbb{R}^N$, the family B^i_a, $i \in I$, is a stratification of $\mathbb{R}^n$.*
3. *For any $k \in K$, there exists $I_k \subseteq I$ such that $E^k_a = \bigcup_{i\in I_k} B^i_a$ for every $a \in \mathbb{R}^N$.*

Here a *standard L-regular cell in $\mathbb{R}^n$ with constant $C = C(n)$* (which is terminology used in [6]) is by definition nothing else than a Λ_1-cell with constant $C = C(n)$.

Definition 2.15 (Uniform Λ_p-stratification)**.** Let $(E_a)_{a\in A}$ be a definable family of sets $E_a \subseteq \mathbb{R}^n$ and let $E \subseteq \mathbb{R}^N \times \mathbb{R}^n$ be the associated definable set (cf. (2.1)). Let p be a positive integer.

A finite collection $\mathscr{S} = \{S^j\}_{j\in J}$ of disjoint definable sets $S^j \subseteq \mathbb{R}^N \times \mathbb{R}^n$ is called a *uniform Λ_p-stratification of* $(E_a)_{a\in A}$ if

1. There exist linear orthogonal maps $\varphi^j : \mathbb{R}^n \to \mathbb{R}^n$, $j \in J$, such that, for each $a \in A$ and each $j \in J$, $\varphi^j(S^j_a)$ is a Λ_p-cell in $\mathbb{R}^n$ with constant C independent of $a \in A$.
2. For each $a \in A$, the family S^j_a, $j \in J$, is a stratification of E_a.

For all $a \in A$, let us set $\mathscr{S}_a := \{S^j_a\}_{j\in J}$. Abusing notation, we will also say that $(\mathscr{S}_a)_{a\in A}$ *is a uniform Λ_p-stratification of* $(E_a)_{a\in A}$.

Let $(E^i_a)_{a\in A}$, $i \in I$, be a finite collection of definable subfamilies of $(E_a)_{a\in A}$. A uniform Λ_p-stratification $(\mathscr{S}_a)_{a\in A}$ of $(E_a)_{a\in A}$ is said to be *compatible with* $(E^i_a)_{a\in A}$, $i \in I$, if additionally

3. For each $i \in I$, there exists a subset $J_i \subseteq J$ such that $E^i_a = \bigcup_{j\in J_i} S^j_a$ for each $a \in A$.

By Theorem 2.14, there always exist uniform Λ_1-stratifications. We shall see that there exist uniform Λ_p-stratifications for all $p \geq 1$.

Theorem 2.16. *Let $(E_a)_{a\in A}$ be a definable family of sets $E_a \subseteq \mathbb{R}^n$ and $(E_a^i)_{a\in A}$, $i \in I$, a finite collection of definable subfamilies of $(E_a)_{a\in A}$. Let p be a positive integer. Then there exists a uniform Λ_p-stratification $(\mathscr{S}_a)_{a\in A}$ of $(E_a)_{a\in A}$ compatible with $(E_a^i)_{a\in A}$, $i \in I$.*

Proof. Let $k = \max_{a\in A} \dim E_a$. We proceed by induction on k. If $k = 0$, then all E_a are finite and the number of elements of E_a is bounded by a constant independent of a. In that case, the assertion is trivially true.

Suppose that $k > 0$. We claim that there exist a finite collection of disjoint definable sets $T^j \subseteq \mathbb{R}^N \times \mathbb{R}^n$, $j \in J$, and linear orthogonal maps $\varphi^j : \mathbb{R}^n \to \mathbb{R}^n$, $j \in J$, such that, for each $a \in A$ and each $j \in J$,

- T_a^j is either empty or open in E_a and compatible with E_a^i, $i \in I$,
- if $T_a^j \neq \emptyset$ then $\varphi^j(T_a^j)$ is a k-dimensional Λ_p-cell in $\mathbb{R}^n$ with constant C independent of $a \in A$, and
- $\dim E_a \setminus \bigcup_{j\in J} T_a^j < k$.

We allow $T_a^j = \emptyset$ to account for the case $\dim E_a < k$.

Then we can use the induction hypothesis for the definable family $(E'_a)_{a\in A}$, where $E'_a := E_a \setminus \bigcup_{j\in J} T_a^j$, and the definable subfamilies $(E_a^i \cap E'_a)_{a\in A}$, $i \in I$, and $((\overline{T}_a^j \setminus T_a^j) \cap E'_a)_{a\in A}$, $j \in J$. The statement follows.

Let us prove the claim. Theorem 2.14 implies that the claim holds for $p = 1$: let T^j, $j \in J$, be the corresponding sets with all the properties as listed in the claim. Now we apply Corollary 2.5 and induction on the dimension. In fact, for each fixed $j \in J$, $(T_a^j)_{a\in A}$ is a definable family of k-dimensional Λ_1-cells T_a^j in $\mathbb{R}^n$ that are open in E_a. After the change of coordinates φ^j, we may assume that T_a^j is a Λ_1-cell with constant C independent of $a \in A$. By Corollary 2.5, there is a definable family $(Z_a^j)_{a\in A}$ of closed definable sets $Z_a^j \subseteq T_a^j$, $\dim Z_a^j < k$, such that the functions defining the cell T_a^j are Λ_p-regular with uniform constants independent of $a \in A$ in the complement of Z_a^j. Thus there exists a definable family $(S_a^j)_{a\in A}$ of subsets $S_a^j \subseteq T_a^j$ such that, for all $a \in A$, S_a^j is a finite disjoint union of k-dimensional definable Λ_p-cells $S_a^{j,\ell}$ that are open in E_a with constant C independent of $a \in A$ and $\dim T_a^j \setminus S_a^j < k$. Thus the number of connected components $S_a^{j,\ell}$ of S_a^j is uniformly bounded by a constant independent of $a \in A$. This implies the claim. □

2.9. **Consequences.** We may use Theorem 2.16 in order to refine Proposition 2.4, Corollary 2.5, and Proposition 2.7.

Corollary 2.17. *Let $(U_a)_{a\in A}$ be a definable family of open sets $U_a \subseteq \mathbb{R}^k$. Let $(\varphi_a)_{a\in A}$ be a definable family of functions $\varphi_a : U_a \to \mathbb{R}$. Let p be a nonnegative integer. There exists a uniform Λ_p-stratification $(\mathscr{S}_a)_{a\in A}$ of $(U_a)_{a\in A}$ such that, for all $a \in A$ and each open stratum $S_a \in \mathscr{S}_a$, φ_a is C^p on S_a and*

$$|\partial^\gamma \varphi_a(u)| \le C(k,p)\, \frac{\sup\{|\varphi_a(v)| : v \in S_a,\ |v-u| < d(u,\partial S_a)\}}{d(u,\partial S_a)^{|\gamma|}}, \quad u \in S_a,\ |\gamma| \le p.$$

Proof. This follows from Theorem 2.16 and Proposition 2.4. □

Corollary 2.18. *Let $(U_a)_{a\in A}$ be a definable family of open sets $U_a \subseteq \mathbb{R}^k$. Let $(\varphi_a)_{a\in A}$ be a definable family of C^1-functions $\varphi_a : U_a \to \mathbb{R}$. Suppose that there is a constant $M > 0$ such that*

$$|\partial_j \varphi_a(u)| \le M, \quad a \in A,\ u \in U_a,\ j = 1, \dots, k.$$

Let p be a positive integer. There exists a uniform Λ_p-stratification $(\mathscr{S}_a)_{a\in A}$ of $(U_a)_{a\in A}$ such that, for all $a \in A$ and each open stratum $S_a \in \mathscr{S}_a$, φ_a is C^p on S_a and

$$|\partial^\gamma \varphi_a(u)| \le C(k,p)\, M\, d(u, \partial S_a)^{1-|\gamma|}, \quad u \in S_a,\ 1 \le |\gamma| \le p.$$

Proof. This follows from Theorem 2.16 and Corollary 2.5. □

Proposition 2.19. *Let $(U_a)_{a\in A}$ be a definable family of open sets $U_a \subseteq \mathbb{R}^k$. Let $(\varphi_a)_{a\in A}$ be a definable family of continuous functions $\varphi_a : U_a \to \mathbb{R}$. Let p be a positive integer. There exists a uniform Λ_p-stratification $(\mathscr{S}_a)_{a\in A}$ of $(U_a)_{a\in A}$ such that, for all $a \in A$ and each open stratum $S_a \in \mathscr{S}_a$, φ_a is C^p on S_a and*

$$|\partial^\gamma \varphi_a(u)| \le C(k,p)\, \frac{\omega(d(u, \partial S_a))}{d(u, \partial S_a)^{|\gamma|}}, \quad u \in S_a,\ 1 \le |\gamma| \le p,$$

where ω is a modulus of continuity for φ_a.

Proof. This follows from Theorem 2.16 and Proposition 2.7. □

We will need another uniform fact:

Proposition 2.20. *Let $(U_a)_{a\in A}$ be a definable family of open sets $U_a \subseteq \mathbb{R}^k$. Let $(\varphi_a)_{a\in A}$ a definable family of functions $\varphi_a : U_a \to \mathbb{R}$. Let p be a positive integer. There exists a uniform Λ_p-stratification $(\mathscr{S}_a)_{a\in A}$ of $(U_a)_{a\in A}$ such that, for all $a \in A$ and each open stratum $S_a \in \mathscr{S}_a$, φ_a is C^p on S_a and, for all $j = 1, \ldots, k$,*

$$\text{either} \quad |\partial_j \varphi_a| \ge 1 \text{ on } S_a \quad \text{or} \quad |\partial_j \varphi_a| < 1 \text{ on } S_a. \tag{2.11}$$

Proof. Let $U \subseteq \mathbb{R}^N \times \mathbb{R}^n$ and $\varphi : U \to \mathbb{R}$ be the associated definable set and function (cf. (2.1) and (2.2)). Let $X \subseteq U$ be the set defined in the proof of Proposition 2.4.

For $j = 1, \ldots, k$, let $\partial_j \varphi(a,u) := \partial_j \varphi_a(u)$ and set

$$Y_j := \bigcup_{a \in A} \{(a,u) \in U \setminus X : \exists \epsilon > 0\, \forall v \in B(u,\epsilon) : \partial_j \varphi(a,v) = 1\},$$
$$Z_j := \{(a,u) \in U \setminus X : \partial_j \varphi(a,u) = 1\} \setminus Y_j,$$

and

$$Z := X \cup \bigcup_{j=1}^k Z_j.$$

Then Z is a definable subset of U and, for all $a \in A$, Z_a is closed in U_a and $\dim Z_a < k$. Now the statement follows from Theorem 2.16. □

3. Bounded definable families of Whitney jets

Recall that a modulus of continuity ω is by definition a positive, continuous, increasing, and concave function $\omega : (0,\infty) \to (0,\infty)$ such that $\omega(t) \to 0$ as $t \to 0$.

3.1. **$C^{m,\omega}$-functions.** Let ω be a modulus of continuity. Let $U \subseteq \mathbb{R}^n$ be an open set. Let $C^{0,\omega}(U)$ be the set of all continuous bounded functions $f : U \to \mathbb{R}$ such that

$$|f|_{C^{0,\omega}(U)} := \inf\{C > 0 : |f(x) - f(y)| \le C\, \omega(|x-y|) \text{ for all } x, y \in U\} < \infty.$$

For a nonnegative integer m, the set $C^{m,\omega}(U)$ consists of all C^m-functions such that $\partial^\alpha f$ is globally bounded for all $|\alpha| \le m$ and $\partial^\alpha f \in C^{0,\omega}(U)$ for all $|\alpha| = m$. Then $C^{m,\omega}(U)$ is a Banach space with the norm

$$\|f\|_{C^{m,\omega}(U)} := \sup_{x\in U} \sup_{|\alpha|\le m} |\partial^\alpha f(x)| + \sup_{|\alpha|=m} |\partial^\alpha f|_{C^{0,\omega}(U)}.$$

We say that $f \in C^{m,\omega}(U)$ is *m-flat* outside an open set $V \subseteq U$ if all $\partial^\alpha f$, $|\alpha| \le m$, vanish on $U \setminus V$.

Assume that the open set $U \subseteq \mathbb{R}^n$ is definable. We denote by $C^{m,\omega}_{\text{def}}(U)$ the subspace of $C^{m,\omega}(U)$ consisting of the definable functions in the latter space. Note that the normed space $C^{m,\omega}_{\text{def}}(U)$ is not complete.

