Cohomology groups for N-fold tilings

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Let E be a 2-dimensional vectorial plane of $\mathbb{R}^n$ and let $\gamma \in \mathbb{R}^n$, with $n \geq 3$.

Consider the two following orthogonal projections: one on E and one on $E^\perp$. Denote them by π and π'.

The **window** W_γ is the projection of $\gamma + [0,1]^n$ on $E^\perp$.

Tiling

The vertices of the tiling are the projections of certain points of $\mathbb{Z}^n$ onto E. Specifically we will look at those v such that $\pi'(v)$ is inside the window W_γ:

$$\{\pi(v) \mid v \in \mathbb{Z}^n, \pi'(v) \in W_\gamma\}$$

The rhombuses are the images by π of the two-dimensional faces of $\mathbb{Z}^n$.

This is called a $n \to 2$ tiling by cut and projection (or model set).

We could also make a tiling if E is not of dimension two...

Some examples

$2 \to 1$

$4 \to 2$

$7 \to 2$

Example $2 \to 1$

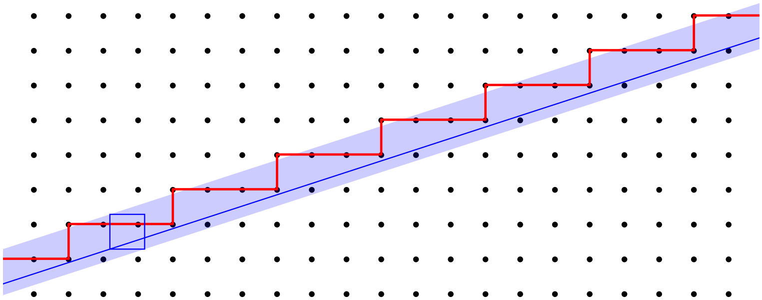
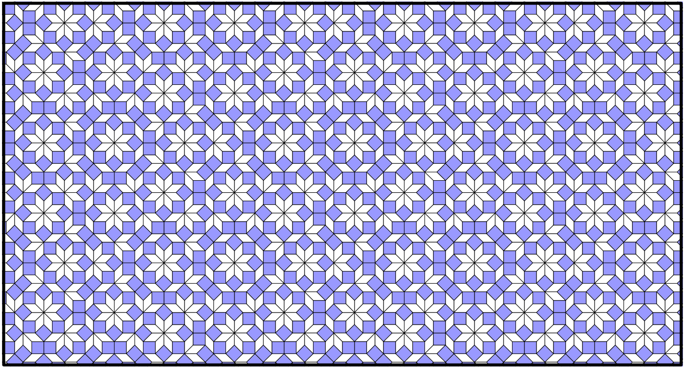


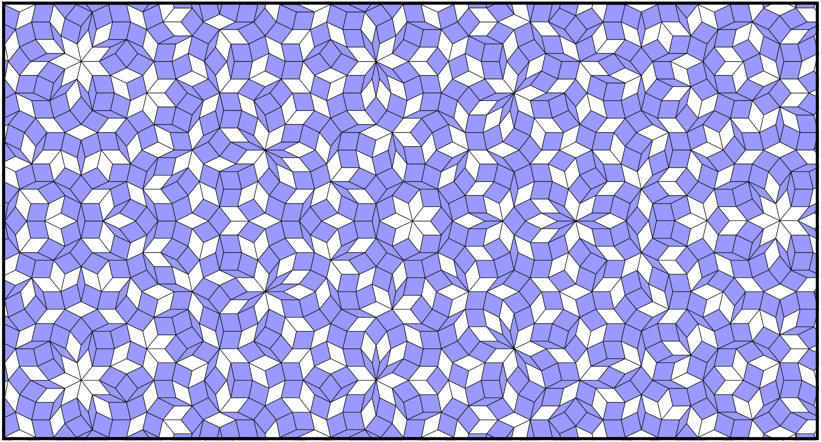
Figure: The square $[0,1]^2 + \gamma$ and E.

Link with sturmian word.

Example $4 \to 2$



Example $7 \to 2$



Topology of tilings

Fix the set of proto tiles.
Define a distance on the set of tilings of E:

Two tilings are **close** if they agree on a **big** ball centered at the origin, up to a **small** translation.

Open sets

For this topology, the open sets are elements of the form: open set of $\mathbb{R}^2$ times a Cantor set.

Properties

- The space of tilings of E is connected,
- it is non path connected,
- The connected components are contractible.

Main object

Consider the tiling T of E obtained by cut and projection for a parameter γ.

Let us denote Ω_E^γ the closure of the orbit of the tiling T under the action of the group of translations of E.

