When Topology Forces Dynamics
Some Applications

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Main themes

- The class of “Topology implies Dynamics” theorems here start with a model map with known, complicated dynamics. This dynamics is shown to be preserved under large perturbations constrained by topological considerations, eg. in the same homotopy class or on the same manifold.
- Such theorems go under various names “Homotopy stability of dynamics” or “Topological persistence”.
- These theorems only require coarse, topological data about systems and so can be valuable in real world applications where information is just known approximately.
- We mainly focus here on applications of the basic theorems.

Main tools

- Hyperbolicity implies dynamical stability. Hyperbolicity in action on H_1, locally with fixed point index, hyperbolic metrics, ...
- Exponential growth rates
- Use of covering spaces to unravel the dynamics.
- Morphism from perturbed system to model system is often of low regularity.

First example: angle doubling on the circle

Basic Dynamics

- A dynamical system here is a pair (X, f) with $f : X \to X$ continuous and X compact, metric usually a manifold.
- An orbit is $o(x, f) = \{\ldots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \ldots\}$ when f is invertible and $o^+(x, f) = \{x, f(x), f^2(x), \ldots\}$ when it isn't and f^n is repeated composition n-times.
- A morphism in the category is called a semi-conjugacy from (Y, g) to (X, f) is a continuous, onto map $\alpha : Y \to X$ with

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y \\ \alpha \downarrow & & \downarrow \alpha \\ X & \xrightarrow{f} & X \end{array}$$

In this case the dynamics of g are at least as complicated as those of f

Dynamics of angle-doubling

- *Angle doubling* is $d : S^1 \to S^1$ via $d(\theta) = 2\theta$ where $S^1 = \mathbb{R}/\mathbb{Z}$, or on the unit interval $x \mapsto 2x \mod 1$.
- It has all the features of a complicated or chaotic dynamical system.
- The set $\{\theta : o(\theta, d) \text{ is dense}\}$ is dense, G_δ and has full Lebesgue measure.
- The set $\{\theta : o(\theta, d) \text{ is periodic}\}$ is dense.
- The topological entropy of d is $h_{top}(d) = \log(2)$, and so uncertainty about position grows like $\exp(\log(2)n) = 2^n$ under iteration.
- The dynamics of d can be coded by the shift on $\{0, 1\}^{\mathbb{N}}$ and so are as complicated and random as a sequence of coin tosses (Bernoulli process).

First example: angle doubling

- Now assume that $g : S^1 \to S^1$ is homotopic to d.
- Lifting to the universal cover $\mathbb{R}$ yields $\tilde{d}(\tilde{x}) = 2\tilde{x}$ and $\tilde{g}(\tilde{x}) = 2\tilde{x} + \varphi(\tilde{x})$ with $\varphi(\tilde{x}+1) = \varphi(\tilde{x})$
- Now iterate, $\tilde{d}^n(\tilde{x}) = 2^n\tilde{x}$ and an easy induction yields

$$\tilde{g}^n(\tilde{x}) = 2^n\tilde{x} + \sum_{i=0}^{n-1} 2^{n-i-1}\varphi(\tilde{g}^i(\tilde{x}))$$

- Thus for most $\tilde{x}$, $|\tilde{g}(\tilde{x})| \to \infty$ at the rate of 2^n.

The semi-conjugacy

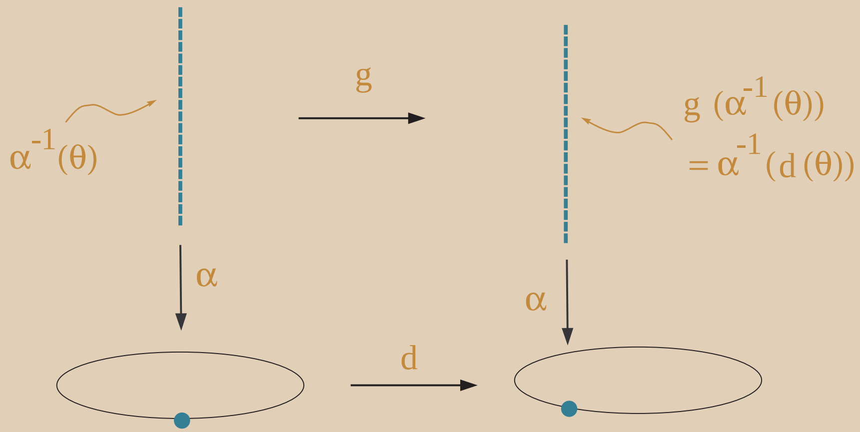
- So we normalize by this rate and obtain

$$\lim_{n\to\infty} \frac{\tilde{g}^n(\tilde{x})}{2^n} = \tilde{x} + \sum_{i=0}^{\infty} \frac{\varphi(\tilde{g}^i(\tilde{x}))}{2^{i+1}}$$

  which converges by the Weierstrass M-test uniformly to a continuous function $\tilde{\alpha}$ with $\tilde{\alpha}(\tilde{x}+1) = \tilde{\alpha}(\tilde{x}) + 1$.

- By construction $\tilde{\alpha} \circ \tilde{g} = \tilde{d} \circ \tilde{\alpha}$ and so projecting to the base

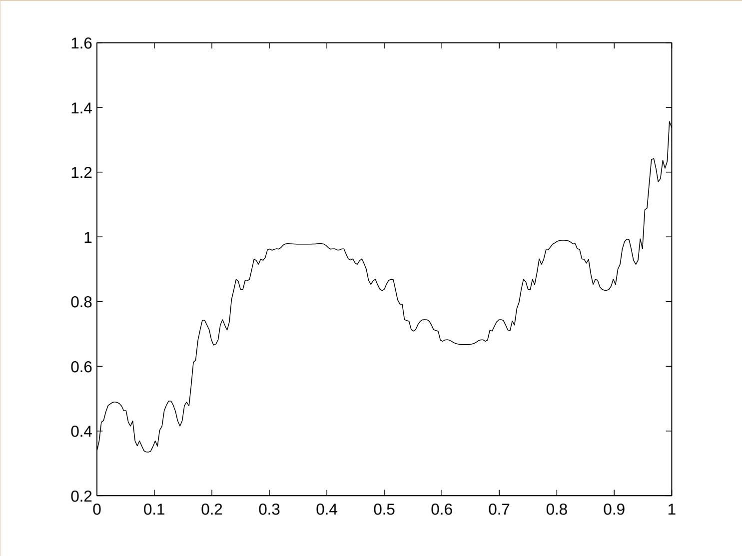
$$\begin{array}{ccc} S^1 & \xrightarrow{g} & S^1 \\ \alpha \downarrow & & \downarrow \alpha \\ S^1 & \xrightarrow{d} & S^1 \end{array}$$

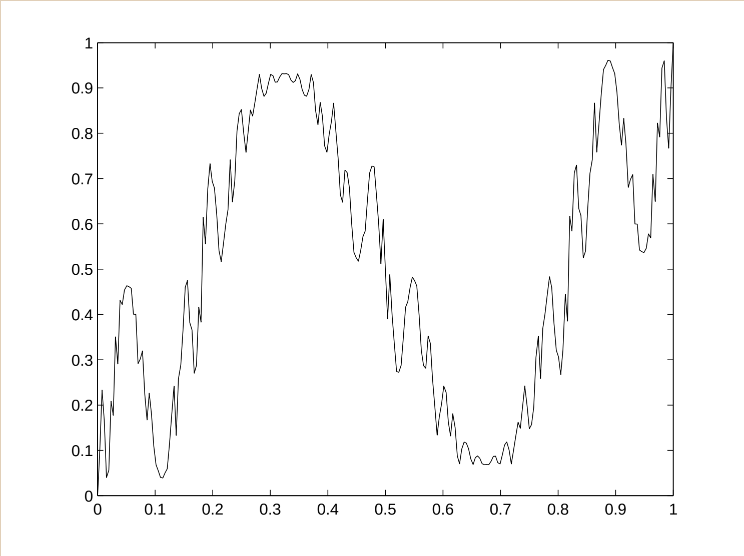


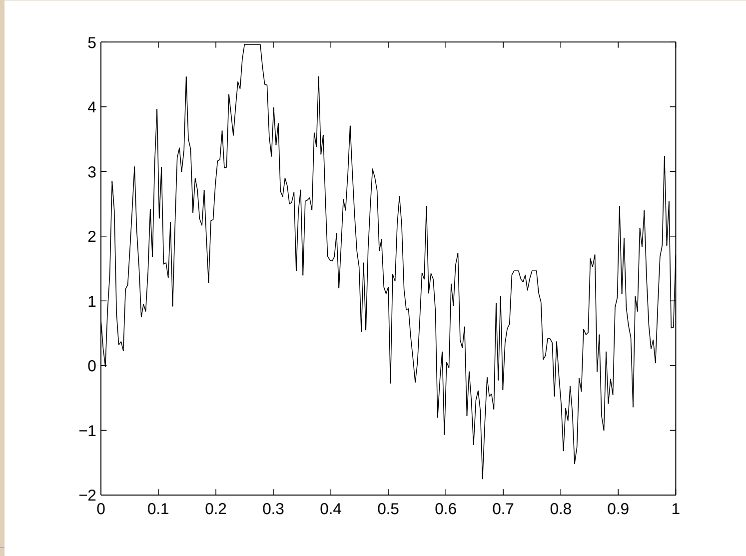
The semi-conjugacy

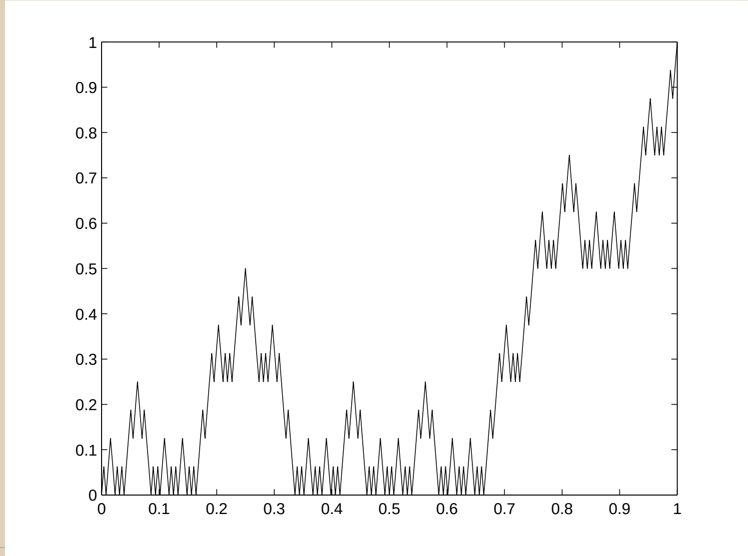
- So we conclude: If g acts on $H_1(S^1)$ as $k \mapsto 2k$, then g is semiconjugate to angle doubling, so its dynamics are at least as complicated as doubling.
- What can we say about the semiconjugacy and its fibers? The fact that formula looks like the definition of a Weierstrass nowhere differentiable function is a hint.
- The figures on the next page show numerical approximations of the semiconjugacies by computing $\tilde{g}^7/2^7$ for various g