Definition 3.1 (Bounded families of $C^{m,\omega}$-functions)**.** Let $m \in \mathbb{N}$ and ω a modulus of continuity. A family $(f_a)_{a\in A}$ of $C^{m,\omega}$-functions $f_a : U_a \to \mathbb{R}$, where $U_a \subseteq \mathbb{R}^n$ is open, is said to be a *bounded family of $C^{m,\omega}$-functions* if

$$\sup_{a\in A} \|f_a\|_{C^{m,\omega}(U_a)} < \infty.$$

We say that $(f_a)_{a\in A}$ is a *definable bounded family of $C^{m,\omega}$-functions* if it is a bounded family of $C^{m,\omega}$-functions and, additionally, the families $(U_a)_{a\in A}$ and $(f_a)_{a\in A}$ are definable. Moreover, $(f_a)_{a\in A}$ is called *m-flat outside* $(V_a)_{a\in A}$ if, for each $a \in A$, $V_a \subseteq U_a$ is open and f_a is m-flat on $U_a \setminus V_a$. We will say that $(f_a)_{a\in A}$ is *C^p outside* $(E_a)_{a\in A}$ if, for each $a \in A$, $E_a \subseteq U_a$ is closed and f_a is C^p on $U_a \setminus E_a$.

3.2. **Whitney jets of class $C^{m,\omega}$.** Let E be a locally closed subset of $\mathbb{R}^n$. An *m-jet on E* is a collection $F = (F^\alpha)_{|\alpha|\le m}$ of continuous functions $F^\alpha : E \to \mathbb{R}$. An m-jet $F = (F^\alpha)_{|\alpha|\le m}$ on E is said to be *flat* on a subset $E' \subseteq E$ if all functions F^α, $|\alpha| \le m$, vanish on E'.

For $a \in E$ we denote by $T^m_a F$ the Taylor polynomial

$$T^m_a F(x) = \sum_{|\alpha|\le m} \frac{1}{\alpha!} F^\alpha(a)(x-a)^\alpha, \quad x \in \mathbb{R}^n,$$

and define the m-jet

$$R^m_a F := F - J^m_E(T^m_a F),$$

where $J^m_E(f) := (\partial^\alpha f|_E)_{|\alpha|\le m}$ for $f \in C^m(\mathbb{R}^n)$.

A $C^{m,\omega}$ (or C^m) function $f : \mathbb{R}^n \to \mathbb{R}$ is an *extension to $\mathbb{R}^n$ of F* if $J^m_E(f) = F$. A necessary and sufficient condition for an m-jet to have a $C^{m,\omega}$-extension to $\mathbb{R}^n$ is to be a Whitney jet of class $C^{m,\omega}$ ([16], [4]):

An m-jet $F = (F^\alpha)_{|\alpha|\le m}$ on E is a *Whitney jet of class $C^{m,\omega}$ on E* if there exists $C > 0$ such that

$$\sup_{x\in E} \sup_{|\alpha|\le m} |F^\alpha(x)| \le C \tag{3.1}$$

and, for all $x, y \in E$ and $|\alpha| \le m$,

$$|(R^m_x F)^\alpha(y)| \le C\,\omega(|x-y|)|x-y|^{m-|\alpha|}. \tag{3.2}$$

Remark 3.2. A condition equivalent to (3.2) is

$$|T^m_x F(z) - T^m_y F(z)| \le C'\,\omega(|x-y|)(|z-x|^m + |z-y|^m) \tag{3.3}$$

for all $x, y \in E$ and $z \in \mathbb{R}^n$; see [13, Proposition IV.1.5]. Moreover, (3.2) holds if and only if

$$|F^0(x) - T^m_y F(x)| \le C\,\omega(|x-y|)|x-y|^m \quad \text{for all } x, y \in E, \tag{3.4}$$

and, if $m \geq 1$,

(3.5)

$$\partial_i F := (F^{\alpha+(i)})_{|\alpha|\le m-1} \text{ is a Whitney jet of class } C^{m-1,\omega} \text{ for all } i = 1, \ldots, n.$$

If E is quasiconvex with constant C' (see Definition 2.9), then (3.2) follows from

$$|F^\alpha(x) - F^\alpha(y)| \le C''\,\omega(|x-y|), \quad x, y \in E,\ |\alpha| = m,$$

cf. [13, IV (2.5.1)]; then the constant C in (3.2) depends only on n, m, C', and C''.

It is not hard to see that the set of all Whitney jets of class $C^{m,\omega}$ on E with the natural addition and the multiplication $FG := J^m_E(T^m F \cdot T^m G)$ is an $\mathbb{R}$-algebra.

Let F be an m-jet on $E \subseteq \mathbb{R}^n$. Let $G_1, \ldots, G_n$ be m-jets on $A \subseteq \mathbb{R}^k$ such that $(G_1^0, \ldots, G_n^0)(A) \subseteq E$. The *composite* of F and G is the m-jet $F \circ G = F \circ (G_1, \ldots, G_n)$ on A defined by

$$(F \circ G)(x) := J^m_A(T^m_{G^0(x)} F \circ T^m_x G)(x).$$

Note that

$$T^m_y(F \circ G)(x) = \pi_m\big(T^m_{G^0(y)} F(T^m_y G(x))\big),$$

where π_m is the natural truncation operator (which truncates monomials of order $> m$). One can show (using Remark 3.2) that, for $m \ge 1$, the composite $F \circ G$ is a Whitney jet of class $C^{m,\omega}$ if F and G are Whitney jets of class $C^{m,\omega}$. We will not use this fact, but the pullback of Whitney jets of class $C^{m,\omega}$ along a Λ_m-regular map will be crucial; see Proposition 3.5.

Definition 3.3 (Bounded families of Whitney jets of class $C^{m,\omega}$)**.** A family $(F_a)_{a\in A}$ of Whitney jets F_a of class $C^{m,\omega}$ on $E_a \subseteq \mathbb{R}^n$ is said to be a *bounded family of Whitney jets of class* $C^{m,\omega}$ if the constant $C > 0$ in (3.1) and (3.2) can be chosen independent of $a \in A$, that is,

$$\sup_{a\in A} \sup_{x\in E_a} \sup_{|\gamma|\le m} |F_a^\gamma(x)| < \infty \tag{3.6}$$

and

$$\sup_{a\in A} \sup_{x\ne y\in E_a} \sup_{|\gamma|\le m} \frac{|(R^m_x F_a)^\gamma(y)|}{\omega(|x-y|)|x-y|^{m-|\gamma|}} < \infty. \tag{3.7}$$

We say that $(F_a)_{a\in A}$ is a *definable bounded family of Whitney jets of class* $C^{m,\omega}$ if it is a bounded family of Whitney jets of class $C^{m,\omega}$ and, additionally, the families $(E_a)_{a\in A}$ and $(F_a^\gamma)_{a\in A}$, $|\gamma| \le m$, are definable. We say that it is *flat* on a subfamily $(E'_a)_{a\in A}$ of $(E_a)_{a\in A}$ if F_a is flat on E'_a for all $a \in A$.

A (definable bounded) family $(f_a)_{a\in A}$ of $C^{m,\omega}$-functions $f_a : \mathbb{R}^n \to \mathbb{R}$ is called a *(definable bounded) family of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of* $(F_a)_{a\in A}$ if f_a is a $C^{m,\omega}$-extension of F_a to $\mathbb{R}^n$, for each $a \in A$.

3.3. **Separation.** Let X, Y, Z be subsets of $\mathbb{R}^n$. Recall that X and Y are said to be *Z-separated* if there exists $C > 0$ such that

$$d(x, Y) \ge C\, d(x, Z), \quad x \in X, \tag{3.8}$$

or equivalently, if there is $C' > 0$ such that

$$d(x, X) + d(x, Y) \ge C'\, d(x, Z), \quad x \in \mathbb{R}^n.$$

If X and Y are $X\cap Y$-separated, then we will simply say that X and Y are *separated*.

Definition 3.4 (Uniformly separated families of sets)**.** Let $(X_a)_{a\in A}$, $(Y_a)_{a\in A}$, and $(Z_a)_{a\in A}$ be definable families of subsets of $\mathbb{R}^n$. Then $(X_a)_{a\in A}$ and $(Y_a)_{a\in A}$ are said to be *uniformly* $(Z_a)_{a\in A}$*-separated* if, for all $a \in A$, X_a and Y_a are Z_a-separated with a constant $C > 0$ (in (3.8)) independent of $a \in A$. We will say that $(X_a)_{a\in A}$ and $(Y_a)_{a\in A}$ are *uniformly separated* if they are uniformly $(X_a \cap Y_a)_{a\in A}$-separated.

3.4. **Pullback along a definable family of Λ_p-regular maps.** Let $\varphi : U \to \mathbb{R}^\ell$ be a Λ_{m+1}-regular map, where $U \subseteq \mathbb{R}^k$ is open and quasiconvex. Let $\overline{\varphi} : \overline{U} \to \mathbb{R}^\ell$ the continuous extension of φ; cf. Section 2.4. Consider

$$\varphi_+ : U \times \mathbb{R}^\ell \to U \times \mathbb{R}^\ell, \quad (u, w) \mapsto (u, w + \varphi(u)),$$

and

$$\overline{\varphi}_+ : \overline{U} \times \mathbb{R}^\ell \to \overline{U} \times \mathbb{R}^\ell, \quad (u, w) \mapsto (u, w + \overline{\varphi}(u)).$$

Let M be a closed subset of $U \times \mathbb{R}^\ell$ and let F be an m-jet on M. The *pullback of* F *along* φ_+ is the m-jet

$$\varphi_+^* F := F \circ J_N^m(\varphi_+)$$

on $N := \varphi_+^{-1}(M) = \{(u, w) \in U \times \mathbb{R}^\ell : (u, w + \varphi(u)) \in M\}$.

We shall need the following result on the pullback of a definable bounded family of Whitney jets of class $C^{m,\omega}$ along a definable family $(\varphi_a)_{a\in A}$ of Λ_{m+1}-regular maps. For each $a \in A$, let $\varphi_{a,+}$ and $\overline{\varphi}_{a,+}$ be defined in analogy to φ_+ and $\overline{\varphi}_+$.

Proposition 3.5 ([10, Proposition 4.3])**.** *Let $(U_a)_{a\in A}$ be a definable family of open quasiconvex sets $U_a \subseteq \mathbb{R}^k$ with constant (in Definition 2.9) independent of $a \in A$. Let $(\varphi_a)_{a\in A}$ be a definable family of Λ_{m+1}-regular maps $\varphi_a : U_a \to \mathbb{R}^\ell$ with constant (in (2.5)) independent of $a \in A$. Let $(M_a)_{a\in A}$ be a definable family of closed quasiconvex subsets M_a of $U_a \times \mathbb{R}^\ell$ such that $(\overline{M}_a)_{a\in A}$ and $(\partial U_a \times \mathbb{R}^\ell)_{a\in A}$ are uniformly separated.*

If $(F_a)_{a\in A}$ is a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(\overline{M}_a)_{a\in A}$ which is flat on $(\partial M_a)_{a\in A}$, then $(G_a)_{a\in A}$ is a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(N_a)_{a\in A}$, where $G_a := \varphi_{a,+}^* F_a$ and $N_a := \varphi_{a,+}^{-1}(M_a)$. Moreover, if, for each $a \in A$, $t_a : U_a \to (0, \infty)$ is a function satisfying $t_a(u) \le d(u, \partial U_a)$ for $u \in U_a$ and

$$|F_a^\kappa(u, w)| \le C_1\, \omega(t_a(u)) t_a(u)^{m-|\kappa|}, \quad (u, w) \in M_a,\ |\kappa| \le m,$$

with $C_1 > 0$ independent of $a \in A$, then, for each $a \in A$,

$$|G_a^\kappa(u, w)| \le C_2\, \omega(t_a(u)) t_a(u)^{m-|\kappa|}, \quad (u, w) \in N_a,\ |\kappa| \le m,$$

with $C_2 > 0$ independent of $a \in A$.

Proof. Follows from the proof of [10, Proposition 4.3]. □

We will be interested in the case that $M_a = \Gamma(\varphi_a)$, $a \in A$. Then $(G_a)_{a\in A}$ extends to a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(\overline{N}_a = \overline{U}_a \times 0)_{a\in A}$ which is flat on $(\partial N_a = \partial U_a \times 0)_{a\in A}$. This follows from the following lemma and Hestenes' lemma (e.g. [12, Theorem 1.10]); cf. [10, Remark 4.2].

