Recall that if we try to solve

$$\dot{x} = ax(1-x) \qquad x(0) = x_0$$

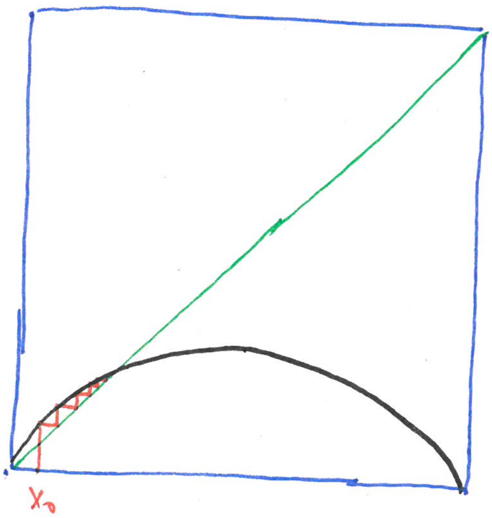
numerically (Euler's method), then we actually iterate a quadratic map:

$$T(y) = y(1 + ah - ahy) \qquad h = \text{stepsize}$$

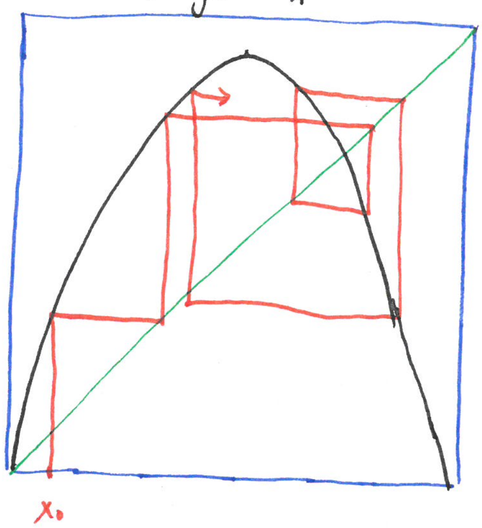
which is conjugate to:

