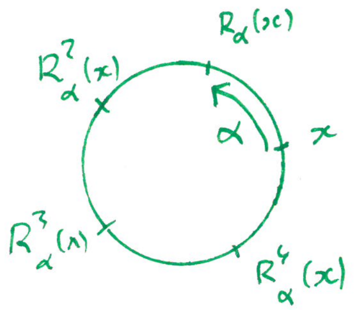
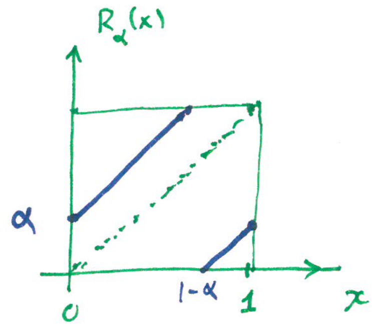
Circle Rotations

$R_\alpha : S^1 \to S^1$ $\quad S^1 = \mathbb{R}/\mathbb{Z} = [0,1]/_{0\sim 1} = \text{circle}$

$x \mapsto x+\alpha \mod 1$





Theorem 1. If $\alpha \in \mathbb{Q}$ then every $x \in S^1$ is periodic

If $\alpha \notin \mathbb{Q}$ then every $x \in S^1$ has a dense orbit.

Proof Do it yourself, Exercise 14.

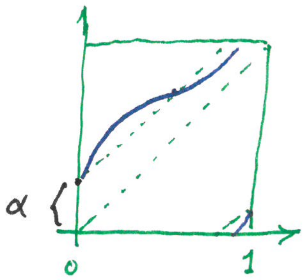
Corollary Circle rotations are not chaotic in the sense of Devaney or any other sense.

(in fact, no sensitive dependence on initial conditions because R_α is an isometry)

Perturbations of Circle Rotations.

<u>Example</u> A well-known family of perturbed circle rotations is the <u>Arnold family</u>

$$f_{\alpha,\varepsilon}(x) = x + \alpha + \varepsilon \sin 2\pi x .$$



NB: if $|\varepsilon| < \frac{1}{2\pi}$, then $f_{\alpha,\varepsilon}$ is a diffeomorphism and

$$1 + 2\pi|\varepsilon| \geq |f'_{\alpha,\varepsilon}(x)| > 1 - 2\pi|\varepsilon| > 0 .$$

<u>Question</u> Is $f_{\alpha,\varepsilon} : S^1 \to S^1$ conjugate to $R_\alpha : S^1 \to S^1$?

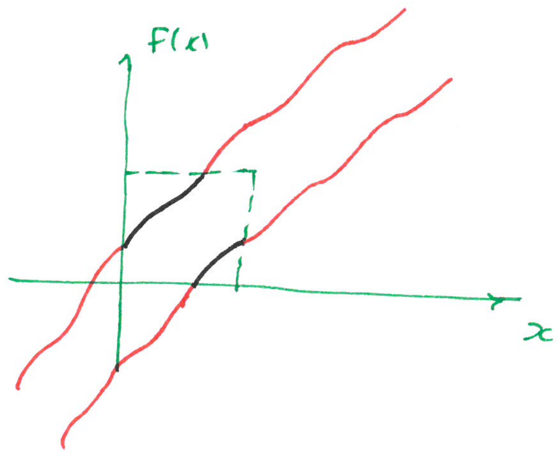
Def A circle homeomorphism $f: S^1 \to S^1$ is order preserving if $x < y < z$ implies $f(x) < f(y) < f(z)$ for all $x, y, z \in S^1$

Here $<$ is a chosen circular order on S^1 (clockwise or counterclockwise, your choice).

Def A map $F: \mathbb{R} \to \mathbb{R}$ is a lift of a circle homeomorphism $f: S^1 \to S^1$ if

- F is continuous
- $\boxed{F(x) \bmod 1 = f(x \bmod 1)}$ for all $x \in \mathbb{R}$

We think specifically of f orientation preserving, and then F is increasing.



