Nonperiodic leaves of codimension one foliations

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What about higher codimension nonleaves? There exist manifolds which are leaves in codimension two but nonleaves in codimension one?

Leaf topology: some chosen results

- Every open surface can be realized as a (nonproper) leaf of some foliation of (every) closed 3-manifold [Cantwell-Conlon 86].

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There exists a codimension one foliation on a compact manifold with a **nonperiodic proper leaf with finitely many ends**? A necessary condition is regularity C^r with $r < 2$.

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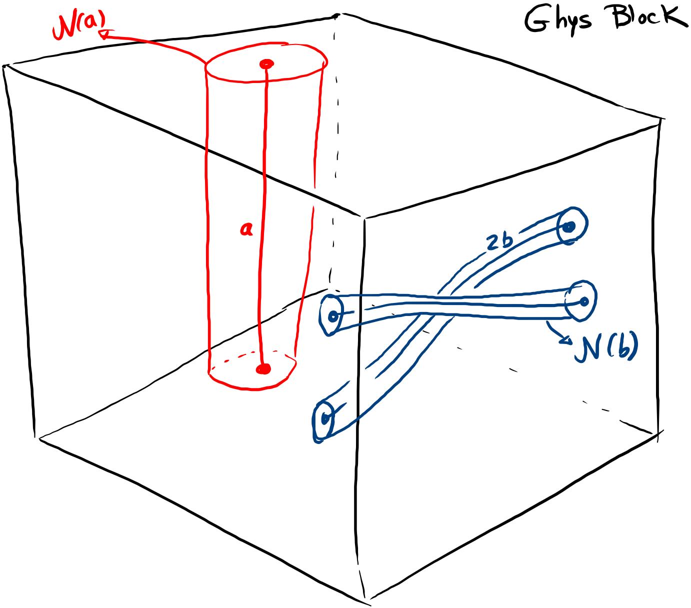

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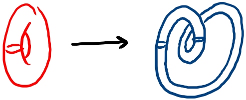
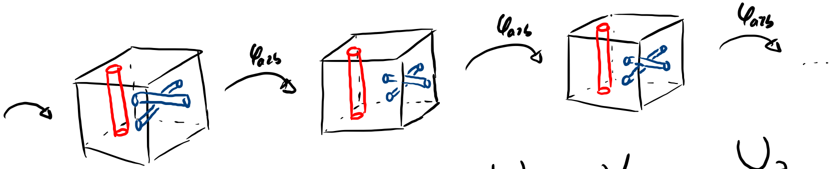
Quoting E. Ghys: "*Il est probablement possible de construire un exemple sur une varieté de dimension* 4*, en codimension* 1*, en se fondant sur la même idée*"

Ghys Manifolds

$\omega \in \{0,1\}^{\mathbb{Z}}$

φ_{a2b} : 



$V_{\omega(0)} \cup_\partial V_{\omega(1)} \cup_\partial V_{\omega(2)} \cup_\partial \cdots$

Ghys Manifold M_ω

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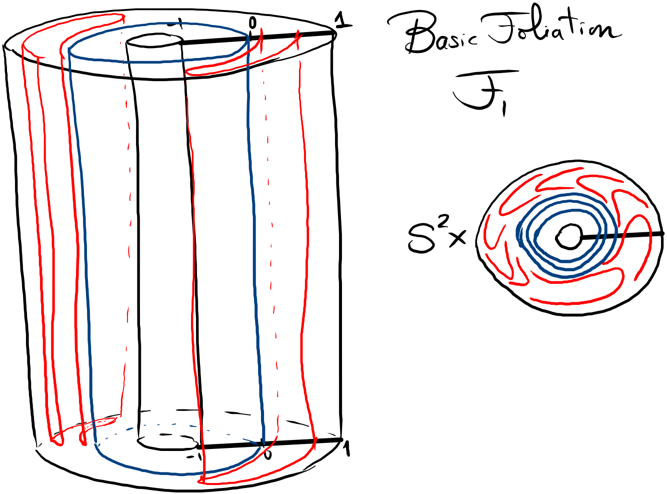

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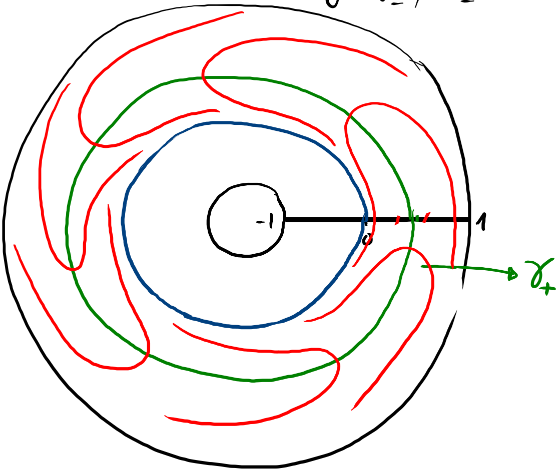
Turbulization along $\gamma_\pm, \omega_\pm$



