# On dynamics of Lorenz maps – Renormalizations and primary $n(k)$-cycles

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AGH University of Science and Technology

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# The talk is based on joint works with Piotr Oprocha


- Ł. Cholewa, P. Oprocha, *On $\alpha$-limit sets in Lorenz maps*, Entropy, **23(9)** (2021), article id: 1153.
- Ł. Cholewa, P. Oprocha, *Renormalization in Lorenz maps – completely invariant sets and periodic orbits*, preprint, arXiv:2104.00110.

# Presentation plan

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- Introduction

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- Theory of Yiming Ding: Renormalizations and invariant sets
- Primary $n(k)$-cycles
- Locally eventually onto Lorenz maps and the matching property

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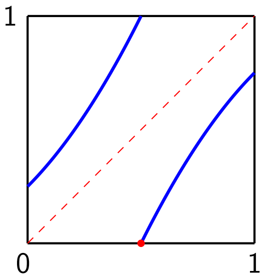


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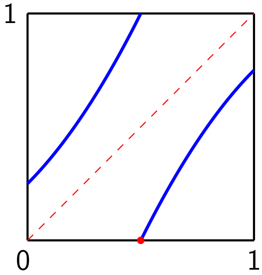


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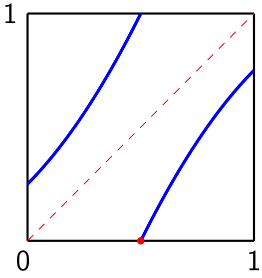


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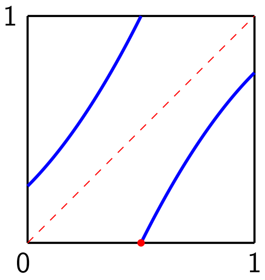


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**Remark**

*The last condition implies that the set* $\bigcup_{n\in\mathbb{N}_0} f^{-n}(c)$ *is dense in* $[0,1]$.

# Motivation: Geometric models of Lorenz attractor

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- Poincaré maps in geometric models of Lorenz attractor.


V. S.Afraĭmovich, V. V. Bykov, L. P. Shil'nikov, *On attracting structurally unstable limit sets of Lorenz attractor type*. (in Russian) Trudy Moskov. Mat. Obshch. **44** (1982), 150–212.

J. Guckenheimer, *A strange, strange attractor*, in: J. E. Marsden and M. McCracken (eds.), The Hopf Bifurcation Theorem and its Applications, Springer, 1976, pp. 368–381.

R. F. Williams, *The structure of Lorenz attractors*. Inst. Hautes Ètudes Sci. Publ. Math. No. **50**, (1979), 73–99.

# Motivation: Number theory

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- Expansions of real numbers in non-integer bases.

W. Parry, *On the $\beta$-expansions of real numbers*. Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416.

A. Rényi, *Representations for real numbers and their ergodic properties*. Acta Math. Acad. Sci. Hungar. **8** (1957), 477–493.

# Motivation: Fractal geometry

- Applications in fractal geometry.

M. F. Barnsley, *Transformations between self-referential sets*. Amer. Math. Monthly **116** (2009), no.4, 291–304.

M. F. Barnsley, B. Harding, A. Vince, *The entropy of a special overlapping dynamical system*. Ergodic Theory Dynam. Systems **34** (2014), no.2, 469–486.

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- Denote $p_i := T^i(0)$ and $q_i := T^i(1)$.

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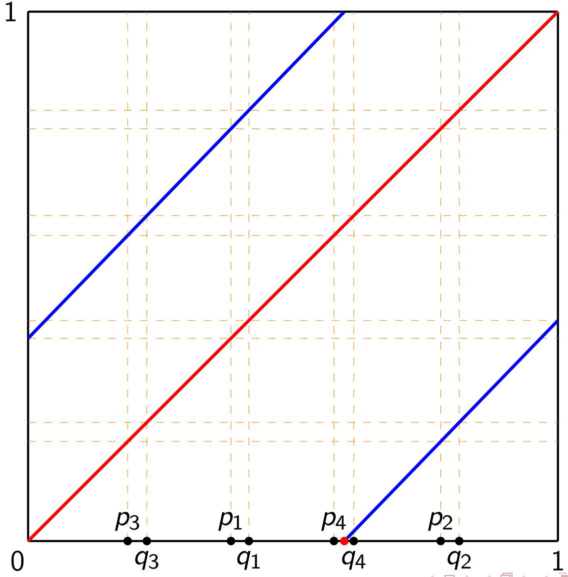