Graphs of $\tilde{g}^7/2^{\approx}\alpha$





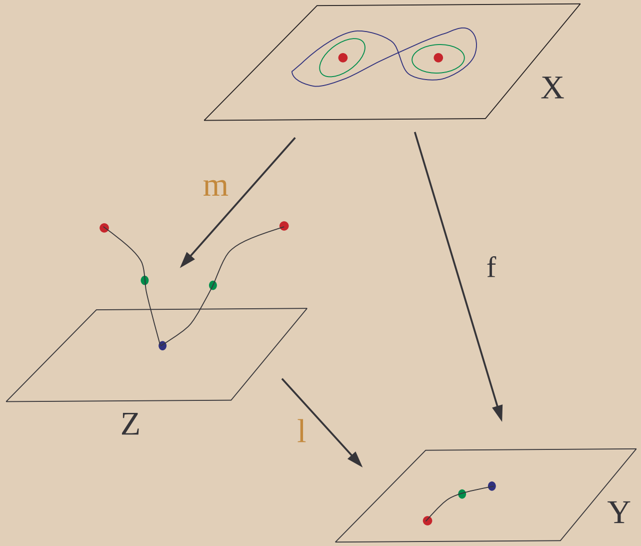




The monotone-light factorization

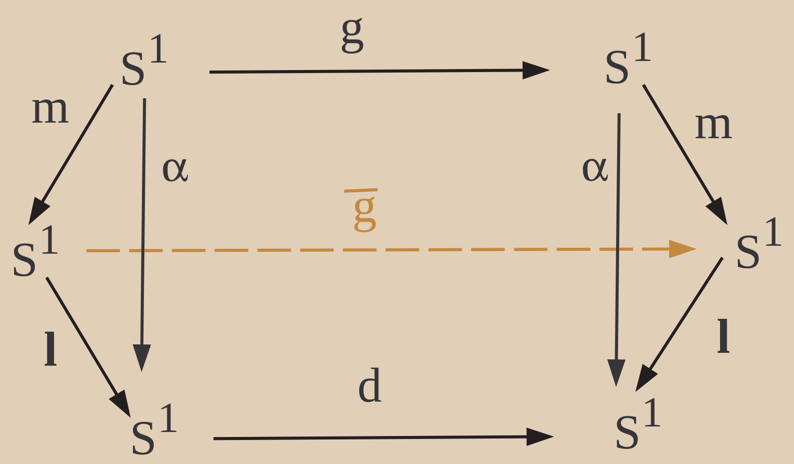
- Now α can have intervals in its point inverses. This implies that g has periodic intervals, or wandering intervals, or smashes intervals to points. This part of the dynamics can be studied independently, so we “collapse it out”.
- A map m is *monotone* if all point inverses are connected and ℓ is *light* if all point inverses are completely disconnected.

Whyburn & Eilenberg (independently): Any $f : X \to Y$ of compact metric spaces can be factored $f = \ell \circ m$.



The monotone-light factorization

- The space Z is the identification space that collapses each component of f^{-1} to a point.
- In the case at hand, $X = Y = S^1$ and so each component of $\alpha^{-1}(x)$ is an interval and so $Z = S^1$ also.
- So to simplify matters we just consider the light part of α and then glue the intervals back in to get α.



Theorem

Assume g is a continuous, degree-two circle map with a light semiconjugacy α. The following are equivalent:

(a) The map $\tilde{g}$ is not injective,

(b) The map α is not injective,

(c) There exists a full measure, dense, G_δ-set $\Lambda \subset S^1$ so that $\theta \in \Lambda$ implies that $\alpha^{-1}(\theta)$ is completely disconnected and uncountable and thus contains a Cantor set.

(d) The topological entropy of g satisfies $h_{top}(g) > \log(2)$,

(e) For all nontrivial intervals $J \subset S^1$, the map $\alpha|_J$ is not of bounded variation (BV).

Remarks on Theorem

- If any of the conditions are negated, then α is a homeomorphism and so the dynamics of g and angle-doubling d are the same up to conjugacy, i.e. change of coordinates.
- Since g and d are always semiconjugate, $h_{top}(g) \geq h_{top}(d)$; the content of (d) is the strict inequality. It means that there is nontrivial dynamics in the way g sends fibers $\alpha^{-1}(\theta)$ to $\alpha^{-1}(d(\theta))$
- Using the summation formula if g is λ-Lipschitz then α is Hölder with exponent $\nu = \log(2)/\log(\lambda)$.

Generalizing to other manifolds

Generalizing to other manifolds

- For simplicity of exposition we assume M is a smooth, compact, connected manifold with $H_1(M;\mathbb{Z})$ torsion-free and first betti-number b.
- Terminology: The hyperbolic part of the spectrum of a matrix A is the eigenvalues off the unit circle and A is called hyperbolic if its entire spectrum is hyperbolic.
- Since we consider action on homology, the appropriate cover is the universal Abelian cover.
- The universal Abelian cover, $\tilde{M}_{Ab}$, has deck or covering group $H_1(M,\mathbb{Z}) \cong \mathbb{Z}^b$ and is obtained by moding out the universal cover by the action of the commutator subgroup of $\pi_1(M)$.
- Every $f : M \to M$ lifts to $\tilde{M}_{Ab}$ since the action on π_1 preserves commutators.

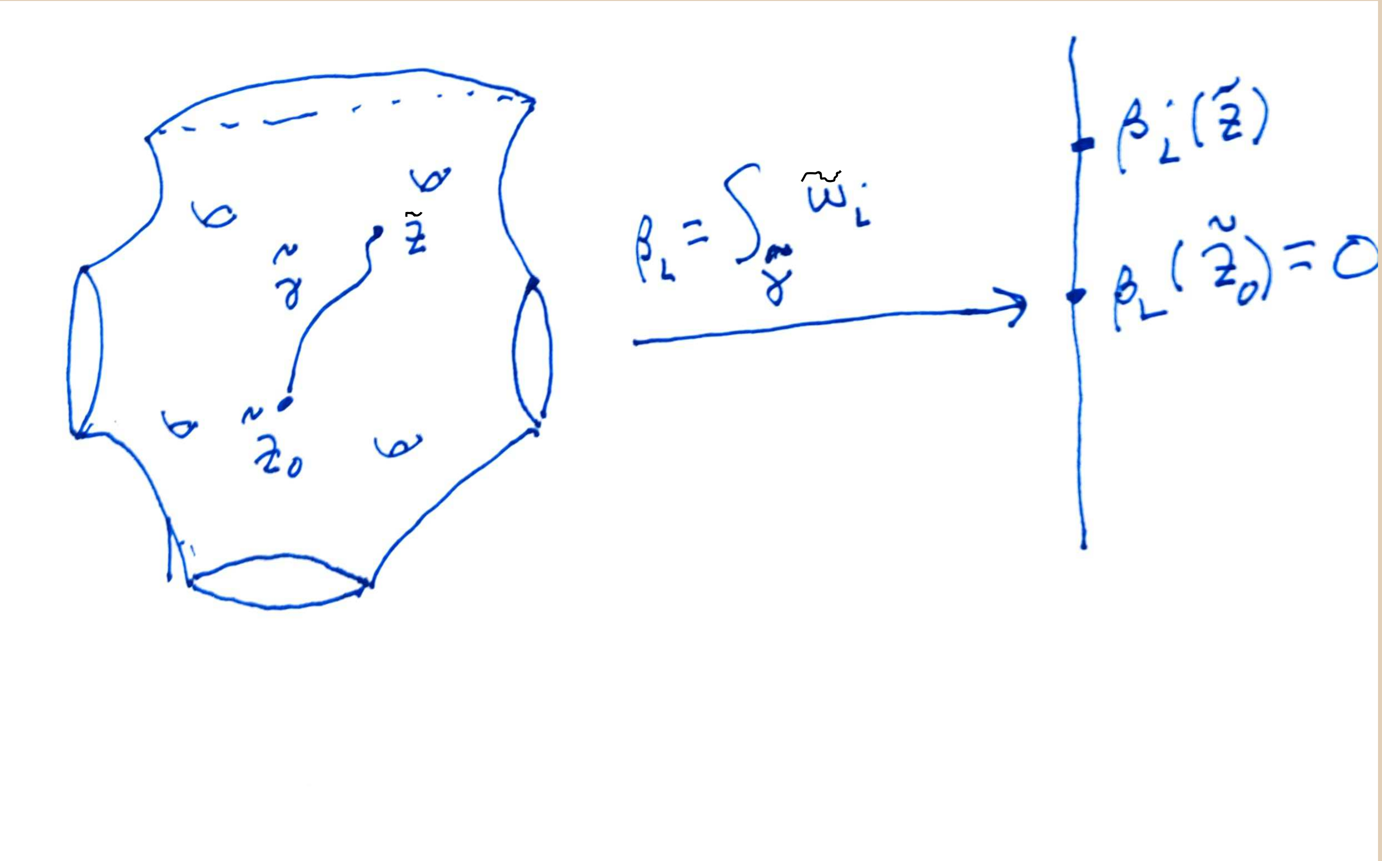
The universal Abelian cover

- It is useful to have a concrete realization of $\tilde{M}_{Ab}$ inside $\mathbb{R}^b$.
- This is commonly done (*cf.* Jacobi) using a collection of closed one-forms ω_i whose classes form a basis for de Rham cohomology $H^1_{dR}(M, \mathbb{R})$ and are dual to a basis of $H_1(M, \mathbb{Z})$.
- Now lift the forms to $\tilde{\omega}_i$ and pick a basepoint $\tilde{z}_0 \in \tilde{M}_{Ab}$ and for each $\tilde{z}$ define the i^{th} component of $\beta : \tilde{M}_{Ab} \to \mathbb{R}^b$ as

$$\beta_i(\tilde{z}) = \int_{\tilde{\gamma}} \tilde{\omega}_i,$$

where $\tilde{\gamma}$ is any smooth path connecting $\tilde{z}_0$ to $\tilde{z}$. This is well defined since forms are closed.

The universal Abelian cover



Then $\beta : \tilde{M}_{Ab} \to \mathbb{R}^b$ is the desired embedding and henceforth $\tilde{M}_{Ab}$ means this embedded copy in $\mathbb{R}^b$.

The universal Abelian cover

- Further, for a homeomorphism $f : M \to M$, its lift $\tilde{f} : \tilde{M}_{Ab} \to \tilde{M}_{Ab}$ acts as

$$\tilde{f}(\tilde{z}) = A\tilde{z} + \phi(z)$$

  where $A = f_* : H_1(M) \to H_1(M)$, and $\phi : M \to \mathbb{R}^b$ is a continuous function, while z is the projection of $\tilde{z}$ down to M.
- So, up to a bounded error (or coarsely or on large scales) the lift of f acts like the action of f on homology.

Averages and semi-conjugacies

- Assume now that $f_* : H_1(M, \mathbb{R}) \to H_1(M, \mathbb{R})$ has a real eigenvalue $\mu > 1$ with eigenvector $\vec{v}$ and fix a lift $\tilde{f} : \tilde{M}_{Ab} \to \tilde{M}_{Ab}$.
- We know there will be motion under $\tilde{f}^n$ at rate μ^n in the direction $\vec{v}$, so we normalize the rate out and attempt to compute

$$\lim_{n\to\infty} \frac{\pi_\mu(\tilde{f}^n(\tilde{z}))}{\mu^n},$$

  where $\pi_\mu : \mathbb{R}^b \to \vec{v}$ is the projection onto the eigen-direction.
- Recall now that $\tilde{f}(\tilde{z}) = A\tilde{x} + \phi(x)$ and so a simple induction yields

$$\tilde{f}^n \tilde{z} = A^n \; \tilde{z} + \sum_{i=1}^{n} A^{n-i} \; \phi(f^{i-1}(z))$$

Averages and semiconjugacies

- So projecting onto $\vec{v}$ we have

$$\pi_\mu(\tilde{f}^n(\tilde{z})) = \mu^n \pi_\mu(\tilde{z}) + \sum_{i=1}^{n} \mu^{n-i} \pi_\mu(\tilde{\phi}(\tilde{f}^{i-1}(\tilde{z})))$$

- Thus letting ϕ_μ denote $\pi_\mu \circ \tilde{\phi}$ pushed to the base, our desired average

$$\lim_{n\to\infty} \frac{\pi_\mu(\tilde{f}^n(\tilde{z}))}{\mu^n} = \pi_\mu(\tilde{z}) + \sum_{i=0}^{\infty} \frac{\phi_\mu(f^i(z))}{\mu^{i+1}},$$

  converges uniformly by the Weierstrass M-test.

