1) A simple ODE for population growth

$$\dot{x} = ax = f(x)$$

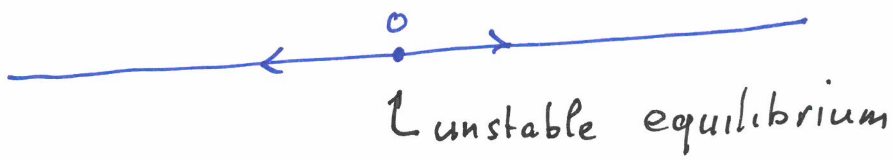
linear growth, proportional to the size of the population

Solution $\quad x(t) = \varphi^t(x_0) = x_0 e^{at}$

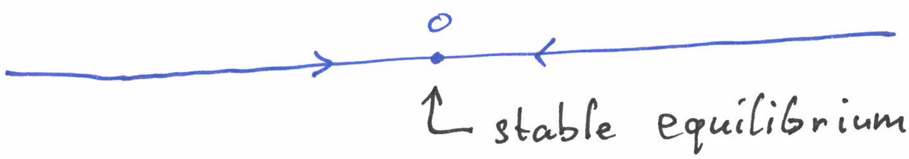
Equilibrium or stationary point at $x_0 = 0$.

Phase portraits:

$a > 0$



$a < 0$



2) Versions of Stability (on metric space (X,d))

An equilibrium y is:

Lyapunov stable if

$$\forall \varepsilon>0 \ \exists \delta>0 \quad d(x_0, y)<\delta \Rightarrow d(x(t), y)<\varepsilon \ \forall t\geq 0$$

Asymptotically stable if

$$\exists \delta>0 \quad d(x_0, y)<\delta \Rightarrow \lim_{t\to 0} d(x(t), y)=0$$

Exponentially stable if

$\exists \alpha>0, C>0, \delta>0$

$$d(x_0, y) \leq \delta \Rightarrow d(x(t), y) \leq C e^{-\alpha t} d(x_0, y) \ \forall t\geq 0$$

Remarks: Exp. stable $\Rightarrow$ Asymp. stable & Lyap stable

In dimension 1: $\left.\begin{array}{l} f(y)=0 \\ f'(y)<0 \end{array}\right\} \Rightarrow$ Exp. stable

Asymp stable $\not\Rightarrow$ Lyap. stable nor Exp. stable

3) Nonlinear ODE for population growth

$$(*) \quad \begin{cases} \dot{x} = a x(1-x) =: f(x) \\ x(0) = x_0 \end{cases} \qquad \text{assume } a>0$$

Equilibria where $f(x)=0$, here at $x=0$ and $x=1$

Phase portrait

compute $f'(x) = a(1-2x) = \begin{cases} a>0 & x=0 \\ -a<0 & x=1 \end{cases}$

<u>Exercises</u>: Draw phase portraits and find exact solutions to

(a) $\dot{x} = x^2$, (b) $\dot{x} = x^3$.

Find the exact solution of $(*)$

4) Topological conjugacy

Use new coordinates $\begin{cases} \dfrac{1+ah}{ah}\, x = y = \psi(x) \\ Q(x) = (1+ah)\, x(1-x) \end{cases}$

Then
$$T(y) = y\,(1+ah - ahy)$$
$$= \frac{1+ah}{ah}\, x \left(1+ah - ah\,\frac{1+ah}{ah}\, x\right)$$
$$= \frac{1+ah}{ah}\,(1+ah)\, x\,(1-x)$$
$$= \psi \circ Q(x) = \psi \circ Q \circ \psi^{-1}(y).$$

$$\begin{tikzcd} x \in \mathbb{R} \arrow[r, "Q"] \arrow[d, "\psi"] & \mathbb{R} \arrow[d, "\psi"] \\ y \in \mathbb{R} \arrow[r, "T"] & \mathbb{R} \end{tikzcd}$$
commutes

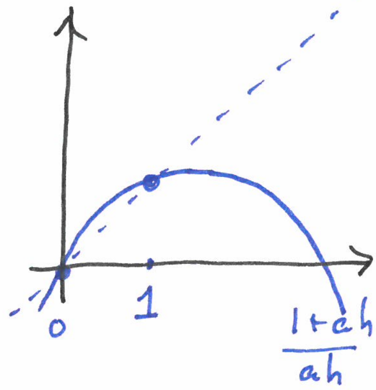
<u>Def</u> Two maps $f: X \to X$ and $g: Y \to Y$ are <u>topologically</u> <u>conjugate</u> if there is a homeomorphism $\psi: X \to Y$ such that $g(y) = \psi \circ f \circ \psi^{-1}(y) \quad \forall y \in Y$.