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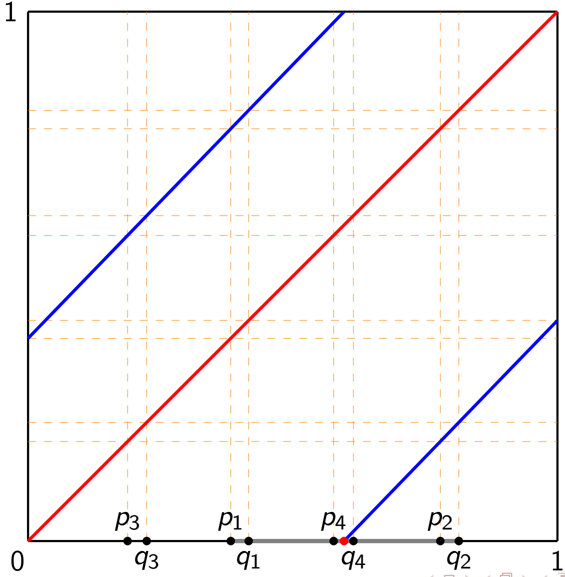

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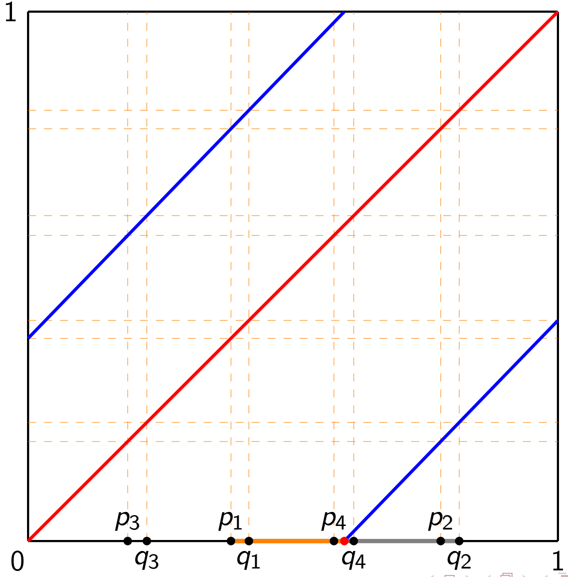

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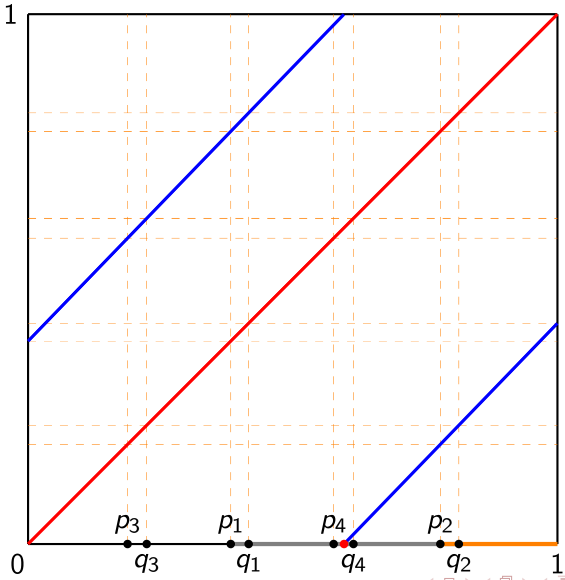

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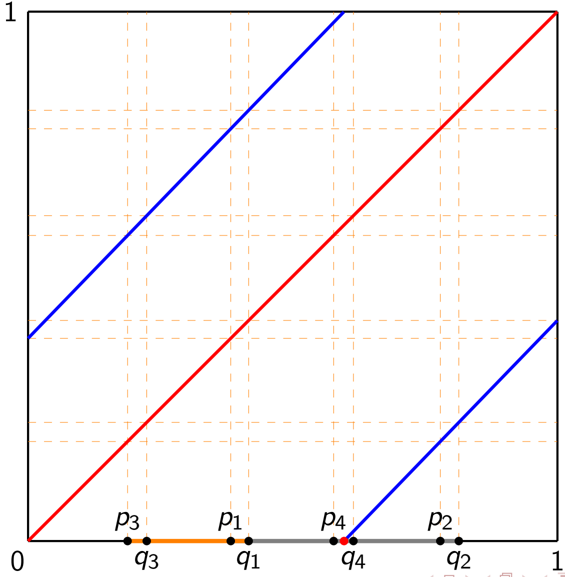