We want to understand the topology of this space.

One way: Compute the cohomology groups of Ω_E^γ with integer coefficients.

Remark that simplicial and singular cohomologies are not usefull.
We use Cech cohomology.

We need to find three groups since $dimE = 2$:

$$H^0(\Omega_E^\gamma), H^1(\Omega_E^\gamma), H^2(\Omega_E^\gamma).$$

By definition $H^0(\Omega_E^\gamma) = \mathbb{Z}$.

In all the following E will denote a plane not containing a vector of $\mathbb{Z}^n$.

We can think that $n = 4 \ldots$

We fix an integer n. Then we are in $\mathbb{R}^N$ ($\mathbb{R}^n$ or $\mathbb{R}^{n/2}$ depending on the parity of n) and we consider the plane E_n spanned by the two following N dimensional vectors

$$(\cos \frac{2k\pi}{n})_{0\leq k\leq N-1}, \quad \text{and} \quad (\sin \frac{2k\pi}{n})_{0\leq k\leq N-1}$$

If $\gamma = 0$, then we call the tilings N-fold tilings. If $\gamma \neq 0$ we call them *generalized n-fold tiling*.

Particular names

- $n = 8$ Ammann-Beenker.
- $n = 5$ Penrose.
- $n = 12$ Socolar or hexagonal.

Theorem (Gahler-Hunton-Kellendonk-. . . 2013)

We have

	$H^0(\Omega^{\gamma}_{E_n})$	$H^1(\Omega^{\gamma}_{E_n})$	$H^2(\Omega^{\gamma}_{E_n})$
$n = 8$	$\mathbb{Z}$	$\mathbb{Z}^5$	$\mathbb{Z}^9$
$n = 5, \gamma \in \mathbb{Z}[\varphi]$	$\mathbb{Z}$	$\mathbb{Z}^5$	$\mathbb{Z}^8$
$n = 5, \gamma \notin \mathbb{Z}[\varphi]$	$\mathbb{Z}$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{34}$
$n = 12, \gamma = 0$	$\mathbb{Z}$	$\mathbb{Z}^7$	$\mathbb{Z}^{28}$
$n = 7, \gamma = 0$	$\maltese$	$\maltese$	$\maltese$

Remark

If $\gamma = 0$, there is in fact another method for the previous examples, since the tilings are substitutive. However the computations are not substantially easier.

If $n = 7$, then the groups are not finitely generated.

Our result

Complete description of the cohomology groups of the generalized 12-fold tilings.

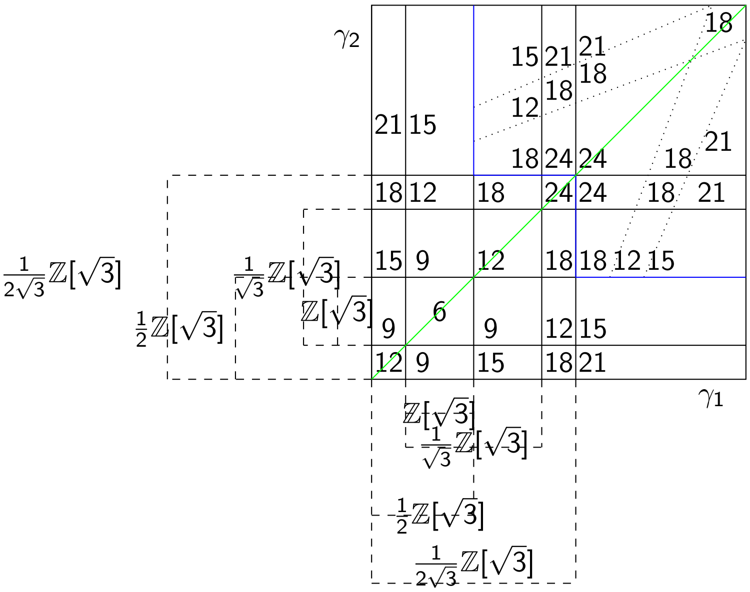
All values for H^1.

All values up to 2 orbits by direction, for H^2 obtained by computer program.

For more orbits by direction, too hard ...

