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On complemented copies of c_0 in spaces of vector-valued continuous function with the pointwise topology

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joint work with Jerzy Kąkol and Damian Sobota

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Setting the stage

Let X be a Tychonoff space and E be a locally convex space.

We consider the space $C(X, E)$ of continuous functions $f : X \to E$.

We equip it with the following topologies

- $C_k(X, E)$ … topology of uniform convergence on compact subsets of X
- $C_p(X, E)$ … topology of pointwise convergence, i.e. the subspace topology induced by E^X.

If X is compact and E is a Banach space, $C_k(X, E)$ is a Banach space.

If X is hemicompact and locally compact and E is a Fréchet space, $C_k(X, E)$ is a Fréchet space.

In the case $E = \mathbb{R}$, we simply write $C_k(X) = C_k(X, \mathbb{R})$ and $C_p(X) = C_p(X, \mathbb{R})$.

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Motivation

Theorem (Cembranos, Freniche)

For an infinite compact space X and an infinite-dimensional Banach space E, the Banach space $C(X, E)$ contains a complemented copy of the Banach space c_0.

Theorem (Domański–Drewnowski)

Let X be a Tychonoff space which contains an infinite compact subset and E a Fréchet space which is not Montel. Then the Fréchet space $C_k(X, E)$ contains a complemented copy of the space c_0.

Under suitable assumptions on X and Y we obtain that

$$C_k(X \times Y) \approx C_k(X, C_k(Y))$$

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Motivation: The Grothendieck property

A Banach space X has the Grothendieck property if every weak*-convergent sequence is weakly convergent.

- Complemented subspaces of a Banach space with the Grothendieck property have the Grothendieck property.
- Separable Banach spaces have the Grothendieck property if and only if they are reflexive.
- c_0 is separable and not reflexive.
- A Banach space which contains c_0 as a complemented subspace does not have the Grothendieck property.
- The Banach space $C(K)$, K compact, has the Grothendieck property if and only if it does not contain a complemented copy of c_0.

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Let $(c_0)_p = \{x \in \mathbb{R}^\omega : \ x(n) \to 0\}$ be endowed with the pointwise topology inherited from $\mathbb{R}^\omega$.

Question

When does a given space $C_p(X,E)$ contain a complemented copy of the space $(c_0)_p$?

The proofs for $C_k(X,E)$ use a variant of the Josefson-Nissenzweig theorem.

Theorem (Josefson-Nissenzweig)

Let X be an infinite dimensional Banach space. There is a sequence $(x_n)_{n\in\omega}$ in X^* which is a weak*-null sequence and $\|x_n\| = 1$ for all $n \in \omega$.

Goal: A corresponding result for $C_p(X, E)$-spaces

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The dual of $C_p(X)$

Let X be a Tychonoff space. Every element of the dual space $\varphi \in L_p(X) = C_p(X)'$ has a unique representation as a finite linear combination of point measures

$$\varphi = \sum a_x \delta_x.$$

We set

$$\|\varphi\| = \sum_{x \in \operatorname{supp} \varphi} |a_x|.$$

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Definition (Banakh and Gabriyelyan)

A locally convex space E has *the Josefson–Nissenzweig Property* (JNP) if the identity map $(E', \sigma(E', E)) \to (E', \beta^*(E', E))$ on the topological dual E' of E is not sequentially continuous.

The topology $\beta^*(E', E)$ is the of uniform convergence on barrel-bounded subsets of E, i.e. on sets which are absorbed by all barrels. If E is barrelled we have $\beta^*(E', E) = \beta(E', E)$.

Proposition (Banakh and Gabriyelyan)

For every Tychonoff space X, the space $C_p(X)$ has the JNP if and only if there exists a sequence of functionals $(\varphi_n)_{n\in\omega}$ in the dual space $L_p(X)$ of $C_p(X)$ such that $\varphi_n(f) \to 0$ for every $f \in C(X)$ and $\|\varphi_n\| = 1$ for every $n \in \omega$.

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Theorem (Banakh–Kąkol–Śliwa)

Let X be an infinite Tychonoff space. Then, the following conditions are equivalent:

1. *$C_p(X)$ has the JNP;*
2. *$C_p(X)$ contains a complemented copy of $(c_0)_p$;*
3. *$C_p(X)$ admits a continuous linear surjection onto $(c_0)_p$.*

- $C_p(\beta\mathbb{N})$ where $\beta\mathbb{N}$ is the Stone-Čech compactification of $\mathbb{N}$ does not have the Josefson-Nissenzweig property.
- If X contains a nontrivial convergent sequence $x_n \to x$ the space $C_p(X)$ has the Josefson-Nissenzweig property: take $\varphi_n = \frac{1}{2}(\delta_{x_n} - \delta_x)$.

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JNP and complemented copies of $(c_0)_p$ in $C_p(X)$

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Complemented copies of $(c_0)_p$ in $C_p(X, E)$

Theorem

Let E be a barrelled locally convex space with the Josefson–Nissenzweig property and let X be a Tychonoff space containing an infinite compact set. Then $C_p(X, E)$ contains a complemented copy of $(c_0)_p$.