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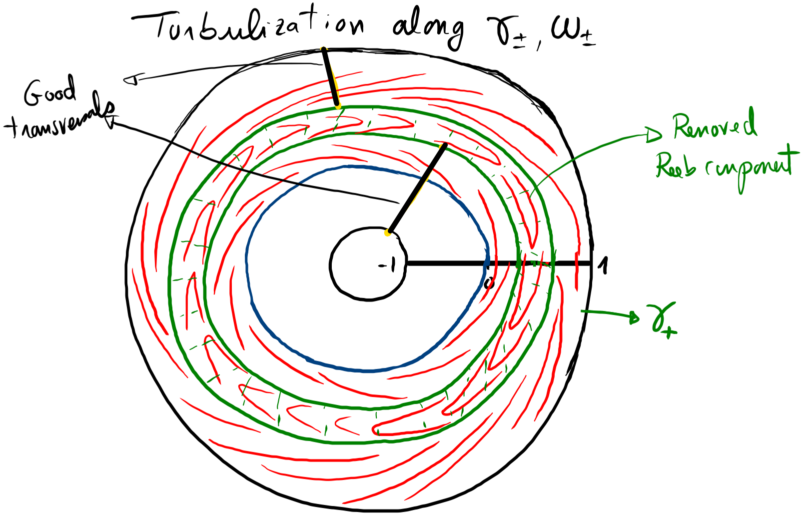

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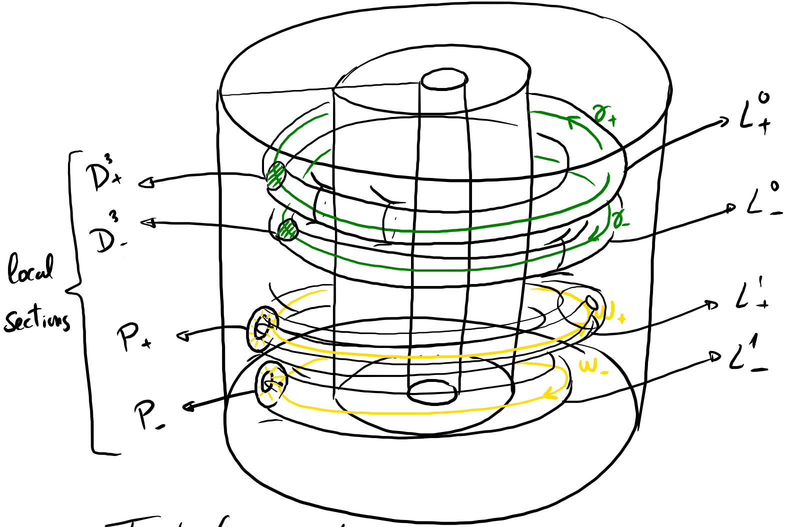

Turbulizing to $\mathcal{F}_2$

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Remark

Let $V_0^{\pm}$ and $V_1^{\pm}$ be the leaves of $\mathcal{F}_4$ obtained from the leaves $T^2 \times L^i_{\pm}$ after removing the previous neighborhoods. These are 5-dimensional Ghys blocks of rank 3 and 5.

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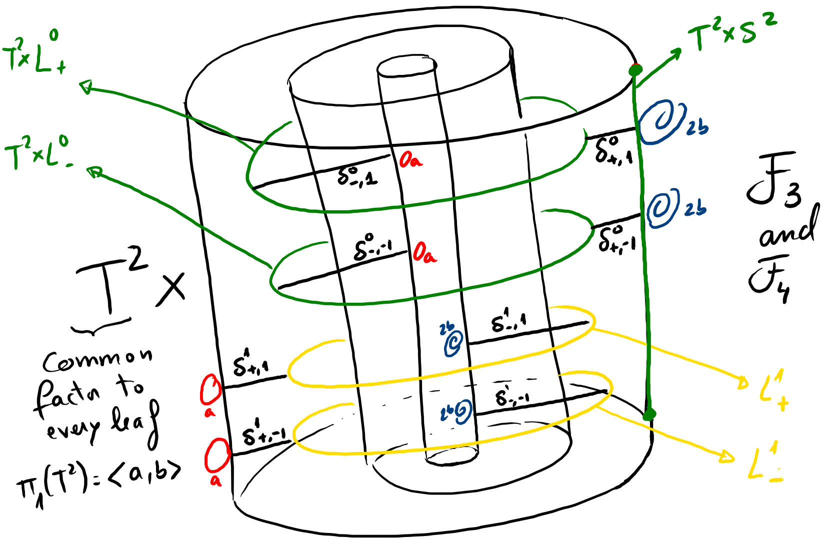