Lemma 3.6. *Let $(E_a)_{a\in A}$ be a family of locally closed, quasiconvex sets $E_a \subseteq \mathbb{R}^n$ with constant (in Definition 2.9) independent of $a \in A$. Suppose that $(F_a)_{a\in A}$ is a bounded family of Whitney jets of class $C^{m,\omega}$ on $(E_a)_{a\in A}$ such that all F_a^α, $a \in A$, $|\alpha| \le m$, have a continuous extension $\overline{F}_a^\alpha$ to $\overline{E}_a$. Then $(\overline{F}_a)_{a\in A}$ is a bounded family of Whitney jets of class $C^{m,\omega}$ on $(\overline{E}_a)_{a\in A}$.*

Proof. Let $K \subseteq \mathbb{R}^n$ be compact and connected by rectifiable paths. Let δ be the inner metric on K. Let $F = (F^\alpha)_{|\alpha|\le m}$ be a Whitney jet of class C^m on K. Then, by [13, IV (2.5.1)], for all $x, y \in K$ and $|\alpha| \le m$,

$$|(R^m_x F)^\alpha(y)| \le n^{\frac{m-|\alpha|}{2}} \delta(x,y)^{m-|\alpha|} \sup_{\xi\in K} \sup_{|\beta|=m} |F^\beta(\xi) - F^\beta(x)|.$$

Let $x, y \in \overline{E}_a$. There exist sequences $(x_k), (y_k) \subseteq E_a$ such that $x_k \to x$ and $y_k \to y$. By assumption, there exist a constant $C_1 > 0$, independent of $a \in A$ and of k, and a rectifiable path σ_k joining x_k and y_k in E_a such that for the length of σ_k we have

$$\ell(\sigma_k) \le C_1\,|x_k - y_k|.$$

Applying the above inequality (to $K = \sigma_k$), we get

$$|(R^m_{x_k} F_a)^\alpha(y_k)| \le n^{\frac{m-|\alpha|}{2}} C_1^{m-|\alpha|} |x_k - y_k|^{m-|\alpha|} \sup_{\xi\in \sigma_k} \sup_{|\beta|=m} |F_a^\beta(\xi) - F_a^\beta(x_k)|.$$

We may assume that σ_k is parameterized by $t \in [0,1]$ with $\sigma_k(0) = x_k$ and $\sigma_k(1) = y_k$. By (3.7), for $t \in [0,1]$,

$$\begin{aligned}\sup_{|\beta|=m} |F^\beta(\sigma_k(t)) - F^\beta(x_k)| &\le C_2\,\omega(|\sigma_k(t) - x_k|) \le C_2\,\omega(\ell(\sigma_k|_{[0,t]}))\\ &\le C_2\,\omega(\ell(\sigma_k)) \le C_2\,\omega(C_1|x_k - y_k|) \le C_3\,\omega(|x_k - y_k|),\end{aligned}$$

for constants $C_i > 0$ independent of $a \in A$. Thus

$$|(R^m_{x_k} F_a)^\alpha(y_k)| \le n^{\frac{m-|\alpha|}{2}} C_1^{m-|\alpha|} |x_k - y_k|^{m-|\alpha|}\, C_3\,\omega(|x_k - y_k|)$$

and letting $k \to \infty$ shows that (3.7) is satisfied for $(\overline{F}_a)_{a\in A}$. That also (3.6) is satisfied is clear. □

3.5. **Cutoff.** We finish this section with a technical cutoff result which will be used in the proof of Theorem 1.3.

Proposition 3.7 ([10, Proposition 3.9])**.** *Assume that there are constants C_i, $i = 0, \ldots, 3$, such that the following holds. Let $(U_a)_{a\in A}$ be a definable family of open quasiconvex sets $U_a \subseteq \mathbb{R}^k$ with constant (in Definition 2.9) independent of $a \in A$. Let $(h_a)_{a\in A}$ be a definable family of C^m-functions $h_a : U_a \times \mathbb{R}^\ell \to \mathbb{R}$ and $(\rho_a)_{a\in A}$ a definable family of C^{m+1}-functions $\rho_a : U_a \to \mathbb{R}$. Let $(t_a)_{a\in A}$ be a definable family of positive C_0-Lipschitz functions $t_a : U_a \to (0,\infty)$ such that $t_a(u) \le d(u, \partial U_a)$ for all $a \in A$, $u \in U_a$. For $\epsilon > 0$ consider the definable family $(\Delta_a^\epsilon)_{a\in A}$, where*

$$\Delta_a^\epsilon := \{(u,w) \in U_a \times \mathbb{R}^\ell : |w| < \epsilon\, t_a(u)\}.$$

Assume that, for all $a \in A$,

$$|\partial^\alpha(1/\rho_a)(u)| \le C_1\, t_a(u)^{-|\alpha|-1}, \quad u \in U_a,\ |\alpha| \le m+1. \tag{3.9}$$

Let $\xi : \mathbb{R} \to \mathbb{R}$ be a definable C^m-function with compact support, fix $1 \le i \le \ell$, and set, for all $a \in A$,

$$f_a(u,w) := \xi\Big(\frac{w_i}{\rho_a(u)}\Big) h_a(u,w), \quad (u,w) \in U_a \times \mathbb{R}^\ell.$$

If $(h_a)_{a\in A}$ is a definable bounded family of $C^{m,\omega}$-functions on $(\Delta_a^\epsilon)_{a\in A}$ such that, for each $a \in A$,

$$|\partial^\gamma h_a(u,w)| \le C_2\,\omega(t_a(u)) t_a(u)^{m-|\gamma|}, \quad (u,w) \in \Delta_a^\epsilon,\ |\gamma| \le m,$$

then $(f_a)_{a\in A}$ *is a definable bounded family of* $C^{m,\omega}$*-functions on* $(\Delta_a^\epsilon)_{a\in A}$ *such that, for each* $a \in A$,

$$|\partial^\gamma f_a(u,w)| \le C_3\, \omega(t_a(u)) t_a(u)^{m-|\gamma|}, \quad (u,w) \in \Delta_a^\epsilon,\ |\gamma| \le m.$$

Proof. It suffices to repeat the proof of Proposition 3.9 in [10] (as well as Lemma 3.5 and Proposition 3.6 which are used in the proof). □

Remark 3.8. Proposition 3.5 and Proposition 3.7 remain true if we remove everywhere the attribute "definable".

4. Bounded definable extension of Whitney jets

This section is devoted to the proof of Theorem 1.3. Let us recall the setup.

Let $0 \le m \le p$ *be integers and* ω *a modulus of continuity. Let* $(E_a)_{a\in A}$ *be a definable family of closed subsets of* $\mathbb{R}^n$*. Let* $(F_a)_{a\in A}$ *be a definable bounded family of Whitney jets of class* $C^{m,\omega}$ *on* $(E_a)_{a\in A}$*. We will show that there exists a definable bounded family* $(f_a)_{a\in A}$ *of* $C^{m,\omega}$*-extensions to* $\mathbb{R}^n$ *of* $(F_a)_{a\in A}$ *that is* C^p *outside* $(E_a)_{a\in A}$.

For each $a \in A$, let $\operatorname{supp} F_a$ denote the closure of $\bigcup_{|\kappa|\le m}\{x \in E_a : F_a^\kappa(x) \ne 0\}$ and let $(E'_a)_{a\in A}$ be a definable subfamily of $(E_a)_{a\in A}$ consisting of closed subsets E'_a of E_a such that $\operatorname{supp} F_a \subseteq E'_a$.

Let $A' := \{a \in A : \operatorname{supp} F_a = \emptyset\}$. The family $(F_a)_{a\in A'}$ can be extended by $(0)_{a\in A'}$ to $\mathbb{R}^n$. So we may assume that, for all $a \in A$, $\operatorname{supp} F_a \ne \emptyset$ and thus $E'_a \ne \emptyset$.

We proceed by induction on $k := \max_{a\in A} \dim E'_a$ and show:

(I$_k$) *Let* $(E_a)_{a\in A}$ *be a definable family of closed subsets* E_a *of* $\mathbb{R}^n$ *and* $(F_a)_{a\in A}$ *a definable bounded family of Whitney jets of class* $C^{m,\omega}$ *on* $(E_a)_{a\in A}$*. Let* $(E'_a)_{a\in A}$ *be a definable subfamily of* $(E_a)_{a\in A}$ *of closed subsets* E'_a *of* E_a *such that* $\operatorname{supp} F_a \subseteq E'_a$ *and* $\dim E'_a \le k$*, for all* $a \in A$*. Then there exists a definable bounded family* $(f_a)_{a\in A}$ *of* $C^{m,\omega}$*-extensions to* $\mathbb{R}^n$ *of* $(F_a)_{a\in A}$ *that is* C^p *outside* $(E'_a)_{a\in A}$.

Let us fix an integer $p \ge m+1$.

Induction basis (I$_0$). If $k = 0$, then each E'_a is a finite set (but E_a might be infinite) and there is a constant which bounds the number $|E'_a|$ of elements of E'_a independently of $a \in A$.

Let us make induction on $s := \max_{a\in A} |E'_a|$. The base case $s = 1$ is treated in the following lemma.

Lemma 4.1. *Let* $(E_a)_{a\in A}$ *be a definable family of closed subsets* E_a *of* $\mathbb{R}^n$ *and* $(E'_a)_{a\in A}$ *a definable subfamily of* $(E_a)_{a\in A}$ *such that, for each* $a \in A$, $E'_a = \{x_a\}$*. Let* $(F_a)_{a\in A}$ *be a definable bounded family of Whitney jets of class* $C^{m,\omega}$ *on* $(E_a)_{a\in A}$ *such that* $\operatorname{supp} F_a \subseteq \{x_a\}$*, for all* $a \in A$*. Then there exists a definable bounded family* $(f_a)_{a\in A}$ *of* $C^{m,\omega}$*-extensions to* $\mathbb{R}^n$ *of* $(F_a)_{a\in A}$ *that is* C^p *outside* $(E'_a)_{a\in A}$.

Proof. Note that, for each $a \in A$, x_a is an isolated point of E_a, by continuity of F_a.

Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be a definable C^p-function that equals 1 in a neighborhood of 0 and has support contained in the unit ball. For each $a \in A$, set

$$d_a := \begin{cases} \min\{1, d(x_a, E_a \setminus \{x_a\})\} & \text{if } E_a \setminus \{x_a\} \ne \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

Define, for each $a \in A$,

$$f_a(x) := \chi\Big(\frac{x-x_a}{d_a}\Big) \cdot T^m_{x_a} F_a(x), \quad x \in \mathbb{R}^n.$$

Then $(f_a)_{a\in A}$ is a definable family of C^p-functions $f_a : \mathbb{R}^n \to \mathbb{R}$ such that each f_a has support contained in the ball $B_a := B(x_a, d_a)$ with radius d_a around x_a and extends the jet F_a. We will prove that the family $(f_a)_{a\in A}$ is bounded in $C^{m,\omega}(\mathbb{R}^n)$.