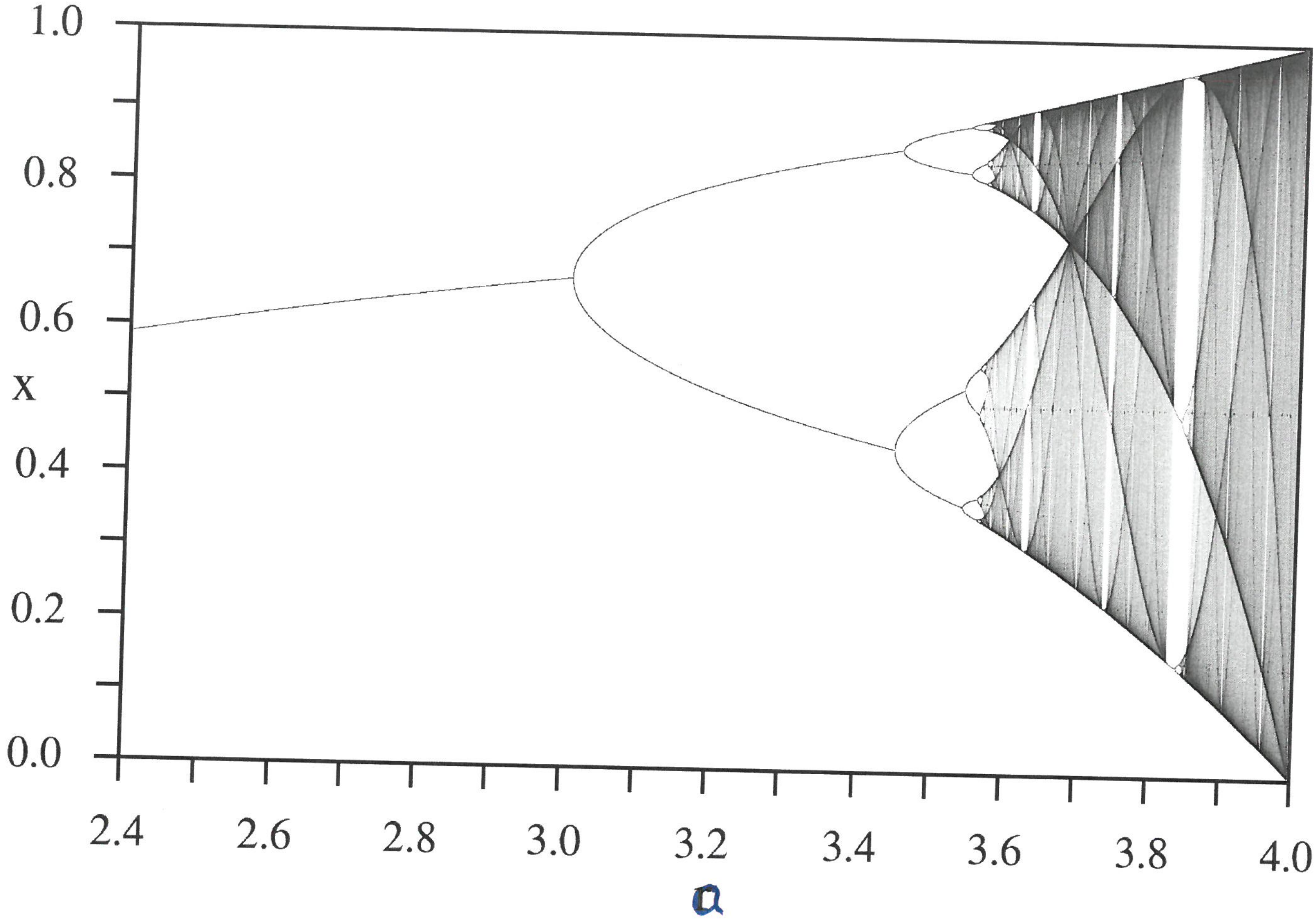
$$Q(x) = (1+ah)\,x(1-x)$$

small h



large h

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x
0.4
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0.0
2.4
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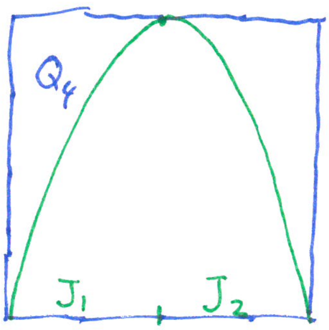
3.6
3.8
4.0
a

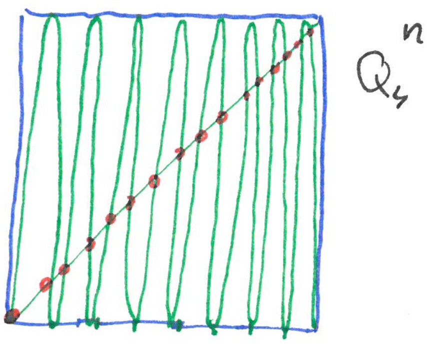
Lemma The "full" quadratic map

$$Q_4(x) = 4x(1-x)$$

has 2^n periodic points of (not necessarily smallest) period n.

Proof: Q_4 itself has two branches that are "onto": we can partition $[0,1]$ into two intervals $J_1 \cup J_2$ such that $J_1 \cap J_2$ is at most one point and $Q_4 : J_i \to [0,1]$ is monotone onto





By induction, Q_4^n has 2^n onto branches. Each such branch intersects the diagonal once, giving one fixed point of Q_4^n, i.e. one periodic n point of Q_4 ☐

We consider the quadratic family

$$Q_a : [0,1] \to [0,1]$$
$$x \longmapsto ax(1-x)$$

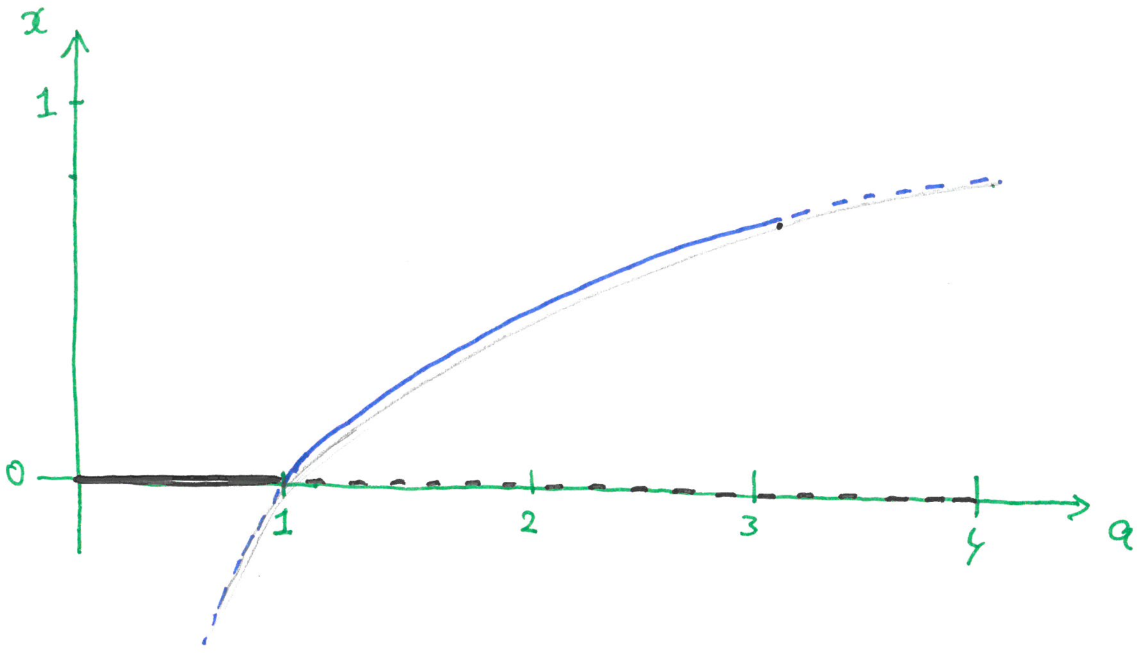
We want to examine its periodic points and their stability, so its helps to also have the derivative