Two different lifts of the same circle homeomorphism

NB: If F and G are both lifts of f, then they are integer translations of each other:

$$\exists k \in \mathbb{Z} \quad \forall x \in \mathbb{R} \qquad F(x) = G(x) + k.$$

Def The rotation number of a circle homeomorphism is its "average displacement":

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(x) - x}{n} \mod 1 .$$

- $\rho(R_\alpha) = \alpha$
- This limit exists and is independent of x, and of F.

a) For $x, y \in \mathbb{R}$ take $m \in \mathbb{Z}$ s.t $F^m(x) \le y < F^{m+1}(x)$.
Then (if this fails, then $F(p) = p$ for some p, and $\rho(f) = 0$)

$$\underbrace{\frac{m+n}{n}}_{\to 1} \cdot \underbrace{\frac{F^{m+n}(x) - x + x - y}{m+n}}_{\to \rho(f) \text{ at } x} \le \underbrace{\frac{F^n(y) - y}{n}}_{\to \rho(f) \text{ at } y} < \underbrace{\frac{F^{m+n+1}(x) - x + x - y}{m+n+1}}_{\to \rho(f) \text{ at } x} \cdot \underbrace{\frac{m+n+1}{n}}_{\to 1}$$

so the limit, if it exists, does not depend on x.

b) If $F^q(x) - x = p \in \mathbb{Z}$ (so x is periodic for f) then

$$p = F^q(x) - x = F^{2q}(x) - F^q(x) = F^{3q}(x) - F^{2q}(x) = \cdots$$

so

$$\lim_{n\to\infty} \frac{F^{nq}(x) - x}{nq} = \lim_{n\to\infty} \frac{F^{nq}(x) - F^{(n-1)q}(x) + \cdots + F^q(x) - x}{nq}$$

$$= \lim_{n\to\infty} \frac{np}{nq} = \frac{p}{q} \text{ exists and } \in \mathbb{Q}$$

If $F^n(x) - x \notin \mathbb{Z}$ for any $x \in \mathbb{R}$ and any $n \in \mathbb{N}$

then by continuity there is $k \in \mathbb{Z}$ such that

NB: k depends on n.

$$k < F^n(x) - x < k+1 \quad \text{for all } x \in \mathbb{R}$$

Hence

$$F^{mn}(x) - x = F^{mn}(x) - F^{(m-1)n}(x) + F^{(m-1)n}(x) - F^{(m-2)n}(x) + \cdots$$

$$\cdots + F^n(x) - x$$

lies between mk and $m(k+1)$.

Therefore

$$\frac{mk}{mn} < \frac{F^{mn}(x) - x}{mn} < \frac{m(k+1)}{mn} \quad \text{and} \quad \frac{k}{n} < \frac{F^n(x) - x}{n} < \frac{k+1}{n}$$

and

$$\left| \frac{F^{mn}(x) - x}{mn} - \frac{F^n(x) - x}{n} \right| < \frac{1}{n}$$

Hence $\left(\frac{F^n(x) - x}{n} \right)_{n \in \mathbb{N}}$ is a Cauchy sequence

and it must converge in $\mathbb{R}$.

- If $F = G + k$ $k \in \mathbb{Z}$, then

$$\frac{F^n(x) - x}{n} = \frac{G^n(x) + kn - x}{n} = \frac{G^n(x) - x}{n} + k$$

Hence the "mod 1" in the definition of rotation number

Theorem 2. Let $f: S^1 \to S^1$ be a circle homeomorphism. Then

$\rho(f) \in \mathbb{Q}$ if and only if f has a periodic point.

Proof If $f^q(x) = x$, then $F^q(x) = x + p$ for a lift $F: \mathbb{R} \to \mathbb{R}$ and some $p \in \mathbb{Z}$.

We saw before that in this case

$$\rho(f) = \lim_{n\to\infty} \frac{F^{nq}(x) - x}{nq} = \frac{p}{q} \in \mathbb{Q}$$

in lowest terms.