Let $\gamma \in \mathbb{N}^n$, $|\gamma| \le m$. Then

$$\partial^\gamma f_a(x) = \sum_{\alpha+\beta=\gamma} \binom{\gamma}{\alpha} d_a^{-|\alpha|}\, \partial^\alpha \chi\Big(\frac{x-x_a}{d_a}\Big) \partial^\beta T^m_{x_a} F_a(x).$$

By (3.7) (for $y \in E_a \setminus \{x_a\}$ with $d_a = |x_a - y|$ if $d_a < 1$) and (3.6) (if $d_a = 1$),

$$|F_a^\beta(x_a)| \le C\, \omega(d_a) d_a^{m-|\beta|}, \quad |\beta| \le m,$$

for a constant $C > 0$ independent of $a \in A$. For the rest of the proof, C will denote a constant independent of $a \in A$; its actual value may change. Thus, for $x \in B_a$,

$$\begin{aligned} |\partial^\beta T^m_{x_a} F_a(x)| &= \Big| \sum_{|\kappa| \le m-|\beta|} \frac{1}{\kappa!} F_a^{\kappa+\beta}(x_a)(x-x_a)^\kappa \Big| \\ &\le C\, \omega(d_a) d_a^{m-|\beta|}, \quad |\beta| \le m. \end{aligned}$$

It follows that, for all $x \in \mathbb{R}^n$,

$$|\partial^\gamma f_a(x)| \le C\, \omega(d_a) d_a^{m-|\gamma|} \le C\, \omega(1), \quad |\gamma| \le m. \tag{4.1}$$

Now assume that $|\gamma| = m$. To see that $|\partial^\gamma f_a|_{C^{0,\omega}(\mathbb{R}^n)}$ is bounded by a constant independent of $a \in A$, it suffices to estimate, for $\alpha + \beta = \gamma$,

$$D(x,y) := \Big| d_a^{-|\alpha|}\, \partial^\alpha \chi\Big(\frac{x-x_a}{d_a}\Big) \partial^\beta T^m_{x_a} F_a(x) - d_a^{-|\alpha|}\, \partial^\alpha \chi\Big(\frac{y-x_a}{d_a}\Big) \partial^\beta T^m_{x_a} F_a(y) \Big|.$$

Let us first assume that $x, y \in \overline{B}_a$. Then

$$\begin{aligned} &d_a^{-|\alpha|} \Big| \partial^\alpha \chi\Big(\frac{x-x_a}{d_a}\Big) - \partial^\alpha \chi\Big(\frac{y-x_a}{d_a}\Big) \Big| |\partial^\beta T^m_{x_a} F_a(x)| \\ &\le C\, \frac{\omega(d_a)}{d_a} |x-y| \le 2C\, \omega(|x-y|), \end{aligned}$$

since ω is concave and $|x - y| \le 2d_a$. On the other hand,

$$\begin{aligned} &d_a^{-|\alpha|} \Big| \partial^\alpha \chi\Big(\frac{y-x_a}{d_a}\Big) \Big| |\partial^\beta T^m_{x_a} F_a(x) - \partial^\beta T^m_{x_a} F_a(y)| \\ &\le C\, d_a^{-|\alpha|} \sum_{|\kappa| \le m-|\beta|} \frac{1}{\kappa!} |F_a^{\kappa+\beta}(x_a)| |(x-x_a)^\kappa - (y-x_a)^\kappa| \\ &\le C'\, \frac{\omega(d_a)}{d_a}\, |x-y| \le 2C'\, \omega(|x-y|). \end{aligned}$$

So $D(x, y) \le C\, \omega(|x - y|)$ for a constant $C > 0$ independent of $a \in A$.

If x and y lie outside of B_a, then $D(x, y) = 0$. If $x \in B_a$ and $y \notin \overline{B}_a$ and z is the point, where the line segment $[x, y]$ meets ∂B_a, then

$$D(x,y) = D(x,z) \le C\, \omega(|x-z|) \le C\, \omega(|x-y|).$$

This ends the proof. $\square$

Now assume that $s > 1$. For each $a \in A$, choose a numbering of the elements of $E'_a = \{x_{a,1}, \dots, x_{a,s_a}\}$, where $s_a \le s$. By the induction hypothesis, $(F_a|_{E_a \setminus \{x_{a,2},\dots,x_{a,s_a}\}})_{a\in A}$ admits a definable bounded family $(f_a^1)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ that is C^p outside $(\{x_{a,1}\})_{a\in A}$. Then $(F_a - J^m_{E_a}(f^1_a))_{a\in A}$ is a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(E_a)_{a\in A}$ which is flat on $(E_a \setminus \{x_{a,2}, \dots, x_{a,s_a}\})_{a\in A}$ and has a definable bounded family $(f^2_a)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ that is C^p outside $(\{x_{a,2}, \dots, x_{a,s_a}\})_{a\in A}$, again by the induction hypothesis. Thus, $(f^1_a + f^2_a)_{a\in A}$ is the desired definable bounded family of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(F_a)_{a\in A}$.

This ends the induction on s and the base case $(\mathbf{I}_0)$ of the induction on k.

Setup for the induction step. Let $k > 0$ and suppose that $(\mathbf{I}_{k-1})$ holds. We will prove $(\mathbf{I}_k)$. This will be accomplished by showing Proposition 4.2 below, but first we make a few preparatory reductions.

By $(\mathbf{I}_{k-1})$, we may assume that $\dim E'_a = k$ for all $a \in A$.

By Theorem 2.16, there is a uniform Λ_p-stratification $(\mathscr{S}_a)_{a\in A}$ of $(E_a)_{a\in A}$ compatible with $(E'_a)_{a\in A}$ such that, for each $a \in A$ and each $|\kappa| \le m$, F^κ_a is of class C^p on the strata in $\mathscr{S}_a$.

By $(\mathbf{I}_{k-1})$, there is a definable bounded $C^{m,\omega}$-extension $(f^0_a)_{a\in A}$ to $\mathbb{R}^n$ of the restriction of $(F_a)_{a\in A}$ to $(E_a \setminus P_a)_{a\in A}$, where

$$P_a = \bigcup \{S_a \in \mathscr{S}_a : S_a \subseteq E'_a, \dim S_a = k\}.$$

Replacing F_a by $F_a - J^m_{E_a}(f^0_a)$, for each $a \in A$, we may assume that F_a is flat on all strata $S_a \in \mathscr{S}_a$, $S_a \subseteq E'_a$, with $\dim S_a < k$ and also on $E_a \setminus E'_a$.

Let us now see that we may furthermore reduce to the case that, for each $a \in A$, E'_a is the closure of just one k-dimensional stratum S_a and that F_a is flat on its frontier. Indeed, the number s_a of k-dimensional strata of E'_a is uniformly bounded by a constant not depending on $a \in A$. We may use induction on $s := \max_{a\in A} s_a$ of which the above statement is the base case that we take for granted for the moment. The induction step works just as for finite sets E'_a: for each $a \in A$, let $S_{a,1}, \dots, S_{a,s_a}$ be a numbering of the k-dimensional strata of E'_a. By the induction hypothesis, $(F_a|_{E_a\setminus R_a})_{a\in A}$, where $R_a := \bigcup_{i\ge 2} S_{a,i}$, admits a definable bounded family $(f^1_a)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ that is C^p outside $(E'_a \setminus R_a)_{a\in A}$. Then $(F_a - J^m_{E_a}(f^1_a))_{a\in A}$ is a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(E_a)_{a\in A}$ which is flat on $(E_a \setminus R_a)_{a\in A}$ and has a definable bounded family $(f^2_a)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ that is C^p outside $(R_a)_{a\in A}$, again by the induction hypothesis. Thus, $(f^1_a + f^2_a)_{a\in A}$ is the desired definable bounded family of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(F_a)_{a\in A}$.

In the case that $k = n$, S_a is open in $\mathbb{R}^n$ and extending F_a by 0 outside S_a, for all $a \in A$, yields a definable bounded family $(F_a)_{a\in A}$ of Whitney jets of class $C^{m,\omega}$ on $(\mathbb{R}^n)_{a\in A}$ so that $(F^0_a)_{a\in A}$ is the desired family of $C^{m,\omega}$-extensions. This follows from Hestenes' lemma (e.g. [12, Theorem 1.10]); indeed, if $x \in S_a$ and $y \notin \overline{S}_a$ and z is the point, where the line segment $[x, y]$ meets ∂S_a, then, by (3.7) and Remark 3.2, for any $u \in \mathbb{R}^n$,

$$\begin{aligned} |T^m_x F_a(u) - T^m_y F_a(u)| &= |T^m_x F_a(u)| = |T^m_x F_a(u) - T^m_z F_a(u)| \\ &\le C\,\omega(|x - z|)(|u - x|^m + |u - z|^m) \\ &\le 2C\,\omega(|x - y|)(|u - x|^m + |u - y|^m), \end{aligned}$$

for a constant $C > 0$ independent of $a \in A$, since $|u - z| \le \max\{|u - x|, |u - y|\}$.

Consequently, we may assume that $\ell := n - k > 0$.

We reduced the proof to showing the following. (We may assume that S_a is a Λ_p-cell in a fixed orthogonal system of coordinates of $\mathbb{R}^n$, which is independent of $a \in A$, thanks to Theorem 2.16.)

Proposition 4.2. *Let $(E_a)_{a\in A}$ be a definable family of closed sets E_a in $\mathbb{R}^n$. Let $(E'_a)_{a\in A}$ be a definable subfamily of $(E_a)_{a\in A}$ of closed subsets E'_a of E_a with $\dim E'_a = k$ such that $E'_a = \overline{S}_a$, where*

$$S_a = \{(u, \varphi_a(u)) \in \mathbb{R}^k \times \mathbb{R}^\ell : u \in T_a\} = \Gamma(\varphi_a),$$

and $(\varphi_a)_{a\in A}$ is a definable family of Λ_p-regular maps $\varphi_a : T_a \to \mathbb{R}^\ell$, T_a an open Λ_p-cell in $\mathbb{R}^k$, and all constants in the definition of T_a and φ_a are independent of $a \in A$. Then any definable bounded family $(F_a)_{a\in A}$ of Whitney jets of class $C^{m,\omega}$ on $(E_a)_{a\in A}$ such that, for all $a \in A$, $\operatorname{supp} F_a \subseteq E'_a$, F_a is flat on ∂S_a, and F^κ_a, $|\kappa| \le m$, is C^p on S_a, admits a definable bounded family $(f_a)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ that is C^p outside $(E'_a)_{a\in A}$.

The proposition is proved in three gradually more general steps.

Step 1: $\varphi_a \equiv 0$ and $E'_a = E_a$ for all $a \in A$.
Step 2: $\varphi_a \equiv 0$ for all $a \in A$.
Step 3: The general case.

Step 1. For all $a \in A$, $E_a = E'_a = \overline{T}_a \times 0$, where $T_a \subseteq \mathbb{R}^k$ is an open Λ_p-cell with constant C independent of $a \in A$. We will prove Proposition 4.2 in this special case with the additional property that $(f_a)_{a\in A}$ is m-flat outside $(\Delta(T_a \times 0))_{a\in A}$, where

$$\Delta(T_a \times 0) := \{(u, w) \in T_a \times \mathbb{R}^\ell : |w| < \min\{1, d(u, \partial T_a)\}\}. \tag{4.2}$$

For each $a \in A$, we write

$$F_a = (F_a^{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbb{N}^k\times\mathbb{N}^\ell, |\alpha|+|\beta|\le m}.$$

Fix $\beta \in \mathbb{N}^\ell$ with $|\beta| \le m$. Let $F_{a,\beta}$ be the m-jet which results from F_a by setting all $F_a^{(\alpha,\beta')}$ equal to 0 whenever $\beta' \ne \beta$. Then $(F_{a,\beta})_{a\in A}$ is a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(\overline{T}_a \times 0)_{a\in A}$. Indeed, for each $a \in A$, the definable Whitney jet F_a of class $C^{m,\omega}$ on $\overline{T}_a \times 0$ can be identified with a collection $\tilde{F}_{a,\beta}$, $|\beta| \le m$, where $\tilde{F}_{a,\beta}$ is a definable $C^{m-|\beta|,\omega}$-function on $\overline{T}_a$ such that $\partial^\alpha \tilde{F}_{a,\beta}(u) = F_a^{(\alpha,\beta)}(u, 0)$ for all $u \in \overline{T}_a$ and $\alpha \in \mathbb{N}^k$, $|\alpha| \le m - |\beta|$; see [8, Remark 5] and [4, pp. 87-88]. It suffices to prove that, for each β, $(F_{a,\beta})_{a\in A}$ admits a definable bounded family of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ that is m-flat outside $(\Delta(T_a \times 0))_{a\in A}$ and C^p outside $(\overline{T}_a \times 0)_{a\in A}$. Thus, we may suppose that, for each $a \in A$, $F_a^{(\alpha,\beta')} = 0$ whenever $\beta' \ne \beta$. By assumption, $F_a^{(\alpha,\beta)}$ is C^p on $T_a \times 0$.

By Theorem 2.16, Corollary 2.17, and Proposition 2.19, there is a uniform Λ_p-stratification $(\mathscr{D}_a)_{a\in A}$ of $(\overline{T}_a)_{a\in A}$ such that, for all $a \in A$, each open k-dimensional $D_a \in \mathscr{D}_a$, and all α, β, $F_a^{(\alpha,\beta)}$ is C^p on $D_a \times 0$, and, for all $u \in D_a$ and $\gamma \in \mathbb{N}^k$ with $|\gamma| \le p$, we have

$$|\partial^\gamma F_a^{(\alpha,\beta)}(u, 0)| \le L \, \frac{\sup\{|F_a^{(\alpha,\beta)}(v, 0)| : v \in D_a,\ |v - u| < d(u, \partial D_a)\}}{d(u, \partial D_a)^{|\gamma|}}, \tag{4.3}$$

and, if $1 \le |\gamma| \le p$,

$$|\partial^\gamma F_a^{(\alpha,\beta)}(u,0)| \le L\,\frac{\omega(d(u,\partial D_a))}{d(u,\partial D_a)^{|\gamma|}}, \tag{4.4}$$

where $L > 0$ is a constant independent of $a \in A$.