$$Q_a'(x) = a(1-2x)$$

1. Fixed points:

$$Q_a(x) = x \iff Q_a(x) - x = 0$$
$$\iff ax(1-x) - x = 0$$
$$\iff x(ax + a - 1) = 0$$
$$\iff x = 0 \ \vee \ x = 1 - \frac{1}{a}$$

$$Q_a'(0) = a, \qquad Q_a'(1-\tfrac{1}{a}) = 2 - a$$



- 0 is always a fixed point and it is stable for $a \in [0, 1]$

- $1 - \frac{1}{a}$ is a fixed point for $a \in [1, 4]$ and it is stable for $a \in [1, 3]$.

2. Period two points

$$Q_a^2(x) = x \iff Q_a(Q_a(x)) - x = 0$$

Fixed points are also points of period 2, so $Q_a(x) - x$ must be a proper divisor.

Tedious computation gives:

$$Q_a(Q_a(x)) - x = \underbrace{\left(Q_a(x) - x\right)}_{\text{2 fixed points}} \cdot \underbrace{\left(a^2x^2 - (a^2+a)x + a + 1\right)}_{\text{1 period 2 orbit}}$$

real iff $a \geq 3$

$$a^2x^2 - \underbrace{(a^2+a)}_{a^2(x_1+x_2)}x + \underbrace{a+1}_{a^2x_1 \cdot x_2} = 0 \Rightarrow x_{1,2} = \frac{a+1 \pm \sqrt{a^2-2a-3}}{2a}$$

