Harmonic measures and rigidity for surface group actions on the circle

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18th July 2022

36th Summer Topology Conference

The Milnor–Wood inequality and Matsumoto's rigidity theorem

Let Σ be an orientable closed surface of genus $g > 1$. Write $\Gamma := \pi_1(\Sigma)$. Consider a Γ-action on the circle, $\rho\colon \Gamma \to \mathrm{Homeo}_+(S^1)$.

Equip Σ a hyperbolic metric and regard $\Sigma = \Gamma\backslash\mathbb{D}$, where $\mathbb{D}$ is the Poincaré disk. This induces a *Fuchsian action*, $\rho_0\colon \Gamma \to PSU(1,1) \to \mathrm{Homeo}_+(S^1)$.

The *Euler number* of ρ is denoted by $e(\rho) \in \mathbb{Z}$, which can be defined as the Euler number of the suspension S^1-bundle $\Sigma \times_\rho S^1$ over Σ,

$$\Sigma \times_\rho S^1 := \Gamma\backslash(\mathbb{D} \times S^1), \quad \gamma \cdot (z,t) = (\gamma z, \rho(\gamma)t).$$

Theorem (Milnor '58, Wood '71 / Matsumoto '87)

1. The Euler number is bounded: $|e(\rho)| \leq -e(\Sigma) = 2g-2$.
2. The equality $e(\rho) = e(\Sigma)$ holds if and only if ρ is semiconjugate to ρ_0.

ρ is *semiconjugate* to ρ_0 if there exists a non-decreasing continuous map $\psi\colon S^1 \to S^1$ of mapping degree one which is (ρ,ρ_0)-equivariant,

$$\psi \circ \rho(\gamma) = \rho_0(\gamma) \circ \psi \quad (\forall \gamma \in \Gamma).$$

Burger–Iozzi–Weinhard's generalization

Let Σ be an orientable surface of genus g with m punctures, $2g + m - 2 > 0$. Write $\Gamma := \pi_1(\Sigma)$ and consider a Γ-action on the circle, $\rho\colon \Gamma \to \mathrm{Homeo}_+(S^1)$.

Equip Σ a complete hyperbolic metric with finite volume so that punctures are cusps. Regard $\Sigma = \Gamma \backslash \mathbb{D}$. Again, let $\rho_0\colon \Gamma \to PSU(1,1) \to \mathrm{Homeo}_+(S^1)$.

Burger–Iozzi–Weinhard defined the ***Euler number*** of ρ, denoted by $e(\rho) \in \mathbb{R}$, based on bounded cohomology. Note that the suspension bundle $\Sigma \times_\rho S^1$ is topologically trivial and classical definition does not apply when $m > 0$.

Theorem (Burger–Iozzi–Weinhard '14)

0. The Euler number is expressed as $e(\rho) = -\sum_{i=1}^{m} \tau(\tilde{\rho}(c_i))$.
1. The Euler number is bounded: $|e(\rho)| \leq -e(\Sigma) = 2g + m - 2$.
2. The equality $e(\rho) = e(\Sigma)$ holds if and only if ρ is semiconjugate to ρ_0.

$\tilde{\rho}\colon \Gamma \to \widetilde{\mathrm{Homeo}}_+(S^1)$ is a homomorphism lift of ρ, and $\tau\colon \widetilde{\mathrm{Homeo}}_+(S^1) \to \mathbb{R}$ the translation number, where $\widetilde{\mathrm{Homeo}}_+(S^1)$ denotes the universal cover group of $\mathrm{Homeo}_+(S^1)$.

Aim of this study

We shall give an alternative proof for Burger–Iozzi–Weinhard's theorem using *foliated harmonic measure*. For closed Σ, such an approach was proposed independently by Frankel and Thurston in 1990s.

Definition (Garnett '83)

Let M be a smoothly foliated manifold equipped with leafwise Riemannian metric. A Borel measure μ on M is called *harmonic* if

$$\int_M \Delta f(x)\mu(dx) = 0$$

for every compactly supported leafwise C^2 function f on M such that Δf is continuous on M, where Δ denotes the leafwise Laplace–Beltrami operator.

We equip the suspension foliation $M = \Sigma \times_\rho S^1$ with leafwise hyperbolic metric.

$$\Sigma \times_\rho S^1 := \Gamma \backslash (\mathbb{D} \times S^1), \quad \gamma \cdot (z, t) = (\gamma z, \rho(\gamma) t).$$

When Σ is closed, a harmonic probability measure exists from Garnett '83. Even when Σ has cusps, a harmonic probability measure on $\Sigma \times_\rho S^1$ exists thanks to a theorem by Alvarez '12.

Sketch of the proof

Γ: a torsion-free lattice in $PSU(1,1)$, $\Sigma = \Gamma\backslash\mathbb{D}$. $\rho\colon \Gamma \to \mathrm{Homeo}_+(S^1)$.
Assume $\rho(\Gamma)$ has no finite orbit in S^1.

