On certain ”differently” characterized subgroups of the circle group

Pratulananda Das

Department of Mathematics, Jadavpur University, West Bengal

36th Summer Topology Conference, July 18-22, 2022,
University of Vienna

Throughout $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and ω will stand for the set of all real numbers, the set of all rational numbers, the set of all integers and the set of all natural numbers respectively. The first three are equipped with their usual abelian group structure and the circle group $\mathbb{T}$ is identified with the quotient group $\mathbb{R}/\mathcal{Z}$ of $\mathbb{R}$ endowed with its usual compact topology. For $x \in \mathbb{R}$ we denote by $\{x\}$ the difference $x - [x]$ (the fractional part) and $\|x\|$ the distance from the integers i.e. $\min\{\{x\}, 1 - \{x\}\}$.

- The motivation to study the so called ”characterized subgroups” can be traced back to the distribution of sequences of multiples of a given real number mod 1.

- The motivation to study the so called "characterized subgroups" can be traced back to the distribution of sequences of multiples of a given real number mod 1.
- Recall that a sequence of real numbers (x_n) is said to be uniformly distributed mod 1, if for every $[a, b] \subseteq [0, 1)$ one has

$$\lim_{n\to\infty} \frac{|\{j : 0 \leq j < n,\ \{x_j\} \in [a, b]\}|}{n} = b - a$$

where $\{x_j\}$ is the fractional part of x_j. In his celebrated results proved in 1916, H. Weyl had investigated the set

$$W_{\mathbf{u}} = \{x \in [0, 1] : (u_n x) \text{ is uniformly distributed mod } 1\}$$

where $\mathbf{u} = (u_n) \in \mathbb{Z}^\omega$.

- Note that for every number $\alpha \in [0,1] \setminus \mathbb{Q}$, $\alpha \notin W_{\mathbf{u}}$ for an appropriate choice of $\mathbf{u}$. Indeed, to this end one can consider the convergents $\frac{r_n}{u_n}$ of the continued fraction expansion of α and as $||u_n\alpha||_{\mathbb{Z}} \to 0$ (where $||.||_{\mathbb{Z}}$ is the distance from the integers), conclude that $\alpha \notin W_{\mathbf{u}}$.

- Note that for every number $\alpha \in [0,1] \setminus \mathbb{Q}$, $\alpha \notin W_{\mathbf{u}}$ for an appropriate choice of **u**. Indeed, to this end one can consider the convergents $\frac{r_n}{u_n}$ of the continued fraction expansion of α and as $||u_n\alpha||_{\mathbb{Z}} \to 0$ (where $||.||_{\mathbb{Z}}$ is the distance from the integers), conclude that $\alpha \notin W_{\mathbf{u}}$.
- In a really impressive observation, [Larcher, PAMS, 1988] proved that if the continued fraction expansion of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is bounded then

$$\{\beta \in \mathbb{R} : ||u_n\beta||_{\mathbb{Z}} \to 0\} = \langle\alpha\rangle + \mathbb{Z}, \tag{1}$$

the subgroup of $\mathbb{R}$ generated by α mudulo 1. Instead of using the fractional part $\{x_j\}$ or working modulo 1, one can conveniently work in the circle group $\mathbb{R} \setminus \mathbb{Z} = \mathbb{T}$

- Recall that an element x of an abelian group is torsion if there exists $k \in \omega$ such that $kx = 0$

- Recall that an element x of an abelian group is torsion if there exists $k \in \omega$ such that $kx = 0$
- [Braconnier, CRAMSP, 1944] An element x of an abelian topological group G is :
  (i) topologically torsion if $n!x \to 0$;
  (ii) topologically p-torsion, for a prime p, if $p^n x \to 0$.

- Recall that an element x of an abelian group is torsion if there exists $k \in \omega$ such that $kx = 0$
- [Braconnier, CRAMSP, 1944] An element x of an abelian topological group G is :
  (i) topologically torsion if $n!x \to 0$;
  (ii) topologically p-torsion, for a prime p, if $p^n x \to 0$.
- It is obvious that any p-torsion element is topologically p-torsion.

- Recall that an element x of an abelian group is torsion if there exists $k \in \omega$ such that $kx = 0$
- [Braconnier, CRAMSP, 1944] An element x of an abelian topological group G is :
  - (i) topologically torsion if $n!x \to 0$;
  - (ii) topologically p-torsion, for a prime p, if $p^n x \to 0$.
- It is obvious that any p-torsion element is topologically p-torsion.
- [Armacost, 1981] defined the subgroups

$$X_p = \{x \in X : p^n x \to 0\} \text{ and } X! = \{x \in X : n!x \to 0\}$$

of an abelian topological group X, and started their investigation

Definition

Let (a_n) be a sequence of integers, the subgroup

$$t_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \text{ in } \mathbb{T}\}$$

of $\mathbb{T}$ is called a characterized (by (a_n)) *subgroup* of $\mathbb{T}$.

