William Rowan Hamilton
Irish mathematician
1805-1865

Hamiltonian Dynamics

in general coordinates q = position
p = usually momentum

Hamiltonian equations

gradient

$$\begin{pmatrix} \dot q \\ \dot p \end{pmatrix} = X_H \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix} = \underbrace{\begin{pmatrix} \nabla_p H \\ -\nabla_q H \end{pmatrix}}_{\text{vector notation}}$$

Hamiltonian vector field

for the Hamiltonian function $H : \mathbb{R}^{2n} \to \mathbb{R}$.

Throughout we assume that H is C^2- smooth

H is preserved along orbits:

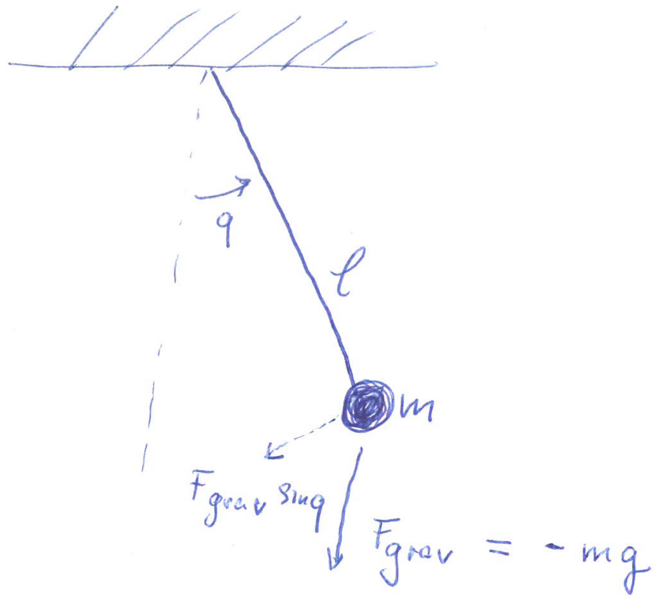
• stands for the inner product

$$\dot H = \nabla_q H \bullet \dot q + \nabla_p H \bullet \dot q$$

$$= \nabla_q H \, \nabla_p H - \nabla_p H \, \nabla_q H = 0$$

Usually H represents the total energy,
but other preserved quantities (momentum,
angular momentum, ...) are possible too.

Example of pendulum



Newton equation of motion

$$\boxed{\ddot{q} + \frac{g}{\ell} \sin q = 0}$$

$g \approx 9.8 \frac{m}{sec^2}$ = gravitational constant

q = angle

$p = m\ell^2 \dot{q}$ ← See page 8 for the reason of this choice of p. Roughly, since $q = \text{position}/\ell$, "momentum" p has extra factor ℓ.

$$E_{kin} = \frac{m(\ell\dot{q})^2}{2} = \frac{p^2}{2m\ell^2} \qquad E_{pot} = -mg\ell \cos q$$

$$H(p, q) = E_{kin} + E_{pot} = \frac{p^2}{2m\ell^2} - mg\ell \cos q$$

$$\frac{\partial H}{\partial p} = \frac{p}{m\ell^2} = \dot{q}$$

$$-\frac{\partial H}{\partial q} = -mg\ell \sin q \overset{\text{Newton eq.}}{=} \ell^2 m \ddot{q} = \dot{p}$$

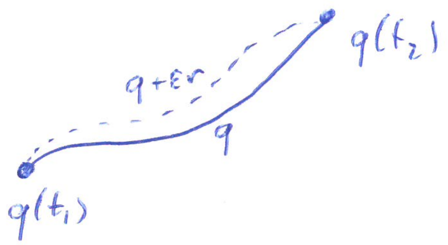
Joseph-Louis Lagrange
Italian mathematician
1736-1813, worked
and died in Paris

Lagrangian Dynamics

in general coordinates $(v, q) \in \mathbb{R}^{2n}$ $\quad q$ position, $\quad v = \dot{q}$

