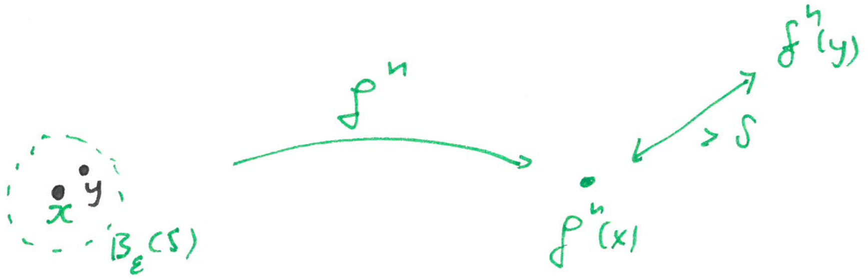
Mathematical Chaos

- Sensitive dependence on initial conditions
- Lyapunov exponents
- Mathematical chaos
  - in the sense of Devaney
  - in the sense of Auslander-Yorke
  - in the sense of Li-Yorke
- Topologically transitive

The importance of "sensitive dependence" was realized by the US meteorologist Edward Lorenz in the 1960's.
Henri Poincaré certainly knew about it from his work on celestial mechanics.

The idea is that we always make measuring/round-off/model errors when computing orbit, but that, regardless how small, they tend to blow up to serious size, and thus make the system unpredictable

Def A map $f: X \to X$ on a metric space (X, d) has sensitive dependence on initial conditions if $\exists \delta > 0 \; \forall \varepsilon > 0 \; \forall x \in X \; \exists y \in B_\varepsilon(x) \; \exists n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) > \delta$



Example $f: S^1 \to S^1$, $x \mapsto 2x \mod 1$ is the doubling map. We show it has sensitive dependence.

Take $\delta = \frac{1}{4}$ and $x \in S^1$, $\varepsilon > 0$ arbitrary.

Take $y = x + \frac{\varepsilon}{2} \in B_\varepsilon(x)$,

Then $d(f^n(x), f^n(y)) = 2^n \cdot d(x,y) = 2^{n-1} \varepsilon$

Euclidean metric

Take first n such that $2^{n-1} \varepsilon > \delta$

To quantify the speed at which error blow up, we define Lyapunov exponents.

Def Let $f: X \to X$ be a differentiable map. The Lyapunov exponent of $x \in X$ is

$$\lambda(x) = \lim_{n \to \infty} \frac{1}{n} \log |(f^n)'(x)|$$

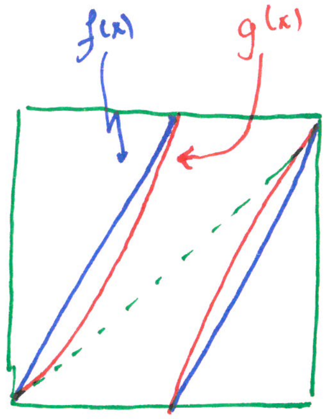
Then by Taylor expansion

$$\underbrace{|f^n(x) - f^n(y)|}_{\text{error after } n \text{ iterations}} \approx |(f^n)'(x)| \underbrace{|x-y|}_{\text{initial error}} \approx \underbrace{e^{\lambda(x)\cdot n}}_{\text{error blows up exponentially for positive } \lambda(x).} |x-y|$$

Remarks

- NB $\lambda(x)$ need <u>not</u> exist as a limit.
- For higher dimension maps, $x \in X$ tends to have several Lyapunov exponents, depending on direction.
- $\lambda(x) > 0 \ \forall x \in X$ seems to imply sensitive dependence on initial conditions. However, there are rather construed counter-examples.

Examples.



$$\boxed{f(x) = 2x \mod 1} \quad \text{on } S^1$$

Every $x \in S^1$ has $\lambda(x) = \log 2$ for f.

$$\boxed{g(x) = 2x - \frac{1}{2\pi}\sin 2\pi x \mod 1} \quad \text{on } S^1$$

- $\lambda(0) = \lim_n \frac{1}{n} \log |\underbrace{(g^n)'(0)}_{1^n}| = 0$

- Points that stay mostly away from 0 have positive Lyapunov exponents

- Points that stay mostly very close to 0 have Lyapunov exponent $= 0$

- Points that oscillate too much between "very close to 0" and "away from 0" may not have a Lyapunov exponent: the limit does not exist.