Conversely, if $\rho(f) = \frac{p}{q} \in \mathbb{Q}$, then consider the lift $F: \mathbb{R} \to \mathbb{R}$ with $F(0) \in [0,1)$.

Assume by contradiction that f^q has no fixed point.

Then $\exists k \in \mathbb{Z}$ such that

$$F^q(x) - x \in (k, k+1) \quad \text{for all } x \in \mathbb{R}. \quad (*)$$

If $k \leq p-1$, then $F^{nq}(x) - x \leq n(p-1)$ $\forall n \in \mathbb{N}$,

so $\frac{F^{nq}(x) - x}{nq} \leq \frac{p-1}{q} \not\to \frac{p}{q}$, a contradiction

A similar contradiction is obtained from assuming that $k \geq p+1$.

So $k = p$.

Then (*) implies that

$$F^q(x) - p > x \qquad \forall x \in \mathbb{R}$$

and therefore $\boxed{\{F^{nq}(x) - np\}_{n \geq 1}}$

is an increasing sequence.

- If this sequence is bounded, then it must converge, to $y \in \mathbb{R}$ say.
  But then $\boxed{F^q(y) - p = y}$ and we found a periodic point for f.

- If the sequence is unbounded, then there is $n_0 \in \mathbb{N}$ such that

$$F^{n_0 q}(x) - n_0 p > x + 1 \qquad \forall x \in \mathbb{R}.$$

Then also

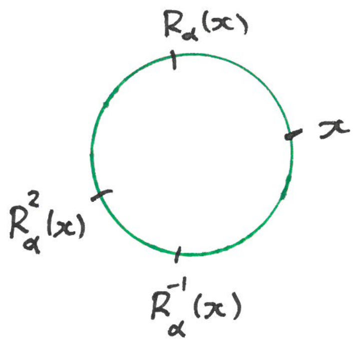
$$\frac{F^{m n_0 q}(x) - x}{m n_0 q} \geq \frac{m(1 + n_0 p)}{m n_0 q} = \frac{p}{q} + \frac{1}{n_0 q} > \rho(f)$$

and this contradicts that the rotation number $\rho(f) = \frac{p}{q}$ $\square$

unlike R_α

However, if $\rho(f) \notin \mathbb{Q}$, this does **NOT** imply that every orbit is dense.

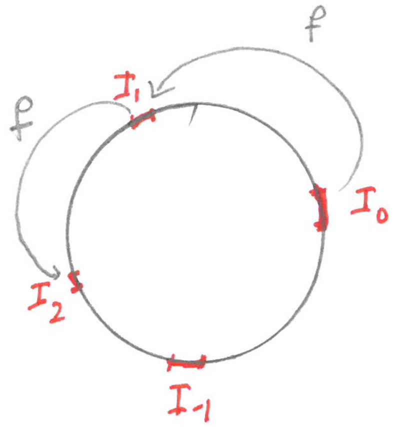
The construction of a counter-example is due to Denjoy. (Arnaud Denjoy, 1884-1974)



blow up every point $R_\alpha^n(x)$, $n \in \mathbb{Z}$

to an interval I_n of length $\varepsilon 2^{-|n|}$

so $\sum_{n \in \mathbb{Z}} |I_n| < \infty$



Then map I_n onto I_{n+1} in some continuous (for example linearly), and leave R_α unchanged outside $\bigcup_{n \in \mathbb{Z}} I_n$.

<u>However</u>

Theorem (Denjoy): If $f : S^1 \to S^1$ (with $\rho(f) \notin \mathbb{Q}$) is a circle diffeomorphism such that f' has bounded variation (and C^2 smoothness is enough for this), then

f is conjugate to R_α for $\alpha \in \rho(f)$

In particular, every orbit is dense in the circle.

