Homomorphisms between multidimensional substitutive subshifts

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Let $(X, S, \mathbb{Z}^d)$ and $(Y, S, \mathbb{Z}^d)$ be two subshifts.

Let $M \in GL(d, \mathbb{Z})$. **Homomorphism associated with** M: Surjective continuous map $\phi : X \to Y$ s.t. $\forall \mathbf{n} \in \mathbb{Z}^d,\ \phi \circ S^{\mathbf{n}} = S^{M\mathbf{n}} \circ \phi$.

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Question

How to characterize the homomorphisms between two subshifts?

Examples: If $M = \mathrm{id}_{\mathbb{R}^d}$.

- **Factor**: Surjective continuous map $\phi : X \to Y$ commuting with the action, i.e., $\forall \mathbf{n} \in \mathbb{Z}^d,\ S^{\mathbf{n}} \circ \phi = \phi \circ S^{\mathbf{n}}$.

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The notion of homomorphisms generalizes the notion of factors via $GL(d, \mathbb{Z})$-conjugacies.

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We have $\langle S \rangle \trianglelefteq \mathrm{Aut}(X, S, \mathbb{Z}^d) \trianglelefteq N^*(X, S, \mathbb{Z}^d)$, and

$N^*(X, S, \mathbb{Z}^d)/\mathrm{Aut}(X, S, \mathbb{Z}^d) \cong \vec{N}^*(X, S, \mathbb{Z}^d) \leq GL(d, \mathbb{Z})$.

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Question

How to characterize the symmetry semigroup of a subshift?

Theorem (M. Curtis, G. Hedlund, R. Lyndon (1969))

If $\phi : (X, S, \mathbb{Z}^d) \to (Y, S, \mathbb{Z}^d)$ *is a factor, there exists* $r > 0$ *and a map* $\Phi : \mathcal{L}_{B(\mathbf{0},r)}(X) \to \mathcal{A}_Y$ *such that for all* $\mathbf{n} \in \mathbb{Z}^d$, $\phi(x)_{\mathbf{n}} = \Phi(x|_{\mathbf{n}+B(\mathbf{0},r)\cap\mathbb{Z}^d})$.

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For homomorphisms we have a similar result: If $\phi : X \to Y$ is a homomorphism associated with a matrix $M \in GL(d, \mathbb{Z})$, then there exists $r > 0$ and a map $\Phi : \mathcal{L}_{B(\mathbf{0},r)}(X) \to \mathcal{A}_Y$ such that $\phi(x)_{\mathbf{n}} = \Phi(x|_{M^{-1}\mathbf{n}+B(\mathbf{0},r)\cap\mathbb{Z}^d})$.

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In particular the semigroup $N(X, S, \mathbb{Z}^d)$ is countable and $N^*(X, S, \mathbb{Z}^d)$ is a discrete subset in $\text{Homeo}(X)$.

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- Let $F \subseteq \mathbb{Z}^d$ be a fundamental domain of $L(\mathbb{Z}^d)$ in $\mathbb{Z}^d$, i.e., a set of representatives classes of $\mathbb{Z}^d / L(\mathbb{Z}^d)$ with $\mathbf{0} \in F$.

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- Let $\mathcal{A}$ be a finite alphabet.

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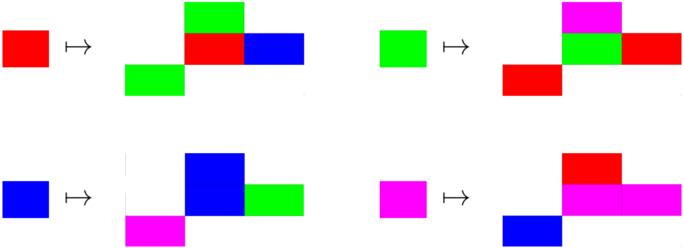
Example of a multidimensional constant-shape substitution:

$$L = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \; F = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}:$$



For any $n > 0$ the n-th iteration is defined as $\zeta^n : \mathcal{A} \to \mathcal{A}^{F_n^\zeta}$, with $F_{n+1}^\zeta = L_\zeta(F_n^\zeta) + F_1^\zeta$.

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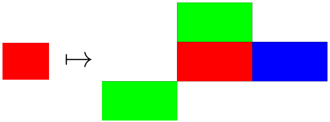
Remark

We assume the sequence (F_n^ζ) is a Følner[a] sequence, i.e., for all $\mathbf{n} \in \mathbb{Z}^d$ we have

$$\lim_{n\to\infty} \frac{|F_n \Delta(\mathbf{n} + F_n)|}{|F_n|} = 0.$$

[a]This is not the exact definition of Følner sequence in group theory.