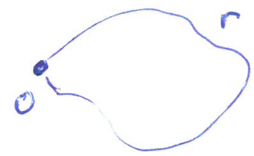
and Lagrangian $L : \mathbb{R}^{2n} \to \mathbb{R}$ twice differentiable

The task is to minimize the action of the Lagrangian over all C^1-paths, i.e.,



minimize

$$I(q) = \int_{t_1}^{t_2} L(\dot{q}(t), q(t))\, dt$$



Fix $r(t_1) = r(t_2) = 0$ $\quad$ ($q(t_1)$ and $q(t_2)$ are fixed!)

and compute the Gateaux derivative

$$\frac{d}{d\varepsilon} I(q+\varepsilon r)\Big|_{\varepsilon=0} = \int_{t_1}^{t_2} \nabla_q L(\dot{q}+\varepsilon\dot{r}, q+\varepsilon r) \circ r + \nabla_v L(\dot{q}+\varepsilon\dot{r}, q+\varepsilon r) \circ \dot{r}\, dt\Big|_{\varepsilon=0}$$

$$\underset{\text{by parts}}{\overset{\text{Integrate 2nd term}}{=}} \int_{t_1}^{t_2} \left(\nabla_q L(\dot{q}, q) - \frac{d}{dt} \nabla_v L(\dot{q}, q) \right) \circ r\, dt$$

NB: constant terms disappear because $r(t_1) = r(t_2) = 0$.

For $\frac{d}{d\varepsilon} I(q+\varepsilon r)\Big|_{\varepsilon=0} = 0$ for all paths r we need

Euler-Lagrange equation

$$\boxed{\nabla_q L(\dot{q}, q) = \frac{d}{dt} \nabla_v L(\dot{q}, q)}$$

Example pendulum continued

Lagrangian $L(v, q) = L(\dot{q}, q) = E_{kin} - E_{pot}$

$$= \frac{m \ell^2 \dot{q}^2}{2} + mg\ell \cos q$$

Euler-Lagrange:

$$0 = \nabla_q L - \frac{d}{dt} \nabla_v L = -mg\ell \sin q - \frac{d}{dt} m \ell^2 \dot{q}$$

$$\Longleftrightarrow \quad 0 = \frac{g}{\ell} \sin q + \ddot{q}$$

which is Newton's equation of motion.

How are the Lagrangian and the Hamiltonian related?

$L = E_{kin} - E_{pot}$ and $H = E_{kin} + E_{pot}$

is a too simple view.

Legendre Transform

Adrien-Marie Legendre
French mathematician
1752 – 1833

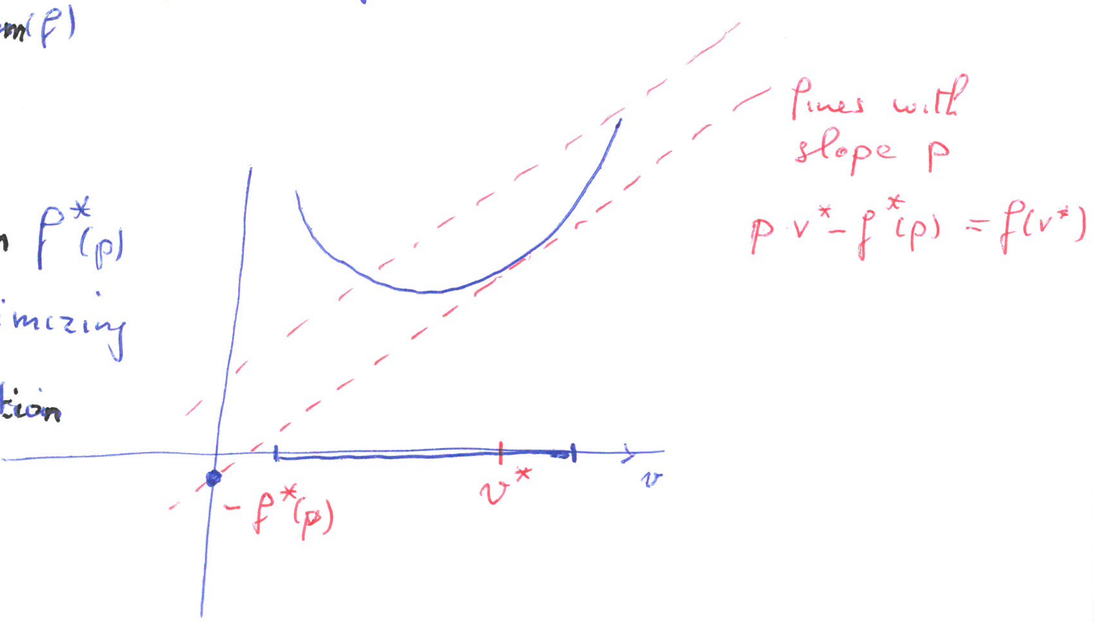
Let $f : \mathrm{Dom}(f) \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be C^2 & strictly convex.