Thm 3 For the Arnold family

$$f_{\alpha,\varepsilon}(x) = x + \alpha + \varepsilon \sin 2\pi x \mod 1$$

$\varepsilon \in (0, \frac{1}{2\pi})$ fixed

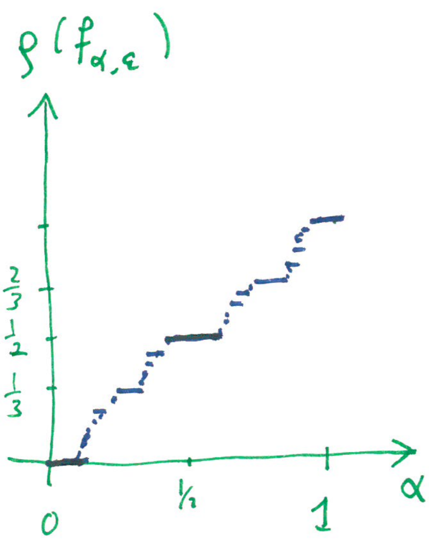
so $f_{\alpha,\varepsilon}$ is C^∞ - diffeo

we have that $\alpha \mapsto \rho(f_{\alpha,\varepsilon})$

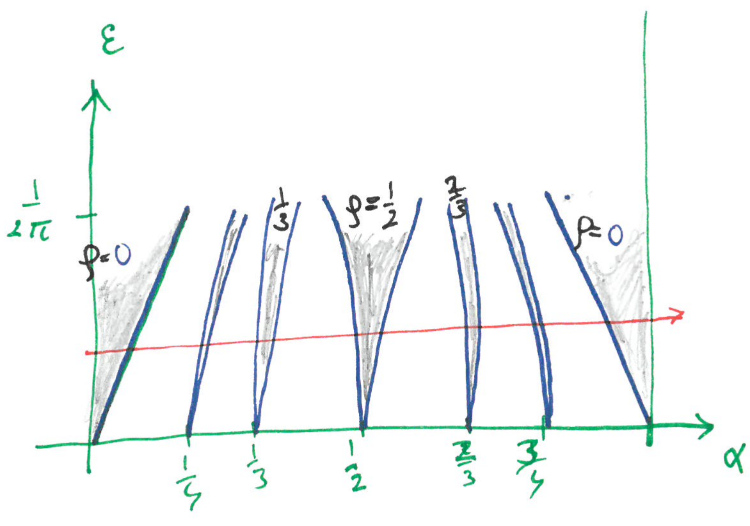
a) is continuous

b) is monotone increasing

c) but not strict because it is locally constant whenever $\rho(f_{\alpha,\varepsilon}) \in \mathbb{Q}$.



Devil's Staircase.



Arnold tongues or resonance tongues, where $\rho(f_{\alpha,\varepsilon})$ is constant $\in \mathbb{Q}$

Proof of Theorem

a) Let $q \in \mathbb{N}$ be arbitrary and take $p \in \mathbb{Z}$ such

continuity that $\rho(f_{\alpha,\varepsilon}) \in \left(\frac{p-1}{q}, \frac{p+1}{q}\right)$.

By the previous proofs we conclude that

$$p-2 \le F^{q}_{\alpha,\varepsilon}(x) - x \le p+2 \qquad \forall x \in \mathbb{R}$$

Now choose α' so close to α such that

$$p-3 \le F^{q}_{\alpha,\varepsilon'}(x) - x \le p+3 \qquad \forall x \in \mathbb{R}$$

Therefore $\rho(f_{\alpha',\varepsilon}) \in \left[\frac{p-3}{q}, \frac{p+3}{q}\right]$ and hence

$$|\rho(f_{\alpha',\varepsilon}) - \rho(f_{\alpha,\varepsilon})| \le \frac{5}{q}$$

This is the continuity of $\alpha \mapsto \rho(f_{\alpha,\varepsilon})$.