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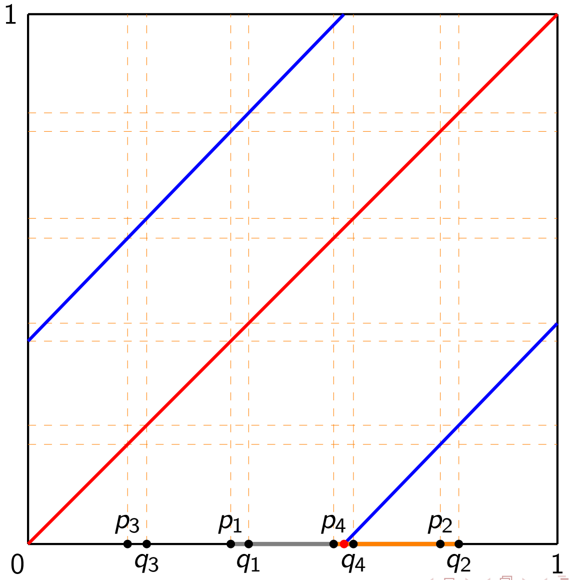

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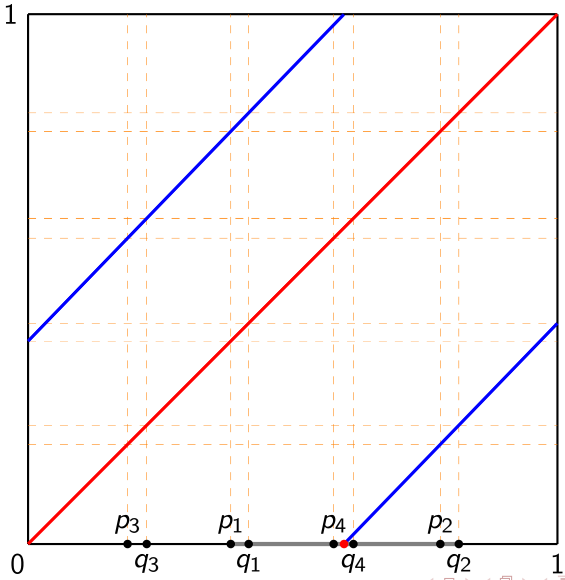

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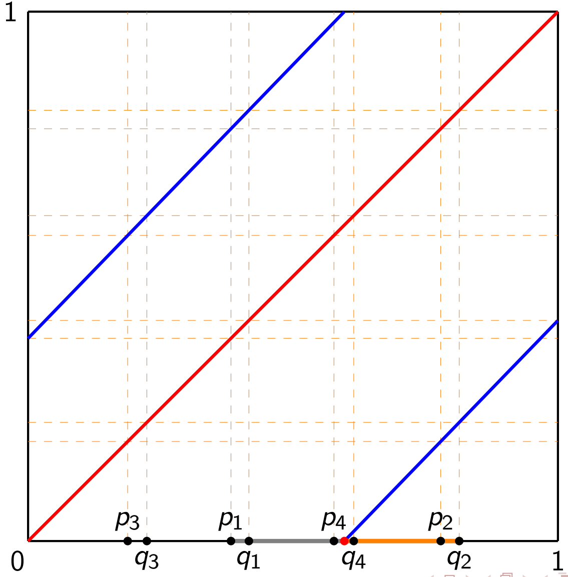

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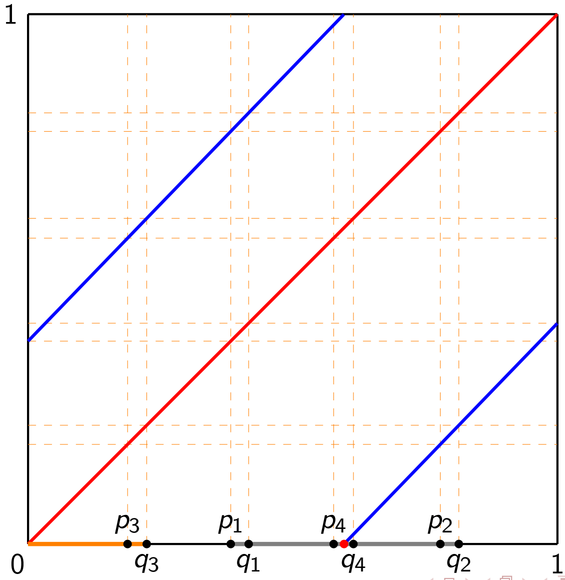

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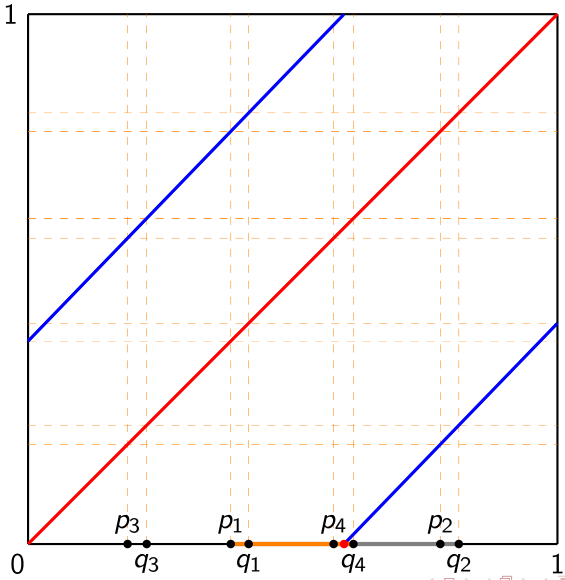