For each $a \in A$, let $Z_a := \bigcup\{D_a \in \mathscr{D}_a : \dim D_a < k\}$. Setting

$$G_a(x) := \begin{cases} F_a(x) & \text{if } x \in Z_a \times 0,\\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Delta(T_a \times 0),\end{cases}$$

defines a definable bounded family $(G_a)_{a\in A}$ of Whitney jets of class $C^{m,\omega}$ on $((Z_a \times 0) \cup (\mathbb{R}^n \setminus \Delta(T_a \times 0)))_{a\in A}$. This follows from Hestenes' lemma (e.g. [12, Theorem 1.10]) and the following reasoning. Clearly, $(G_a)_{a\in A}$ satisfies (3.6). To see (3.7) it suffices to consider the case that $x \in Z_a \times 0$ and $y \in \mathbb{R}^n \setminus \Delta(T_a \times 0)$ and to show that

$$|F_a^\kappa(x)| \le C\,\omega(|x-y|)|x-y|^{m-|\kappa|}, \quad |\kappa| \le m, \tag{4.5}$$

for a constant $C >$ independent of $a \in A$. We have

$$|F_a^\kappa(x)| \le C\,\omega(d(u,\partial T_a))d(u,\partial T_a)^{m-|\kappa|}, \quad |\kappa| \le m,$$

by (3.7), since $(F_a)_{a\in A}$ is flat on $(\partial T_a \times 0)_{a\in A}$, and

$$|F_a^\kappa(x)| \le C = \frac{C}{\omega(1)} \cdot \omega(1)1^{m-|\kappa|}, \quad |\kappa| \le m, \tag{4.6}$$

by (3.6). Then (4.5) follows, since we have $|x-y| \ge c\,\min\{1, d(u,\partial T_a)\}$ for a universal constant $c > 0$, by (4.2).

By $(\mathbf{I}_{k-1})$, there exists a definable bounded family $(g_a)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(G_a)_{a\in A}$ that is C^p outside $(Z_a \times 0)_{a\in A}$. So, instead of $(F_a)_{a\in A}$, it is enough to consider $(F_a - J^m_{E_a}(g_a))_{a\in A}$.

If D_a and D'_a are distinct open k-dimensional strata in $\mathscr{D}_a$, then $\Delta(D_a \times 0) \subseteq \Delta(T_a \times 0)$ and $\overline{\Delta(D_a \times 0)} \cap \overline{\Delta(D'_a \times 0)} \subseteq Z_a \times 0$. Thus it suffices to find, separately for each $(D_a)_{a\in A}$, a definable bounded family $(f_a)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $((F_a - J^m_{E_a}(g_a))|_{\overline{D}_a \times 0})_{a\in A}$ that is m-flat outside $(\Delta(D_a \times 0))_{a\in A}$ and C^p outside $(\overline{D}_a \times 0)_{a\in A}$.

For each $a \in A$, set

$$h_a(u,w) := \frac{1}{\beta!} F_a^{(0,\beta)}(u,0) w^\beta - g_a(u,w),$$

and define $f_a : \mathbb{R}^n \to \mathbb{R}$ by

$$f_a(u,w) := \begin{cases} r_a(u,w)h_a(u,w) & \text{if } u \in D_a,\\ 0 & \text{otherwise,}\end{cases}$$

where

$$r_a(u,w) := \prod_{i=1}^{\ell} \prod_{j=0}^{2k} \xi\Big(C\sqrt{\ell}\frac{w_i}{\rho_{a,j}(u)}\Big) \tag{4.7}$$

with $\xi : \mathbb{R} \to \mathbb{R}$ a semialgebraic C^p-function which is 1 near 0 and vanishes outside $(-1,1)$, $\rho_{a,0}, \rho_{a,1}, \dots, \rho_{a,2k}$ the functions associated with the open Λ_p-cell D_a (cf. Section 2.6), and C is the constant from (2.7) which may be taken independent from $a \in A$, since it is determined by the constants in the definition of the Λ_p-cells

D_a, $a \in A$; see Remark 2.12. Note that the m-jet of h_a at $(u,0)$ coincides with $(F_a - J^m_{E_a}(g_a))(u,0)$ for all $u \in D_a$.

By construction, $(f_a)_{a\in A}$ is a definable family. We will see that it is a bounded family of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $((F_a - J^m_{E_a}(g_a))|_{\overline{D}_a \times 0})_{a\in A}$. It is m-flat outside $(\Delta(D_a\times 0))_{a\in A}$, thanks to the properties of r_a, and it is C^p outside $(\overline{D}_a \times 0)_{a\in A}$. Indeed, if $(u,w) \in (D_a \times \mathbb{R}^\ell) \setminus \Delta(D_a \times 0)$, then, by (2.7),

$$\sqrt{\ell} \max_{1\le i\le \ell} |w_i| \ge |w| \ge \min\{1, d(u,\partial D_a\} \ge \frac{1}{C} \min_{0\le j\le 2k} \rho_{a,j}(u)$$

so that r_a is identically zero on $(D_a \times \mathbb{R}^\ell) \setminus \Delta(D_a \times 0)$. It remains to check that the family $(f_a)_{a\in A}$ is contained and bounded in $C^{m,\omega}(\mathbb{R}^n)$. To this end, we need two lemmas.

Lemma 4.3. *For each $a \in A$, h_a is of class $C^{m,\omega}$ on $\Delta(D_a\times 0)$ and the $C^{m,\omega}$-norm of h_a on $\Delta(D_a \times 0)$ is bounded by a constant independent of $a \in A$.*

Proof. By construction, each h_a is of class C^m. Since $(g_a)_{a\in A}$ is a bounded family of $C^{m,\omega}$-functions on $\mathbb{R}^n$, it suffices it consider $(u,w) \mapsto F_a^{(0,\beta)}(u,0)w^\beta$. We have to check that there is a constant $C > 0$ such that, for all $a \in A$, all $\kappa = (\sigma,\tau) \in \mathbb{N}^k\times\mathbb{N}^\ell$, $|\kappa| \le m$, and all $(u,w) \in \Delta(D_a \times 0)$,

$$|\partial^\kappa\big(F_a^{(0,\beta)}(u,0)w^\beta\big)| \le C, \tag{4.8}$$

and, if $|\kappa| = m$, for all $x_i = (u_i,w_i) \in \Delta(D_a \times 0)$, $i = 1,2$,

$$|\partial^\kappa\big(F_a^{(0,\beta)}(u_1,0)w_1^\beta\big) - \partial^\kappa\big(F_a^{(0,\beta)}(u_2,0)w_2^\beta\big)| \le C\,\omega(|x_1 - x_2|). \tag{4.9}$$

We may assume that $\tau \le \beta$. Let $\alpha,\gamma \in \mathbb{N}^k$ be such that $\alpha+\gamma = \sigma$ and $|\alpha|+|\beta| \le m$ is maximal. To see (4.8), observe that, by (4.3), (3.6), and $|w| < \min\{1, d(u,\partial D_a)\}$,

$$\begin{aligned}|\partial^\gamma F_a^{(\alpha,\beta)}(u,0)w^{\beta-\tau}| &\le L\,\frac{\sup\{|F_a^{(\alpha,\beta)}(v,0)| : v \in D_a,\ |v-u| < d(u,\partial D_a)\}}{d(u,\partial D_a)^{|\gamma|}}|w|^{|\beta-\tau|}\\ &\le CL\,|w|^{|\beta-\tau|-|\gamma|} \le CL,\end{aligned}$$

where $C > 0$ is the supremum in (3.6); indeed, if $\gamma \ne 0$ then $|\alpha|+|\beta| = m$ and thus $|\beta-\tau| \ge |\gamma|$.

Let us prove (4.9). Now $|\kappa| = m$ and $|\alpha|+|\beta| = m$, whence $|\beta-\tau| = |\gamma|$. Then it is enough to show

$$|\partial^\gamma F_a^{(\alpha,\beta)}(u_1,0)w_1^{\beta-\tau} - \partial^\gamma F_a^{(\alpha,\beta)}(u_2,0)w_2^{\beta-\tau}| \le C\,\omega(|x_1-x_2|). \tag{4.10}$$

If $\gamma = 0$, this follows from (3.7). So let us assume that $|\gamma| \ge 1$.

Set $t_a(u) := \frac{1}{2}d(u,\partial D_a)$. Then

$$|t_a(u_1) - t_a(u_2)| \le \tfrac{1}{2}|u_1-u_2|. \tag{4.11}$$

Note that, for $i = 1,2$,

$$|w_i| < d(u_i,\partial D_a) = 2t_a(u_i). \tag{4.12}$$

We consider two cases.

Case 1. Suppose that $t_a(u_i) \le |x_1-x_2|$ for $i = 1,2$. Then, by (4.4) and (4.12),

$$|\partial^\gamma F_a^{(\alpha,\beta)}(u_i,0)w_i^{\beta-\tau}| \le L\,\omega(2t_a(u_i)) \le 2L\,\omega(|x_1-x_2|),$$

since ω is concave and increasing.

Case 2. Assume (without loss of generality) that $t_a(u_1) > |x_1-x_2|$. Then $|u_1-u_2| \le |x_1-x_2| < t_a(u_1) = \frac{1}{2}d(u_1, \partial D_a)$ so that the line segment $[x_1, x_2]$ is contained in $D_a \times \mathbb{R}^\ell$. Furthermore, if $u \in [u_1, u_2]$ then, by (4.11),

$$|t_a(u_1) - t_a(u)| \le \tfrac{1}{2}|u_1 - u| \le \tfrac{1}{2}|x_1 - x_2| < \tfrac{1}{2}t_a(u_1),$$

whence

$$\tfrac{1}{2}t_a(u_1) < t_a(u) < \tfrac{3}{2}t_a(u_1), \quad u \in [u_1, u_2].$$

The left-hand side of (4.10) is bounded by

$$|\partial^\gamma F_a^{(\alpha,\beta)}(u_1, 0) - \partial^\gamma F_a^{(\alpha,\beta)}(u_2, 0)||w_1|^{|\beta-\tau|} + |\partial^\gamma F_a^{(\alpha,\beta)}(u_2, 0)||w_1^{\beta-\tau} - w_2^{\beta-\tau}|.$$

By (4.4) and (4.12),

$$\begin{aligned}
&|\partial^\gamma F_a^{(\alpha,\beta)}(u_1, 0) - \partial^\gamma F_a^{(\alpha,\beta)}(u_2, 0)||w_1|^{|\beta-\tau|} \\
&\quad\lesssim \sup_{u\in[u_1,u_2]} \sum_{j=1}^{k} |\partial^{\gamma+(j)} F_a^{(\alpha,\beta)}(u, 0)||u_1 - u_2|\, t_a(u_1)^{|\gamma|} \\
&\quad\lesssim \sup_{u\in[u_1,u_2]} \frac{\omega(2t_a(u))}{t_a(u)^{|\gamma|+1}}|u_1 - u_2|\, t_a(u_1)^{|\gamma|} \\
&\quad\lesssim \frac{\omega(t_a(u_1))}{t_a(u_1)}|x_1 - x_2| \\
&\quad\le \omega(|x_1 - x_2|),
\end{aligned}$$

since ω is concave. Again, by (4.4) and (4.12),

$$\begin{aligned}
&|\partial^\gamma F_a^{(\alpha,\beta)}(u_2, 0)||w_1^{\beta-\tau} - w_2^{\beta-\tau}| \\
&\quad\lesssim \frac{\omega(2t_a(u_2))}{t_a(u_2)^{|\gamma|}}|w_1 - w_2|\, t_a(u_1)^{|\gamma|-1} \lesssim \frac{\omega(t_a(u_1))}{t_a(u_1)}|x_1 - x_2| \le \omega(|x_1 - x_2|).
\end{aligned}$$

The proof is complete. □

The proof shows that (4.9) actually holds on the larger set $\{(u, w) \in D_a \times \mathbb{R}^\ell : |w| < d(u, \partial D_a)\}$.

Lemma 4.4. *For each $a \in A$,*

$$|\partial^\kappa h_a(u, w)| \le C\,\omega(d(u, \partial D_a))d(u, \partial D_a)^{m-|\kappa|}, \tag{4.13}$$

for all $(u, w) \in \Delta(D_a \times 0)$, all $\kappa \in \mathbb{N}^n$, $|\kappa| \le m$, and a constant $C > 0$ independent of $a \in A$.