- Let $\tilde{\alpha} : \tilde{M}_{Ab} \to \mathbb{R}$ be the continuous function defined by this sum.

Averages and semiconjugacies

- Now that we have convergence of

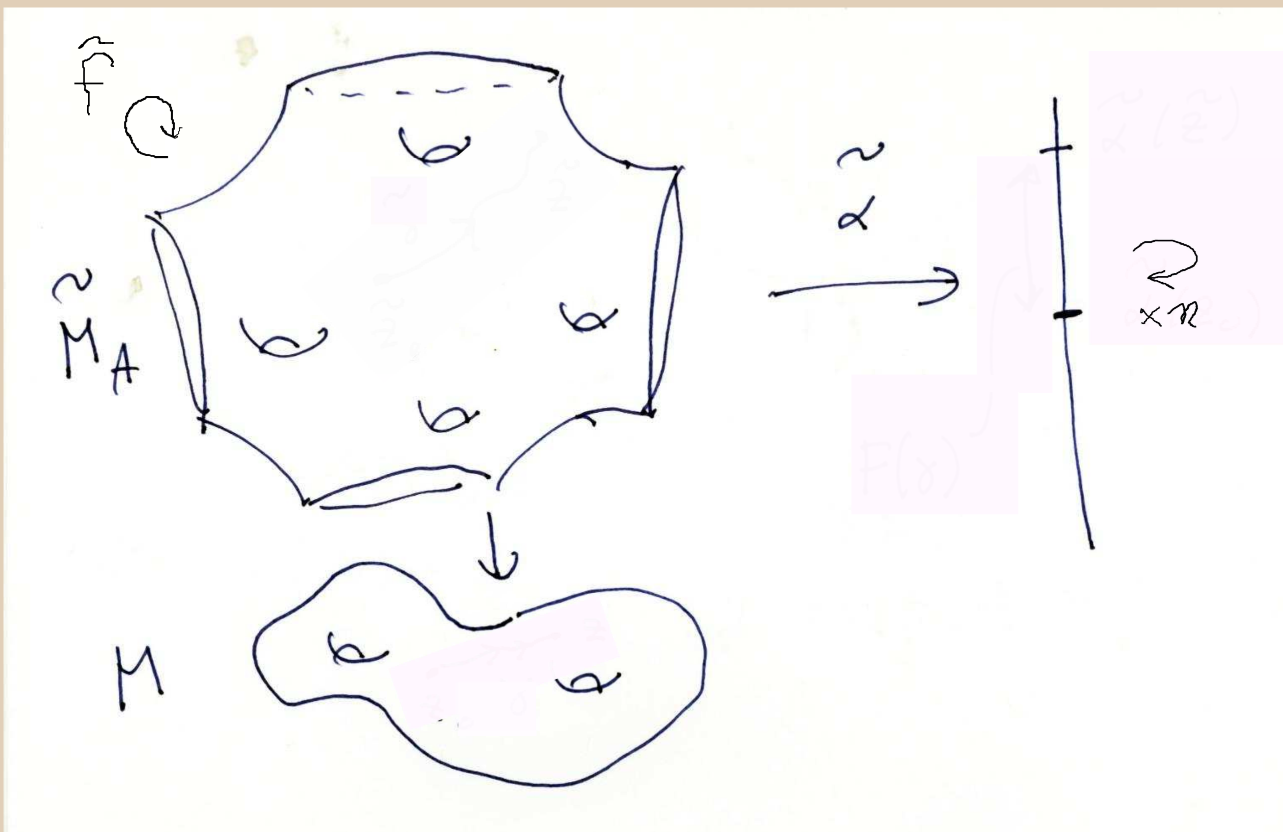
$$\tilde{\alpha}(\tilde{z}) = \lim_{n\to\infty} \frac{\pi_\mu(\tilde{f}^n(\tilde{z}))}{\mu^n},$$

  it follows directly that

$$\begin{array}{ccc} \tilde{M}_{Ab} & \xrightarrow{\tilde{f}} & \tilde{M}_{Ab} \\ \tilde{\alpha}\downarrow & & \downarrow\tilde{\alpha} \\ \mathbb{R} & \xrightarrow{\times\mu} & \mathbb{R} \end{array}$$

Averages and semiconjugacies

So in dynamical language, we have obtained a semiconjugacy between $\tilde{f}$ acting on $\tilde{M}_{Ab}$ and multiplication by μ on $\mathbb{R}$. Its existence depends just on the action of f on H_1.



Franks-Shub Theorem:

Franks-Shub Theorem: Assume that $f : M \to M$ is a continuous map of the smooth, connected manifold M and $\mu \in \mathbb{R}$ is a simple real eigenvalue of $f_* : H_1(M;\mathbb{Z}) \to H_1(M;\mathbb{Z})$ with $|\mu| > 1$. For each lift $\tilde{f} : \tilde{M}_{Ab} \to \tilde{M}_{Ab}$ of f to the universal Abelian cover $\tilde{M}_{Ab}$, there exists a unique map $\tilde{\alpha}_\mu : \tilde{M}_{Ab} \to \mathbb{R}$ with

$$\tilde{\alpha}_\mu \circ \tilde{f} = \mu\,\tilde{\alpha}_\mu$$

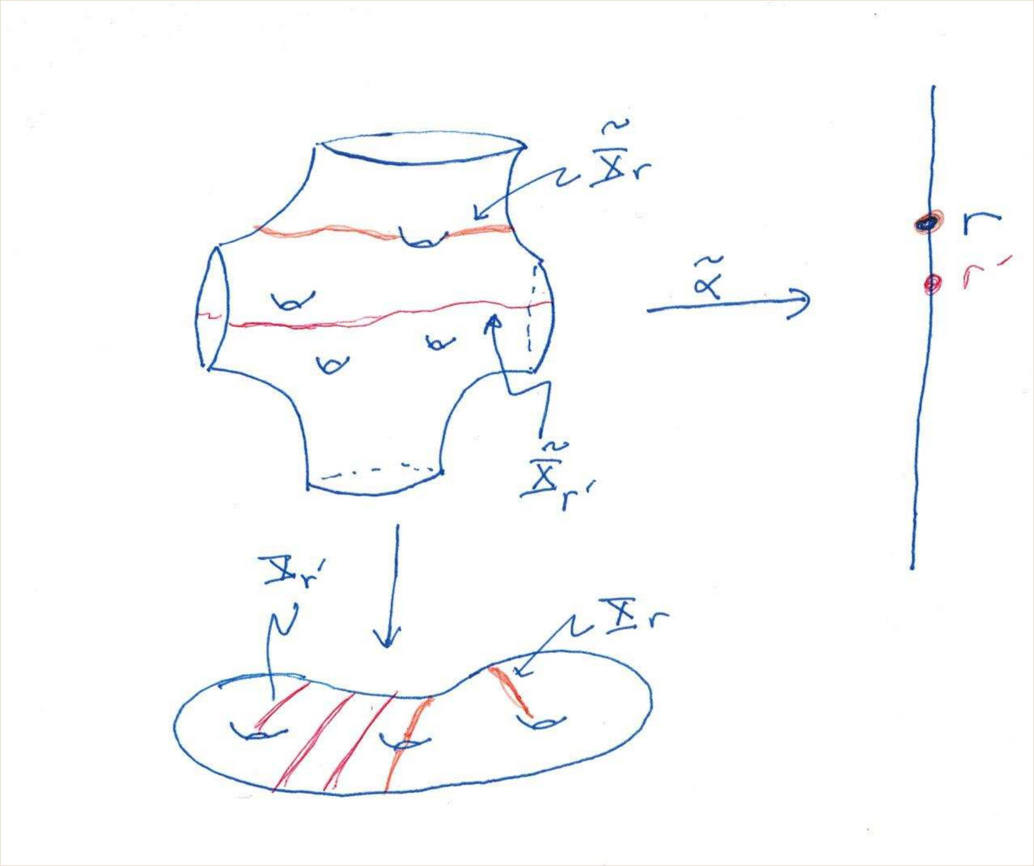
- The above argument is an variant of Franks (68) doing one eigenvalue at a time or a translation of Shub (78) from Alexander cocycles.
- The cases of complex and generalized eigenvalues are easily dealt with.

Additional structures

- Each eigenvalue $|\mu| > 1$ of the action of f on H_1 yields a semiconjugacy $\tilde{\alpha}_\mu$.
- If f is invertible and $0 < |\mu| < 1$ then using f^{-1} and μ^{-1} also yields a semiconjugacy.
- In turn, each semiconjugacy yields an invariant decomposition and an arc cocyle as explained next.

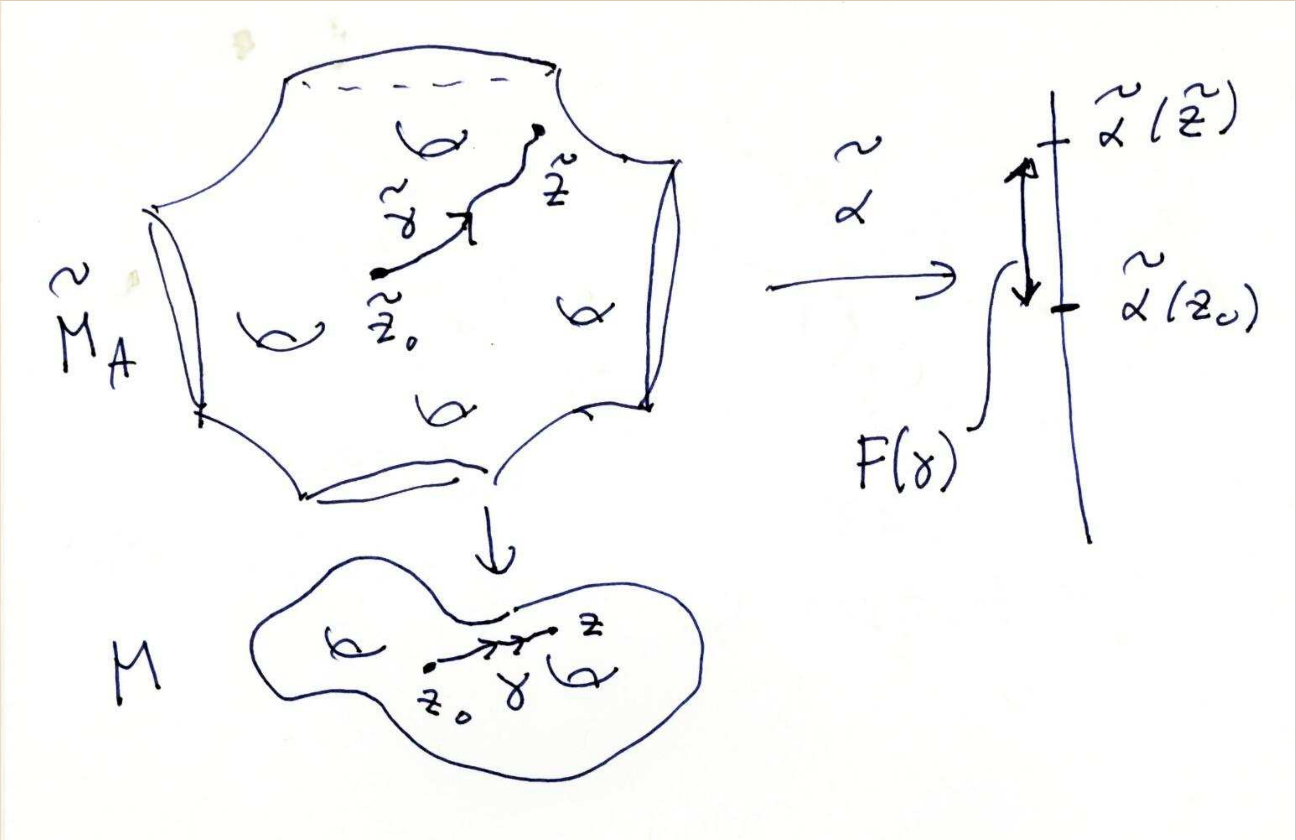
Additional structures

First, a $\tilde{f}$-invariant continuous decomposition of $\tilde{M}_{Ab}$ into level sets $\tilde{X}_r = \tilde{\alpha}_\mu^{-1}(r)$. Since, $\tilde{\alpha}_\mu \circ \tilde{f} = \mu\,\tilde{\alpha}_\mu$, we have $\tilde{f}(\tilde{X}_r) = \tilde{X}_{\mu r}$. This descends to a f-invariant decomposition $\{X_r\}$ of M.