i.e. $\nabla^2 f = \Delta f$ is positive definite.

Define the Legendre transform

$$f^*(p) = \sup_{v \in \mathrm{Dom}(f)} \; p \cdot v - f(v)$$

The supremum in $f^*(p)$ translates to minimizing $-f^*(p)$ = intersection of lines of slope p with vertical axis, provided the line intersects the graph of f.



This is achieved at the tangent line of slope p.

Let $v^*(p)$ the value where the supremum is assumed. Then

$$\begin{cases} p = \nabla_v f(v^*(p)) & (*) \\[2ex] f^*(p) = p \cdot v^*(p) - f(v^*(p)) & (**) \end{cases}$$

by inserting $v^*(p)$ in def of Legendre transform

Thm a) $f^{**} = f$

b) The Legendre transform of a C^2 strictly convex function is C^2 and strictly convex.

Proof a)

$$\nabla_p f^*(p) \overset{\circledast\circledast}{=} \nabla_p \left(p \bullet v^*(p) - f(v^*(p)) \right)$$

$$= v^*(p) + p \bullet \nabla_p v^*(p) - \nabla_v f \bullet \nabla_p v^*(p)$$

$= \nabla_v f(v^*(p))$ by $\circledast$

$$= v^*(p),$$

so $\circledast$ holds for f^* instead of f. But then $\circledast\circledast$ follows as well, and hence $f^{**} = f$.

(1) so $\nabla_p f^*(p) = v$, $\nabla_v f(v) = p$ so $\nabla_v f$ and $\nabla_p f^*$ are each others inverse

Now f is C^2 & strictly convex

iff the Jacobian $J(\nabla f)$ is positive definite

iff $J(\nabla f)^{-1} = J(\nabla f^*)$ is positive definite

∇f and ∇f^* are each others inverse by (1)

iff f^* is C^2-smooth and strictly convex.

Recall, a matrix is (strictly) positive definite if $Av \bullet v > 0 \quad \forall v \in \mathbb{R}^n \setminus \{0\}$

$\boxed{/\!/}$

<u>Thm</u> Let (for fixed q) $p = \nabla_v L(v^*, q)$ and

$$H(p,q) = L^*(v,q) = p \cdot v^* - L(v^*, q)$$

That is, the Hamiltonian is the Legendre transform of the Lagrangian.
Then the Euler-Lagrange equations are equivalent to the Hamiltonian equations.

<u>Proof</u> Since also $L = H^*$, using ⊛ with $f = H$ and v and p swapped, we find

$$\dot{q} = v = \nabla_p H. \quad \leftarrow$$ one of the Hamiltonian equations

Also

$$\nabla_q H = \nabla_q \left(p \cdot v^*(p,q) - L(v^*(p,q), q) \right)$$

$$= p \cdot \nabla_q v^* - \nabla_v L \cdot \nabla_q v^* - \nabla_q L$$

$$\overset{\circledast \text{ with } f=L}{=} p \cdot \nabla_q v^* - p \cdot \nabla_q v^* - \nabla_q L$$

$$= -\nabla_q L$$

$$\overset{\text{Euler-Lagrange}}{=} -\frac{d}{dt} \nabla_v L$$

$$\overset{\circledast \text{ with } f=L}{=} -\frac{d}{dt} p = -\dot{p}$$

the other Hamiltonian equation ▨

Example pendulum continued

Take $v = \dot{q}$ and $L(v, q) = \frac{m\ell^2 v^2}{2} + mg\ell \cos q$

Take the Legendre transform to obtain the Hamiltonian:

$$H(p, q) = \sup_{v \in \mathbb{R}} p \cdot v - \left(\frac{m\ell^2 v^2}{2} + mg\ell \cos q \right).$$