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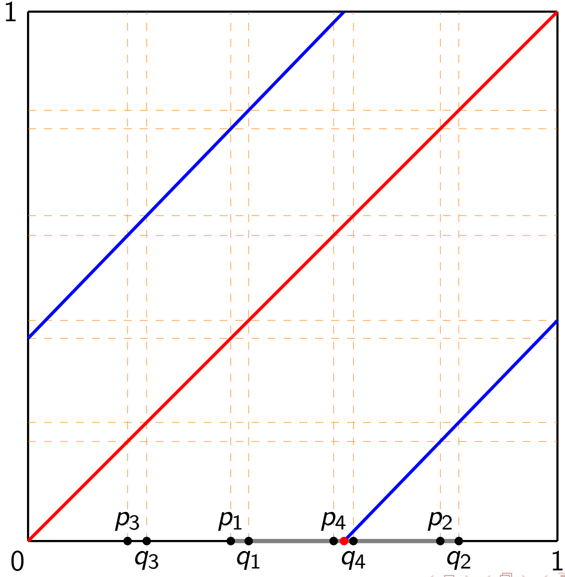

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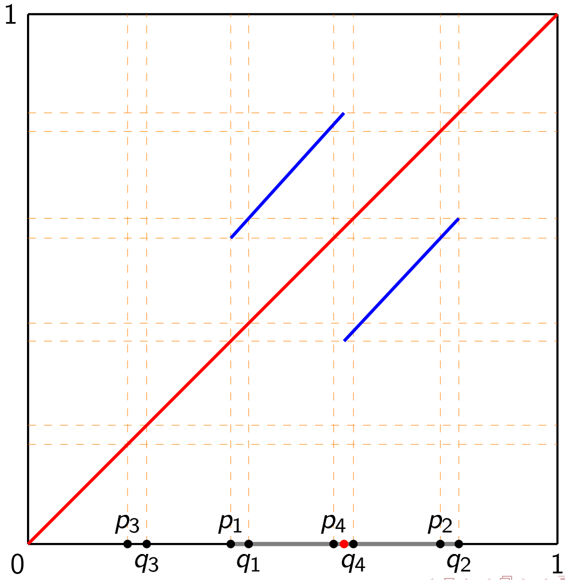

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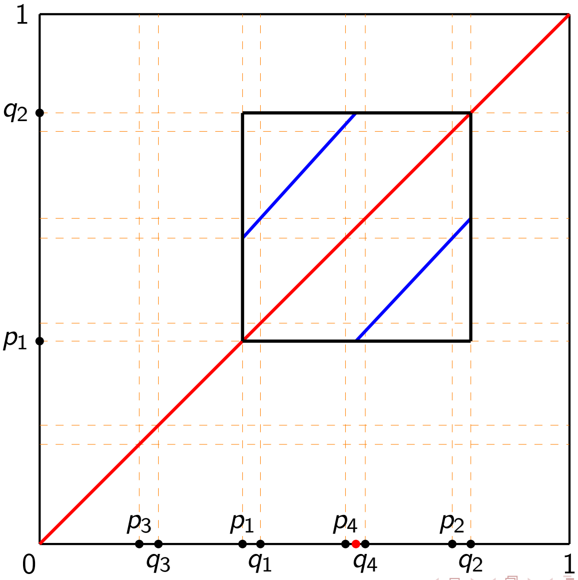
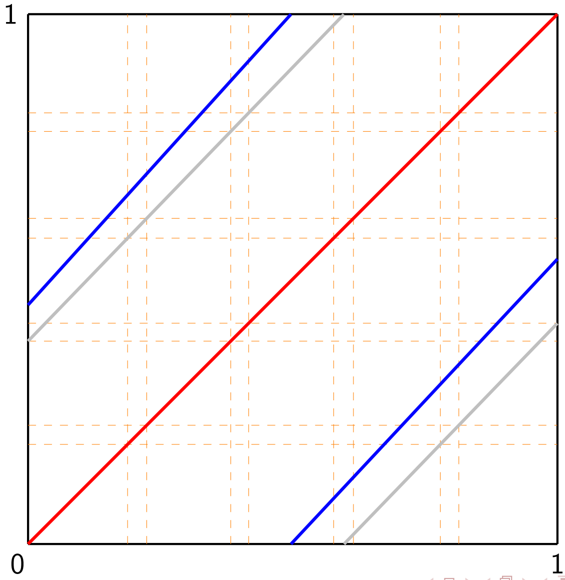


# Example: Graph of the map $G$ after rescaling



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Let $U \subset [0,1]$ be an open set. By $N(U)$ we denote the smallest integer $n \geqslant 0$ such that $c \in f^n(U)$.