Additional structures

Second, a “transverse structure” to the decomposition defined on paths in the cover $\tilde{M}_{Ab}$ as $\tilde{F}_\mu(\tilde{\gamma}) = \tilde{\alpha}(\tilde{\gamma}(1)) - \tilde{\alpha}(\tilde{\gamma}(0))$. $\tilde{F}_\mu$ descends to F_μ a way of assigning numbers to paths in M which is eigen.



Additional structures

- These structures are quite general and depend just on the action of f on H_1. With more hypothesis on f (like pseudoAnsosv) they interact with invariant foliations yielding transverse Hölder distributions, etc.

Example: linear Anosov map

- Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

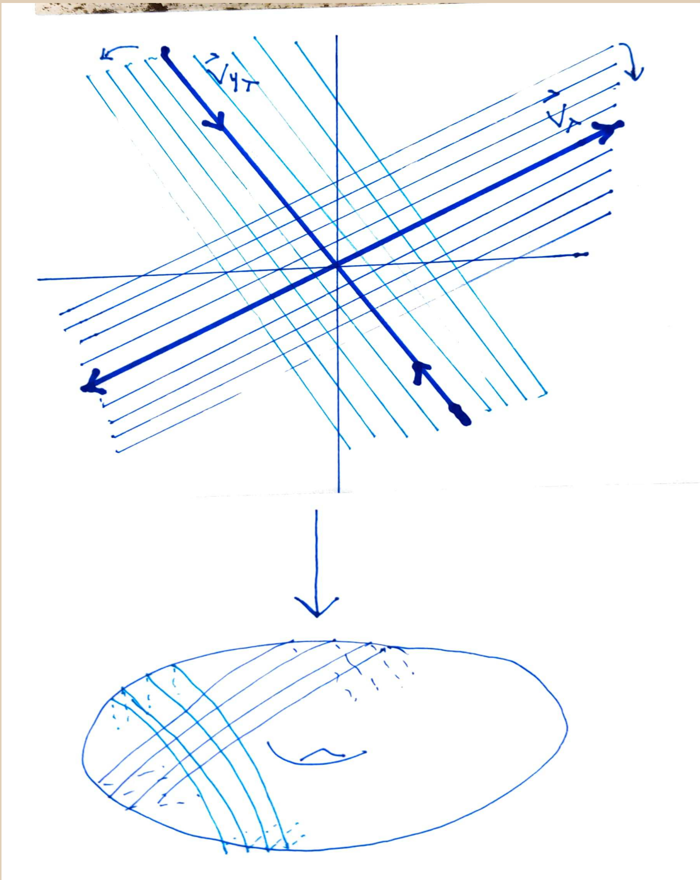
- A induces a homeomorphism ϕ_A of $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$,

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{T}^2 & \xrightarrow{\phi_A} & \mathbb{T}^2 \end{array}$$

- The map ϕ_A is an example of a *linear Anosov map*.

Example: linear Anosov map

The eigenvalues of A are $\lambda > 1$ and λ^{-1}. The eigenvectors have irrational slope and project down to dense wrappings of $\mathbb{T}^2$. The foliations of the plane in the stable and unstable directions project down to a pair of invariant transverse foliations, one expanding and one contracting.



These are the two decompositions discussed above for this special case of ϕ_A and ϕ_A^{-1}

Example: linear Anosov map

- The map ϕ_A
  - is ergodic and mixing w.r.t. Lebesgue measure,
  - has dense orbits (is transitive) and the set of periodic point is dense,
  - has a Markov partition which yields a (semiconjugacy) to a mixing subshift of finite type,
  - has topological entropy and everywhere Lyapunov exponents of $\log(|\lambda|)$, where $|\lambda| > 1$ is the largest eigenvalue of A.
- Now we allow perturbations of ϕ_A but remain in the same isotopy class.

Application to linear Anosovs on tori

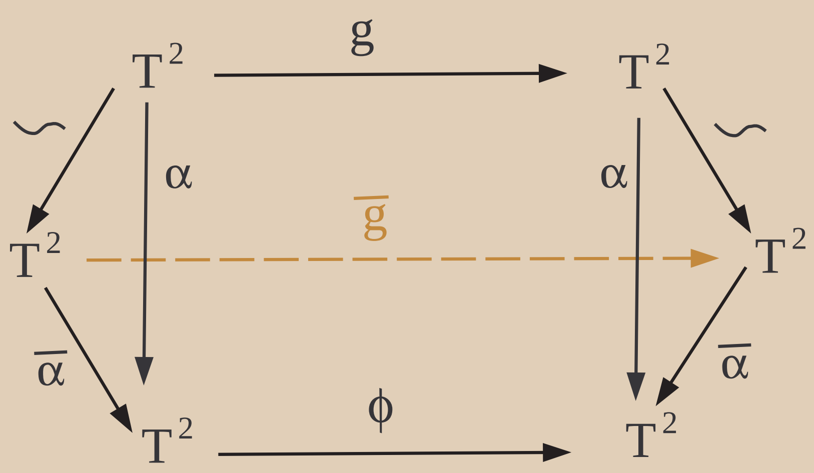
- Theorem: Franks If g is isotopic to ϕ_A then there is a semiconjugacy

$$\begin{array}{ccc} \mathbb{T}^2 & \xrightarrow{g} & \mathbb{T}^2 \\ \alpha \downarrow & & \downarrow \alpha \\ \mathbb{T}^2 & \xrightarrow{\phi_A} & \mathbb{T}^2 \end{array}$$

- So any homeomorphism isotopic to a ϕ_A has at least its dynamics.
- The proof goes by applying the basic semiconjugacy theorem in $\tilde{M}_{Ab}$ to $\lambda > 1$ and then to $\lambda^{-1} < 1$ using g^{-1} and ϕ_A^{-1} and then projecting to the base.
- We can obtain more information about the semiconjugacy using the monotone-light decompositions.

Reduction to a light semiconjugacy

- Lemma: Every component of a point inverse $\alpha^{-1}(z)$ of the semiconjugacy α is cell-like (non-separating).
- Thus by Bing-Moore, if $\sim$ corresponds to collapsing components of point inverses to points, then $\mathbb{T}^2/\sim = \mathbb{T}^2$.
- Further, g descends to a homeomorphism $\overline{g}$ whose semiconjugacy $\overline{\alpha}$ has point inverses all of whose components are points, thus each $\overline{\alpha}^{-1}(z)$ is completely disconnected.



Theorem on the semi-conjugacy

Theorem: Assume g is a homeomorphism of two-dimensional $\mathbb{T}^2$ that is isotopic to the linear Anosov ϕ and has a light semiconjugacy α. The following are equivalent:

(a) The map α is not injective,

(b) There exists a full measure, dense, G_δ-set $\Lambda \subset \mathbb{T}^2$ so that $z \in \Lambda$ implies that its point inverse $\alpha^{-1}(z)$ is completely disconnected and uncountable and thus contains a Cantor set,

(c) The topological entropy of g satisfies $h_{top}(g) > h_{top}(\phi)$.

Theorem on the semi-conjugacy

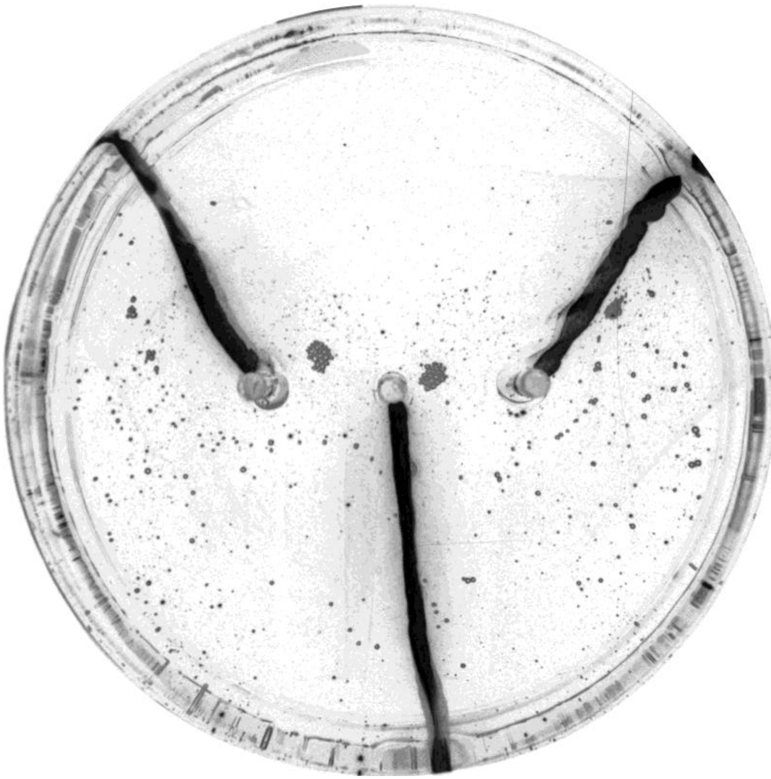
- So if α isn't injective, there is entropy carried in the fibers of the semiconjugacy.
- The proof uses two results of Blokh, Oversteegen and Tymchatyn
- First, the image of an open set under a light map has interior. This implies that anything light-semiconjugate to a transitive map is transitive (transitive $=$ has a dense orbit)
- Thus if α is not locally $1-1$ somewhere, it is not locally $1-1$ everywhere
- Second, a light, nowhere locally injective map between manifolds has the property that the topologically generic point has a Cantor set as its point inverse.

General hyperbolic actions

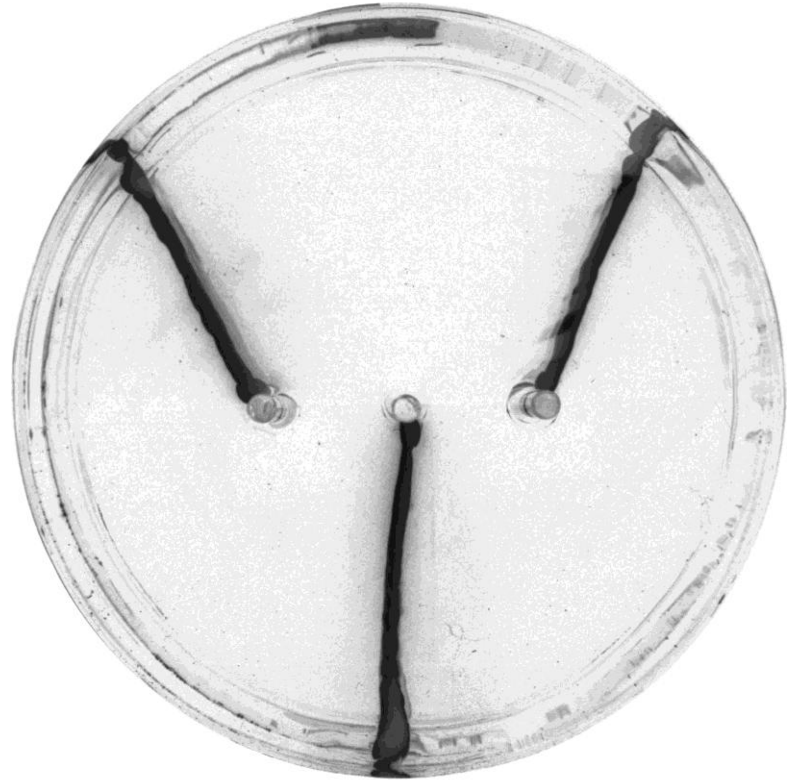
- The general version of Franks theorem starts with a compact, connected CW-complex X and a map $g : X \to X$ for which g_* acting on $H_1(X, \mathbb{R}) = \mathbb{R}^b$ is hyperbolic.
- The target is the linear Anosov on $\mathbb{T}^b$ defined by matrix A.
- In the general case, the semiconjugacy is into $\mathbb{T}^b$ not onto.
- For the proof use all the semiconjugacies with $|\mu| \neq 1$ from $\tilde{M}_{Ab}$ to $\mathbb{R}^n$ and they descend to $M \to \mathbb{T}^n$.
- An example application is maps on a wedge of circles homotopic to the map induced by a hyperbolic free group endomorphism.

