A characterization of productive cellularity

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This is a joint work with Lucia Junqueira and Renan Mezabarba.

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In the topological side, just consider the family of all non-empty sets $\tau_{\neq\emptyset}$ ordered by inclusion (therefore, an antichain is a family of mutually disjoint non-empty sets).

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So the general preservation of the ccc property cannot be decided in ZFC, but it is possible to decide it for specific classes of spaces.

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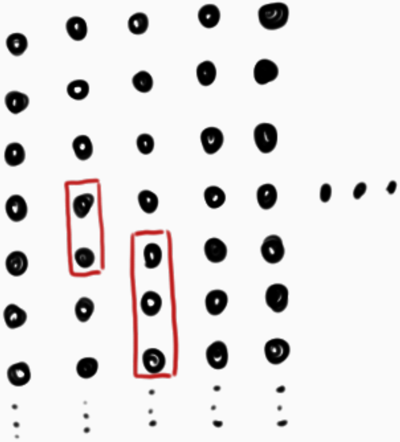

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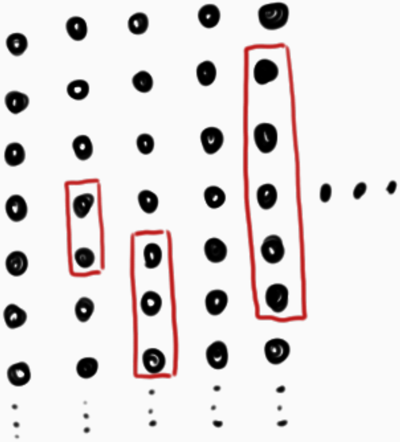

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Lemma
Let X be a space and $\mathcal{A}$ a large collection of antichains. Suppose that $A \subset \bigcup \mathcal{A}$ is a linked set. Then $\{\{a\} : a \in A\}$ is an antichain in $\mathbb{F}(\mathcal{A})$.

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Just note that, given $a, b \in A$ with $a \neq b$, a, b are compatible. Therefore, $\{a, b\}$ is not a subset of any antichain. $\square$

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Corollary
If X has the Knaster property, X is productively ccc.

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A regular space X is productively Lindelöf space if, and only if, $X \times \mathcal{Y}(X)$ for every "reasonable hyperspace" $\mathcal{Y}(X)$.

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Trying to do the same thing for the ccc property, we obtain:

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It turns out that "$X \times \mathcal{Y}(X)$ being ccc for $\mathcal{Y}(X)$ reasonable" is equivalent to $\mathbb{F}(\mathcal{A})$ being non-ccc. And everything get much more simple.

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Let Y be a ccc ordered set such that $X \times Y$ is not ccc. Let $W \subset X \times Y$ be an uncountable antichain.

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We will prove that $\mathcal{A} = \{A_y : y \in Y\}$ is the desired collection.

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- $\mathcal{R}$ is uncountable. Since Y is ccc, there are $y, y_F, y_G \in Y$, $F, G \in \mathcal{F}$ such that $y \leq y_F, y_G$.

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Thus $\mathbb{F}(\mathcal{A})$ is ccc.

Why not all the finite antichains of $\bigcup \mathcal{A}$?

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(so saying that X is ccc is the same as saying that $c(X) = \aleph_0$)

We can ask the same kind of question as before, and work with this more general setting, asking for which cardinals we can bound $c(X \times Y)$.

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With this notation, asking if X is productively ccc is the same as asking if $\aleph_0 \in Sp(X)$.

(Basically we are copying what Arhangel'skii did in 1981 to the tightness property)

The characterization

The ccc case characterization works the same way in the general case:

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Theorem
Given an ordered space X and $\kappa \geq \aleph_0$, $\kappa \in Sp(X)$ if, and only if, $c(\mathbb{F}(\mathcal{A})) > \kappa$ for every κ-large collection $\mathcal{A}$.

Where a collection $\mathcal{A}$ of antichains is κ-large if $|\bigcup \mathcal{A}| > \kappa$.

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And, by the previous observation, $pc(X) \leq |X|$.

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Which is a generalization of

Theorem (Fremlin)
Every product of separable spaces is productively ccc.

Back to Knaster

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Following this notation, is convenient to define $< \omega$-linked as centered (i.e. for every $F \in [X]^{<\omega}$, there is a $p \in X$ such that $p \leq q$ for all $q \in F$). And, call $K_{<\omega}$ the property to have $\aleph_1$-precaliber (i.e. every uncountable subset has an uncountable $< \omega$-linked subset).