However, g has sensitive dependence on initial conditions.

Measures of mathematical chaos

- sensitive dependence on initial conditions
- all Lyapunov exponent are positive
  (too strong a condition)
- positive entropy (= positive exponential growth rate of "essentially different" orbits.)
- chaos in the sense of Devaney
  - (i) sensitive dependence
  - (ii) $\exists$ dense orbit
  - (iii) dense set of periodic orbits
- chaos in the sense of Auslander-Yorke
  - (i) sensitive dependence
  - (ii) $\exists$ dense orbit.
- chaos in the sense of Li-Yorke
- -- many more --

Remarks 1) The map $T: \mathbb{R} \to \mathbb{R}$, $x \mapsto 2x$
has only positive Lyapunov exponents, sensitive dependence, yet is quite predictable: every non-fixed point tends to $\pm\infty$ under iteration. Not chaotic.

2) In "Devaney chaos" (ii) & (iii) $\Rightarrow$ (i)
if the space is more than a single periodic orbit.

3) Consider the rotation $\boxed{R_\alpha : S^1 \to S^1, \quad x \mapsto x+\alpha \mod 1}$

If $\alpha = \frac{p}{q} \in \mathbb{Q}$ then every orbit is periodic.
If $\alpha \notin \mathbb{Q}$ then every orbit is dense.
No sensitive dependence, Lyapunov exponents $= 0$,
No chaos

4) Define $f: \mathbb{T}^2 \to \mathbb{T}^2$

$$(x,y) \mapsto (x+\alpha, 2y) \mod 1$$

If $\alpha \notin \mathbb{Q}$ then f is chaotic in the sense of Auslander-Yorke but not in the sense of Devaney.

Let us continue our example of the doubling map $\boxed{f(x) = 2x \mod 1}$

- Periodic orbits $\quad f^n(x) = x \iff 2^n x \equiv x \mod 1$

$$\iff (2^n - 1)x = p \in \mathbb{Z}$$

$$\iff x = \frac{p}{2^n - 1} \qquad n \in \mathbb{N}, \; 0 \leq p < 2^n$$

Therefore the set of periodic points is dense.

- Dense orbit $\quad$ How to prove that f has a dense orbit?

Def $\quad$ Let $f: X \to X$ be a map on a metric space (X, d). If $\forall\, U, V \subset X$ open there is $n \in \mathbb{N}$ such that $U \cap T^{-n}(V) \neq \emptyset$ then f is called topologically transitive

- For the doubling map, take $\mathcal{U} \subset S^1$ open and take $J = [\frac{p}{2^n}, \frac{p+1}{2^n}) \subset \mathcal{U}$.

Then $f^n(J) = S^1$, so certainly $U \cap f^{-n}(V) \neq \emptyset$.

The doubling map is topologically transitive.

Theorem Let $T: X \to X$ be a continuous map on a 2^{nd} countable locally compact metric space (X,d) without isolated points. Then T is topologically transitive if and only if T has a dense orbit.

Proof $\Leftarrow$ Suppose $orb(x)$ is dense in X.

Let $\mathcal{U}$ and V be arbitrary open subsets of X.

Thus there are $i < j \in \mathbb{N}$ s.t. $T^i(x) \in \mathcal{U}$, $T^j(x) \in V$