Isotopy stability on surfaces: back to the experiment

Initial state



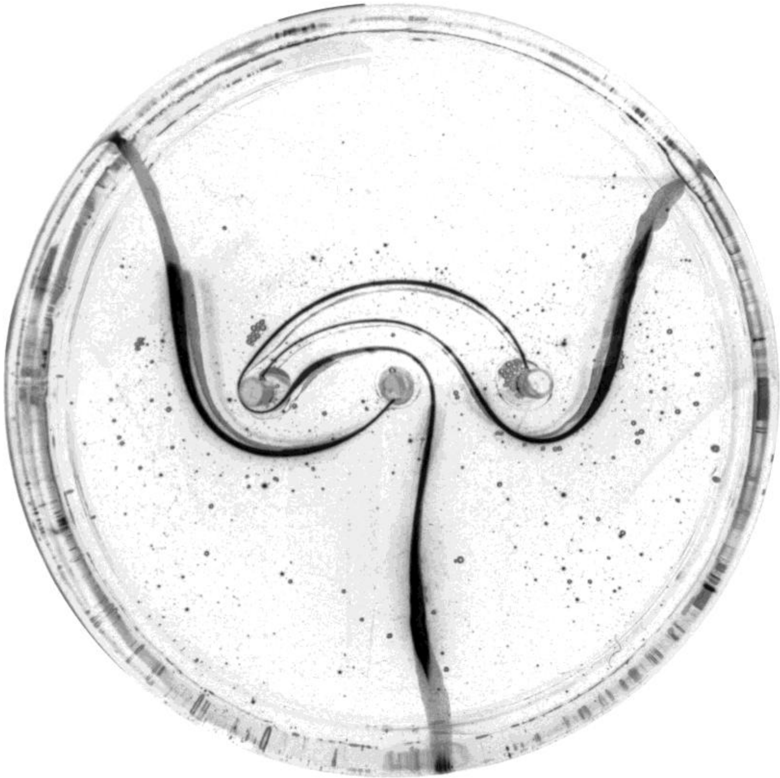
PseudoAnosov



Finite Order

Experiment by Mark Stremler, see Boyland, P., Aref, H. and Stremler, M., Topological fluid mechanics of stirring, *J. Fluid Mech*., **403**, 277--304, 2000.

1 iterate



PseudoAnosov

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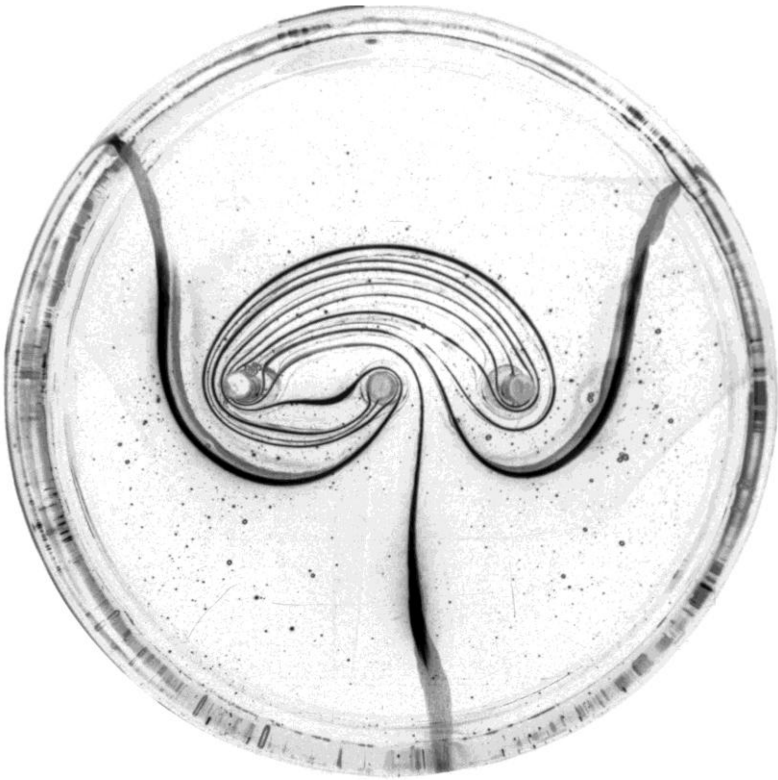
2 iterates



PseudoAnosov

Finite Order

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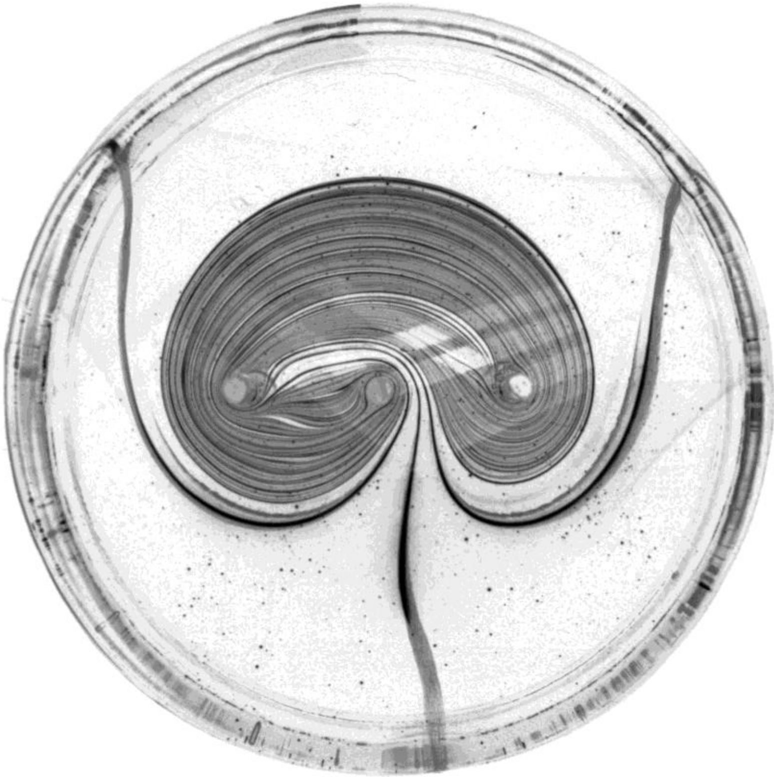
9 iterates



PseudoAnosov

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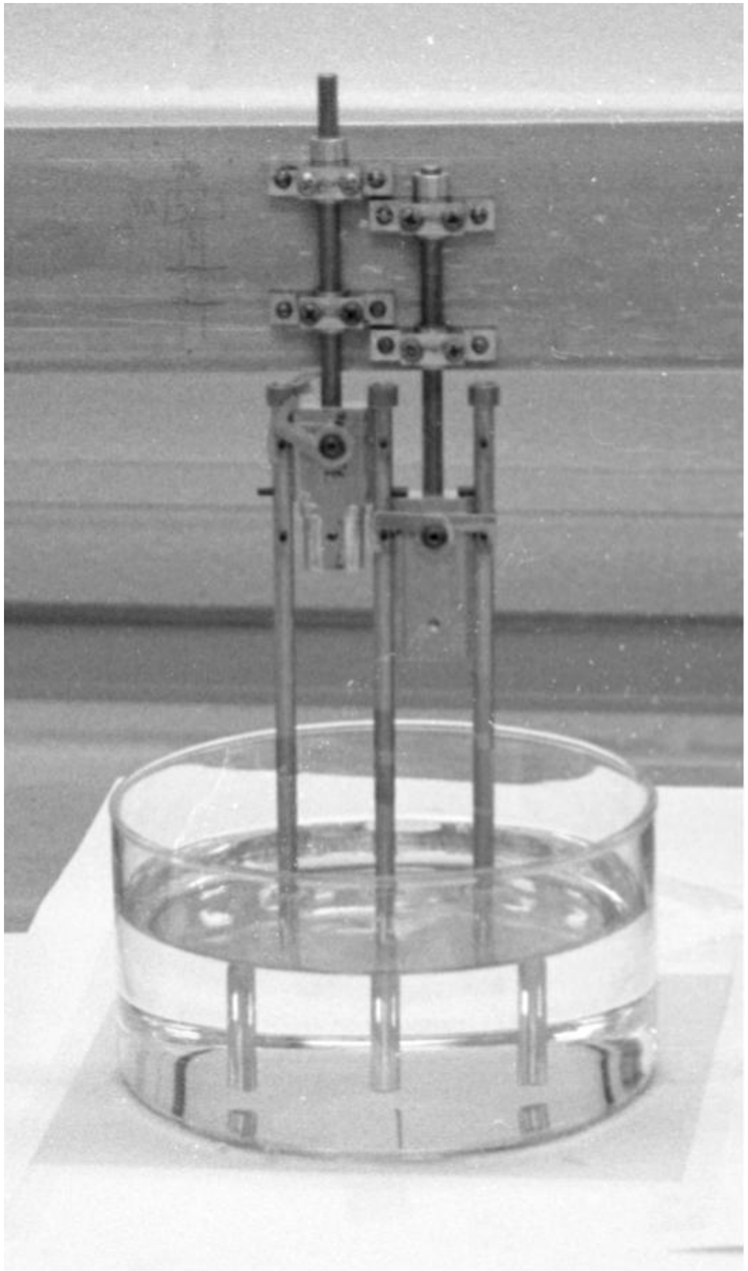