Example of iterations of a substitution:



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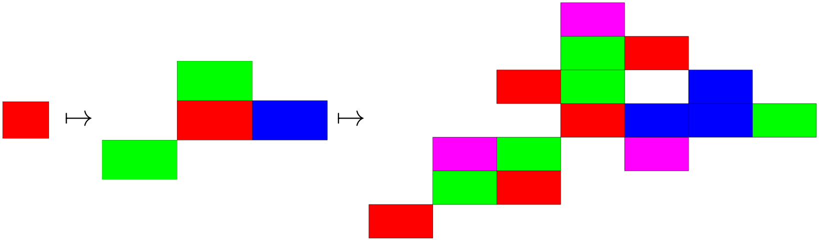

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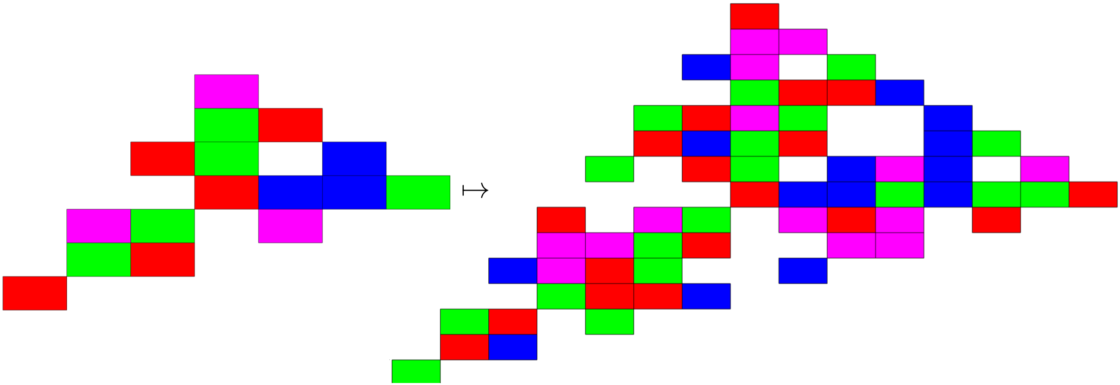

Figure: Second and third iteration of a substitution.

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The **language** of a substitution is the set

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With the language we define the subshift $X_\zeta \subseteq \mathcal{A}^{\mathbb{Z}^d}$ as the set of all sequences $x \in \mathcal{A}^{\mathbb{Z}^d}$ such that every pattern of x is in $\mathcal{L}_\zeta$.

The action of $\mathbb{Z}^d$ on X_ζ is defined by shifts:

$$\forall \mathbf{n} \in \mathbb{Z}^d,\ (S^{\mathbf{n}}x)_{\mathbf{k}} = x_{\mathbf{n}+\mathbf{k}}, \forall \mathbf{k} \in \mathbb{Z}^d.$$

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We say that the substitution ζ is **aperiodic** if there are no non-trivial periods, i.e., if $S^{\mathbf{p}}x = x$ for some $x \in X_\zeta$, then $\mathbf{p} = 0$.

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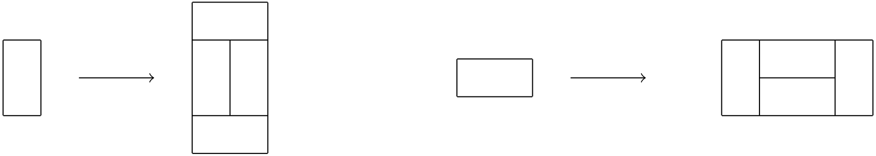
Let ζ be an aperiodic primitive substitution **satisfying certain geometric and dynamical properties**. *Then*

1. *Any homomorphism in the normalizer semigroup $N(X_\zeta, S, \mathbb{Z}^d)$ is invertible.*
2. *The group $N(X_\zeta, S, \mathbb{Z}^d)/\langle S\rangle$ is finite.*

A. Bustos, D. Luz, N. Mañibo (2021): Point (2) for bijective block substitutions.

The table tiling:



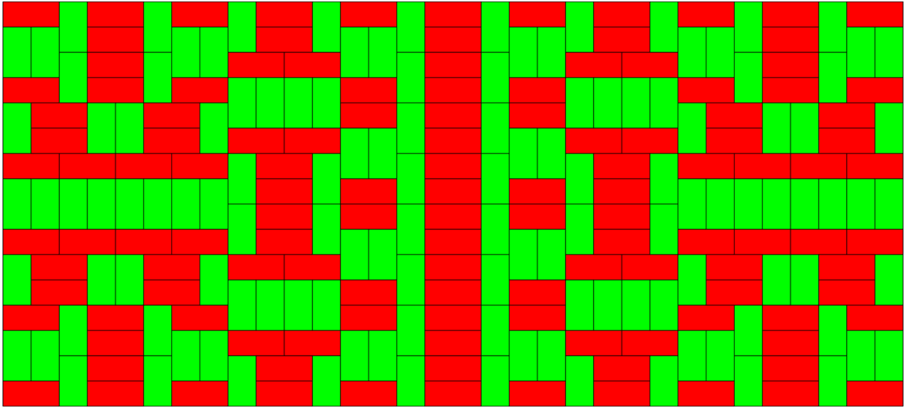


Figure: A pattern of the table tiling

The table tiling: $L_t = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $F_1^t = [0,1]^2 \cap \mathbb{Z}^2$

$$0 \mapsto \begin{matrix} 3 & 0 \\ 1 & 0 \end{matrix}, \qquad 1 \mapsto \begin{matrix} 1 & 1 \\ 0 & 2 \end{matrix}, \qquad 2 \mapsto \begin{matrix} 2 & 3 \\ 2 & 1 \end{matrix}, \qquad 3 \mapsto \begin{matrix} 0 & 2 \\ 3 & 3 \end{matrix}.$$

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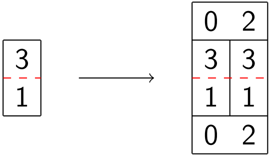

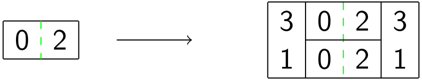

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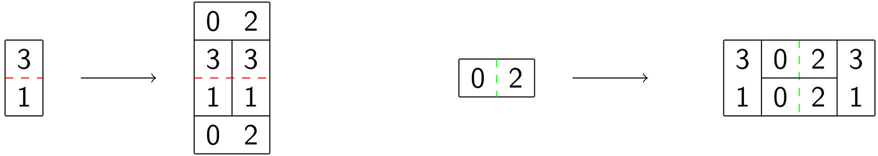

Proposition

For the table tiling, we have $N(X_t, S, \mathbb{Z}^2) \cong \mathbb{Z}^2 \rtimes D_4$.

Geometric property: Polytope

The matrix L_ζ^{-1} is a contraction in $\mathbb{R}^d$. We define a contraction in the collection $\mathcal{C}(\mathbb{R}^d)$ of nonempty compact subsets of $\mathbb{R}^d$ equipped with the Hausdorff metric h

$$\begin{array}{rccc} F: & (\mathcal{C}(\mathbb{R}^d), h) & \to & (\mathcal{C}(\mathbb{R}^d), h) \\ & A & \mapsto & \bigcup\limits_{\mathbf{g}\in F_1^\zeta} (L_\zeta^{-1}(A) + \mathbf{g}), \end{array}$$

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We can approximate the digit tile: $T_\zeta = \lim_{n\to\infty} L_\zeta^{-n}(F_n^\zeta)$.

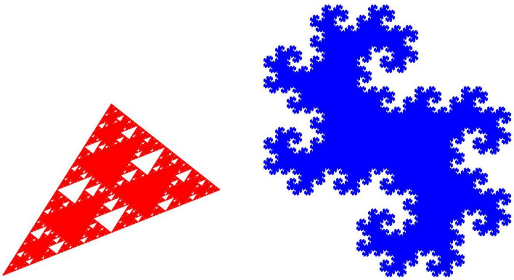

Figure: Approximation of some digit tiles: (a) Gasket, (b) Twin Dragon.

- (a) $$L_{\zeta_{(a)}} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$ $$F_1^{\zeta_{(a)}} = \{(0,0),(1,0),(0,1),(-1,-1)\}.$$

- (b) $$L_{\zeta_{(b)}} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$ $$F_1^{\zeta_{(b)}} = \{(0,0),(1,0)\}.$$

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The first condition only depend on the expansion matrix and the support.

$$L = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \text{ and } F_1 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}.$$

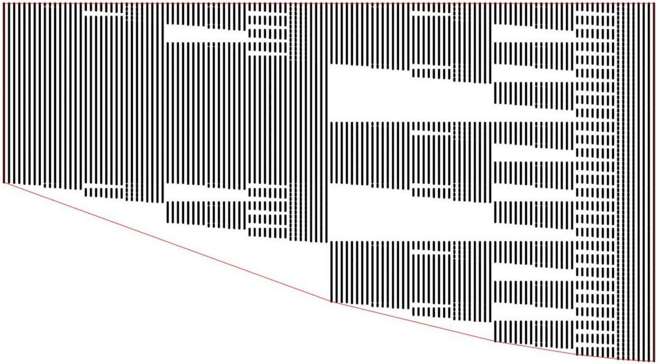

Figure: Example conv(T_ζ) **not** a poytope.