Proof. Fix $x = (u, w) \in \Delta(D_a \times 0)$. If $d(u, \partial D_a) < d(u, \partial T_a)$, then let $u' \in \partial D_a$ be such that $|u - u'| = d(u, \partial D_a)$ and set $x' = (u', 0)$. The open line segment (x, x') is contained in $\Delta(D_a \times 0)$. Since $u' \in T_a$, where $F_a^{(0,\beta)}$ is of class C^p, and h_a is of class $C^{m,\omega}$ on $\Delta(D_a \times 0)$ with $C^{m,\omega}$-norm bounded by a constant independent of $a \in A$, by Lemma 4.3, we may conclude the assertion from Taylor's theorem.

So we assume that $d(u, \partial D_a) = d(u, \partial T_a)$. Let $u' \in \partial T_a$ such that $|u - u'| = d(u, \partial T_a)$. Let us first assume that $\kappa = (\sigma, \tau) \in \mathbb{N}^k \times \mathbb{N}^\ell$ with $|\kappa| = m$. By construction, $\partial^\kappa g_a(u', 0) = 0$ so that

$$|\partial^\kappa g_a(u, w)| = |\partial^\kappa g_a(u, w) - \partial^\kappa g_a(u', 0)| \lesssim \omega(|u - u'|) = \omega(d(u, \partial D_a)),$$

where we used $|w| < d(u, \partial D_a) = |u - u'|$. Hence it suffices to consider $\partial^\kappa\big(F_a^{(0,\beta)}(u, 0)w^\beta\big)$ or equivalently $\partial^\gamma F_a^{(\alpha,\beta)}(u, 0)w^{\beta-\tau}$, where $\alpha, \gamma \in \mathbb{N}^k$ are such that $\alpha+\gamma=\sigma$, $|\alpha|+|\beta|=m$, and $\tau \le \beta$. Thus $|\beta-\tau|=|\gamma|$. If $|\gamma|\ge 1$, (4.4) implies

$$|\partial^\gamma F_a^{(\alpha,\beta)}(u,0)w^{\beta-\tau}| \le L\,\frac{\omega(d(u,\partial D_a))}{d(u,\partial D_a)^{|\gamma|}}|w|^{|\gamma|} \le L\,\omega(d(u,\partial D_a)),$$

and, if $\gamma=0$, (3.7) gives

$$|\partial^\gamma F_a^{(\alpha,\beta)}(u,0)w^{\beta-\tau}| = |F_a^{(\alpha,\beta)}(u,0)| \lesssim \omega(d(u,\partial T_a)) = \omega(d(u,\partial D_a)),$$

since $(F_a)_{a\in A}$ is flat on $(\partial T_a\times 0)_{a\in A}$.

To prove the statement for $|\kappa|<m$, we proceed by induction on $m-|\kappa|$. Suppose that the assertion is already shown for every $\lambda\in\mathbb{N}^n$ with $|\kappa|<|\lambda|\le m$. Since the open line segment (x,x') connecting $x=(u,w)$ and $x'=(u',0)$ is contained in $\Delta(D_a\times 0)$, we have, by induction hypothesis, where $x''=(u'',w'')$,

$$\begin{aligned}
|\partial^\kappa h_a(u,w)| &\le \sup_{x''\in(x,x')} \sum_{j=1}^n |\partial^{\kappa+(j)} h_a(u'',w'')||x-x'| \\
&\lesssim \sup_{x''\in(x,x')} \omega(d(u'',\partial D_a)) d(u'',\partial D_a)^{m-|\kappa|-1}\, d(u,\partial D_a) \\
&\lesssim \omega(d(u,\partial D_a)) d(u,\partial D_a)^{m-|\kappa|},
\end{aligned}$$

since $d(u'',\partial D_a)\le d(u,\partial D_a)$. □

It follows from Lemma 4.3 and Lemma 4.4 that, for each $a\in A$,

$$|\partial^\kappa h_a(u,w)| \le C\,\omega(\min\{1,d(u,\partial D_a)\})\min\{1,d(u,\partial D_a)\}^{m-|\kappa|}, \tag{4.14}$$

for all $(u,w)\in\Delta(D_a\times 0)$, all $\kappa\in\mathbb{N}^n$, $|\kappa|\le m$, and a constant $C>0$ independent of $a\in A$. Indeed, by Lemma 4.3,

$$|\partial^\kappa h_a(u,w)| \le C = \frac{C}{\omega(1)}\cdot\omega(1)1^{m-|\kappa|}, \quad |\kappa|\le m,$$

for all $(u,w)\in\Delta(D_a\times 0)$, which, together with (4.13), gives (4.14).

Now Proposition 3.7 (see also Remark 3.8) implies that the family $(f_a)_{a\in A}$ is bounded in $C^{m,\omega}(\mathbb{R}^n)$. Indeed, Lemma 4.4, Lemma 4.3, and (4.14) guarantee that the assumptions of Proposition 3.7 are satisfied, where $t_a(u)=\min\{1,d(u,\partial D_a)\}$. Condition (3.9) holds thanks to (2.9) and Remark 2.12. We also get

$$|\partial^\kappa f_a(u,w)| \le C\,\omega(d(u,\partial D_a))d(u,\partial D_a)^{m-|\kappa|}, \tag{4.15}$$

for all $(u,w)\in\Delta(D_a\times 0)$, all $\kappa\in\mathbb{N}^n$, $|\kappa|\le m$, and a constant $C>0$ independent of $a\in A$.

Step 2. For all $a\in A$, $E_a'=\overline{S}_a=\overline{T}_a\times 0$, but possibly E_a' is a proper subset of E_a for some $a\in A$. Consider the definable family $(r_a)_{a\in A}$ of functions $r_a: T_a\to(0,\infty)$ given by

$$r_a(u) := \begin{cases} \inf\{|w| : (u,w)\in E_a\setminus S_a\} & \text{if } \{w : (u,w)\in E_a\setminus S_a\}\ne\emptyset, \\ 1 & \text{otherwise.}\end{cases}$$

Since F_a is flat on $E_a\setminus S_a$ we have (by (3.6) and (3.7))

$$|F_a^\kappa(u,0)| \le C\,\omega(r_a(u))r_a(u)^{m-|\kappa|} \tag{4.16}$$

for all $u \in T_a$, all $\kappa \in \mathbb{N}^n$, $|\kappa| \le m$, and a constant $C > 0$ independent of $a \in A$. (In the case that $\{w : (u,w) \in E_a \setminus S_a\} = \emptyset$, it follows from (3.6) and we have to replace C by $C/\omega(1)$.)

By Theorem 2.16 and Proposition 2.20, there is a uniform Λ_p-stratification of $(\overline{T}_a)_{a\in A}$ such that

$$\overline{T}_a = Q_{a,1} \cup \cdots \cup Q_{a,s} \cup Z_a,$$

where, for each $a \in A$ and each $i = 1,\dots,s$, Z_a is closed with $\dim Z_a < k$, each $Q_{a,i}$ is an open k-dimensional Λ_p-cell with constant independent of $a \in A$, r_a is C^p on $Q_{a,i}$, and either

(i) $|\partial_j r_a| \le 1$, $j = 1,\dots,k$, on $Q_{a,i}$, in which case we may assume that $|\partial^\alpha r_a(u)| d(u, \partial Q_{a,i})^{|\alpha|-1}$, $1 \le |\alpha| \le p$, is bounded on $Q_{a,i}$ by a constant independent of $a \in A$, by Corollary 2.18, or

(ii) $|\partial_j r_a(u)| > 1$ for some j on $Q_{a,i}$.

By $(\mathbf{I}_{k-1})$, we may assume that $(F_a)_{a\in A}$ is flat on $(Z_a \times 0)_{a\in A}$ and hence on $(\partial Q_{a,i} \times 0)_{a\in A}$ for each $i = 1,\dots,s$.

Now it is enough to show that, for every $i = 1,\dots,s$, $(F_a|_{E_a \cap (\overline{Q}_{a,i} \times \mathbb{R}^\ell)})_{a\in A}$ admits a definable bounded family $(f_{a,i})_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ that is m-flat outside $(\Delta(Q_{a,i} \times 0))_{a\in A}$ and C^p outside $(\overline{Q}_{a,i} \times 0)_{a\in A}$. To this end, we fix i and drop it from the notation.

Step 1 gives a definable bounded family $(g_a)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(F_a|_{\overline{Q}_a \times 0})_{a\in A}$ that is m-flat outside $(\Delta(Q_a \times 0)_{a\in A}$ and C^p outside $(\overline{Q}_a \times 0)_{a\in A}$. By Taylor's formula and (4.16), for each $a \in A$,

$$|\partial^\kappa g_a(u,w)| \le C\, \omega(r_a(u))\, r_a(u)^{m-|\kappa|} \tag{4.17}$$

for all $(u,w) \in Q_a \times \mathbb{R}^\ell$, $|w| < C'\, r_a(u)$, and all $\kappa \in \mathbb{N}^n$, $|\kappa| \le m$, where $C, C' > 0$ are independent of $a \in A$. Similarly, we have

$$|\partial^\kappa g_a(u,w)| \le C\, \omega(d(u,\partial Q_a))\, d(u,\partial Q_a)^{m-|\kappa|} \tag{4.18}$$

for all $(u,w) \in Q_a \times \mathbb{R}^\ell$, $|w| < C'\, d(u,\partial Q_a)$, and all $\kappa \in \mathbb{N}^n$, $|\kappa| \le m$, where $C, C' > 0$ are independent of $a \in A$.

In Case (ii), one can easily see (cf. [8, p. 94]) that, for each $a \in A$, $r_a(u) \ge d(u,\partial Q_a)$ for $u \in Q_a$, so that $E_a \setminus S_a \subseteq \mathbb{R}^n \setminus \Delta(Q_a \times 0)$. That means that g_a is a $C^{m,\omega}$-extension to $\mathbb{R}^n$ of $F_a|_{E_a \cap (\overline{Q}_a \times 0)}$, and we are done.

In Case (i), a modification is necessary: we define, for each $a \in A$,

$$f_a(u,w) := \begin{cases} \prod_{i=1}^\ell \xi\Big(\sqrt{\ell}\,\frac{w_i}{r_a(u)}\Big) \cdot g_a(u,w) & \text{if } u \in Q_a, \\ 0 & \text{otherwise,} \end{cases}$$

where $\xi : \mathbb{R} \to \mathbb{R}$ is a semialgebraic C^p-function that is 1 near 0 and vanishes outside $(-1,1)$. Note that $(f_a)_{a\in A}$ is a definable family of functions $f_a : \mathbb{R}^n \to \mathbb{R}$. Moreover, we set

$$\Delta'(Q_a \times 0) := \{(u,w) \in Q_a \times \mathbb{R}^\ell : |w| < t_a(u)\}$$

with

$$t_a(u) := \min\{r_a(u), d(u,\partial Q_a)\},$$

and claim that, for each $a \in A$, f_a is of class $C^{m,\omega}$ on $\Delta'(Q_a \times 0)$ with $C^{m,\omega}$-norm bounded by a constant independent of $a \in A$, and

$$|\partial^\kappa f_a(u,w)| \le C\, \omega(t_a(u))\, t_a(u)^{m-|\kappa|} \tag{4.19}$$

for $(u,w) \in \Delta'(Q_a \times 0)$ and all $\kappa \in \mathbb{N}^n$, $|\kappa| \le m$, where $C > 0$ is independent of $a \in A$.

To see this, let us first assume that $r_a(u) < d(u, \partial Q_a)$ so that $t_a(u) = r_a(u)$. Since we are in Case (i), we find that, thanks to (2.10),

$$|\partial^\alpha (1/r_a)(u)| \le C\, r_a(u)^{-|\alpha|-1}, \quad u \in Q_a,\, |\alpha| \le p,$$

for a constant $C > 0$ independent of $a \in A$. Thus, the claim follows from (4.17) and Proposition 3.7.

If $r_a(u) \ge d(u, \partial Q_a)$ (that is $t_a(u) = d(u, \partial Q_a)$), then similarly

$$|\partial^\alpha (1/r_a)(u)| \le C\, d(u, \partial Q_a)^{-|\alpha|-1}, \quad u \in Q_a,\, |\alpha| \le p,$$

Then we infer the claim from (4.18) and Proposition 3.7.

We conclude that $(f_a)_{a\in A}$ is the required family of definable bounded $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(F_a|_{E_a \cap (\overline{Q}_a \times \mathbb{R}^\ell)})_{a\in A}$ that is m-flat outside $(\Delta(Q_a \times 0))_{a\in A}$ and C^p outside $(\overline{Q}_a \times 0)_{a\in A}$. This ends Step 2.

