Schwarzian Derivative

Named after geometer Karl Hermann Schwarz 1843-1921

Def Given a C^3 map $f : \mathbb{R} \to \mathbb{R}$, the Schwarzian derivative is

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

where we simply ignore points with $f'(x) = 0$.

Lemma If $f, g : \mathbb{R} \to \mathbb{R}$ are both C^3 maps the

$$S(f \circ g) = (Sf) \circ g + Sg$$

Corollary If $Sf < 0$, then $Sf^n < 0$ for every iterate $n \geq 1$.

Proof Exercise

Fact If $f(x) = \dfrac{ax+b}{cx+d}$, $a, b, c, d \in \mathbb{R}$ is a Möbius transformation, then

$$Sf = 0$$

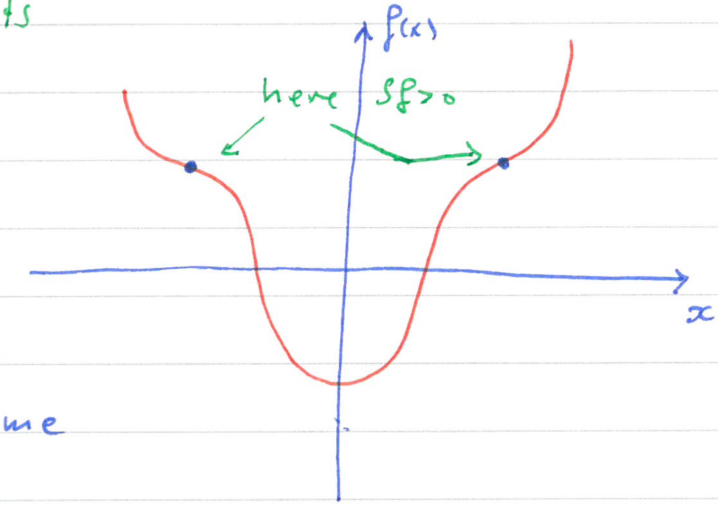
If $f(x)$ is a quadratic polynomial, then $Sf < 0$.

Prop: If $f : \mathbb{R} \to \mathbb{R}$ is a C^3 map with $Sf < 0$, then f' has

a) no positive local minimum

b) no negative local maximum

This implies that the graph of f cannot have inflection points



Proof of a).

By contradiction assume that f' has a positive local minimum at x. Then

1) $f'(x) > 0$ the minimum of f' is positive

2) $f''(x) = 0$ f' has an extremum at x

3) $f'''(x) \geq 0$ the extremum is a minimum

In conclusion $$Sf(x) = \underbrace{\frac{f'''(x)}{f'(x)}}_{\geq 0} - \frac{3}{2}\underbrace{\left(\frac{f''(x)}{f'(x)}\right)^2}_{=0 \text{ because } f''(x)=0} \geq 0$$

But this contradicts that $Sf < 0$, ending the proof □

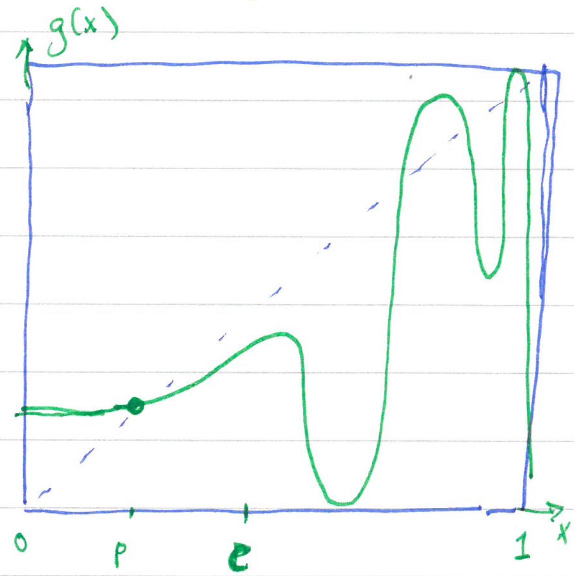
Cor If $f_a : \mathbb{R} \to \mathbb{R}$ undergoes a pitchfork bifurcation at $x \in \mathbb{R}$ for $a = 0$, then $Sf_0(x) \geq 0$

Proof Check Sf_a for the normal form $f_a(x) = x + ax^3$.

Prop If $f:[0,1]\to[0,1]$ is a C^3 map and $Sf<0$, then if p is an attracting periodic point, then it attracts a critical point c, or a boundary point.

Note: c is called a critical point of f if $f'(c)=0$. If $\tilde{c}$ is a critical point of f^n, the by the chain rule, there is $0\le j<n$ and a critical point c of f such that $c=f^j(\tilde{c})$.

Proof of the Proposition



Let p be an attracting periodic point, say of period n. Then p is a fixed point of $g:=f^{2n}$ and $0\le g'(p)\le 1$.
Assume $g'(p)<1$ (the case $g'(p)=1$ is left as an exercise)

Let $a^+=\min\{x>p : g'(x)\ge 1 \text{ or } x=1\}$,
$a^-=\max\{x<p : g'(x)\ge 1 \text{ or } x=0\}$.

a) If $g'(x)>0$ $\forall x\in[a^-,a^+]$, and $0<a^-<a^+<1$, then g has a positive local minimum for some $y\in[a^-,a^+]$ and $Sg(y)\ge 0$

b) If $g'(c)=0$ for some $c\in[a^-,a^+]$, then we take such c closest to p and find $g^k(c)\to p$ as $k\to\infty$ (See Figure)

c) If no such critical point exits, then $a^-=0$ (or $a^+=1$) and $g^k(0)\to p$ (or $g^k(1)\to p$) as $k\to\infty$.

Consequences for the quadratic family

$Q_a : [0,1] \to [0,1]$, $\quad Q_a(x) = ax(1-x)$, $\quad SQ_a(x) = -\frac{3}{2}\left(\frac{2a}{2ax-1}\right)^2 < 0$

1) The family has no pitchfork bifurcation.

2) The only critical point is $c = \frac{1}{2}$.
Hence, if $\tilde{c}$ is a critical point for Q_a^n, then $Q_a^m(\tilde{c}) = c$ for some $0 \leq m < n$.

3) $Q_a(1) = Q_a(0) = 0$, and this fixed point is unstable for $a > 1$.

4) If $p \neq 0$ is an attracting periodic point, say of period n, then it cannot attract a boundary point (because of part 3)).
Hence it must attract a critical point $\tilde{c}$ of Q_a^{2n} (by the previous proposition).
But $c \in \mathrm{orb}(\tilde{c})$, so p must attract c.

5) A single critical point can only be attracted to single attracting periodic orbit.
Therefore, for any parameter a, Q_a can have at most one attracting periodic orbit, and this orbit attracts $c = 1/2$.