# Local Entropy and Continua

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# Overview

# Chaos



Throughout, $X$ is a compact metric space without isolated points and $T : X \to X$ is a continuous function. $(X, T)$ is called Topological Dynamical System (TDS).

**Definition**

*We say that $T$ is Li-Yorke chaotic if there is an uncountable set $U \subseteq X$, such that for all $x, y \in U$, $x \neq y$,*

$$\underline{\lim}\, d(T^n(x), T^n(y)) = 0 \quad \& \quad \overline{\lim}\, d(T^n(x), T^n(y)) > 0.$$

Set $U$ can be made a Cantor set. (GGM, PAMS 2021)

**Definition**

*We say that $T$ is Devaney chaotic (or just chaotic) if*

- *$T$ is transitive, i.e., $T$ has a dense forward orbit,*
- *$per(T)$, the set of periodic points of $T$, is dense in $X$.*

# Topological Entropy

Let $(X, T)$ be a TDS and $\mathcal{U}, \mathcal{V}$ open covers of $X$. We denote the smallest cardinality of a subcover of $\mathcal{U}$ with $N(\mathcal{U})$, and

$$\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}.$$

## Definition

*We define the entropy of $(X, T)$ with respect to $\mathcal{U}$ as*

$$h(X, T, \mathcal{U}) = \lim_{n\to\infty} \frac{1}{n} \log N(\vee_{m=1}^{n} T^{-m}(\mathcal{U})).$$

*The (topological) entropy of $(X, T)$ is defined as*

$$h(T) = \sup_{\mathcal{U}} h(X, T, \mathcal{U}).$$



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# Relationship: Chaos & Entropy

**Theorem**

*Let $(X, T)$ be a TDS.*

- *(HY, 2002, Mai 2004) $T$ is Devaney chaotic implies that $T$ is Li-Yorke chaotic.*
- *(BGKM, 2002) $h(T) > 0 \Rightarrow T$ is Li-Yorke chaotic.*

**Theorem (Classical by now, See Ruette for references)**

*Let $(I, T)$ be a TDS. Then, the following are equivalent.*

- *$h(T) > 0$.*
- *$T$ is Devaney chaotic on a subsystem.*
- *$T$ has a periodic point not a power of 2.*

# Indecomposability

**Theorem**

*Let $(X, T)$ be a TDS.*

- *(Barge-Martin, 1985) If $X = [0, 1]$ and $T$ has positive entropy, then $\varprojlim(X, T)$ contains an indecomposable continua.*
- *(Mouron, 2011) If $X$ is arc-like and $T$ has positive entropy, then $\varprojlim(X, T)$ contains an indecomposable continua.*
- *(D-Kato 2017) If $X$ is $G$-like for some graph $G$, and $T$ has positive entropy, then $\varprojlim(X, T)$ contains an indecomposable continua.*


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# Limitations



Henderson's example from 1964 show that $(I, T)$ indecomposable (or even hereditarily indecomposable) does not imply any dynamical properties of $T$.

For restricted graph maps, such as those which are strictly piecewise monotone, having positive entropy is equivalent to the inverse limit containing an indecomposable continuum. (Lu-Xiong-Ye, 2000)

Boronski and Oprocha (2015) showed that one cannot weaken positive entropy to Li-Yorke chaos in the above theorem.



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# Sequence Entropy

Let $(X, T)$ be a TDS and $A = \{a_1 < a_2 < \ldots\}$ be a sequence of positive integers.

## Definition

*Then,*

$$h^A(X, T, \mathcal{U}) = \lim_{n \to \infty} \frac{1}{n} \log N(\vee_{m=1}^{n} T^{-a_m}(\mathcal{U})).$$

*The entropy of $(X, T)$ along sequence $A$ is defined as*

$$h(T) = \sup_{\mathcal{U}} h^A(X, T, \mathcal{U}).$$

# Sequence Entropy & Li-Yorke chaos



**Theorem (Li-Oprocha-Yang-Zeng, 2017)**

*A graph map is Li-Yorke chaotic if and only it has positive sequence entropy.*

As mentioned earlier, apparently no connection to indecomposability.

It may be interesting to investigate what happens on $G$-like continua, $G$ a graph. The result may hold for hereditarily decomposable continua. Perhaps there some "meta" theorem which allows transfer of general dynamics on graph to dynamics on hereditarily decomposable $G$-like continua. Minc-Transue (1989) obtained such result for Sharkovskii's result. Ye(Top&App 1995) contains some related ideas.

# UPE and CPE

In 1990's, Blanchard defined what he called UPE (Uniform Positive Entropy) and CPE in topological dynamics to capture the notion of K-automorphism.

