Chaotic dynamics in foliated spaces

R. Barral Lijó [1]

[1]Ritsumeikan University, Japan

18 July 2022

Outline

Preliminaries on chaos

Preliminaries on foliated spaces

Results

Definition

(X, d): metric space, $f : X \to X$ is *sensitive to initial conditions (SIC)* if there is $c > 0$ so that, for every $x \in X$, $r > 0$, there are $y \in X$ and $n \in \mathbb{N}$ s.t.

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  'Chaos: When the present determines the future, but the approximate present does not approximately determine the future.' E. Lorenz

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3. f is sensitive to initial conditions.

Theorem (J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey [1])

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- This result was later generalized to group and semigroup actions.
- Note that this holds even when X is NOT compact.

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- the system is *almost equicontinuous* if the points of equicontinuity are dense.

Classical result:

Theorem (Auslander-Yorke dichotomy)

Let (X, d) be a compact metric space, and let $f : X \to X$ be topologically transitive. Then, either f is sensitive to initial conditions, or it is almost equicontinuous. If f is minimal, then it is either SIC or equicontinuous.

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- The omission is motivated because (TT) and (DPO) imply (SIC) for semigroup actions.
- Main motivation: to learn *whether and when this omission is warranted*.

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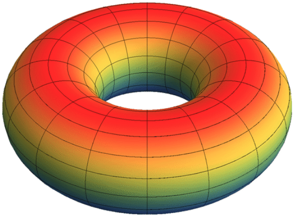

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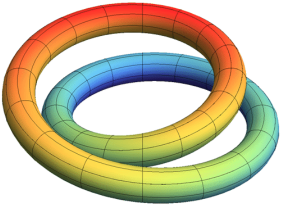

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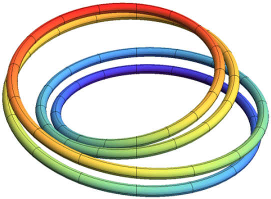

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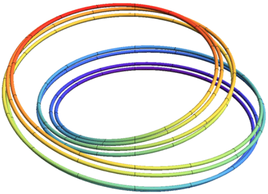

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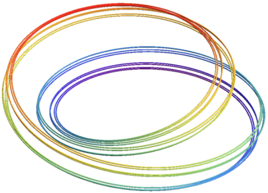

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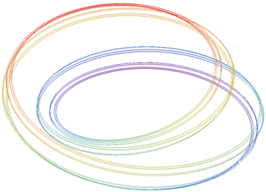

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- Guiding principle: foliated spaces are generalized dynamical systems.
- The leaves $\approx$ the "orbits".
- Using the words of Churchill, they are dynamical systems "in the absence of time."

Holonomy pseudogroup

To model foliated dynamics, we use pseudogroups:

Definition

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- Foliated spaces have *holonomy pseudogroups*.

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- There is a 1-to-1 correspondence between leaves of X and orbits in the holonomy pseudogroup.

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- We are following known ideas by Hector and Hirsch, Matsumoto [5], Álvarez and Candel. . .

Results on SIC

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If G is a f.g. group acting on a compact space X, then the action $G \curvearrowright X$ is sensitive if and only if the induced pseudogroup $\mathcal{G}$ is.

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Results on SIC 2

- Let $G \curvearrowright X$ be an action where G is not f.g. or X is not compact.
- Let $\mathcal{G}$ be the pseudogroup generated by the action.
- In general, $\mathcal{G}$ being sensitive is a stronger condition than G being sensitive.
- We manage to produce an explicit example where the action is sensitive but the pseudogroup is not for
  - $F_2 \curvearrowright T^2 \times \mathbb{Z}$, and
  - a non f.g. group acting on T^2.

Devaney chaos for foliated space

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Definition

A foliated space X is SIC, equicontinuous or almost equicontinuous if its holonomy pseudogroup is.

Theorem (Auslander-Yorke dichotomy, (4th key point))

A compact and transitive foliated space X is either SIC or almost equicontinuous. If X is minimal, then it is either SIC or equicontinuous.

- Examples: a Riemannian or Lie foliation is equicontinuous.
- The geodesic foliation on the unit tangent bundle of a compact hyperbolic surface is SIC.
- The Reeb foliation on the solid torus is almost equicontinuous.

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Definition (Devaney chaos (2nd key point))

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Does (1) + (2) imply (3)? (3rd key point)

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So the "compact" case works as for group actions. What about non-compact foliations?

Counterexample

Theorem

There is an affine foliation by surfaces on a non-compact 4-manifold satisfying:

- *it is topologically transitive (there is a dense leaf),*
- *the set of compact leaves is dense, but*
- *the foliation is not sensitive.*

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- metrics d_i on the transversals $\cong \mathbb{R}^2$,
- and a closed non-compact leaf L such that
- every holonomy transformation defined on L is an isometry.
- So there is no "butterfly effect" around L.

Based on the contents of
Chaos for foliated spaces and pseudogroups, ____ (2022),
arXiv:2202.09983

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