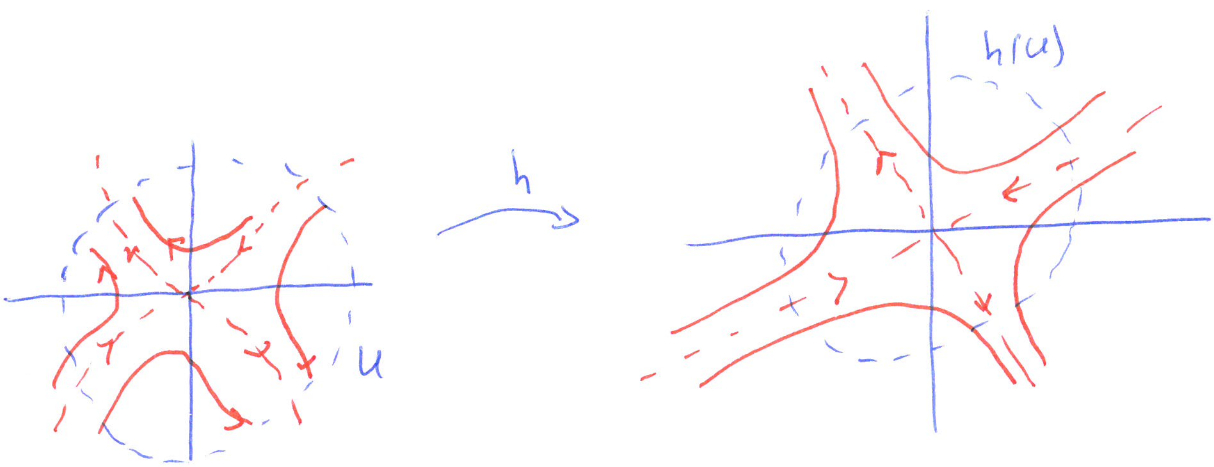
The **Hartman-Grobman Theorem** shows that, at least **locally** near **hyperbolic** equilibria or fixed points, the behaviour of a non-linear system is conjugate to its linearization:

Hartman-Grobman Theorem for discrete time: If $f : \mathbb{R}^d \to \mathbb{R}^d$ is a C^1 map with a **hyperbolic** fixed point x_0 and derivative matrix $Df(x_0) = B$, then there is a neighbourhood U of x_0 and a **unique** near-identity homeomorphisms h such that on U, $h \circ f(x) = x_0 + B(h(x) - x_0)$.

Hartman-Grobman Theorem for discrete time: If $f : \mathbb{R}^d \to \mathbb{R}^d$ is a C^1 vector field with a **hyperbolic** equilibrium x_0 and $Df(x_0) = A$, then there is a neighbourhood U of x_0 and a **unique** near-identity homeomorphisms h such that $h \circ \phi^t(x) = x_0 + e^{At}(h(x) - x_0)$ for all $x \in U$ and t such that $\phi^t(x) \in \mathcal{U}$.

Of course, the formulas become easier if we translate x_0 to 0:

$$h \circ f = Bh \qquad \text{and} \qquad h \circ \phi^t = e^{At}h.$$



Sketch of Proof (Hartman - Grobman)

1) Discrete time: For small nbh $\mathcal{U} \ni x_0$ we can write

$$f(x) = f(x_0) + \underset{\substack{\uparrow \\ Df(x_0)}}{B}(x - x_0) + g(x) \qquad g(x) = \mathcal{O}(|x-x_0|^2)$$

WLOG $x_0 = 0$, $f(x_0) = 0$ (via translation $L: x \mapsto x - x_0$, $L^{-1}: x \mapsto x + x_0$

$$f(x) = Bx + g(x)$$

We hope for a near-identity solution: $h^{-1}(x) = Id + v(x)$
for $v(x) = \mathcal{O}(|x|^2)$. $= x + v(x)$.

Remark The near-identity solution will turn out to be unique. Otherwise many solutions. Eg. $f(x) = \frac{1}{2}x \in \mathbb{R}^2$
$h(x) = Rx$ = rotation $f \circ h = h \circ f$ works for every rotation.

Need to solve $\quad h \circ f = h \circ (B + g) = B \circ h$

$$(B \circ g) \circ (Id + v) = f \circ h^{-1} = h^{-1} \circ B = (Id + v) \circ B$$

$$\Rightarrow \quad 0 = [(Id + v) \circ B - (B \circ g) \circ (Id + v)] \circ B^{-1}$$

$$= \underbrace{v - B \circ v \circ B^{-1}}_{\mathcal{L}(v)} - g \circ (Id + v) \circ B^{-1} =: \Psi(v, g)$$

$\Psi(v, g) = 0$ iff $v = \mathcal{L}^{-1}\big(\mathcal{L}(v) - \Psi(v, g)\big) =: \Theta(v)$
iff Θ has a fixed point.

One can show that Θ is a contraction on the Banach space $C^k(\mathcal{U}, \mathbb{R}^d)$ with sup-norm.
By Banach Contraction Principle, Θ has a unique fixed point

2. Continuous time: Need to solve (with $x_0 = 0$)

$$h \circ \varphi^t = e^{At} \circ h \qquad \text{i.e.}$$

$$\varphi^t \circ h^{-1} \circ e^{-At} x = h^{-1}(x) \qquad (*)$$

First take time $t=1$ map: $\varphi^1 =: f$, $e^{-A1} = B^{-1}$, $e^{A} = B$

Find h unique! as in part 1. such that

$$\varphi^1 \circ h^{-1} \circ e^{-A} x = h^{-1}(x). \qquad (**)$$

But we also have

$$\varphi^1 \circ \underbrace{[\varphi^t \circ h^{-1} \circ e^{-At}]}_{h_t^{-1}} \circ e^{-A} = \varphi^t \circ \underbrace{[\varphi^1 \circ h^{-1} \circ e^{-A}]}_{h^{-1}} \circ e^{-At}$$

$$= \underbrace{[\varphi^t \circ h^{-1} \circ e^{-At}]}_{h_t^{-1}}$$

But the solution of $(**)$ was unique, so

$$h^{-1} = h_t^{-1} = \varphi^t \circ h^{-1} \circ e^{-At}$$

and $(*)$ is solved

$\blacksquare$