4) Suppose we solve ⊛ numerically with a simple Euler method with stepsize $h>0$

$$x_{n+1} = x_n + h f(x_n) = \underbrace{x_n + h a x_n (1-x_n)}_{T(x_n)}$$

So $\quad T(y) = y(1+ha-hay)$

$\quad\quad T'(y) = 1+ha-2hay$



Fixed points $T(y)=y$ at $y=0$ & $y=1$.

We <u>iterate</u> the map T:

$$x_0,\ T(x_0),\ \underbrace{T\circ T(x_0)}_{T^2(x_0)},\ \underbrace{T\circ T\circ T(x_0)}_{T^3(x_0)},\ \dots$$

Fixed point: $\quad T(y) = y$

Periodic point: $\quad T^p(y) = y$

Preperiodic point: $\quad T^m(x) = y = T^p(y)$

Suppose $f: X \to X$ and $g: Y \to Y$ are topologically conjugate.

Exercises: Show that

1) x is p-periodic for $f \Rightarrow \psi(x)$ is p-periodic for g

2) $\psi(\omega_f(x)) = \omega_g(\psi(x))$

<u>Proposition 1</u> Let $f: \mathbb{R} \to \mathbb{R}$ be a C^2 map and $f(q) = q$ is a fixed point.

Let $\lambda = |f'(q)|$

If $\lambda \begin{cases} < 1 & \text{then } q \text{ is exponentially stable,} \\ = 1 & \text{undecided but } \underline{\text{not}} \text{ exp. stable,} \\ > 1 & \text{the } q \text{ is unstable.} \end{cases}$

<u>Remark</u> If $f^p(q) = q$ is periodic, and $\lambda = |(f^p)'(q)|$. (This is called the <u>multiplier</u> of q) then Proposition 1 holds without changes.

Proof of Proposition 1.

Take x close to q. The Taylor approximation

$$|f(x) - f(q)| = |f'(q)(x-q) + f''(\xi)(x-q)^2|$$

$$\leq \lambda |x-q| + |f''(\xi)| \cdot |x-q|^2$$

If $\lambda < 1$, the because f'' is bounded near q, we can choose x to be so close to q that

$|f''(\xi)| \cdot |x-q| \leq \frac{1-\lambda}{2}$. Then

$$\leq \left(\lambda + \frac{1-\lambda}{2}\right)|x-q| = \underbrace{\frac{1+\lambda}{2}}_{<1} |x-q|$$

By induction

$$|f^n(x) - f^n(q)| \leq \left(\frac{1+\lambda}{2}\right)^n |x-q|$$

so exponential stability follows.

Left as exercises :

Continuous time ODE	Discrete time Iteration of a map.
flow $\varphi^t(x)$ is solution of $\dot{x} = f(x)$	orbit $(x_n)_{n\geq 0}$ or $(x_n)_{n\in\mathbb{Z}}$ $x_{n+1} = f(x_n)$
$\begin{cases} \varphi^0(x) = x \\ \varphi^{s+t}(x) = \varphi^s(\varphi^t(x)) \\ \frac{d}{dt}\varphi^t(x = f(x) \end{cases}$	$x_n = f^n(x_0)$ $x_{-n} = f^{-n}(x_0) = (f^{inv})^n(x_0)$ if f is invertible
stationary point when $f(x) = 0$	fixed point $f(x) = x$.
periodic orbit $\varphi^T(x) = x$ for minimal $T > 0$	periodic orbit $f^p(x) = x$
none, since flows are invertible	preperiodic orbit $f^m(x) = y = f^p(y)$, and $x \notin orb(y)$.

α- and ω-limitsets

$$\omega(x) = \{ y : \exists t_i \to \infty \quad \varphi^{t_i}(x) \to y \}$$

$$\text{or } \{ y : \exists n_i \to \infty \quad f^{n_i}(x) \to y \}$$

$$\alpha(x) = \{ y : \exists t_i \to -\infty \quad \varphi^{t_i}(x) \to y \}$$

$$\text{or } \{ y : \exists n_i \to -\infty : \quad f^{n_i}(x) \to y \}$$

(if f is invertible

$\alpha(x)$ and $\omega(x)$ are

- closed
- fully invariant $f(\omega(x)) = \omega(x)$
  $f^{-1}(\alpha(x)) = \alpha(x)$
- If $\mathrm{orb}(x)$ is bounded, then $\omega(x)$ is compact
- If $\{\varphi^t(x)\}$ is bounded, then $\alpha(x)$ and $\omega(x)$ are compact and <u>connected</u>