The Lotka-Volterra Model

- Predator-Prey model from biology
- ODE in 2D
- Analysis of stationary points
- Nullclines to aid drawing the phase portrait
- Lyapunov functions to prove (Lyapunov/asymptotic) stability

(*) prey $\dot{x} = x(1-y)$, predator $\dot{y} = \alpha(x-1)y$

$$\begin{aligned} \dot{x} &= x(1-y) \\ \dot{y} &= \alpha(x-1)y \end{aligned} = F\begin{pmatrix} x \\ y \end{pmatrix}, \quad \alpha > 0$$

Only $x, y \geq 0$ are of physical relevance.

More general form
$$\begin{aligned} \dot{x} &= (A - By)x \\ \dot{y} &= (Cx - D)y \end{aligned} \qquad A, B, C, D > 0$$

can be reduced to (*) via a change of coordinates

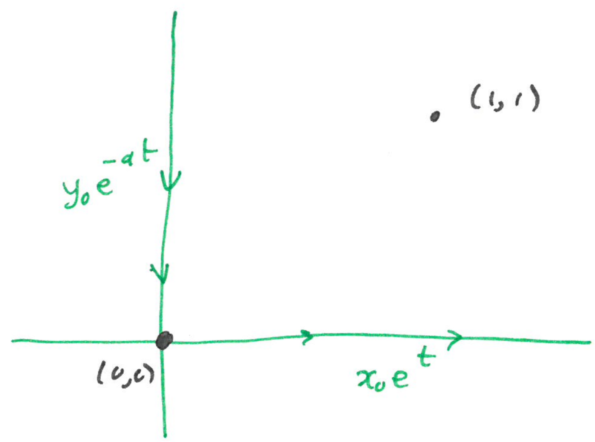
To find stationary points (equilibria)

set $F\begin{pmatrix} x \\ y \end{pmatrix} = 0$

(i) $x = 0 \Rightarrow y = 0$

(ii) $y = 1 \Rightarrow x = 1$

No others.



$$DF_{(x,y)} = \begin{pmatrix} 1-y & -x \\ \alpha y & \alpha(x-1) \end{pmatrix}$$

(i) $DF_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix}$ Eigenvalues $-\alpha < 0 < 1$

Saddle.

Note $x = 0 \Rightarrow \begin{cases} \dot{x} = 0 \Rightarrow x(t) \equiv 0 \quad \forall t \\ \dot{y} = -\alpha y \Rightarrow y(t) = y_0 e^{-\alpha t} \end{cases}$

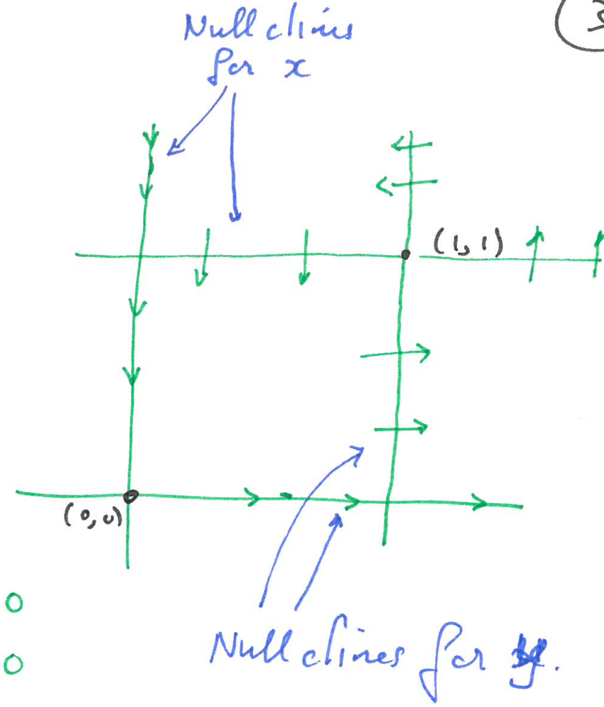
$y = 0 \Rightarrow \begin{cases} \dot{y} = 0 \Rightarrow y(t) \equiv 0 \quad \forall t \\ \dot{x} = x \Rightarrow x(t) = x_0 e^{t} \end{cases}$

(ii) $DF_{(1,1)} = \begin{pmatrix} 0 & -1 \\ \alpha & 0 \end{pmatrix}$ Eigenvalues $\lambda = \pm i\sqrt{\alpha}$

Center

If the ODE was linear, then periodic solutions around (1,1), but in the <u>nonlinear</u> case we don't know yet.

Def A <u>nullcline</u> is a region where one component of a vector field F is $=$ zero.



Here: $x=0 \vee y=1$ for $F_1 = 0$
$y=0 \vee x=1$ for $F_2=0$

This suggest that solutions turn counterclockwise in the 1st quadrant, but we don't know yet if they spiral inwards to $(1,1)$, or outwards away from $(1,1)$, or are periodic.