Step 3. The general case of Proposition 4.2: for all $a \in A$, $S_a = \Gamma(\varphi_a)$, $E'_a = \overline{S}_a \subseteq E_a$, where $\varphi_a : T_a \to \mathbb{R}^\ell$ is not necessarily identically 0. Consider the definable family $(s_a)_{a\in A}$ of functions $s_a : S_a \to (0,\infty)$ given by

$$s_a(x) := \min\{d(x, E_a \setminus S_a), d(x, \partial S_a\}, \quad x \in S_a.$$

For each $a \in A$, let $\overline{\varphi}_a : \overline{T}_a \to \mathbb{R}^\ell$ be the continuous extension of φ_a; cf. Section 2.4. Furthermore, we consider the maps

$$\varphi_{a,\pm} : T_a \times \mathbb{R}^\ell \to T_a \times \mathbb{R}^\ell, \quad (u,w) \mapsto (u, w \pm \varphi_a(u))$$

and

$$\overline{\varphi}_{a,\pm} : \overline{T}_a \times \mathbb{R}^\ell \to \overline{T}_a \times \mathbb{R}^\ell, \quad (u,w) \mapsto (u, w \pm \overline{\varphi}_a(u)).$$

Note that $\overline{\varphi}_{a,+}$ is a bi-Lipschitz homeomorphism with inverse $\overline{\varphi}_{a,-}$ and Lipschitz constants independent of $a \in A$.

Since F_a is flat on $E_a \setminus S_a$ and on ∂S_a, we have (by (3.7))

$$|F_a^\kappa(x)| \le C\, \omega(s_a(x)) s_a(x)^{m-|\kappa|} \tag{4.20}$$

for all $x \in S_a$, all $\kappa \in \mathbb{N}^n$, $|\kappa| \le m$, and a constant $C > 0$ independent of $a \in A$. Setting

$$t_a(u) := s_a(u, \varphi_a(u)), \quad u \in T_a,$$

(4.20) reads as

$$|F_a^\kappa(u, \varphi_a(u))| \le C\, \omega(t_a(u)) t_a(u)^{m-|\kappa|}, \quad u \in T_a,\, |\kappa| \le m. \tag{4.21}$$

The uniformity of the constants in the definition of T_a and φ_a implies that $(\overline{S}_a)_{a\in A}$ and $(\partial T_a \times \mathbb{R}^\ell)_{a\in A}$ are uniformly separated. Observe that (by the definition of s_a) $t_a(u) \le C'\, d(u, \partial T_a)$ for $C' > 0$ independent of $a \in A$, since $\overline{\varphi}_{a,+}$ is a bi-Lipschitz homeomorphism with Lipschitz constants independent of $a \in A$.

Thus Proposition 3.5 (and Lemma 3.6) implies that $(G_a)_{a\in A}$, where $G_a := \varphi_{a,+}^*(F_a|_{S_a})$, is a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(T_a \times 0)_{a\in A}$ and extends to a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(\overline{T}_a \times 0)_{a\in A}$ which is flat on $(\partial T_a \times 0)_{a\in A}$ and such that

$$|G_a^\kappa(u,0)| \le C\, \omega(t_a(u)) t_a(u)^{m-|\kappa|} \tag{4.22}$$

for all $u \in T_a$, all $\kappa \in \mathbb{N}^n$, $|\kappa| \le m$, and a constant $C > 0$ independent of $a \in A$. For each $a \in A$, set $\tilde{E}_a := \overline{\varphi}_{a,-}(E_a \cap (\overline{T}_a \times \mathbb{R}^\ell))$. Since $\overline{\varphi}_{a,+}$ is a bi-Lipschitz homeomorphism with constants independent of $a \in A$, we may conclude

$$|G_a^\kappa(u,0)| \le C\,\omega(d((u,0),\tilde{E}_a \setminus (T_a \times 0)))d((u,0),\tilde{E}_a \setminus (T_a \times 0))^{m-|\kappa|} \tag{4.23}$$

for all $u \in T_a$, all $\kappa \in \mathbb{N}^n$, $|\kappa| \le m$, and a constant $C > 0$ independent of $a \in A$. Thus $(\tilde{G}_a)_{a\in A}$, where

$$\tilde{G}_a(u,w) := \begin{cases} G_a(u,0) & \text{if } (u,w) \in \overline{T}_a \times 0, \\ 0 & \text{if } (u,w) \in \tilde{E}_a \setminus (\overline{T}_a \times 0), \end{cases}$$

is a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(\tilde{E}_a)_{a\in E}$ that is flat on $(\tilde{E}_a \setminus (T_a \times 0))_{a\in A}$.

By Step 2, there exists a definable bounded family $(\tilde{g}_a)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(\tilde{G}_a)_{a\in A}$ that is m-flat on $(\tilde{E}_a \setminus (T_a \times 0))_{a\in A}$ as well as outside $(T_a \times \mathbb{R}^\ell)_{a\in A}$ and C^p outside $(\overline{T}_a \times 0)_{a\in A}$.

For each $a \in A$, define $f_a : \mathbb{R}^n \to \mathbb{R}$ by

$$f_a(u,w) := \begin{cases} (\tilde{g}_a \circ \varphi_{a,-})(u,w) & \text{if } (u,w) \in T_a \times \mathbb{R}^\ell, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(f_a)_{a\in A}$ is a definable bounded family of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(F_a)_{a\in A}$ that is C^p outside $(\overline{S}_a)_{a\in A}$, which follows again from Proposition 3.5 (with $M_a = T_a \times \mathbb{R}^\ell$ and $U_a = T_a$).

This completes the proof of Proposition 4.2, hence of $(\mathbf{I}_k)$, and thus the proof of Theorem 1.3.

5. Further applications

In this section, we present a local version of Theorem 1.3, we discuss the dependence of the bounded extension on the modulus of continuity which leads to the proof of Theorem 1.4, and finally we obtain a definable version of a correspondence between Whitney jets of class $C^{m,\omega}$ and certain Lipschitz maps, which was originally observed by Shvartsman [11].

5.1. **Definable $C^{m,\omega}_{\text{loc}}$-extensions.** Let $U \subseteq \mathbb{R}^n$ be open. We denote by $C^{m,\omega}_{\text{loc}}(U)$ the space of functions $f : U \to \mathbb{R}$ such that $f|_V \in C^{m,\omega}(V)$, for all relatively compact open subsets $V \subseteq U$.

Let $E \subseteq \mathbb{R}^n$ be a closed set. An m-jet F on E is called a *(definable) Whitney jet of class $C^{m,\omega}_{\text{loc}}$ on E* if $F|_K$ is a (definable) Whitney jet of class $C^{m,\omega}$ on K, for all (definable) compact subsets $K \subseteq E$. A $C^{m,\omega}_{\text{loc}}$-function $f : \mathbb{R}^n \to \mathbb{R}$ is a *$C^{m,\omega}_{\text{loc}}$-extension to $\mathbb{R}^n$ of F* if $J^m_E(f) = F$.

Let $(E_a)_{a\in A}$ be a family of closed sets $E_a \subseteq \mathbb{R}^n$. A family $(F_a)_{a\in A}$ of Whitney jets of class $C^{m,\omega}_{\text{loc}}$ on E_a is called a *(definable) bounded family of Whitney jets of class $C^{m,\omega}_{\text{loc}}$* if $(F_a|_{K_a})_{a\in A}$ is a (definable) bounded family of Whitney jets of class $C^{m,\omega}$ for each (definable) subfamily $(K_a)_{a\in A}$ of $(E_a)_{a\in A}$ consisting of (definable) compact sets $K_a \subseteq E_a$.

A family $(f_a)_{a\in A}$ of $C^{m,\omega}_{\text{loc}}$-functions $f_a : \mathbb{R}^n \to \mathbb{R}$ is called a *(definable) bounded family of $C^{m,\omega}_{\text{loc}}$-extensions to $\mathbb{R}^n$ of $(F_a)_{a\in A}$* if f_a is a $C^{m,\omega}_{\text{loc}}$-extension to $\mathbb{R}^n$ of F_a, for each $a \in A$, and, for each (definable) relatively compact subset $V \subseteq \mathbb{R}^n$, $(f_a|_V)_{a\in A}$ is a (definable) bounded family of $C^{m,\omega}$-functions.

Corollary 5.1. *Let $0 \le m \le p$ be integers. Let ω be a modulus of continuity. Let $(E_a)_{a\in A}$ be a definable family of closed subsets E_a of $\mathbb{R}^n$. For any definable bounded family $(F_a)_{a\in A}$ of Whitney jets of class $C^{m,\omega}_{\text{loc}}$ on $(E_a)_{a\in A}$ there exists a definable bounded family $(f_a)_{a\in A}$ of $C^{m,\omega}_{\text{loc}}$-extensions to $\mathbb{R}^n$ of $(F_a)_{a\in A}$ that is C^p outside $(E_a)_{a\in A}$.*

Proof. This follows by standard arguments from Theorem 1.3 and the existence of definable partitions of unity of class C^p. □

Remark 5.2. Note that we do not say that f_a is definable as a global function $f_a : \mathbb{R}^n \to \mathbb{R}$, because the gluing argument (based on the partition of unity) involves an infinite sum.

Nevertheless, global definability of f_a can be achieved by an adaptation of the whole proof to the $C^{m,\omega}_{\text{loc}}$-setting. We leave the details to the reader.

5.2. **Dependence on the modulus of continuity.** The main result, Theorem 1.3, only depends in a weak sense on the modulus of continuity ω, namely, the uniform constant C occasionally must be multiplied by $\omega(1)$ or by $\omega(1)^{-1}$; see (4.1), (4.6), (4.14), and (4.16).

Thus, we can allow in Theorem 1.3 that, for each $a \in A$, F_a is a Whitney jet of class C^{m,ω_a} on E_a, where ω_a is a modulus of continuity and there is a constant $C > 0$ independent of $a \in A$ such that

$$C^{-1} \le \omega_a(1) \le C, \quad a \in A. \tag{5.1}$$

Then the statement is the following:

Theorem 5.3. *Let $0 \le m \le p$ be integers. Let $(\omega_a)_{a\in A}$ be a family of moduli of continuity satisfying (5.1). Let $(E_a)_{a\in A}$ be a definable family of closed subsets E_a of $\mathbb{R}^n$. For any definable family $(F_a)_{a\in A}$ of Whitney jets F_a of class C^{m,ω_a} on E_a such that*

$$\sup_{a\in A}\sup_{x\in E_a}\sup_{|\gamma|\le m} |F_a^\gamma(x)| < \infty, \tag{5.2}$$

and

$$\sup_{a\in A}\sup_{x\ne y\in E_a}\sup_{|\gamma|\le m} \frac{|(R_x^m F_a)^\gamma(y)|}{\omega_a(|x-y|)|x-y|^{m-|\gamma|}} < \infty, \tag{5.3}$$

there exists a definable family $(f_a)_{a\in A}$ of C^{m,ω_a}-extensions f_a to $\mathbb{R}^n$ of F_a such that f_a is of class C^p outside E_a, for all $a \in A$, and

$$\sup_{a\in A} \|f_a\|_{C^{m,\omega_a}(\mathbb{R}^n)} < \infty. \tag{5.4}$$

5.3. **Proof of Theorem 1.4.** That the definable family $(F_a)_{a\in A}$ of Whitney jets of class C^m on $(E_a)_{a\in A}$, where $E_a \subseteq \mathbb{R}^n$ is compact, is *bounded* means that

$$\sup_{a\in A}\sup_{x\in E_a}\sup_{|\gamma|\le m} |F_a^\gamma(x)| < \infty \tag{5.5}$$

and

$$\sup_{a\in A}\sup_{x\ne y\in E_a}\sup_{|\gamma|\le m} \frac{|(R_x^m F_a)^\gamma(y)|}{|x-y|^{m-|\gamma|}} < \infty. \tag{5.6}$$

Proof of Theorem 1.4. We modify slightly an argument used in [13, Proposition IV.1.5]. For each $a \in A$, consider

$$\sigma_a(t) := \sup_{\substack{x \ne y \in E_a \\ |x-y| \le t}} \sup_{|\gamma| \le m} \frac{|(R^m_x F_a)^\gamma(y)|}{|x-y|^{m-|\gamma|}}, \quad t > 0, \quad \sigma_a(0) := 0.$$

Then $\sigma_a : [0,\infty) \to [0,\infty)$ is an increasing function that is continuous at 0 and

$$\sigma_a(t) = \sigma_a(\operatorname{diam} E_a), \quad t \ge \operatorname{diam} E_a.$$

Thus also $\tau_a : [0,\infty) \to [0,\infty)$ defined by

$$\tau_a(t) := \begin{cases} \sigma_a(t) & \text{if } t < 1, \\ \max\{1, \sigma_a(t)\} & \text{if } t \ge 1, \end{cases}$$

is increasing and continuous at 0 with

$$\tau_a(t) \le \max\{1, \sigma_a(\operatorname{diam} E_a)\}, \quad t \ge 0. \tag{5.7}$$

Let ω_a be the least concave majorant of τ_a which is finite, thanks to (5.7). Then ω_a is a modulus of continuity and

$$\sup_{a \in A} \sup_{x \ne y \in E_a} \sup_{|\gamma| \le m} \frac{|(R^m_x F_a)^\gamma(y)|}{\omega_a(|x-y|)|x-y|^{m-|\gamma|}} \le 1.$$

Moreover, $\omega_a(1) \ge 1$ and, by (5.7),

$$\omega_a(t) \le \max\{1, \sigma_a(\operatorname{diam} E_a)\} \le C, \quad t \ge 0,$$

for a constant $C > 0$ independent of $a \in A$, thanks to (5.6). In particular, (5.1) is satisfied.