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Given a space X and $\alpha \in \omega \cup \{< \omega\}$, we define

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Therefore, we have

$$pc(X) \leq K_2(X) \leq \cdots \leq K_n(X) \leq K_{<\omega}(X) \leq d(X).$$

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This is known:

Theorem (Todorcevic, Velickovic (1987))
$MA_{\aleph_1}$ is equivalent to

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Question
What can we say around $c(X) = K_2(X)$ or $c(X) = pc(X)$?

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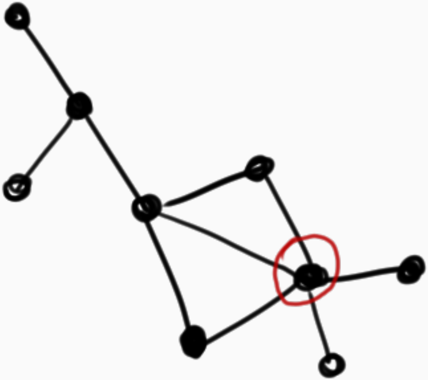
Bonus

And now for something completely different.

Graphs

Graphs



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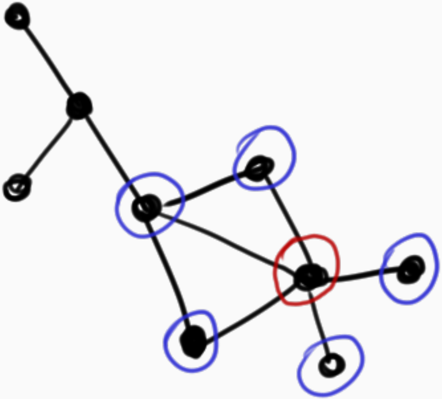

Unfriendly partitions

We say that $\pi : V(G) \to 2$ is unfriendly at $v \in V$ if

$$|\{w \in N(v) : \pi(w) \neq \pi(v)\}| \geq |\{w \in N(v) : \pi(w) = \pi(v)\}|$$

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And we say that π is unfriendly if it is unfriendly for every v.

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- Infinite regular graphs (regular: every vertex has the same degree);
- “Almost” locally finite (non trivial) or “almost” regular graphs.

Counterexamples

In 1990, Milner and Shelah provided two graphs with no vertices of finite degree with no unfriendly partitions:

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In 1990, Milner and Shelah provided two graphs with no vertices of finite degree with no unfriendly partitions:

- If there is an ω_1 generated *p*-point, then there is an example of cardinality $\aleph_\omega$;
- In ZFC, there is an example of cardinality $\mathfrak{c}^{+\omega}$ (the first limit cardinal above $\mathfrak{c}$).

Is there an smaller one?

In 1990, Aharoni, Milner and Prikry proved that if a graph has no vertices with finite degree and there is a finite set F of regular cardinals such that $d(v) \in F$ for every v, then the graph has an unfriendly partition.

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Theorem (A., Real)
The statements:

- $\Gamma = \aleph_\omega$;
- $\Gamma = (\mathfrak{c})^{+\omega}$

are independent from ZFC $+ \aleph_\omega < \mathfrak{c}$.

Big question

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Question
Does every countable graph have an unfriendly partition?

Partial results

There are some partial results (like the ones about "almost regular"): the main one is about graphs that have no "forbidden substructures" (Berger, 2017).

Another one

Theorem (A., Real)
If G is a countable graph such that there are no two adjacent vertices of finite degree, then G has an unfriendly partition.

A decomposition

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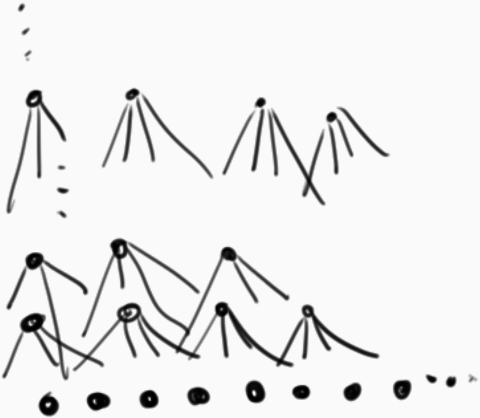

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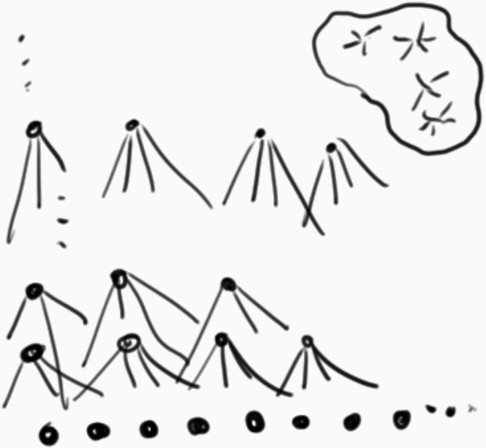

This is a joint work with Lucas Real.

Thank you ($\times 2$)