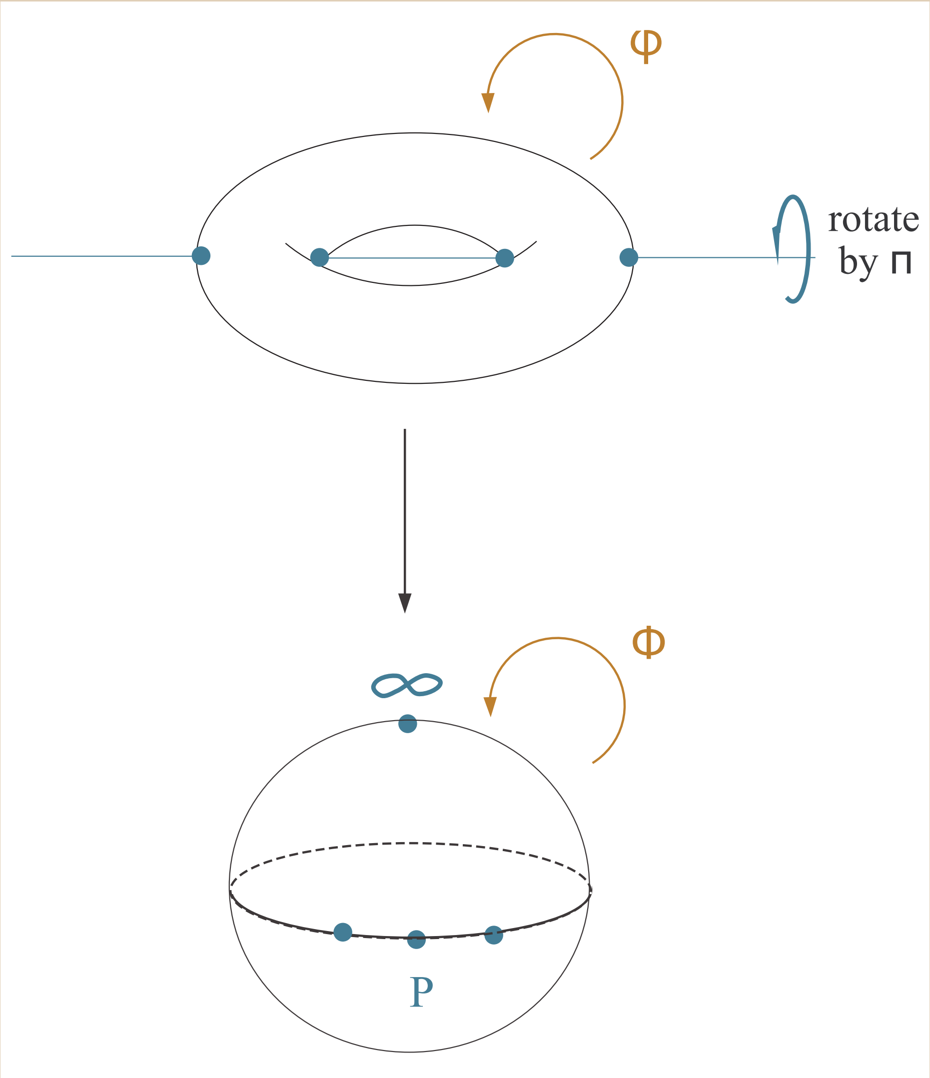
Recall the mixing experiment

- We want to use use the isotopy stability results on the torus to analyze the results of the experiment. Specifically, homotopy stability of dynamics in the isotopy class of the fluid motion rel the stirrers.
- The appropriate tool to connect the torus to the disk minus the stirrers is hyperelliptic involution (Lattés, Birman, Katok . . .).
- This involution is realized in $\mathbb{C}$ by the Weierstrass $\mathcal{P}$-function.
- The linear torus map ϕ_A that is connected to the experiment comes from the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Connection to experiment: the torus to the sphere

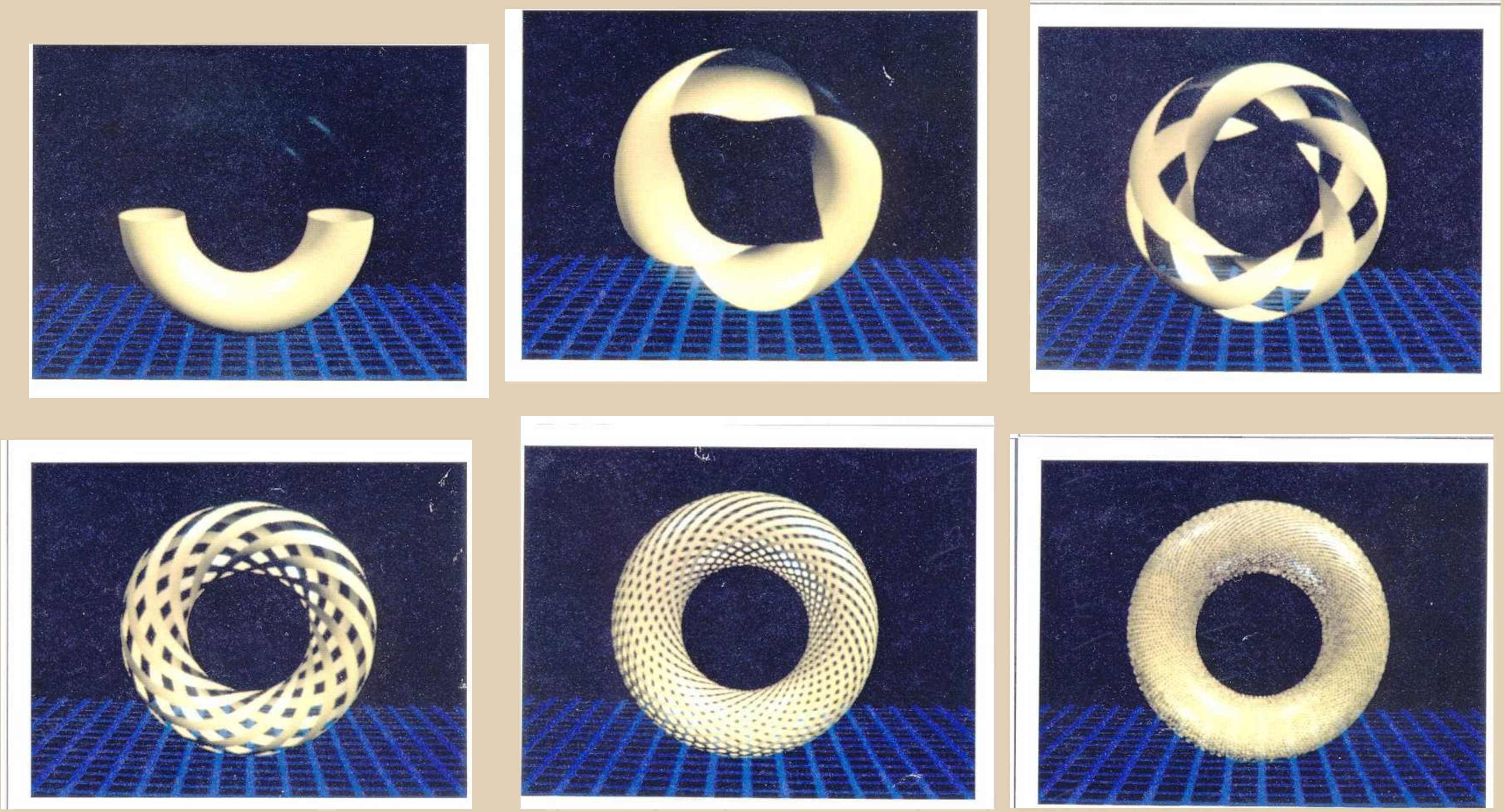
- The linear map A commutes with an involution. Moding out this hyperelliptic involution gives a sphere with 4 branch points. The map ϕ_A descends to a *pseudoAnosov (pA)* map Φ on the sphere.
- One of the branch points ∞ is fixed by Φ and the other three form a periodic orbit P.



Connection to experiment: the torus to the disk

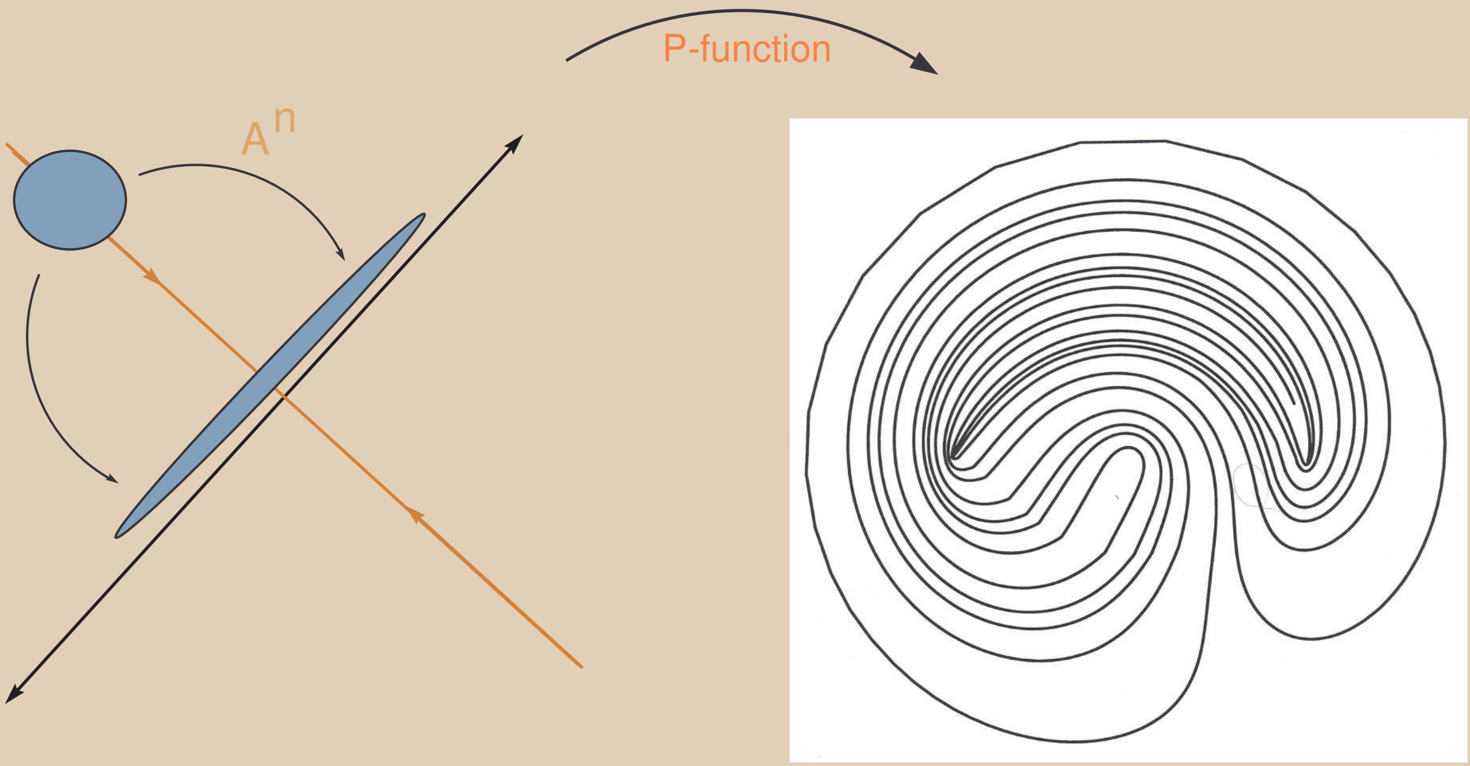
- Now blow up ∞ on the sphere to get a disk. The map is called Φ_A.
- The matrix A was chosen so that the resulting map Φ_A is in the isotopy class which gives rise to the experiment braid but the same construction works for any $A \in \mathrm{SL}(2, \mathbb{Z})$ and yields pA maps when $|\mathrm{trace}(A)| > 2$.
- The pA map Φ_A shares most of the properties of ϕ_A: a Markov partition, mixing, dense periodic points, etc.
- Pushing down Franks Theorem we can that any map isotopic to Φ_A (for example, the fluid motion) has at least its dynamics.
- One particular feature of interest is the emergent structure which we examine first in the torus case. (closely connected to invariant decompositions above).