By the chain rule and the above

$$(Q_a^2)'(x_1) = (Q_a^2)'(x_2) = a(1-2x_1) \cdot a(1-2x_2)$$

$$= a^2(1 - 2(x_1+x_2) + 4x_1x_2)$$

$$= a^2 - 2(a^2+a) + 4(a+1)$$

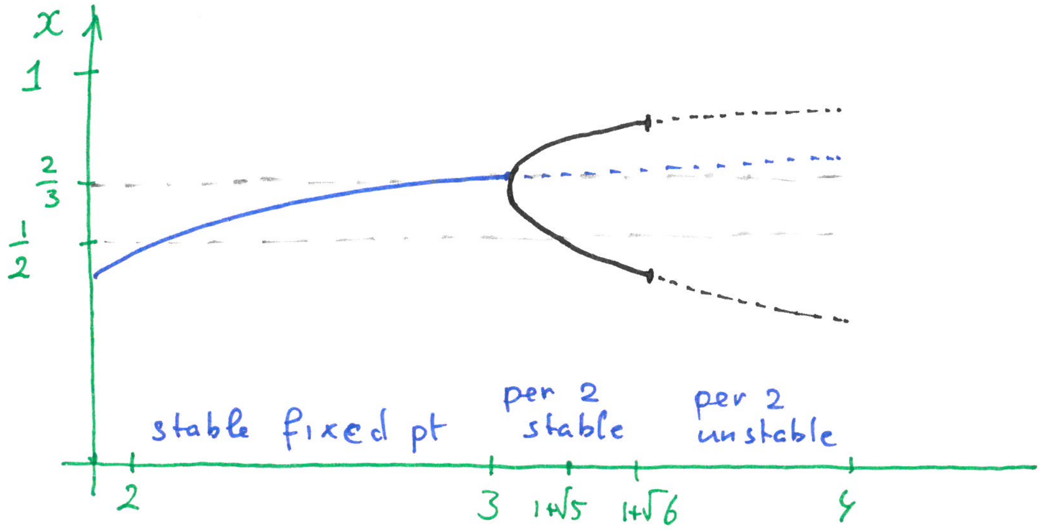
$$= -a^2 + 2a + 4$$

The period 2 orbit is stable if

$$|(Q_a^2)'(x_{1,2})| \leq 1 \iff |-a^2+2a+4| \leq 1 \wedge a \geq 3$$

$$\iff a \in [3, 1+\sqrt{6}]$$

Note: $(Q_a^2)'(x_2) = 0 \iff a = 1+\sqrt{5}$ and then $x_2 = \frac{1}{2}$

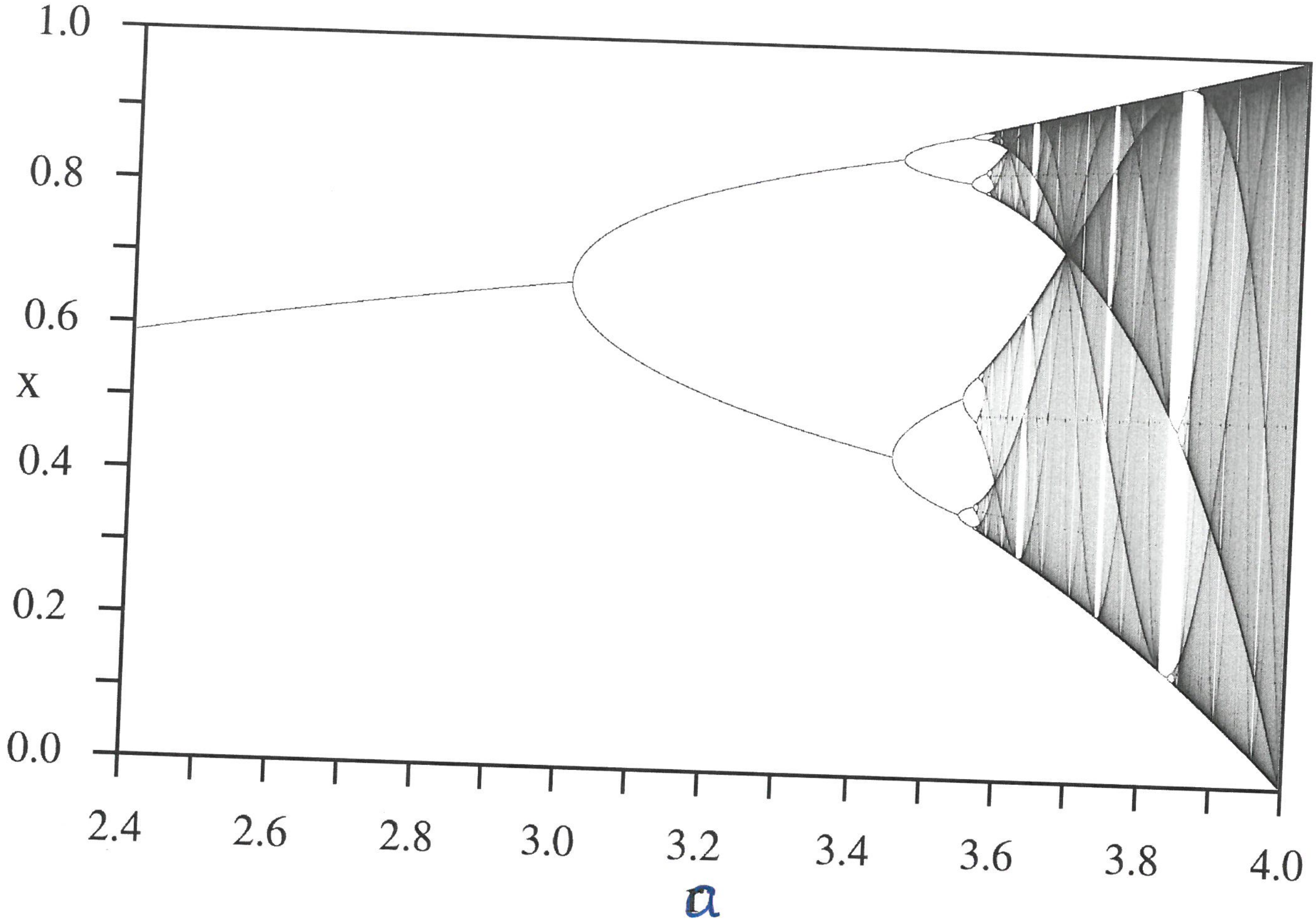


Periodic points of period ≥ 3 are too complicated to find explicitly. (Note: the degree of the polynomial equation is 2^{period})

However, there are some constraints that help you locating them:

i) For polynomial maps f, every stable\neutral periodic orbit has to <u>attract</u> a critical point (i.e. a point $c \in \mathbb{C}$ where $f'(c) = 0$) Quadratic maps have only one critical point, and hence at most one stable orbit at the time.

ii) Periodic points emerge in a <u>bifurcation</u>. The Implicit Function Theorem dictates that at emergence, the multiplier of p: $Df^{\text{period}}(p) = 1$, and then a critical point is attracted. Hence, for the quadratic family, at most one new periodic orbit can emerge at any time (= parameter value).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4.0
a