From
$$\dot{x} = x(1-y)$$
$$\dot{y} = \alpha(x-1)y \qquad (*)$$

try to eliminate time t, and set $y = y(x)$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\dot{y}}{\dot{x}} = \alpha \frac{x-1}{x} \cdot \frac{y}{1-y}$$

Separate variables

$$\frac{1-y}{y}\dot{y} = \alpha \frac{x-1}{x}\dot{x}$$

Integrate

$$\log y - y + \text{Const} = \alpha(x - \log x)$$

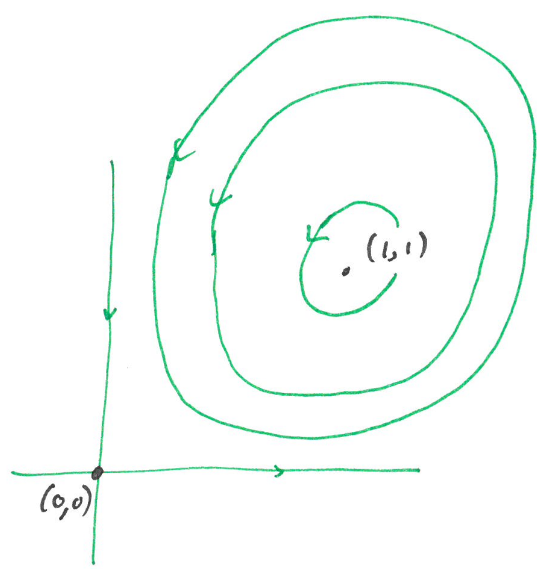
Introduce $f(u) = u - 1 - \log u$

$f(1) = 0$
$f'(s) = 1 - \frac{1}{s} = 0$ at $s = 1$
1 is global minimum of f.

$$L(x,y) = \alpha f(x) + f(y) = \text{Const.}$$

So L is a preserved quantity

Solutions of $(*)$ need to remain on level curves of L



Most solutions are therefore periodic.

Lie derivative

Compute

$$\frac{d}{dt} L(x,y) = \begin{pmatrix} \frac{\partial L}{\partial x} \\ \frac{\partial L}{\partial y} \end{pmatrix} \bullet \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \nabla L \bullet F$$

(∇L: gradient of L; $\bullet$: dot-product; F: vector field)

$$= \begin{pmatrix} \alpha\left(1-\frac{1}{x}\right) \\ 1-\frac{1}{y} \end{pmatrix} \bullet \begin{pmatrix} x(1-y) \\ \alpha(x-1)y \end{pmatrix}$$

$$= \alpha(x-1)(1-y) + (y-1)\cdot\alpha(x-1) = 0.$$

Def A function $L: \mathcal{U} \to \mathbb{R}$ on a nbh of a stationary point p is called Lyapunov function if

1) $L(p) = 0$, $L(x) > 0$ for $x \in \mathcal{U} \setminus \{p\}$

2) $L(\varphi^{t_1}(x)) \leq L(\varphi^{t_0}(x))$ $\forall x \in \mathcal{U}$ $\forall t_1 > t_0$

(if "$<$" for $\forall x \in \mathcal{U} \setminus \{p\}$ $\forall t_1 > t_0$, then it is a strict Lyapunov function)

Remark

If $L(p) = 0$ and the Lie derivative of L is zero on a nbh $\mathcal{U}$ of p, then L is a Lyapunov function.

If the Lie derivative is < 0 on $\mathcal{U} \setminus \{p\}$ the L is a strict Lyapunov function

Theorem a) If an equilibrium p has a Lyapunov function, then p is Lyapunov stable

b) If the Lyapunov function is strict, then p is asymptotically stable.

Remark i) We give the proof for when Lyapunov function L has non-positive / negative Lie derivative.

ii) Recall

Lyapunov stable : $\forall \varepsilon > 0 \ \exists \delta > 0 \quad z(0) \in B_\delta(p) \Rightarrow$
$z(t) \in B_\varepsilon(p) \quad \forall t \geq 0$

Asymptotically stable $\exists \delta > 0 \quad z(0) \in B_\delta(p) \Rightarrow$
$z(t) \to p$ as $t \to \infty$.

Proof a) Since $\frac{d}{dt} L(z(t)) \leq 0$,
we have $L(z(t_1)) \leq L(z(t_0))$ $\forall t_1 \geq t_0$.
L has a local minimum at p, so
$\forall \varepsilon > 0$ $\exists \eta > 0$ such that $\{z : L(z) < \eta\} \subset B_\varepsilon(p)$,
and $\exists \delta > 0$ such that $B_\delta(p) \subset \{z : L(z) < \eta\}$.
Therefore $z(t) \in B_\varepsilon(p)$ for all $z(0) \in B_\delta(p)$.
This is Lyapunov stability.

b) Let $z(0) \in B_\delta(p)$ as before.
Take $\eta > 0$ so small that

$$V = \overline{\{ w \in B_\delta(p) : L(w) \leq \eta \}} \subset B_\delta(p)$$

Then V is invariant under the flow and compact.
Since the Lie derivative < 0 on $B_\delta(p) \setminus \{p\}$,

$$\overline{\{ w(t) : t \geq 1 \}} \not\ni w(0) \qquad \forall w(0) \in B_\delta(p) \setminus \{p\}$$

(because L is continuous). In fact $\exists W \ni w(0)$
such that $\varphi^t(W) \cap W = \emptyset \quad \forall t \geq 1$ (*)

By compactness of V, $\exists w \in V$ and $t_n \to \infty$
such that $z(t_n) \to w$. But then
$w = p$, otherwise (*) fails ▨