Thus Theorem 5.3 implies that there is a definable family $(f_a)_{a\in A}$ such that, for each $a \in A$, f_a is a C^{m,ω_a}-extension to $\mathbb{R}^n$ of F_a, C^p outside E_a, and

$$\sup_{a \in A} \|f_a\|_{C^{m,\omega_a}(\mathbb{R}^n)} < \infty.$$

In particular, $(f_a)_{a\in A}$ is a bounded family of C^m-functions. □

5.4. **Definable Whitney jets as Lipschitz maps.** We end with a few observations on a definable version of a correspondence, due to Shvartsman [11], between Whitney jets of class $C^{m,\omega}$ and certain Lipschitz maps. Here the notation follows closely the one of [11].

Let ω be a modulus of continuity and m a positive integer. For $\alpha \in \mathbb{N}^n$ with $|\alpha| < m$ let ψ_α be the inverse of the (strictly increasing) function $s \mapsto s^{m-|\alpha|}\omega(s)$ and put $\varphi_\alpha := \omega \circ \psi_\alpha$. For $|\alpha| = m$, set $\varphi_\alpha(t) := t$.

Let $\mathcal{P}_m$ denote the space of real polynomials of degree at most m in n variables. For $T_i = (P_i, x_i) \in \mathcal{P}_m \times \mathbb{R}^n$, $i = 1, 2$, define

$$\delta_\omega(T_1, T_2) := \max\Big\{\omega(|x_1 - x_2|), \max_{\substack{|\alpha| \le m \\ i=1,2}} \varphi_\alpha(|\partial^\alpha(P_1 - P_2)(x_i)|)\Big\}.$$

Then we get a metric d_ω on $\mathcal{P}_m \times \mathbb{R}^n$ by setting

$$d_\omega(T, T') := \inf \sum_{j=0}^{k-1} \delta_\omega(T_j, T_{j+1}),$$

where the infimum is taken over all finite sequences $T = T_1, T_2, \ldots, T_k = T'$ in $\mathcal{P}_m \times \mathbb{R}^n$. It turns out (cf. [11, Theorem 2.1]) that

$$d_\omega((P,x),(P',x')) \le \delta_\omega((P,x),(P',x')) \le d_\omega((e^n P,x),(e^n P',x')).$$

Let $\mathcal{T}_{m,n}$ be the metric space $(\mathcal{P}_m \times \mathbb{R}^n, d_\omega)$. For a nonempty subset $X \subseteq \mathbb{R}^n$, we denote by X_ω the metric space $(X, (x,y) \mapsto \omega(|x-y|))$. Let $\mathbf{Lip}(X_\omega, \mathcal{T}_{m,n})$ be the space of Lipschitz maps $T : x \mapsto (P_x, z_x)$ such that $\max_{|\alpha|\le m} \sup_{x\in X} |\partial^\alpha P_x(x)| < \infty$, equipped with the norm

$$\begin{aligned} \|T\|^*_{LO(X)} := \max_{|\alpha|\le m} \sup_{x\in X} |\partial^\alpha P_x(x)| \\ + \inf\{\lambda > 0 : d_\omega(\lambda^{-1}T(x), \lambda^{-1}T(y)) \le \omega(|x-y|) \text{ for all } x,y \in X\}, \end{aligned}$$

where $\lambda^{-1}T(x) := (\lambda^{-1}P_x, z_x)$. Let $T^m_x f$ be the Taylor polynomial of order m at x of a C^m-function f.

Now let us recall a result of [11].

Proposition 5.4 ([11, Proposition 1.9 and Proposition 2.8])**.** *Let $X \subseteq \mathbb{R}^n$ be a closed set. Given a family of polynomials $\{P_x \in \mathcal{P}_m : x \in X\}$, there exists $f \in C^{m,\omega}(\mathbb{R}^n)$ such that $T^m_x f = P_x$ for all $x \in X$ if and only if the map $T : x \mapsto (P_x, x)$ belongs to $\mathbf{Lip}(X_\omega, \mathcal{T}_{m,n})$. We have*

$$\inf\{\|f\|_{C^{m,\omega}(\mathbb{R}^n)} : T^m_x f = P_x \text{ for all } x \in X\} \approx \|T\|^*_{LO(X)}$$

in the sense that either side is bounded by a constant $C(m,n)$ times the other side. Moreover, if $T : x \mapsto (P_x, x)$ belongs to $\mathbf{Lip}(X_\omega, \mathcal{T}_{m,n})$, then T has an extension $\widetilde{T} : x \mapsto (\widetilde{P}_x, x)$ in $\mathbf{Lip}(\mathbb{R}^n_\omega, \mathcal{T}_{m,n})$ satisfying

$$\|\widetilde{T}\|^*_{LO(\mathbb{R}^n)} \le C(m,n)\, \|T\|^*_{LO(X)}.$$

These results are based on the classical extension theorem for Whitney jets of class $C^{m,\omega}$. As a consequence of Theorem 1.2, we may conclude the following definable version, where, provided that X is definable, $\mathbf{Lip}_{\text{def}}(X_\omega, \mathcal{T}_{m,n})$ is the subspace of definable maps $T : x \mapsto (P_x, z_x)$ in $\mathbf{Lip}(X_\omega, \mathcal{T}_{m,n})$, which means that z_x and the coefficients of P_x are definable maps in x. Recall that $C^{m,\omega}_{\text{def}}(\mathbb{R}^n)$ is the subspace of $C^{m,\omega}(\mathbb{R}^n)$ consisting of all definable functions in $C^{m,\omega}(\mathbb{R}^n)$.

Proposition 5.5. *Let $X \subseteq \mathbb{R}^n$ be a definable closed set. Given a definable family of polynomials $\{P_x \in \mathcal{P}_m : x \in X\}$, there exists $f \in C^{m,\omega}_{\text{def}}(\mathbb{R}^n)$ such that $T^m_x f = P_x$ for all $x \in X$ if and only if the map $T : x \mapsto (P_x, x)$ belongs to $\mathbf{Lip}_{\text{def}}(X_\omega, \mathcal{T}_{m,n})$. Moreover, if $T : x \mapsto (P_x, x)$ belongs to $\mathbf{Lip}_{\text{def}}(X_\omega, \mathcal{T}_{m,n})$, then T has an extension $\widetilde{T} : x \mapsto (\widetilde{P}_x, x)$ in $\mathbf{Lip}_{\text{def}}(\mathbb{R}^n_\omega, \mathcal{T}_{m,n})$.*

Concerning the existence of uniform bounds for the norms, remarks similar to the ones in [9, Section 4.4] apply. But Theorem 1.3 implies the following supplement.

Proposition 5.6. *Suppose that in the setting of Proposition 5.5, the family of polynomials depends definably on additional parameters $a \in A$, i.e., a definable family of polynomials $\{P^a_x \in \mathcal{P}_m : x \in X, a \in A\}$ is given. Then there exists a bounded family $(f^a)_{a\in A}$ of definable $C^{m,\omega}$-functions $f^a : \mathbb{R}^n \to \mathbb{R}$ such that*

$$T^m_x f^a = P^a_x, \quad \text{for all } x \in X \text{ and } a \in A,$$

if and only if $(T^a : x \mapsto (P^a_x, x))_{a\in A}$ forms a bounded subset of $\mathbf{Lip}_{\text{def}}(X_\omega, \mathcal{T}_{m,n})$. Moreover, if $(T^a : x \mapsto (P^a_x, x))_{a\in A}$ forms a bounded subset of $\mathbf{Lip}_{\text{def}}(X_\omega, \mathcal{T}_{m,n})$, *then there is a family* $(\widetilde{T}^a : x \mapsto (\widetilde{P}^a_x, x))_{a \in A}$ *of extensions* $\widetilde{T}^a$ *of* T^a *which forms a bounded subset of* $\mathbf{Lip}_{\mathrm{def}}(\mathbb{R}^n_\omega, \mathcal{T}_{m,n})$*.*

Acknowledgements. A large part of the work on this paper has been done at Mathematisches Forschungsinstitut Oberwolfach (*Oberwolfach Research Fellows (OWRF) ID 2244p, October 31 - November 19, 2022*). We are grateful for the support and the excellent working conditions.

References

[1] D. Azagra, E. Le Gruyer, and C. Mudarra, *Explicit formulas for* $C^{1,1}$ *and* $C^{1,\omega}_{\mathrm{conv}}$ *extensions of 1-jets in Hilbert and superreflexive spaces*, J. Funct. Anal. **274** (2018), no. 10, 3003–3032. https://doi.org/10.1016%2Fj.jfa.2017.12.007

[2] R.G. Bartle and L.M. Graves, *Mappings between function spaces*, Trans. Amer. Math. Soc. **72** (1952), 400–413. https://doi.org/10.2307/1990709

[3] A. Brudnyi and Y. Brudnyi, *Methods of geometric analysis in extension and trace problems. Volume 1*, Monographs in Mathematics, vol. 102, Birkhäuser/Springer Basel AG, Basel, 2012.

[4] G. Glaeser, *Étude de quelques algèbres tayloriennes*, J. Analyse Math. **6** (1958), no. 1, 1–124. https://doi.org/10.1007%2Fbf02790231

[5] M. Gromov, *Entropy, homology and semialgebraic geometry [after Y. Yomdin]*, Séminaire Bourbaki. Astérisque (1987), no. 145-146, 5, 225–240.

[6] K. Kurdyka and A. Parusiński, *Quasi-convex decomposition in o-minimal structures. Application to the gradient conjecture*, Singularity theory and its applications, Adv. Stud. Pure Math., vol. 43, Math. Soc. Japan, Tokyo, 2006, pp. 137–177. https://doi.org/10.2969/aspm/04310137

[7] K. Kurdyka and W. Pawłucki, *Subanalytic version of Whitney's extension theorem*, Stud. Math. **124** (1997), no. 3, 269–280. https://doi.org/10.4064/sm-124-3-269-280

[8] ———, *O-minimal version of Whitney's extension theorem*, Stud. Math. **224** (2014), no. 1, 81–96. https://doi.org/10.4064/sm224-1-4

[9] A. Parusiński and A. Rainer, *Definable Lipschitz selections for affine-set valued maps*, arXiv:2306.09155. https://arxiv.org/abs/2306.09155

[10] W. Pawłucki, *A linear extension operator for Whitney fields on closed o-minimal sets*, Annales de l'Institut Fourier **58** (2008), no. 2, 383–404. https://doi.org/10.5802%2Faif.2355

[11] P. Shvartsman, *The Whitney extension problem and Lipschitz selections of set-valued mappings in jet-spaces*, Trans. Amer. Math. Soc. **360** (2008), no. 10, 5529–5550. https://doi.org/10.1090%2Fs0002-9947-08-04469-3

[12] A. Thamrongthanyalak, *Whitney's extension theorem in o-minimal structures*, Ann. Pol. Math. **119** (2017), no. 1, 49–67. https://doi.org/10.4064/ap3940-2-2017

[13] J.-C. Tougeron, *Idéaux de fonctions différentiables*, Springer-Verlag, Berlin, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 71.

[14] L. van den Dries, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, Cambridge, 1998.

[15] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), no. 2, 497–540. https://doi.org/10.1215/S0012-7094-96-08416-1

[16] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), no. 1, 63–89. http://dx.doi.org/10.2307/1989708

Adam Parusiński: Université Côte d'Azur, CNRS, LJAD, UMR 7351, 06108 Nice, France

Email address: adam.parusinski@univ-cotedazur.fr

Armin Rainer: Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Wien, Austria

Email address: armin.rainer@univie.ac.at