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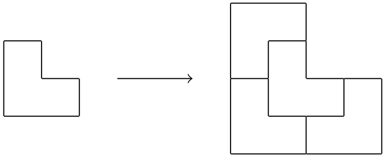

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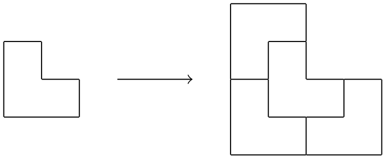

Discrete chair tiling: $L_t = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $F_1^t = [0,1]^2 \cap \mathbb{Z}^2$

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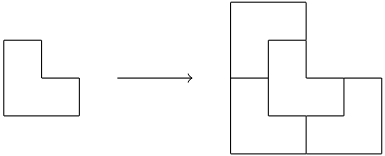

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Example of a **not** bijective substitution.

Dynamical property: Non-deterministic directions for $(X_\zeta, S, \mathbb{Z}^d)$.

A vector $\mathbf{v} \in \mathbb{S}^{d-1}$ is **deterministic** for $(X_\zeta, T, \mathbb{Z}^d)$ if and only if

$$\forall x, y \in X_\zeta : x_{H_\mathbf{v} \cap \mathbb{Z}^d} = y|_{H_\mathbf{v} \cap \mathbb{Z}^d} \implies x = y,$$

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M. Boyle, D. Lind (1997): Introduction of the notion of expansive subdynamics.
V. Cyr, B. Kra (2015): Use of non-expansive directions for a weak version of the Nivat's conjecture.
P. Guillon, J. Kari, C. Zinoviadis (2015): Determinism for 2D-subshifts.

Proposition

Let ζ be an aperiodic primitive constant-shape substitution. Then for all $\mathbf{v} \in \mathrm{ND}(X_\zeta, S, \mathbb{Z}^d)$ and $M \in \vec{N}(X_\zeta, S, \mathbb{Z}^d)$, we have $M^*\mathbf{v}/\|M^*\mathbf{v}\| \in \mathrm{ND}(X_\zeta, S, \mathbb{Z}^d)$.

Theorem (C., 2021)

Let ζ be an aperiodic primitive polytope substitution. The non-deterministic directions $\mathrm{ND}(X_\zeta, S, \mathbb{Z}^d)$ *is the intersection of* $\mathbb{S}^{d-1}$ *with a non-empty union of opposite normal cones of the form* $\hat{N}_{\mathbf{G}}(\mathrm{conv}(T_\zeta))$*, where* $\mathbf{G}$ *is a face of* $\mathrm{conv}(T_\zeta)$.

$$\hat{N}_{\mathbf{G}}(\mathrm{conv}(T_\zeta)) = \{\mathbf{v} \in \mathbb{R}^d \colon \min_{\mathbf{t} \in \mathrm{conv}(T_\zeta)} \langle \mathbf{v}, \mathbf{t} \rangle = \langle \mathbf{v}, \mathbf{p} \rangle\,,\ \forall\ \mathbf{p} \in \mathbf{G}\}.$$

Theorem (C. (2021))

Let ζ be an aperiodic primitive polytope substitution with rank$(\mathrm{ND}(X_\zeta, S, \mathbb{Z}^d)) = d$. *Then*

1. *Any homomorphism in the normalizer semigroup* $N(X_\zeta, S, \mathbb{Z}^d)$ *is invertible.*
2. *The group* $N(X_\zeta, S, \mathbb{Z}^d)/\langle S \rangle$ *is finite.*

Question

Does there exists an aperiodic d-dimensional constant-shape substitution with less than d linearly independent nondeterministic directions? **Partially answered**.

Question

What can be said about the normalizer group for nonpolytope constant-shape substitutions?

Question

What subgroups of $GL(d, \mathbb{Z})$ can be realized as the symmetry group of a constant-shape substitution?

Question

Is it decidable the question of whether a constant-shape substitution satisfies $\mathrm{ND}(X_\zeta, S, \mathbb{Z}^d) = \mathbb{S}^{d-1}$?

and so on...

THANKS