Set derivative $\frac{d}{dv} = 0$: $\quad p - m\ell^2 v = 0 \Leftrightarrow v = \frac{p}{m\ell^2}$

Insert $H(p, q) = \frac{p^2}{m\ell^2} - \frac{m\ell^2 p^2}{2m^2\ell^4} - mg\ell \cos q$

$$= \frac{p^2}{2m\ell^2} - mg\ell \cos q .$$

$$= E_{kin} + E_{pot} .$$

Note that we get from this computation that:

$$\boxed{p = m\ell^2 v = m\ell^2 \dot{q}}$$

Same example as before, but now with Einstein's formula of kinetic energy:

$$E_{kin} = m(v)\,c^2 \qquad m(v) = m_0 \sqrt{1 + \frac{|v|^2}{c^2}}$$

$$E_{pot} = E_{pot}(q)$$

m_0 ↑ rest mass, c ↑ speed of light

Lagrangian $\quad L(v,q) = E_{kin}(v) - E_{pot}(q)$

Euler-Lagrange $\quad \frac{d}{dq} E_{pot}(q) = \frac{d}{dt} \frac{m_0 c\, \dot{q}}{\sqrt{c^2 + |\dot{q}|^2}}, \quad \dot{q} = v$

Hamiltonian via Legendre transform:

$$H(p,q) = L^*(v,q) = \sup_{v \in \mathbb{R}} v \cdot p - \left(m_0 c^2 \sqrt{1 + \frac{|v|^2}{c^2}} - E_{pot}(q) \right)$$

Solve $\frac{d}{dv} = 0$: $\quad p - \frac{m_0 c\, v}{\sqrt{c^2 + |v|^2}} = 0$

gives $\quad |p|^2 (c^2 + |v|^2) = m_0^2 c^2 |v|^2 \iff |v|^2 = \frac{c^2 |p|^2}{c^2 m_0^2 - |p|^2}.$

Insert in $H(p,q)$:

$$H(p,q) = \frac{c\, p^2}{\sqrt{c^2 m_0^2 - p^2}} - m_0 c^2 \sqrt{1 + \frac{p^2}{c^2 m_0^2 - p^2}} + E_{pot}(q)$$

$$= -c\sqrt{c^2 m_0^2 - p^2} + E_{pot}(q).$$

Thm The Hamiltonian flow preserves volume in $\mathbb{R}^{2n}$.

Remark This "Volume" is sometimes called Liouville measure in this context.

Joseph Liouville, French mathematician 1809-1882

Proof Recall $X_H = \begin{pmatrix} \nabla_p H \\ -\nabla_q H \end{pmatrix}$, the Hamiltonian vector field. Let $V(t) = \int_{\varphi^t(\Omega)} dVol$, φ^t_H Hamiltonian flow.

Earlier (Thm 19 in the Schmeiser notes) we have seen

divergence $\operatorname{div} X_H = \sum_{k=1}^{n} \frac{\partial (X_H)_k}{\partial x_k}$

$$\dot{V}(t) = \int_{\varphi^t_H(\Omega)} \operatorname{div} X_H \, dVol,$$

but $\operatorname{div} X_H = \sum_i \frac{\partial}{\partial q_i}\left(\frac{\partial}{\partial p_i} H\right) + \sum_i \frac{\partial}{\partial p_i}\left(-\frac{\partial}{\partial q_i} H\right) = 0.$

Therefore $\dot{V}(t) = 0$, $V(t)$ is constant $\blacksquare$