Definition

Let (a_n) be a sequence of integers, the subgroup

$$t_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \text{ in } \mathbb{T}\}$$

of $\mathbb{T}$ is called a characterized (by (a_n)) *subgroup* of $\mathbb{T}$.

- Even if the notion was inspired by the various (earlier) instances, the term *characterized* appeared much later, coined in [Bíró, Deshouillers, Sós, SSMH, 2001].

Definition

Let (a_n) be a sequence of integers, the subgroup

$$t_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \text{ in } \mathbb{T}\}$$

of $\mathbb{T}$ is called a characterized (by (a_n)) *subgroup* of $\mathbb{T}$.

- Even if the notion was inspired by the various (earlier) instances, the term *characterized* appeared much later, coined in [Bíró, Deshouillers, Sós, SSMH, 2001].

Example

(a) Let p be a prime. For the sequence (a_n), defined by $a_n = p^n$ for every n, obviously $t_{(p^n)}(\mathbb{T})$ contains the Prüfer group $\mathcal{Z}(p^\infty)$. Armacost proved that $t_{(p^n)}(\mathbb{T})$ simply coincides with $\mathcal{Z}(p^\infty)$.
(b) Armacost posed the problem to describe the group $\mathbb{T}! = t_{(n!)}(\mathbb{T})$. It was resolved by [Borel, CM, 1991].

- Historically some of the most interesting cases studied (like the two types of subgroups considered by Armacost) are particular cases of characterized groups which are characterized by arithmetic sequences.

- Historically some of the most interesting cases studied (like the two types of subgroups considered by Armacost) are particular cases of characterized groups which are characterized by arithmetic sequences.
- Precisely a sequence of positive integers (a_n) is an arithmetic sequence if

$$1 = a_0 < a_1 < a_2 < \cdots < a_n < \dots \quad \text{and } a_n | a_{n+1} \text{ for every } n \in \mathbb{N}.$$

- Historically some of the most interesting cases studied (like the two types of subgroups considered by Armacost) are particular cases of characterized groups which are characterized by arithmetic sequences.
- Precisely a sequence of positive integers (a_n) is an arithmetic sequence if

$$1 = a_0 < a_1 < a_2 < \cdots < a_n < \ldots \quad \text{and } a_n | a_{n+1} \text{ for every } n \in \mathbb{N}.$$

- An element x in an abelian topological group G is called $\underline{a}$-torsion element if $a_n x \to 0$. Borel first started studies of $\underline{a}$-torsion elements (null sequences modulo 1) in $\mathbb{R}$. The final results on describing the $\underline{a}$-torsion elements of $\mathbb{T}$ can be found in [Di Santo, Dikranjan, CA, 2004] followed by [Dikranjan, Impieri, CA, 2014].

- Historically some of the most interesting cases studied (like the two types of subgroups considered by Armacost) are particular cases of characterized groups which are characterized by arithmetic sequences.
- Precisely a sequence of positive integers (a_n) is an arithmetic sequence if

$$1 = a_0 < a_1 < a_2 < \dots < a_n < \dots \quad \text{and } a_n | a_{n+1} \text{ for every } n \in \mathbb{N}.$$

- An element x in an abelian topological group G is called $\underline{a}$-torsion element if $a_n x \to 0$. Borel first started studies of $\underline{a}$-torsion elements (null sequences modulo 1) in $\mathbb{R}$. The final results on describing the $\underline{a}$-torsion elements of $\mathbb{T}$ can be found in [Di Santo, Dikranjan, CA, 2004] followed by [Dikranjan, Impieri, CA, 2014].
- It had already been observed by [Eggleston, PLMS, 1952] that the asymptotic behavior of the sequence $q_n := \frac{a_n}{a_{n-1}}$ of ratios has a strong impact on the size of $t_{(a_n)}(\mathbb{T})$:

(E1) $t_{(a_n)}(\mathbb{T})$ is countable if (q_n) is bounded,

(E2) $|t_{(a_n)}(\mathbb{T})| = 2^{\aleph_0}$ if $q_n \to \infty$.

- [Bíró, Deshouillers and Sós, SSMH, 2001] established the important fact that every countable subgroup of $\mathbb{T}$ is characterized.

- [Bíró, Deshouillers and Sós, SSMH, 2001] established the important fact that every countable subgroup of $\mathbb{T}$ is characterized.
- The whole history concerning these investigations along with relevant references can be seen from the excellent survey article on characterized subgroups of $\mathbb{T}$ [Di Santo, Dikranjan, Giordano Bruno, Ric. Mat, 2018]).