b) If $\alpha' > \alpha$, then $F_{\alpha',\varepsilon}(x) > F_{\alpha,\varepsilon}(x) \quad \forall x \in \mathbb{R}$

monotonicity

and

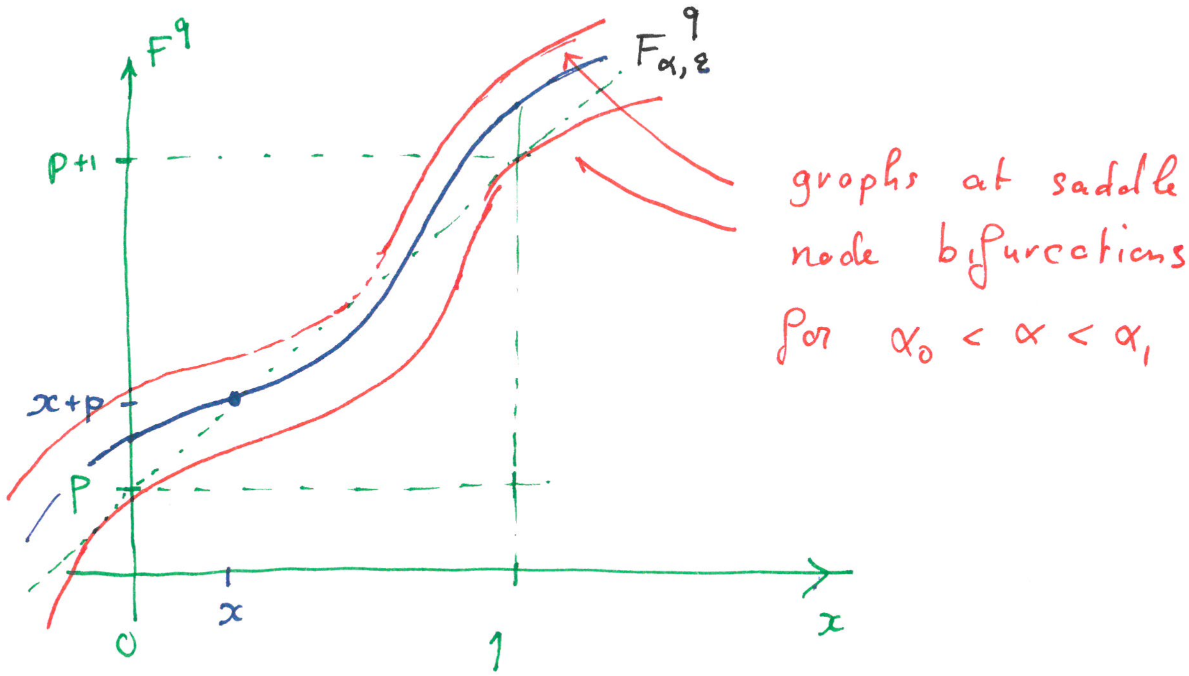
$$\frac{F^{n}_{\alpha',\varepsilon}(x) - x}{n} > \frac{F^{n}_{\alpha,\varepsilon}(x) - x}{n} \qquad \forall x \in \mathbb{R}\ \forall n \in \mathbb{N}.$$

Therefore in the limit $\rho(f_{\alpha',\varepsilon}) \ge \rho(f_{\alpha,\varepsilon})$,

the monotonicity (but not strict monotonicity!)

c) If $f_{\alpha,\varepsilon}$ has a periodic point x of period q, then

$$F^q_{\alpha,\varepsilon}(x) = p + x \qquad \text{for some } p \in \mathbb{Z}$$



By changing α, i.e. moving $F^q_{\alpha,\varepsilon}$ up and down, the "fixed" point x disappears in saddle node bifurcations, at parameters α_0 and α_1. Because the graph of $F^q_{\alpha,\varepsilon}$ is non-linear, $\alpha_0 < \alpha_1$ and in between, $F^q_{\alpha,\varepsilon}$ has a fixed point, so $\rho(f_{\alpha',\varepsilon}) \equiv \frac{p}{q}$ for all $\alpha' \in [\alpha_0, \alpha_1]$ ▨