Proof sketch I

Since E is barrelled and has the JNP, there is a weak* null sequence $(x_n^*)_{n\in\omega}$ in E' which is not convergent for the $\beta(E', E)$-topology. Hence we may choose a bounded sequence $(x_n)_{n\in\omega}$ in E with

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$(x_n^*)_{n\in\omega}$ weak*-null sequence in E', $(x_n)_{n\in\omega}$ in E with $\langle x_n^*, x_n\rangle = 1$.

Sketch of proof II.

Let $K \subset X$ be an infinite compact set. Since X is Tychonoff, we may choose a sequence of continuous functions $\varphi_n\colon X \to [0,1]$ with disjoint support and $\varphi_n(t_n) = 1$ for some $t_n \in K$.

We set

$$J\colon (c_0)_p \to C_p(X,E), \qquad (a_n)_{n\in\omega} \overset{J}{\longmapsto} \sum_{n\in\omega} a_n\varphi_n(t)x_n$$

and

$$P\colon C_p(X,E) \to (c_0)_p, \qquad f \overset{P}{\longmapsto} \big(\langle x_n^*, f(t_n)\rangle\big)_{n\in\omega}.$$

Finally we show that J and P are continuous, J isomorphic embedding, P surjective and $JPJP = JP$. □

Complemented copies of $(c_0)_p$ in $C_p(X, E)$, II

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Corollary

Let X be a Tychonoff space containing an infinite compact subspace and let E be a Fréchet space which is not Montel. If $C_k(X, E)$ is an infinite-dimensional Fréchet space, then $C_k(X, E)$ contains a complemented copy of the Banach space c_0.

Proof.

Follows using the closed graph theory. □

Question

Does the theorem say something for the particular case $E = C_p(Y)$?

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Applications

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$C_p(X, C_p(Y))$ and $C_p(X \times Y)$

Recall that $C_k(X, C_k(Y)) \approx C_k(X \times Y)$ if Y is locally compact. An isomorphism is given by the mapping

$$C_k(X, C_k(Y)) \to C_k(X \times Y), \qquad f \mapsto g$$

where $g(x, y) := f(x)(y)$.

Similarly, consider the case of $C_p(X, C_p(Y))$, i.e.

$$C_p(X, C_p(Y)) \to , \qquad f \mapsto g,$$

where $SC(X \times Y)$ denotes the space of separately continuous functions on $X \times Y$.

Theorem (Folklore)

The above mapping defines an isomorphism between $C_p(X, C_p(Y))$ and $SC_p(X \times Y)$.

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The topology σ on $X \times Y$

Let X and Y be Tychonoff spaces. By σ let us denote the weak topology on $X \times Y$ generated by the family of all separately continuous functions.

Proposition (Henriksen–Woods)

- *$(X \times Y, \sigma)$ is a Tychonoff space*
- *$C(X \times Y, \sigma) = SC(X \times Y)$, i.e. every continuous function on $(X \times Y, \sigma)$ is separately continuous on $X \times Y$ and vice versa*

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- *$C(X \times Y, \sigma) = SC(X \times Y)$, i.e. every continuous function on $(X \times Y, \sigma)$ is separately continuous on $X \times Y$ and vice versa*

The JNP for spaces of the form $C_p(X, C_p(Y))$

Let X and Y be infinite Tychonoff spaces.

Theorem

$C_p(X \times Y, \sigma)$ has the JNP $\Leftrightarrow$ either $C_p(X)$ or $C_p(Y)$ has the JNP.

Proof sketch.

$\Leftarrow$: assume w.l.o.g. $C_p(X)$ has the JNP, i.e. there is a JN-sequence (φ_n) on X with $\varphi_n = \sum_{i=1}^{k_n} \alpha_i^n \delta_{x_i^n}$. Fix $y \in Y$ and define the functional ψ_n on $C_p(X \times Y, \sigma)$ by

$$\psi_n = \sum_{i=1}^{k_n} \alpha_i^n \delta_{(x_i^n, y)}.$$

Then $(\psi_n)_{n \in \omega}$ is a JN-sequence on $(X \times Y, \sigma)$.

$\Rightarrow$ is more involved $\square$

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The dual of $C_p(X, E)$

Let X be a Tychonoff space and let E be a locally convex space. The topological dual space of $C_p(X, E)$ is algebraically isomorphic to the space $L_p(X) \otimes E'$ [Ferrando–Kąkol]

Each functional $\varphi \in C_p(X, E)'$ may be represented as

$$\sum_{i,j} \alpha_{i,j}(\delta_{x_i} \otimes e'_j),$$

where $x_i \in X$ and $e'_j \in E'$. Hence, if $E = C_p(Y)$ for some Tychonoff space Y, then each $\varphi \in C_p(X, E)'$ may be written as a finite sum

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Let us define *the norm* of φ (in $L_p(X) \otimes L_p(Y)$) in the following natural way:

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Proposition

For every Tychonoff spaces X and Y the space $C_p(X \times Y, \sigma)$ has the JNP if and only if there exists a sequence $(\psi_n)_{n\in\omega}$ in the dual $L_p(X) \otimes L_p(Y)$ which is weakly convergent to 0 and such that $\|\psi_n\| = 1$ for every $n \in \omega$.*

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The Josefson-Nissenzweig property of $C_p(X,E)$

Proposition

Let X be a Tychonoff space and let E be a locally convex space. If $C_p(X)$ has the JNP or E has the JNP, then the same is true for the space $C_p(X,E)$.