Grayson, Kitchen & Zetter



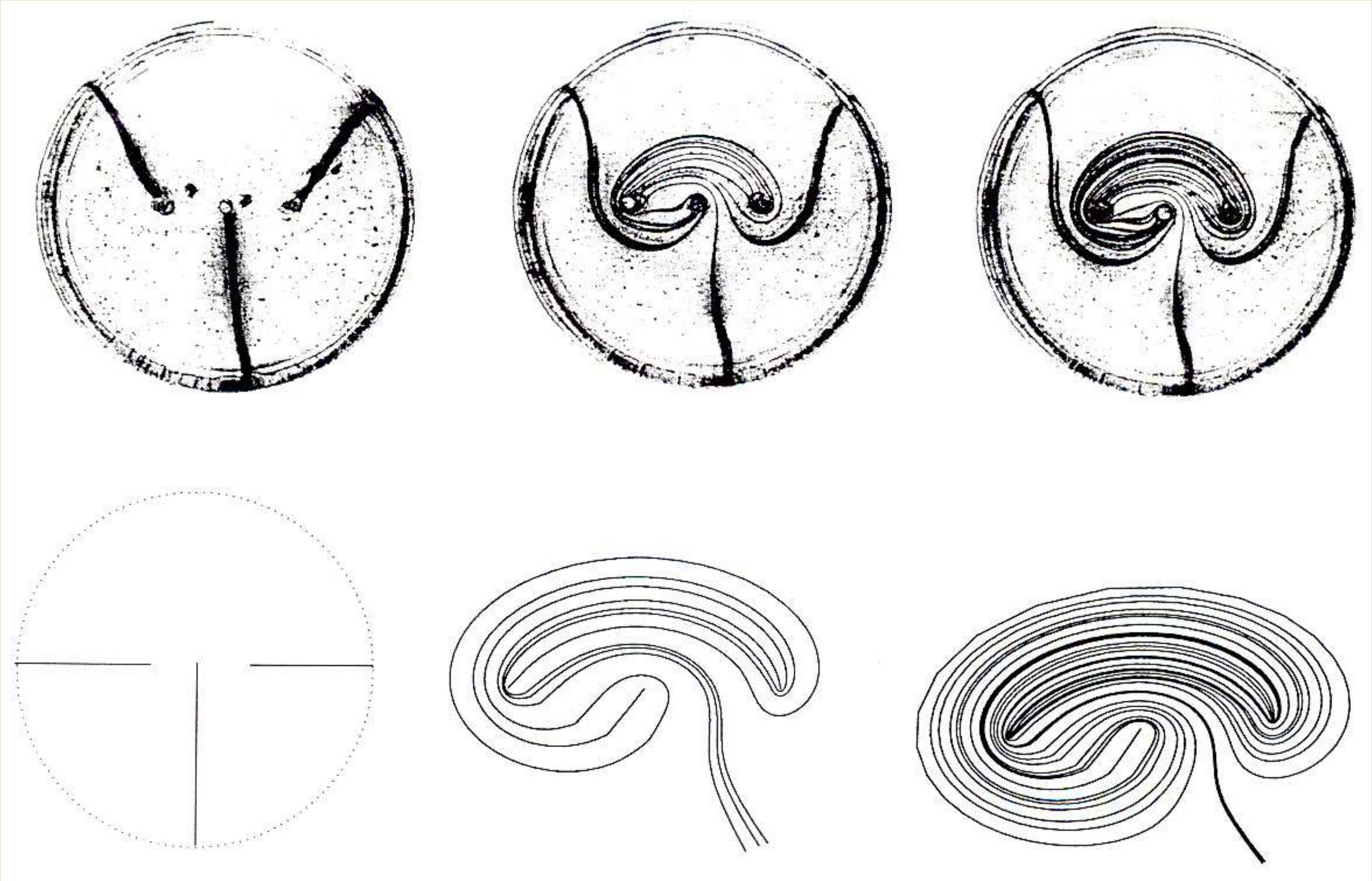
connection to the experiment

In plane, under iteration by the linear map A, open sets converge to the unstable eigen-direction. This line projects by a Weierstrass $\mathcal{P}$-function down to a “labyrinth” in the disk.



Connection to the experiment

The projection of the unstable manifolds of the three periodic points of the linear Anosov gives rise to the experiment's "emergent structure".



Isotopy Stability

- The general theory of isotopy stability of surface dynamics rests on Thurston's classification of surface isotopy classes into pseudoAnosov (pA), finite order and reducible.
- The stability theorems apply to the pA classes.
- In each pA class there is a pA map which has a pair of invariant measured foliations with mild singularities. The pA map has all the nice dynamical properties of linear Anosovs.
- Handel's Theorem says that anything isotopic to a pA map has at least its dynamics.
- The crucial difference is that one doesn't get a global semiconjugacy, but rather all the pA dynamics are present in some invariant set of the general map
- Rather than explicitly state these results we focus on applications.

A tool for applications

- A useful tool for applications is that any map in a pA isotopy class the lengths of topologically nontrivial arcs always grow exponentially fast under iteration at a rate dictated by the pA class.
- This will be applied to fluid motions that satisfy Euler’s equations.
- We first formalize the notion of the topological growth rate of arcs and loops under iteration.

One-dimensional topological growth rate

- Let $L^{top}(\gamma)$ be the least length among curves in $\gamma' s$ homotopy class with respect to some fixed Riemannian metric.
- After n-iterates the normalized length is

$$L_n^{top}(\gamma, g) = \frac{L^{top}(g^n \circ \gamma)}{L^{top}(\gamma)}$$

- So we evolve curve forward for n iterates and then shrink to the least length in homotopy class.

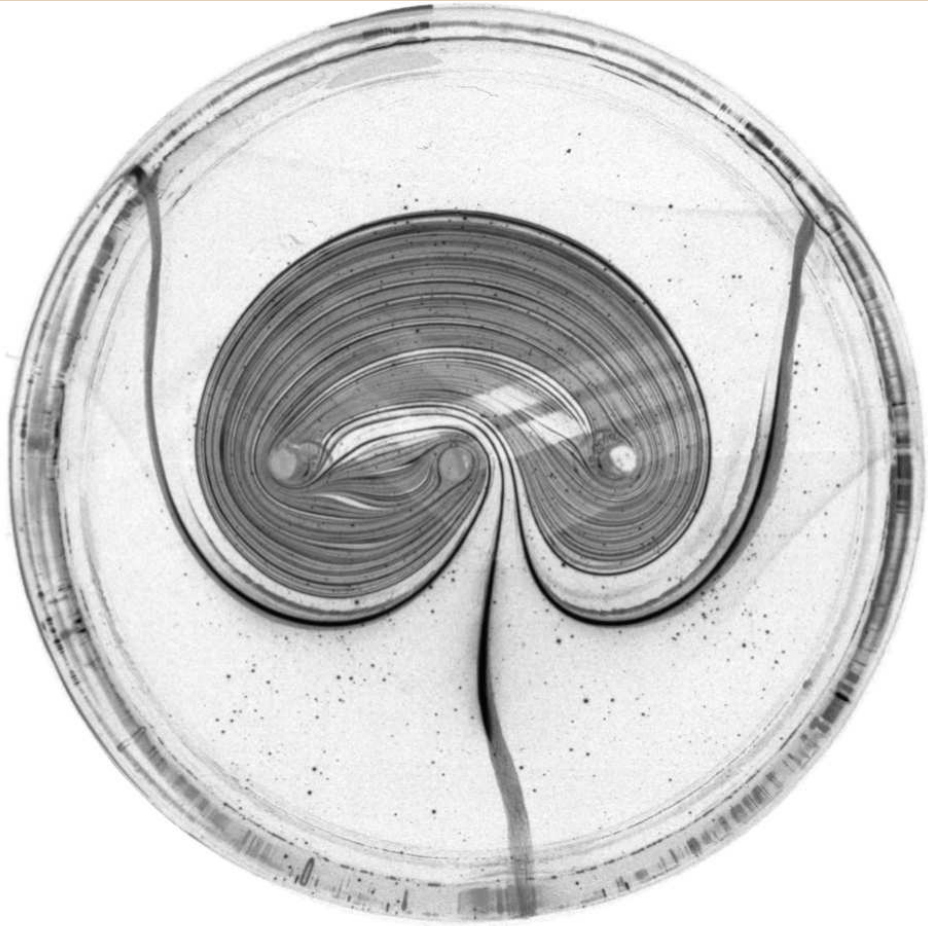
Exponential growth and the dilatation

- (Thurston): Given a pseudoAnosov map Φ there exist constants $\lambda > 1$ (the dilatation) and $0 < C_1 < C_2$ such that for all g isotopic to Φ and for every essential curve or arc γ,

$$C_1 \lambda^n \leq L_n^{top}(\gamma, g) \leq C_2 \lambda^n.$$

- Alternatively, on a punctured surface λ is the exponential word length growth under the free group automorphism induced by g.
- In the experiment $\lambda = (3 + \sqrt{5})/2$.

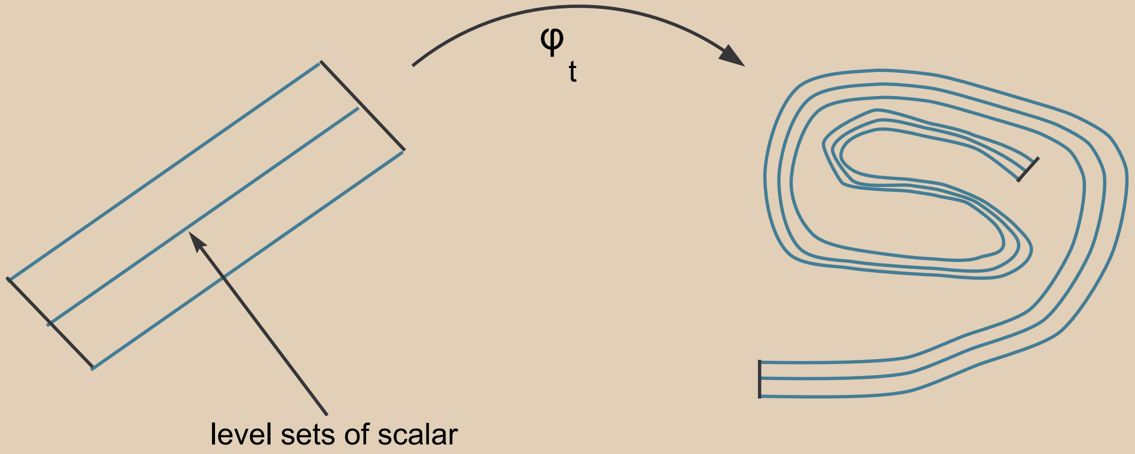
Exponential growth and the dilatation



For this pA stirring protocol, $\lambda \approx 2.62$, and so in this figure after 9 iterates material lines have been stretched by at least $\lambda^9 \approx 5,778$.

Exponential growth for Euler fluid motions

- Now we consider Euler fluid flows stirred by pA motions of the stirrers, so their isotopy classes rel the stirrers are of pA type.
- Under a fluid flow scalar fields (like cream in coffee) are pushed forward (passively advected). Thus by Thurston's Theorem lengths of generic level sets grow exponentially. Since the area is preserved, the level sets get closer together and so the gradient of the scalar field grows exponentially.



Exponential growth for Euler fluid motions

- The Helmholtz-Kelvin Theorem says vorticity is a passively advected scalar for an Euler fluid motion and by the previous observation, the gradient of the vorticity grows exponentially.
- Theorem: Let M_t be a time-periodic stirring protocol of pA type with Euler fluid motion ψ_t. If the initial vorticity ω_0 is a generic C^2-function, there are positive constants c, c' so that

$$\sup_{\mathbf{x}\in M_0} \|\nabla\omega_t(\mathbf{x})\| \geq c\lambda^t \quad \text{and} \quad \int_{M_t} \|\nabla\omega_t(\mathbf{x})\| \geq c'\lambda^t$$

for all $t \in \mathbb{R}$ where $\lambda > 1$ is the dilation of the pA protocol.

