Overview

field	rational numbers $\mathbb{Q}$	Gaussian number field $\mathbb{Q}(i)$	quadratic number field $\mathbb{Q}(\sqrt{d})$, $d \in \mathbb{Z}$ squarefree, $d \neq 0,1$	general number field K, $n := [K:\mathbb{Q}] < \infty$
ring of integers	$\mathbb{Z}$	$\mathbb{Z}[i]$	$\mathbb{Z}[\sqrt{d}]$ $\mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right]$	$\mathcal{O}_K$
	=	=	= =	
$\mathbb{Z}$-basis	$\mathbb{Z}\cdot 1$	$\mathbb{Z}\cdot 1 + \mathbb{Z}\cdot i$	$\mathbb{Z} + \mathbb{Z}\sqrt{d}$ if $d \not\equiv 1 \pmod 4$ $\mathbb{Z} + \mathbb{Z}\frac{1+\sqrt{d}}{2}$ if $d \equiv 1 \pmod 4$	$\exists$ integral basis, i.e., $\mathcal{O}_K = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$ for some $\omega_1, \ldots, \omega_n \in \mathcal{O}_K$
multiplicative structure	unique prime factorization	unique prime factorization $p \equiv 3 \pmod 4$ stay prime $p \equiv 1 \pmod 4$ split into primes (e.g. $5 = 2^2+1^2 = (2+i)(2-i)$) $2 = -i(1+i)^2$	unique prime factorization $d<0$: for $d \in \{-1,-2,-3,-7,-11,-19,-43,-67,-163\}$ $d>0$: for $d \in \{2,3,5,6,7,11,13,\ldots\}$ For all other d there is just a unique factorization of ideals into prime ideals	in general only unique factorization of ideals into prime ideals
invertible elements (group of units)	± 1 $(\cong \mathbb{Z}/2\mathbb{Z})$	$\pm 1, \pm i$ $(\cong \mathbb{Z}/4\mathbb{Z})$	$\pm 1, \pm i$ if $d=-1$ $(\cong \mathbb{Z}/4\mathbb{Z})$ $\pm 1, \frac{\pm 1 \mp i\sqrt{3}}{2}, \frac{\pm 1 \pm i\sqrt{3}}{2}$ if $d=-3$ $(\cong \mathbb{Z}/6\mathbb{Z})$ ± 1 $(\cong \mathbb{Z}/2\mathbb{Z})$ for all other $d<0$ $\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ for all $d>0$ (e.g., $\pm(1+\sqrt{2})^u$, $u \in \mathbb{Z}$ if $d=2$)	Dirichlet's unit theorem

Applications to diophantine equations, e.g., $x^2+1=y^3$ has only $(x,y)=(0,1)$ as a solution in $\mathbb{Z}^2$. (Basic idea: $y^3 = x^2+1 = (x+i)(x-i)$. Use the arithmetical structure of $\mathbb{Z}[i]$ to deduce: $\exists a,b \in \mathbb{Z}: x+i = (a+ib)^3 = a^3 - 3ab^2 + i(3a^2b - b^3)$. Solve the system $\left.\begin{matrix} a^3-3ab^2 = x \\ 3a^2b - b^3 = 1 \end{matrix}\right\}$.)

Applications to Fermat's Last Theorem: $z^n = x^n + y^n = (x+y)(x+\zeta y)\ldots(x+\zeta^{n-1}y)$ with $\zeta = e^{2\pi i/n}$. Work in the ring $\mathbb{Z}[\zeta]$ of the cyclotomic field $\mathbb{Q}(\zeta)$.
(One can prove Fermat's Last Theorem in this way only if $\mathbb{Z}[\zeta]$ is a unique factorization domain.)