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*Then* $f^l(e_-) = e_-$, $f^r(e_+) = e_+$ *and the following map*

$$R_E f(x) = \begin{cases} f^l(x), & x \in [f^{r-1}(0), c) \\ f^r(x), & x \in (c, f^{l-1}(1)] \end{cases}$$

*is a renormalization of* $f$*.*

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**It may happen that the set $J_g$ is empty or not completely invariant!**

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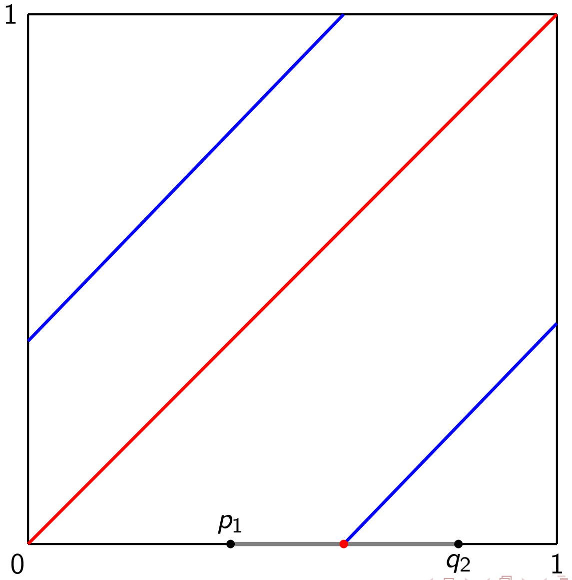

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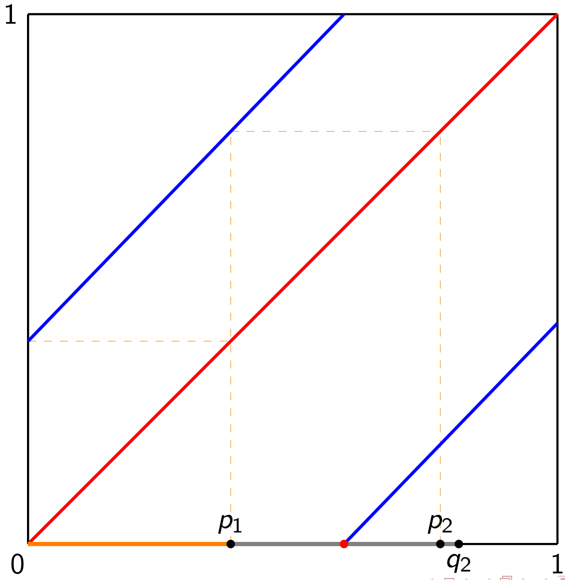

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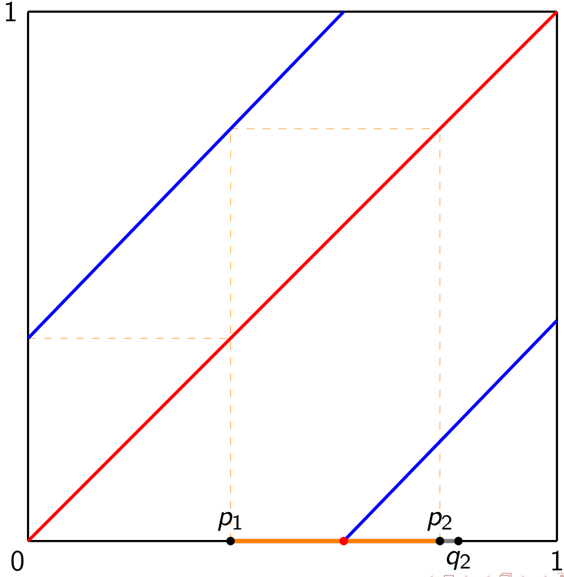

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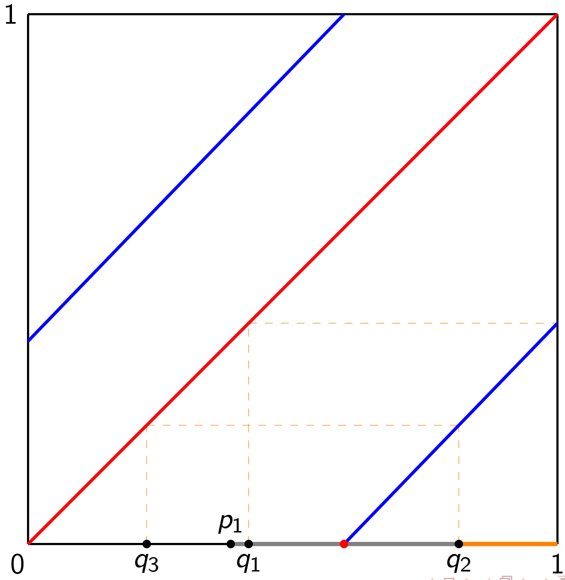

