Toeplitz subshifts and invariant measures.

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Group actions on the Cantor set

We deal with **Cantor dynamical systems** (X, T, G), i.e, :

- X is a Cantor set.
- G is a countable infinite group (Ex: $\mathbb{Z}^d$, $\mathbb{Q}$, $\mathbb{F}_2$).
- T is a continuous action of G on X, where $T^g : X \to X$ is the homeomorphism induced by the action of $g \in G$ on X.
  - $T^{1_G} = id$.
  - $T^g \circ T^h = T^{gh}$ for every $g, h \in G$.

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Remark: The elements of Σ^G can be seen as tilings of G. If $G = \mathbb{Z}^d$ we can also see them as tilings of $\mathbb{R}^d$

Minimal and aperiodic subshifts.

A subshift $X \subseteq \Sigma^G$ is:

- **aperiodic** if $\sigma^g(x) = x$ implies $g = 1_G$, for every $x \in X$.
- **minimal** if every $x \in X$ is repetitive, that is, for every finite set $P \subseteq G$, there exists a finite set $F \subseteq G$ such that $F \cdot T_P(x) = G$, where

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Remark: if $G \neq \mathbb{Z}^d$, the existence of repetitive and aperiodic elements of Σ^G is not obvious.

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For every countable group G, there exists a minimal aperiodic subshift $X \subseteq \{0,1\}^G$.

Remark: For every countable group G, there exists an aperiodic repetitive element in $\{0,1\}^G$.

Example: Toeplitz G-subshifts.

An element $x \in \Sigma^G$ is **Toeplitz*** if for every $g \in G$ there exists a finite index subgroup Γ of G such that

$$x(g) = x(\gamma g) = \sigma^{\gamma^{-1}} x(g) \text{ for every } \gamma \in \Gamma.$$

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The intersection of finite index subgroups of G has to be trivial!!

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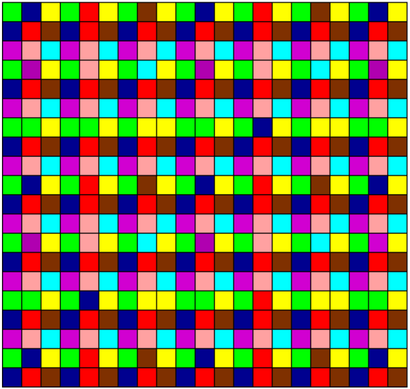
Remark: There are aperiodic Toeplitz subshifts in $\{0,1\}^{\mathbb{Z}^d}$, $\{0,1\}^{\mathbb{F}_2}$ but not in $\{0,1\}^{\mathbb{Q}}$.

Example



An aperiodic Toeplitz element in $\Sigma^{\mathbb{Z}^2}$, where Σ is an alphabet with 8 letters (colors).

Maximal equicontinuous factor of Toeplitz G-subshifts.

Remark: G is residually finite if and only if there exists a decreasing sequence $(\Gamma_n)_{n\geq 0}$ of finite index subgroups of G with trivial intersection.

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Remark: The odometers are exactly the minimal aperiodic equicontinuous systems on the Cantor set (C., Medynets 2016). Then G admits an equicontinuous aperiodic action on the Cantor set if and only if G is residually finite.

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Proposition (C. Petite 2008; Krieger 2007): The Toeplitz G-subshifts are exactly the symbolic minimal almost 1-1 extensions of the G-odometers.

Question:

Given a reasidually finite group G, which are the properties of the Toeplitz G-subshifts?

- **Theo (Krieger 2007):** It is possible to construct a Toeplitz G-subshift having any possible topological entropy.[†]
- **Theo (C., Petite 2014):** for every Choquet simplex K there exists an aperiodic Toeplitz G-subshift whose set of invariant probability measures if affine homeomorphic to K[‡]

[†] For $G = \mathbb{Z}$, Williams 1984.
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Remark: Realization of Choquet simplices is related to topological orbit equivalence classification problems.

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Topological orbit equivalence

- The systems (X, T, G) and (Y, S, Γ) are **topological orbit equivalent** if there exists an homeomorphism $h : X \to Y$ such that $h(o_T(x)) = o_S(h(x))$, for every $x \in X$.
- The **reduced dimension group** of a Cantor system is an invariant for topological orbit equivalence (complete invariant if $G = \mathbb{Z}^d$ and $\Gamma = \mathbb{Z}^m$: Giordano, Matui, Putnam, Skau 2010).
- The spaces of traces of the reduced dimension group is affine homeomorphic to the space of invariant probability measures.
- Realization of dimension group $\Rightarrow$ Realization of simplex.

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Remark: For an arbitrary amenable residually finite group G we also realize dimension groups. Nevertheless the dimension groups that we can get depend on the indices of the finite index subgroup of G (some restrictions could appear).

Final comments:

- If G is congruent **monotilable**[§] (ex: any abelian group) it is possible to construct systems which are *like* Toeplitz, in order to realize any Choquet simplex as the set of invariant measures (C., Cecchi-Bernales, 2019)
- For amenable groups, it is enough to realize the Poulsen simplex to realize any Choquet simplex (Frej, Huczek 2018)
- For every countable group G (not necessarily amenable), it is possible to construct systems with more than one ergodic measure (Elek 2020).
- **Work in progress** (Jaime Gómez, PhD student): properties of Toeplitz G-subshifts for G a non amenable residually finite group.

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