Conjecture on the maximal value for the two groups:

$$\mathbb{Z}^{25}, \mathbb{Z}^{564}.$$


γ_2
γ_1
15 21 21
18
18
12
21 15
18 24 24 18 21
18 12 18 24 24 18 21
15 9 12 18 18 12 15
9 6 9 12 15
12 9 15 18 21
18
$\frac{1}{2\sqrt{3}}\mathbb{Z}[\sqrt{3}]$
$\frac{1}{2}\mathbb{Z}[\sqrt{3}]$
$\frac{1}{\sqrt{3}}\mathbb{Z}[\sqrt{3}]$
$\mathbb{Z}[\sqrt{3}]$
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$\frac{1}{\sqrt{3}}\mathbb{Z}[\sqrt{3}]$
$\frac{1}{2}\mathbb{Z}[\sqrt{3}]$
$\frac{1}{2\sqrt{3}}\mathbb{Z}[\sqrt{3}]$

To be continued ?

Dynamical properties in $\Omega_{E_{12},\gamma}$.

Window of a $2n$-fold, with $n > 6$.

Tiling of $\mathbb{R}^3$. Method to obtain H^3 ?

Information on the tilings ?

General result

Lemma (Julien 2010)

The cohomology groups of Ω_E^γ are finitely generated if and only if

$$\beta = 2 - 4 + rk\Gamma$$

In any case we have $\beta \geq 2 - 4 + rk\Gamma$.

Theorem (Gähler-Hunton-Kellendonk-2013)

The free abelian parts of the cohomology groups of Ω_E^γ, if finitely generated, are given by

- $H^0 = \mathbb{Z}$
- $H^1 = \mathbb{Z}^{4+L_1-R_1}$
- $H^2 = \mathbb{Z}^{3+L_1+e-R_1}$

with $e = -L_0 + \sum_{\alpha \in I_1} L_0^\alpha$.

Let us denote Γ the set $\pi'(\mathbb{Z}^n)$ in the space $E^\perp$.

$$\overline{\Gamma} \sim \mathbb{Z}^k \oplus \mathbb{R}^l$$

We denote Δ the discrete part of $\overline{\Gamma}$ and $F = vect(\Delta)$.

$$E^\perp = F \oplus F^\perp$$

Let P be the collection of spaces parallel to $F^\perp$ defined as

$$P = \bigcup_{\delta \in \Delta} (F^\perp + \delta).$$

Our case

$$\mathbb{R}^6 = E \oplus E' \oplus F.$$

The window W_γ is a four dimensional polytope.

The cut of W_γ by a plane parallel to $F^\perp$ defines a polygon.

We study W_γ and its intersection with P.

We obtain polygons and need to understand the orbit of these lines under the action of Γ.

Works only if $dimF^\perp = 2$.

Definition

The main objects are lines. They are directed by $f_1, \ldots, f_n$ projections of the canonical basis of $\mathbb{R}^n$.

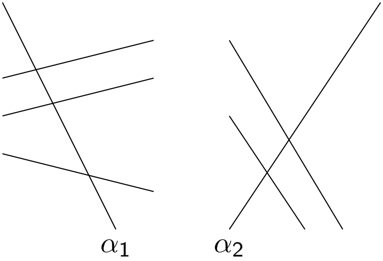
- Action of Γ on these lines. Set I_1 of 1-**singularities**
- Intersection of such lines. Set I_0 of 0-**singularities**.

- We denote Γ^i the stabilizer under Γ of the line $Vect(f_i)$. Its rank is denoted $1+\beta_i$.
- Let us denote R_1 the rank of the module generated by $\Lambda_2\Gamma^\alpha, \alpha \in \ldots$
- The number β is then defined by $\beta = \max\{\sum_I \beta_i, \mid |I| = 2\}$.

- L_1 number of lines,
- L_0 number of points.
- $1 + \beta_1$ number of orbits of points on each line.

Some parallel lines can be in the same Γ orbit ...

Some intersection points on different lines can be in the same Γ orbit ...



$$L_0^{\alpha} = 1 + \beta_1 = 3,$$

$$L_0^{\alpha'} = 1 + \beta_2 = 2$$

$$L_1 = 7$$

$$L_0 \leq 5$$

Restriction on γ:

The action of Γ on lines has two parts: one is discrete, and a continuous one.

Only continuous one is interesting. Thus we can restrict to $\gamma \in F^{\perp}$.

We move the window by γ and it changes the lines.

Window

The polytope has 52 vertices.

They are splitted in points, triangles and hexagons.
$4 * 1 + 8 * 3 + 4 * 6 = 52.$

The faces of dimension three are like cubes.

Window for $\gamma = 0$

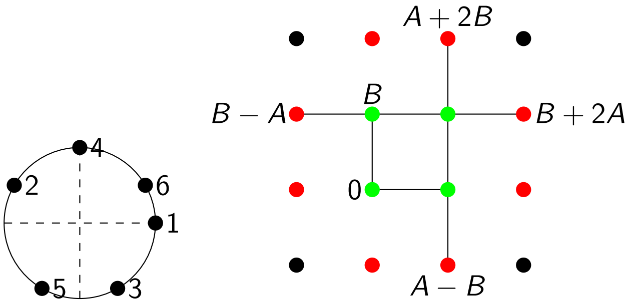


Figure: Black=points, red=triangles, green=hexagons

Window with $\gamma \neq 0$

We want to describe the lines supporting the edges of the polygons.