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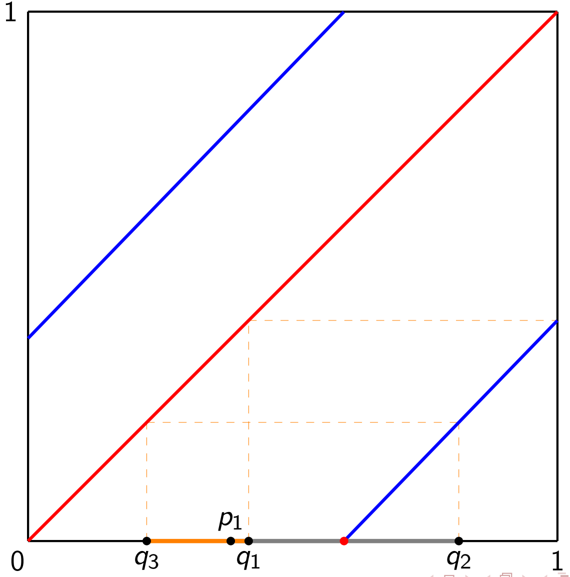

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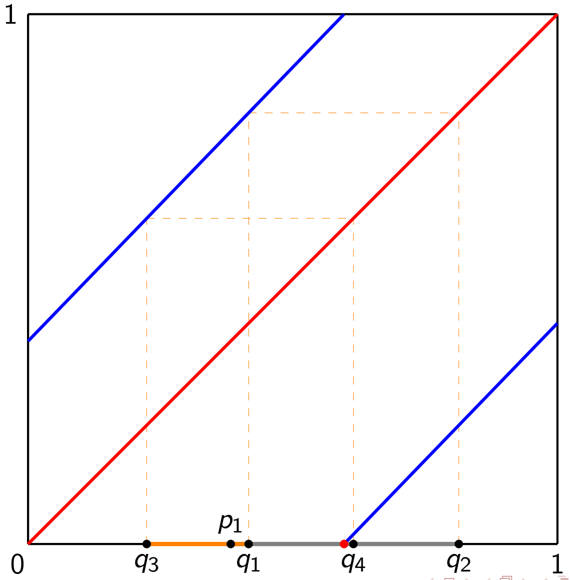

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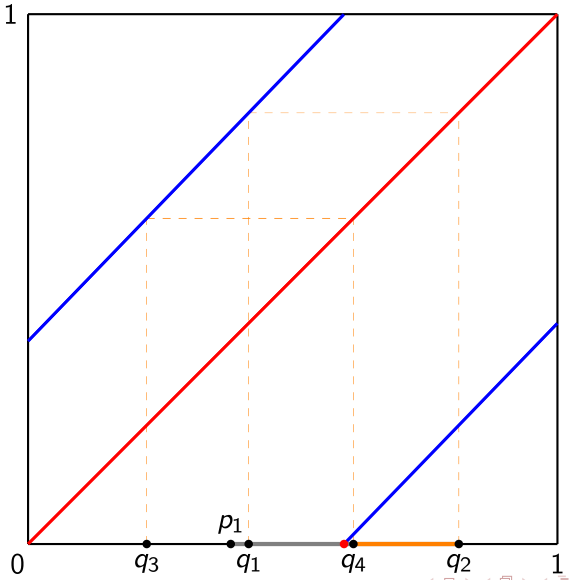

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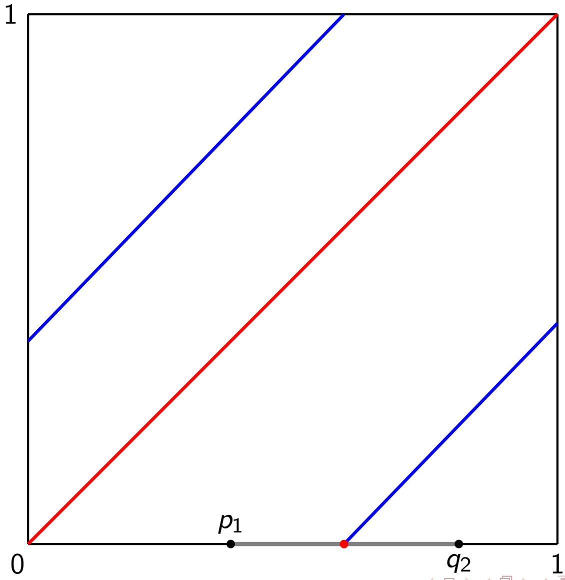

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- $J_G = \{x \in [0,1] : \text{Orb}(x) \cap (p_1, q_2) = \emptyset\} = \emptyset$.
- There is no proper, closed and completely invariant set that defines the renormalization $G$.