A TDS $(X, T)$ has UPE (uniform positive entropy) if for any essential cover $\mathcal{C} = \{U, V\}$ we have that $h(\mathcal{C}, T) > 0$.

A TDS $(X, T)$ has CPE (complete positive entropy) if each non-trivial factor of $T$ has positive topological entropy.

**Theorem (Blanchard)**

*Let $(X, T)$ be a TDS. Then,*

- *UPE implies CPE*
- *UPE implies topological weak mixing.*
- *There are CPE systems which are not UPE.*

# Independence Set



A set $I \subseteq \mathbb{N}$ has positive density if $\liminf_n \frac{|I \cap [1,n]|}{n>0} > 0$.

Given a TDS $(X, T)$ and $\{U, V\} \subset X$, we say $I \subset \mathbb{N}$ is an independence set for $\{U, V\}$ if for all finite $J \subseteq I$, and for all $(Y_j) \in \prod_{j \in J}\{U, V\}$, we have that

$$\cap_{j \in J} T^{-j}(Y_j) \neq \emptyset.$$

# IE-pair

Let $(X, T)$ be a TDS. We say that $(x_1, x_2) \in X \times X$ is an independence entropy pair (IE-pair) of $(X, T)$ if for every pair of open sets $A_1, A_2$, with $x_1 \in A_1$ and $x_2 \in A_2$, there exists an independence set for $\{A_1, A_2\}$ with positive density. The set of IE-pairs of $(X, T)$ is denoted by $E(X, T)$.

**Theorem (Huang-Ye, Kerr-Li)**

*Let $(X, T)$ be a TDS.*

1. *$(X, T)$ has positive entropy if and only if there exists $x \neq y \in X$ with $(x, y) \in E(X, T)$.*
2. *$(X, T)$ has UPE if and only if $E(X, T) = X \times X$.*

**Theorem (Blanchard)**

*$(X, T)$ has CPE if and only if the smallest closed equivalence relation containing $E(X, T) = X \times X$.*

# UPE and Indecomposability



**Theorem (D-Kato, 2017)**

*Suppose that $(X, T)$ is UPE where $X$ is $G$-like for some graph $G$. Then, $\varprojlim(X, T)$ is indecomposable.*

**Corollary**

*Suppose $X$ is a graph. If $(X, T)$ is weak mixing, then $\varprojlim(X, T)$ is indecomposable.*

# Operator Γ (Barbieri, Garcia-Ramos)

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Chaos, Entropy & Indecomposability

Local Entropy, UPE, CPE & Indecomposability

Let $X$ be a compact metric space and $E \subseteq X^2$.

We define $E^+$ as the smallest equivalence relation that contains $E$ and $\Gamma(E) = \overline{E^+}$.

For a successor ordinal $\alpha$,

$$\Gamma^\alpha(E) = \Gamma(\Gamma^{\alpha-1}(E))$$

and for limit ordinal $\alpha$,

$$\Gamma^\alpha(E) = \overline{\cup_{\beta<\alpha}\Gamma^\beta(E)}$$

.

# Complexity of CPE



**Theorem (Barbieri, Garcia-Ramos, 2021)**

*A TDS $(X, T)$ has CPE if and only if there exists a countable ordinal $\alpha$ such that $\Gamma^{\alpha}(E(X, T)) = X \times X$. The least such ordinal $\alpha$ is called the CPE rank of $(X, T)$.*

*For every countable ordinal $\alpha$, there exists a TDS $(X, T)$ with CPE rank $\alpha$ where $X$ is the Cantor space.*

# When CPE and UPE coincide

**Theorem (D-Garcia-Ramos, 2021)**

*A TDE with the shadowing property has CPE if and only if it has UPE.*

**Theorem (D-Garcia-Ramos, 2021)**

*A mixing graph map has CPE if and only if it has UPE.*

**Theorem (D-Garcia-Ramos, 2021)**

*On a large class of spaces (including graph maps), one can construct CPE maps of arbitrarily high CPE rank.*

# Some Questions



Every interval map $T$ with CPE has the property that $\varprojlim(I, T)$ contains an indecomposable continua. In some sense how deeply indecomposable subcontinua are buried depends on the CPE rank of the system.

The set of hereditarily decomposable continua is $\Pi_1^1$-complete (D, 2000) with $\Pi_1^1$ rank given by Krasinkiewicz-Minc crookedness index (1976).

The set of CPE systems on $[0, 1]$ is $\Pi_1^1$-complete (D-Garcia-Ramos, 2021) with the CPE rank.

Is there a relationship between CPE rank and Krasinkiwiecz-Minc crookedness index?

Thank you!