Let μ be a harmonic measure on $M = \Sigma \times_\rho S^1$ with $\mu(M) = 4\pi^2|e(\Sigma)|$.
Note that μ is expressed in $(z,t) \in \mathbb{D} \times S^1$ as

$$\mu = h(z, e^{it})\mathrm{vol}_{\mathrm{hyp}}(z)\nu(t)$$

where $h(z,t)$ is positive harmonic function for ν-a.e. t, $\mathrm{vol}_{\mathrm{hyp}}(z)$ hyperbolic volume measure and $\nu(t)$ a Borel measure on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$,

$$\int_{S^1} h(z, e^{it})\nu(dt) = 2\pi.$$

Step 1. From the assumption ν has no atoms. May assume that ν is the Lebesgue measure λ by replacing ρ up to a semiconjugacy that collapses the complement of the support of μ.

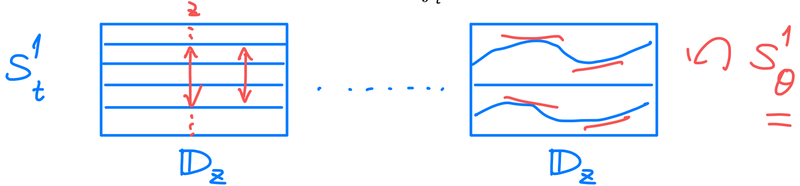
$$\mu = h(z, e^{it})\mathrm{vol}_{\mathrm{hyp}}(z)\lambda(t)$$

Sketch of the proof – continued

$\mu_z = h(z, e^{it})\lambda(t)$

Step 2. Define an S^1-action $\{\phi_\theta\}$ on M and regard M an S^1-principal bundle. The foliation is to be transversely Lipschitz from Harnack's inequality.

$$\phi_\theta(z, e^{it}) := (z, e^{i\tau_\theta(z,t)}), \quad \int_t^{\tau_\theta(z,t)} h(z, e^{is})ds = \theta.$$



Step 3. Take the defining 1-form ω for the foliation such that $\omega(\frac{\partial}{\partial\theta}) = 1$, which gives a continuous connection $M \to \Sigma$. Taking the average yields

$$\overline{\omega} := \frac{1}{2\pi}\int_0^{2\pi} \phi_\theta^*\omega \; d\theta$$

a continuous S^1-connection. From Harnack's inequality and the isoperimetric inequality, we see that $\overline{\omega}$ has curvature K with $|K| \leq 1$.

Sketch of the proof for rigidity

Step 4. We show

$$e(\rho) = \frac{1}{2\pi} \int_{\Sigma} K(z) \mathrm{vol}_{\mathrm{hyp}}(dz).$$

When Σ is closed, this is well-known. When Σ has m cusps, we cut off their neighborhood and obtain a compact surface Σ^s with boundary $\cup_{i=1}^m c_i^s$, where c_i^s is a horocircle that approach to i-th cusp as $s \to \infty$.

$$\frac{1}{2\pi} \int_{\Sigma^s} K(z) \mathrm{vol}_{\mathrm{hyp}}(dz) = \sum_{i=1}^{m} \tau(\widetilde{\mathrm{hol}}_{\overline{\omega}}(c_i^s)) \xrightarrow{s \to \infty} -\sum_{i=1}^{m} \tau(\tilde{\rho}(c_i)).$$

This step completes the proof for $|e(\rho)| \leq -e(\Sigma)$.

Sketch of the proof for rigidity

$$\omega = d\theta - \sum_{j=1}^{2} \omega_j(z,\theta)\, dx^j$$

Step 5. Assume $e(\rho) = e(\Sigma)$. We must have $K = -1$ a.e.. From Step 3,

$$\begin{aligned}
-K(z) &= \frac{(1-|z|^2)^2}{4\pi} \int_0^{2\pi} \frac{1}{2}\left(-\frac{\partial \omega_1}{\partial \theta}(z,\theta)\omega_2(z,\theta) + \frac{\partial \omega_2}{\partial \theta}(z,\theta)\omega_1(z,\theta) \right) d\theta \\
&\overset{\textit{Isoperimetric}}{\leq} \frac{(1-|z|^2)^2}{4\pi} \cdot \frac{1}{4\pi} \left(\int_0^{2\pi} \sqrt{\left(\frac{\partial \log h}{\partial x_1}\right)^2 + \left(\frac{\partial \log h}{\partial x_2}\right)^2} d\theta \right)^2 \\
&= \frac{1}{4\pi^2} \left(\int_0^{2\pi} |d \log h|_{\mathrm{hyp}} d\theta \right)^2 \\
&\overset{\textit{Harnack}}{\leq} 1.
\end{aligned}$$

Since the both equality holds, the harmonic measure must be of the form

$$h(z, e^{it}) = \frac{1-|z|^2}{|m(e^{it}) - z|^2}$$

where $m \in \mathrm{Homeo}_+(S^1)$ is (ρ, ρ_0)-equivariant. □

Thank you for your attention!