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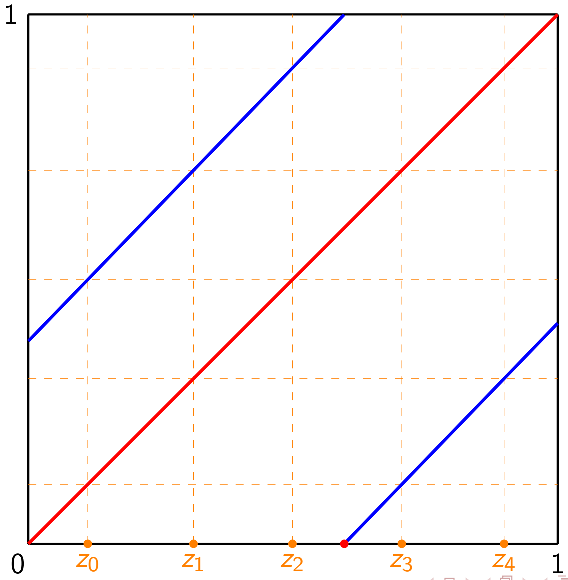

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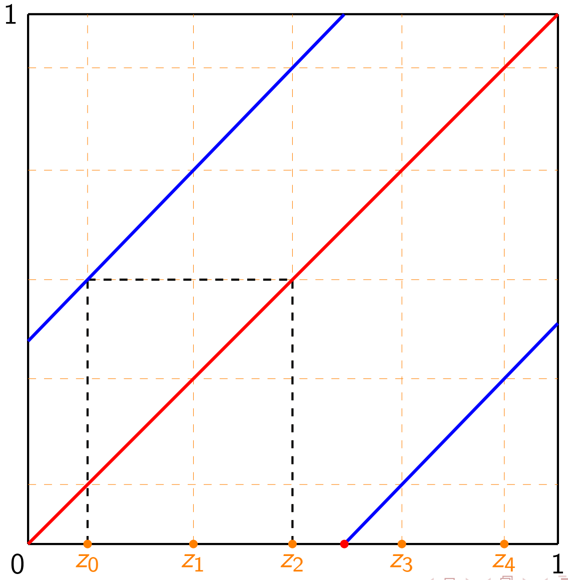

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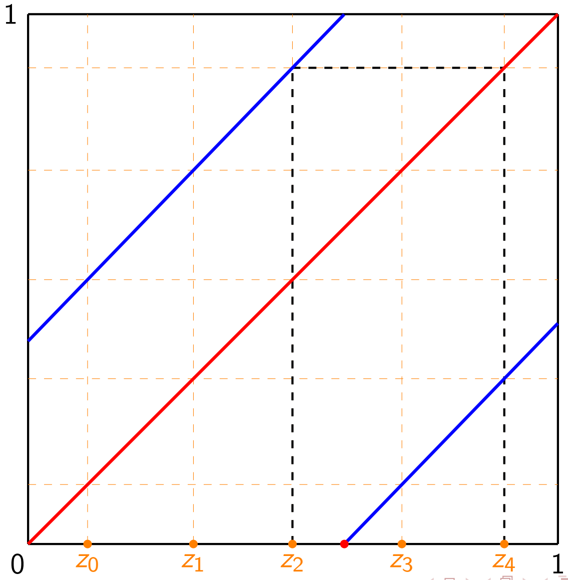

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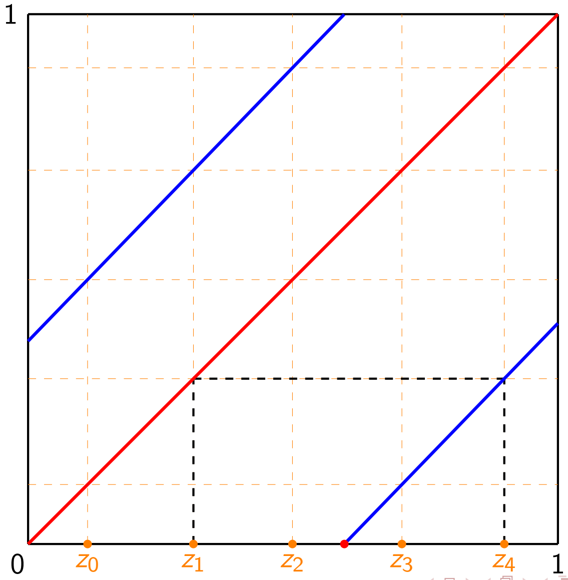