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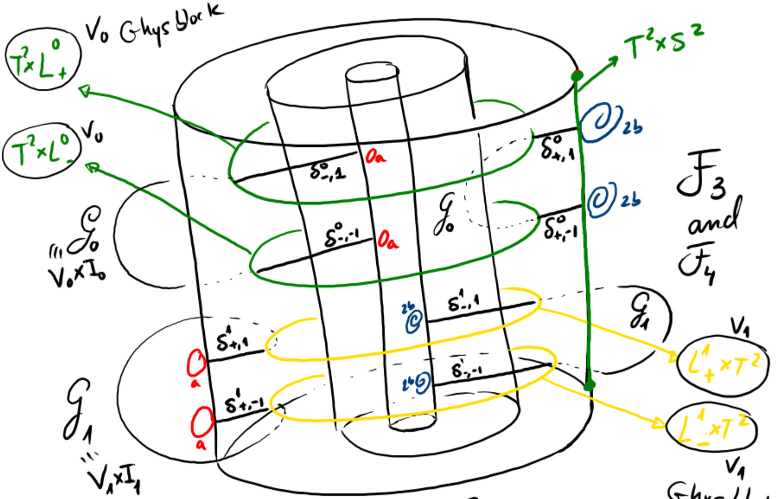
- The saturation of $I_i^{\pm}$ in $\mathcal{F}_5$ is homeomorphic to $\mathcal{G}_i$

Step 5: The bridge foliations



The bridges $\mathcal{G}_0$ and $\mathcal{G}_1$

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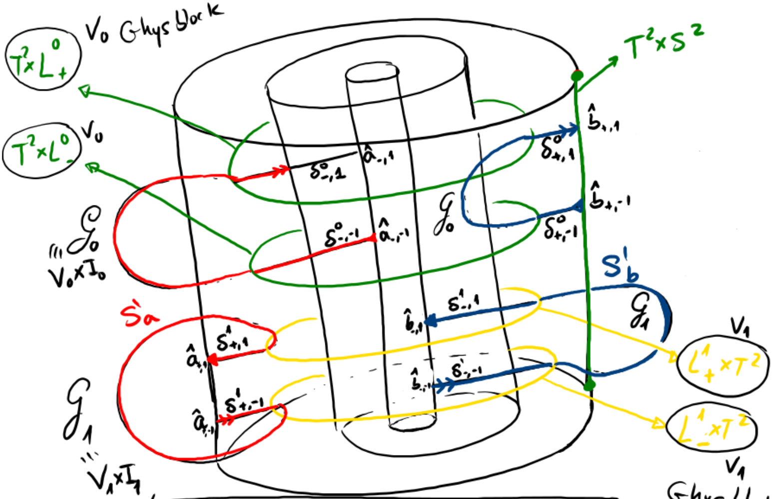

Transverse circles S^1_a and S^1_b

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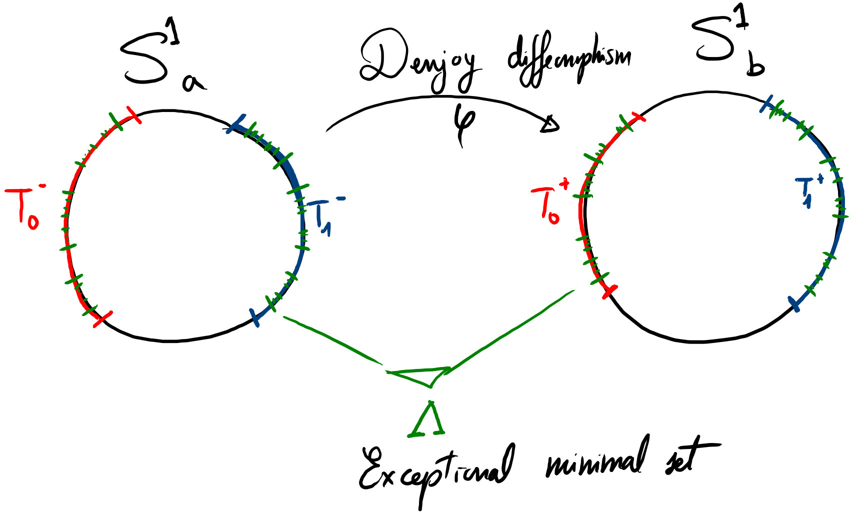

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Proposition

For any Denjoy map φ as above the sequence ω_φ is end repetitive.

Conjecture

[In progress] For every end repetitive ω there exists a Denjoy diffeomorphism such that $\omega_\varphi = \omega$.

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- Thus a (large) period is broken at each close return time q_n. Thus the sequence is (forward) repetitive and nonperiodic. The same argument is used backwards for the rotation number $\mathrm{rot}(\varphi^{-1})$.

Thanks for your attention!