Proof Sketch I

Assume that E has the Josefson–Nissenzweig property. Pick a weak* null sequence $(\varphi_n)_{n\in\omega}$ in E' and a barrel-bounded set $B \subset E$ on which it does not converge to zero uniformly. Let $f \in C_p(X)$ and $x \in X$ be such that $f(x) = 1$. For each $n \in \omega$ set $\psi_n = \delta_x \otimes \varphi_n$; of course, $\psi_n \in L_p(X) \otimes E'$. Note that for every $g \in C_p(X,E)$ we have $\langle \psi_n, g\rangle = \langle \varphi_n, g(x)\rangle \to 0$ as $n \to \infty$, so $(\psi_n)_{n\in\omega}$ is a weak* null sequence in $(C_p(X,E))'$.

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The Josefson-Nissenzweig property of $C_p(X, E)$

- $\psi_n = \delta_x \otimes \varphi_n$ which is a weak* null sequence in $(C_p(X, E))'$.
- $f \in C_p(X)$ and $x \in X$ be such that $f(x) = 1$
- $B \subset E$ barrel-bounded, $\varphi_n \not\to 0$ on B

Proof.

Proof Sketch II

$$C_p(X) \otimes E \ni \sum_i f_i \otimes e_i \longmapsto \Big[x \mapsto \sum_i f_i(x) e_i\Big] \in C_p(X, E).$$

Check that $f \otimes B$ is barrel-bounded in $C_p(X, E)$ and

$$\begin{aligned} \sup_{g \in f \otimes B} |\langle \psi_n, g\rangle| &= \sup_{e \in B} |\langle \delta_x \otimes \varphi_n, f \otimes e\rangle| = \sup_{e \in B} |\langle \varphi_n, f(x) e\rangle| \\ &= \sup_{e \in B} |\langle \varphi_n, e\rangle| \not\to 0 \end{aligned}$$

□

Summary of the case of $E = C_p(Y)$

Theorem

Let X and Y be infinite Tychonoff spaces. Then, the following conditions are equivalent:

1. $C_p(X, C_p(Y))$ *has the JNP;*
2. $C_p(X, C_p(Y))$ *contains a complemented copy of* $(c_0)_p$*;*
3. $C_p(X, C_p(Y))$ *admits a continuous linear surjection onto* $(c_0)_p$*;*
4. $C_p(X)$ *contains a complemented copy of* $(c_0)_p$ *or* $C_p(Y)$ *contains a complemented copy of* $(c_0)_p$*;*
5. $C_p(X)$ *admits a continuous linear surjection onto* $(c_0)_p$ *or* $C_p(Y)$ *admits a continuous linear surjection onto* $(c_0)_p$*;*
6. $C_p(X)$ *has the JNP or* $C_p(Y)$ *has the JNP.*

Open Problems

- Let X be a Tychonoff space (containing an infinite compact subspace) and E a locally convex space. Assume that $C_p(X, E)$ has the JNP. Does it follow that $C_p(X, E)$ contains a complemented copy of $(c_0)_p$?
- Let X be a Tychonoff space and E a locally convex space. Assume that the space $C_p(X, E)$ has the JNP. Does it follow that $C_p(X)$ has the JNP or E has the JNP?
- Can the Josefson–Nissenzweig property of the spaces $C_p(X, E)$ for X Tychonoff and E an arbitrary locally convex space be characterised in terms of some special "JN-sequences" in $L_p(X) \otimes E'$, like it is done in the case of $E = C_p(Y)$ for Y Tychonoff?

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References

C. Bargetz, J. Kąkol, D. Sobota: On complemented copies of the space c_0 in spaces $C_p(X, E)$. Preprint (arxiv:2107.03211)

T. Banakh, S. Gabriyelyan, *The Josefson–Nissenzweig property for locally convex spaces*, preprint (arXiv:2003.06764), 2020.

P. Cembranos, *$C(K, E)$ contains a complemented copy of c_0*, Proc. Amer. Math. Soc. **91** (1984), no. 4, 556–558.

P. Domański, L. Drewnowski, *Fréchet spaces of continuous vector-valued functions. Complementability in dual Fréchet spaces and injectivity*, Studia Math. **102** (1992), 257–267.

F.J. Freniche, *Barrelledness of the space of vector valued and simple functions*, Math. Ann. 267 (1984), 479–486.

M. Henriksen, R.G. Woods, *Separate versus joint continuity: A tale of four topologies*, Topol. Appl. **97** (1999), 175–205.