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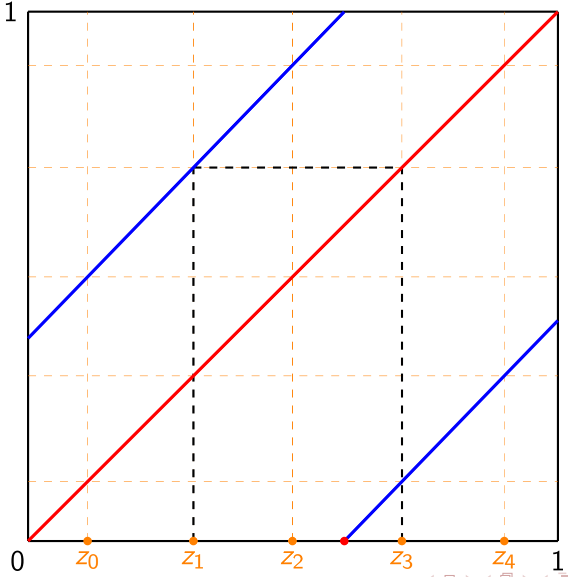

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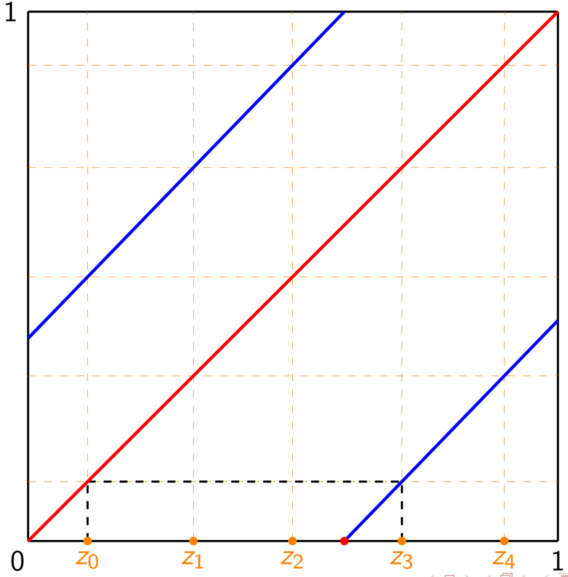

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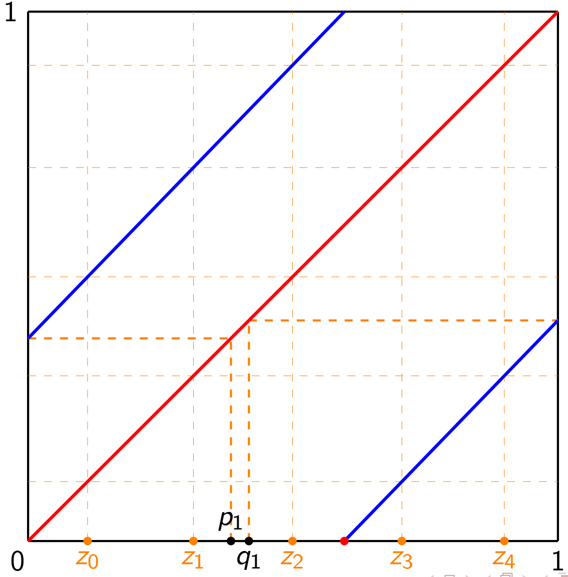

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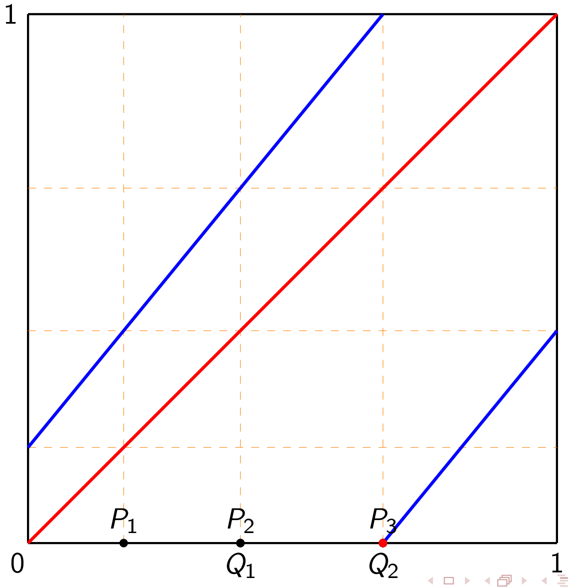

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**Remark**

*Note that the polynomial $x^4 - x - 1$ has two other zeros $\beta_1$ and $\beta_2$ such that*

$$|\beta_1| = |\beta_2| \approx 1.06334,$$

*which implies that $\beta$ is algebraic but non-Pisot and non-Salem number.*

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# The end

**Thank you for your attention!**

**Vielen Dank für Ihre Aufmerksamkeit!**