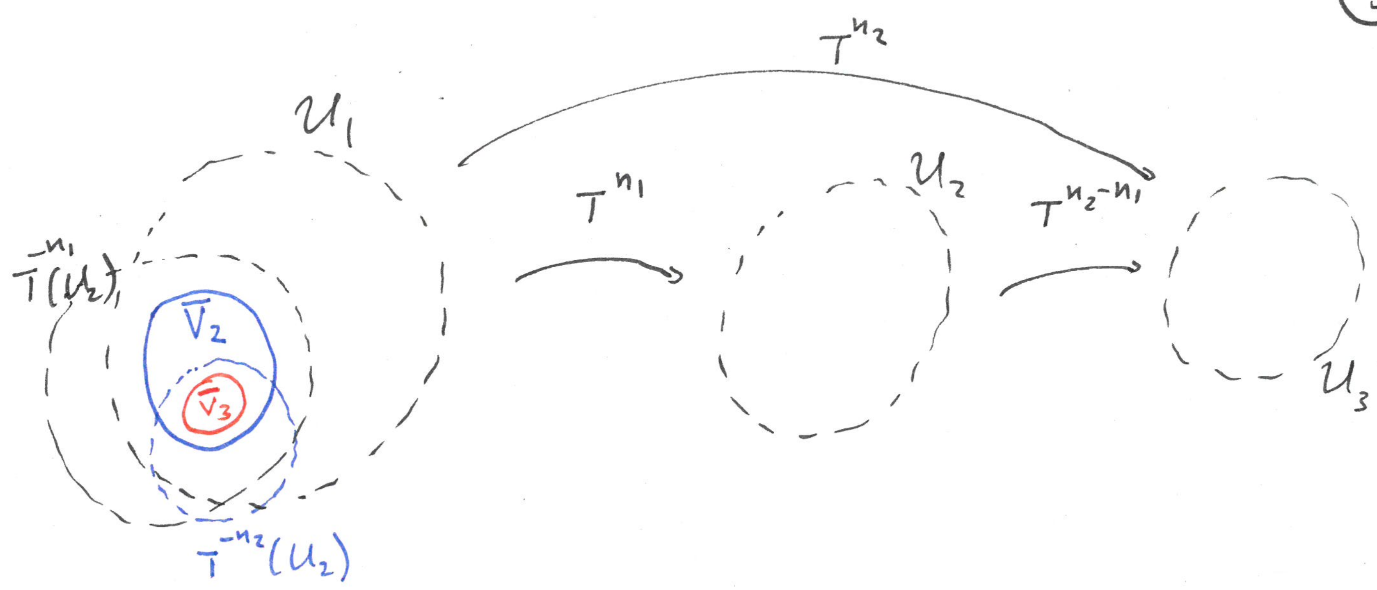
Take $j - i =: n$. Then $\mathcal{U} \cap T^{-n}(V) \neq \phi$ (it contains $T^i(x)$)

you can choose $i<j$ because X has no isolated pts.

$\Rightarrow$ A metric space has a countable basis of the topology, i.e. $\exists \{\mathcal{U}_j\}_{j\in\mathbb{N}}$ of open sets, such that every open set $\mathcal{U}$ can be written as $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_{j_n}$ for some subcollection $\{\mathcal{U}_{j_n}\}_{n\in\mathbb{N}}$.

2^{nd} countable is used here

Assume that T is topologically transitive



Find n_1 such that $\underbrace{\mathcal{U}_1}_{\text{open}} \cap \underbrace{T^{-n_1}(\mathcal{U}_2)}_{\text{open}} \neq \emptyset$

Find V_2 open such that $\overline{V}_2 \subset \mathcal{U}_1 \cap T^{-n_1}(\mathcal{U}_2)$

Find $n_2 > n_1$ such that $\underbrace{V_2}_{\text{open}} \cap \underbrace{T^{-n_2}(\mathcal{U}_3)}_{\text{open}} \neq \emptyset$

Find V_3 open such that $\overline{V}_3 \subset V_2 \cap T^{-n_2}(\mathcal{U}_3)$

Continue this procedure for all $j \in \mathbb{N}$.

Local compactness used here

Now we have a nested sequence

$$\overline{V}_2 \supset \overline{V}_3 \supset \overline{V}_4 \supset \cdots \supset \bigcap_k \overline{V}_k \neq \emptyset$$

For every $x \in \bigcap_k \overline{V}_k$, $T^{n_j}(x) \in \mathcal{U}_{j+1}$,

so x has a dense orbit. $\blacksquare$