A partial order on periodic orbits

- Sharkovski's theorem applies to maps of the real line and says that any map with a periodic orbit of a given period (say 3) implies that it has periodic orbits of other periods (all periods).
- For homeomorphisms of the disk we specify a periodic orbit not by its period, but rather by the isotopy class on its complement.
- If this isotopy class is of pA type (like the fluid) then the isotopy stability theorem for that pA map implies that the given map has all the infinitely many periodic points of the pA map.
- These periodic orbits are then dominated by the original one.
- Roughly speaking, "one braid implies another braid"

A partial order on periodic orbits

- For simplicity we restrict to the closed, two-dimensional disk D^2.
- The main objects are pairs (g, P) where $g : D^2 \to D^2$ an orientation preserving homeomorphism and $P = P_0, \ldots, P_{n-1}$ with $P_i = g^i(P_0)$ mod n in i
- Say that $(g, P) \sim (g', P')$ if there exists an orientation preserving homeomorphism $h : (D^2, P) \to (D^2, P')$ with the following commuting up to isotopy.

$$\begin{array}{ccc} D^2 - P & \xrightarrow{g} & D^2 - P \\ h\downarrow & & \downarrow h \\ D^2 - P' & \xrightarrow{g'} & D^2 - P' \end{array}$$

A partial order on periodic orbits

- So $(g, P) \sim (g', P')$ means that the action on the complement of the orbits have conjugate isotopy classes.
- This is obviously an equivalence relation and the equivalence class of (g, P) is called its braidtype and denoted $\mathrm{bt}(g, P)$.
- Note that $\mathrm{bt}(g, P)$ is naturally identified with a conjugacy class in the braid group B_n where n is the period of P.
- For $g : D^2 \to D^2$, let $\mathrm{bt}(g) = \{\mathrm{bt}(g, P) : P$ is a periodic orbit of $g\}$.
- Let BT be the collection of all braidtypes of all orientation preserving homeomorphisms of the disk.

A partial order on periodic orbits

- For two braidtypes $\beta, \beta' \in \mathrm{BT}$ say that $\beta \succeq \beta'$ if for every g, $\beta \in \mathrm{bt}(g)$ implies $\beta' \in \mathrm{bt}(g)$.
- Thus $\beta \succeq \beta'$ means that any g that has a braidtype β also has one β'.
- Define $h_{top}(\beta) = \inf\{h_{top}(g) : \beta \in \mathrm{bt}(g)\}$. When β is pA, Thurston showed $h_{top}(\beta) = h_{top}(\phi)$ where ϕ is a pA map in the isotopy class of β.

A partial order on periodic orbits

Theorem:

1. $\succeq$ is a partial order on BT
2. If $\beta \neq \beta'$ are pA braidtypes with $\beta \succeq \beta'$ then $h_{top}(\beta) > h_{top}(\beta')$
3. If β is a pA braidtype and ϕ a pA representative, then $\{\beta' : \beta \succeq \beta'\} = \mathrm{bt}(\phi)$.
4. There exist pairs β, β' which are unrelated under $\succeq$ but have the same entropy.
5. If g is $C^{1+\nu}$ then $h_{top}(g) = \sup\{h_{top}(\beta)\} : \beta \in \mathrm{bt}(g)$ using Katok.

A partial order on periodic orbits

- By 3. the order is computable once the dynamics of a pA can be computed which is possible via train tracks.
- The order in general is very complicated and very un-tree-like. There are well understood linear suborders in which the entropy is monotonic (Hall and de Calvalho).
- The order constrains the way in which periodic orbits are built in parameterized families (bifurcation theory).

General hyperbolic actions

- The general version of Franks theorem starts with a compact, connected CW-complex X and a map $g : X \to X$ for which g_* acting on $H_1(X, \mathbb{R}) = \mathbb{R}^b$ is hyperbolic.
- The target is the linear Anosov on $\mathbb{T}^b$ defined by matrix A.
- In the general case, the semiconjugacy is into $\mathbb{T}^b$ not onto.
- For the proof use all the semiconjugacies with $|\mu| \neq 1$ from $\tilde{M}_{Ab}$ to $\mathbb{R}^n$ and they descend to $M \to \mathbb{T}^n$.
- For example, maps on a wedge of circles homotopic to the map induced by a hyperbolic free group endomorphism.

General hyperbolic actions

- The general version of Franks theorem works on a compact, connected CW-complex X and a map $g : X \to X$ for which $g_* = A$ acting on $H_1(X, \mathbb{Z})$ is hyperbolic.
- In this case the target is the linear Anosov defined by the $b \times b$ matrix A and is $\phi_A : \mathbb{T}^b \to \mathbb{T}^b$ which is the descent of $A : \mathbb{R}^b \to \mathbb{R}^b$ to $\mathbb{R}^b/\mathbb{Z}^b$.

General hyperbolic actions

- Theorem: Franks Assume X is a connected, compact CW complex with first betti number b and $g : M \to M$ is such that $A := g_* : H_1(M, \mathbb{Z}) \to H_1(M, \mathbb{Z})$ is a hyperbolic matrix A and $\phi_A : T^b \to \mathbb{T}^b$ is the corresponding linear Anosov map there is map $\alpha : M \to \mathbb{T}^b$ with

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \alpha \big\downarrow & & \big\downarrow \alpha \\ \mathbb{T}^n & \xrightarrow{\phi_A} & \mathbb{T}^n \end{array}$$

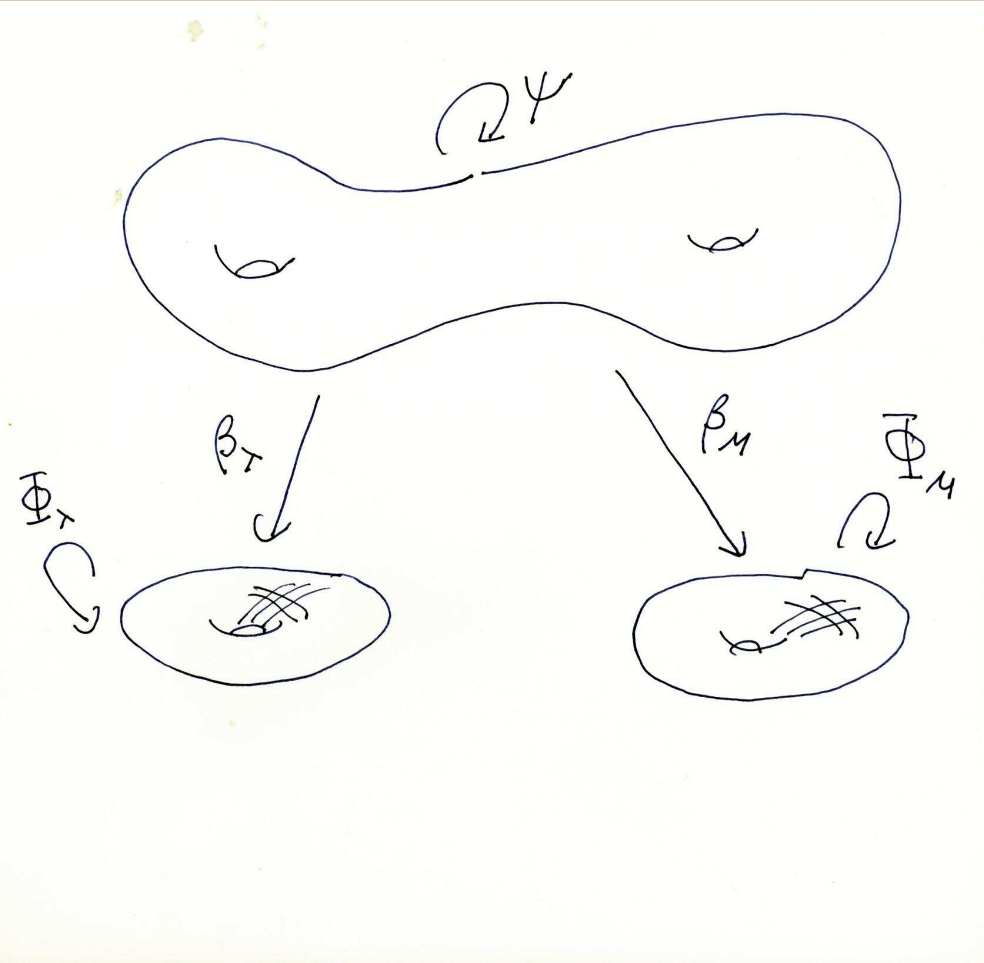
General hyperbolic actions

- For the proof use all the semiconjugacies with $|\mu| \neq 1$ from $\tilde{M}_{Ab}$ to $\mathbb{R}^n$ and they descend to $M \to \mathbb{T}^n$.
- Note that in general, α is no longer onto.
- We give an application of the general theorem to pseudoAnosov maps.
- Another application is to hyperbolic free group endomorphisms treated as expanding maps of a wedge of circles.

Example: Evil Twin

- Let M be a genus two surface and ψ a pseudoAnosov map.
- Assume the characteristic polynomial of ψ acting on $H_1(M;\mathbb{Z}) \cong \mathbb{Z}^4$ splits over the integers into a pair of irreducible quadratic factors with roots $0 < \lambda^{-1} < \mu^{-1} < 1 < \mu < \lambda$ (recall that ψ_* is symplectic).
- The eigenvalues/vectors yield four semi-conjugacies $\tilde{\alpha}_\lambda, \tilde{\alpha}_{\lambda^{-1}}, \tilde{\alpha}_\mu$, and $\tilde{\alpha}_{\mu^{-1}}$.
- Fathi shows that the Franks semiconjugacy into $\mathbb{T}^4$ splits and descends into paired maps $\beta_\lambda := (\alpha_\lambda, \alpha_{\lambda^{-1}})$ and $\beta_\mu := (\alpha_\mu, \alpha_{\mu^{-1}})$, each a semiconjugacy onto a linear, two-dimensional toral automorphism. But one can prove that the characters of the two semiconjugacies are quite different.

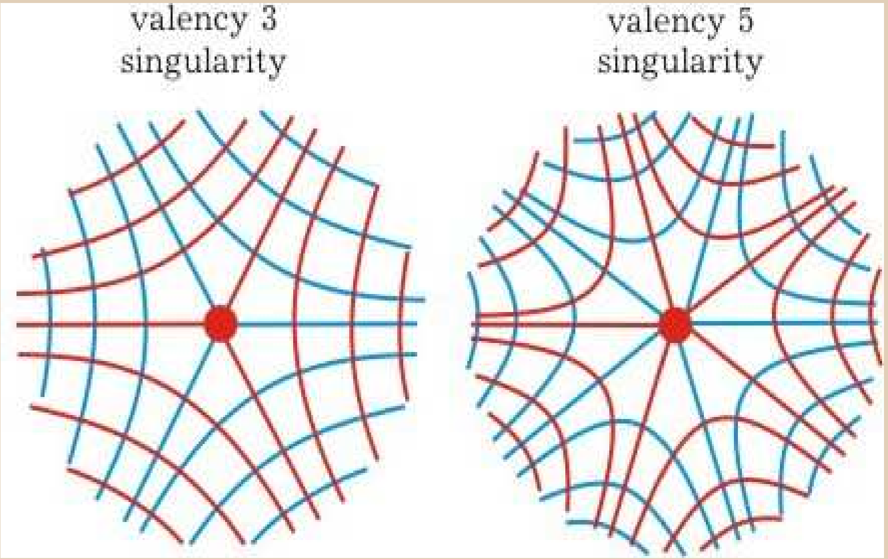
Example:Evil twin



- β_λ is a branched cover (Franks and Ryyken) and so is locally a diffeomorphism at all but finitely many points and point inverses are finite sets.
- β_μ is Hölder with exponent $\nu = \log(\mu)/\log(\lambda)$, but no larger ν's. It is nowhere differentiable and nowhere locally injective or BV. Typical point inverses are Cantor sets.

PseudoAnosov homeomorphisms

- A homeomorphism $\Phi : M^2 \to M^2$ of a compact surface is called pseudoAnosov (pA) if it has a pair of transverse, invariant foliations, one stable and the other unstable.
- The foliations have a finite number of well-behaved singularities



(Figure from A.Yu. Zhirov)