The lines are at intersections of the cube with the plane $F^{\perp} + \gamma$.

In dimension four, a plane can cut a cube without intersection with an edge.

Intersection of cubes with the plane P_γ.

Each intersection is either an hexagon, or a triangle, or a segment.

For each line we know the direction and a point.

We need to compute the number of orbits of lines, and the number of orbits of intersections of points.

Easy case

- $rk(H^1(\Omega_{E_{12},0})) = 4 + 6 - 3.$
- $rk(H^2(\Omega_{E_{12},0})) = 3 + 6 - 14 + 6 * 6 - 3 = 28.$

There are 6 directions. On each direction α, we have $L_0^\alpha = 6$. We also have $L_0 = 14 < 6 * 6$.

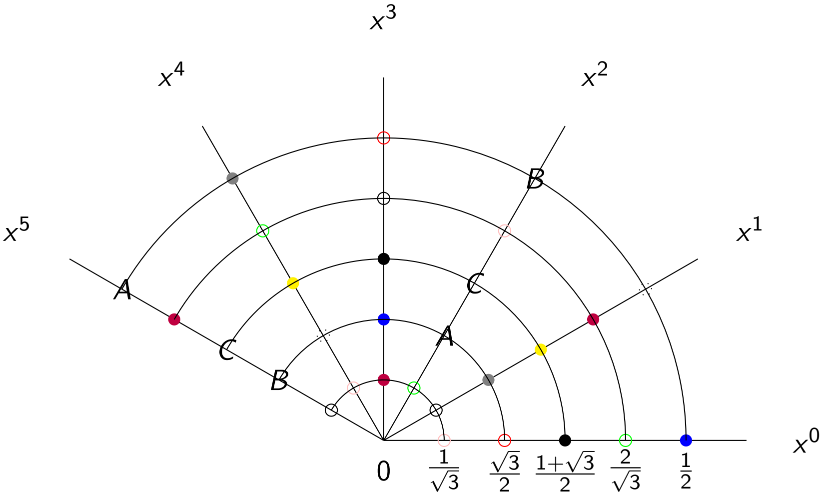
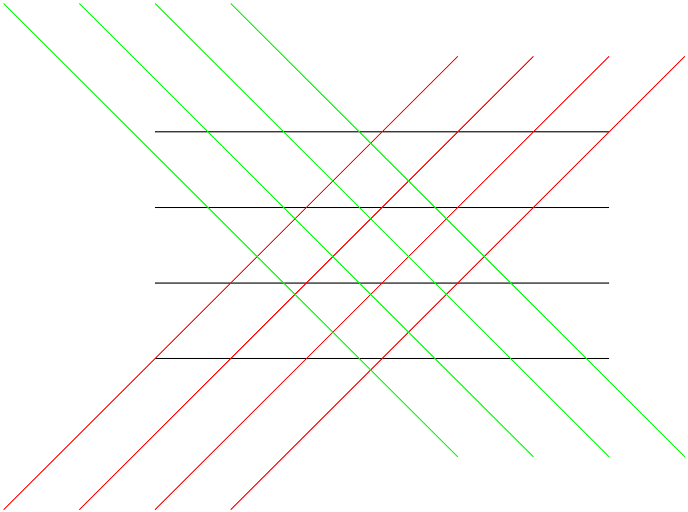

x^3
x^4
x^2
x^5
x^1
x^0
B
C
A
A
C
B
0
$\frac{1}{\sqrt{3}}$
$\frac{\sqrt{3}}{2}$
$\frac{1+\sqrt{3}}{2}$
$\frac{2}{\sqrt{3}}$
$\frac{1}{2}$

Figure: $L_0 = 14$.

Six lines in six directions if $\gamma = 0$.

Otherwise 24 lines in six directions. 4 parallel lines in each direction.

Now γ has two parameters.



The following cases are possible for the number of orbits of lines.

- $6 = 3 + 3$
- $9 = 3 * 2 + 3$
- $15 = 3 * 3 + 3 * 2$
- $18 = 3 * 3 + 3 * 3$
- $21 = 3 * 4 + 3 * 3$
- $24 = 3 * 4 + 3 * 4$

We can have one, two, three up to four lines by direction.

The end