- [Bíró, Deshouillers and Sós, SSMH, 2001] established the important fact that every countable subgroup of $\mathbb{T}$ is characterized.
- The whole history concerning these investigations along with relevant references can be seen from the excellent survey article on characterized subgroups of $\mathbb{T}$ [Di Santo, Dikranjan, Giordano Bruno, Ric. Mat, 2018]).
- Characterized subgroups of compact abelian groups were introduced in [Dikranjan, TP, 2001-2002] followed by [Dikranjan, Milan, Tonolo, JPAA, 2005] and studied later by Hart and Kunen [TA, 2005, 2006], as well as by [Dikranjan, Kunen, JPAA, 2007], [Dikranjan, Impieri, TA, 2016]

- [Bíró, Deshouillers and Sós, SSMH, 2001] established the important fact that every countable subgroup of $\mathbb{T}$ is characterized.
- The whole history concerning these investigations along with relevant references can be seen from the excellent survey article on characterized subgroups of $\mathbb{T}$ [Di Santo, Dikranjan, Giordano Bruno, Ric. Mat, 2018]).
- Characterized subgroups of compact abelian groups were introduced in [Dikranjan, TP, 2001-2002] followed by [Dikranjan, Milan, Tonolo, JPAA, 2005] and studied later by Hart and Kunen [TA, 2005, 2006], as well as by [Dikranjan, Kunen, JPAA, 2007], [Dikranjan, Impieri, TA, 2016]
- For the application of characterized subgroups to the problem of building group topologies with or without convergent sequences, see [Babieri, Dikranjan, Milan, Weber, AGT, 2005].

- **Definition 2.** [Buck, AJM, 1946] By $|A|$ we denote the cardinality of a set A. The lower and the upper natural densities of $A \subset \omega$ are defined by

$$\underline{d}(A) = \liminf_{n \to \infty} \frac{|A \cap [1, n]|}{n} \quad \text{and} \quad \overline{d}(A) = \limsup_{n \to \infty} \frac{|A \cap [1, n]|}{n}.$$

If $\underline{d}(A) = \overline{d}(A)$, we say that the natural density of A exists and it is denoted by $d(A)$.

- **Definition 2.** [Buck, AJM, 1946] By $|A|$ we denote the cardinality of a set A. The lower and the upper natural densities of $A \subset \omega$ are defined by

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap [1,n]|}{n} \text{ and } \overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n}.$$

  If $\underline{d}(A) = \overline{d}(A)$, we say that the natural density of A exists and it is denoted by $d(A)$.
- **Observation:** We say that a subset of ω is "small" if it has natural density zero. We write $\mathcal{I}_d = \{A \subset \omega : d(A) = 0\}$. Evidently $\mathcal{I}_d$ forms an ideal (i.e. $\omega \notin \mathcal{I}_d$, it is hereditary and closed under finite unions).

- **Definition 2.** [Buck, AJM, 1946] By $|A|$ we denote the cardinality of a set A. The lower and the upper natural densities of $A \subset \omega$ are defined by

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap [1,n]|}{n} \text{ and } \overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n}.$$

If $\underline{d}(A) = \overline{d}(A)$, we say that the natural density of A exists and it is denoted by $d(A)$.
- **Observation:** We say that a subset of ω is "small" if it has natural density zero. We write $\mathcal{I}_d = \{A \subset \omega : d(A) = 0\}$. Evidently $\mathcal{I}_d$ forms an ideal (i.e. $\omega \notin \mathcal{I}_d$, it is hereditary and closed under finite unions).
- It is easy to see that the set of odd integers as well as the set of even integers has density $\frac{1}{2}$ whereas the set of all squares has evidently density zero.

- **Definition 2.** [Buck, AJM, 1946] By $|A|$ we denote the cardinality of a set A. The lower and the upper natural densities of $A \subset \omega$ are defined by

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap [1,n]|}{n} \text{ and } \overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n}.$$

  If $\underline{d}(A) = \overline{d}(A)$, we say that the natural density of A exists and it is denoted by $d(A)$.
- **Observation:** We say that a subset of ω is "small" if it has natural density zero. We write $\mathcal{I}_d = \{A \subset \omega : d(A) = 0\}$. Evidently $\mathcal{I}_d$ forms an ideal (i.e. $\omega \notin \mathcal{I}_d$, it is hereditary and closed under finite unions).
- It is easy to see that the set of odd integers as well as the set of even integers has density $\frac{1}{2}$ whereas the set of all squares has evidently density zero.
- The set of all prime numbers has natural density zero while for the set $A = \cup_{n=0}^{\infty}\{2^{2n}, \dots, 2^{2n+1} - 1\}$, $d(A)$ does not exist.

• The following definition was introduced by [Fast, CM, 1951], [Steinhaus, CM, 1951] and before [Zygmund, 1936], all independently, with different names (for sequences of real numbers)

• The following definition was introduced by [Fast, CM, 1951], [Steinhaus, CM, 1951] and before [Zygmund, 1936], all independently, with different names (for sequences of real numbers)

- **Definition 3.** A sequence (x_n) in (X, ρ) is said to be statistically convergent to $x_0 \in X$ if for arbitrary $\varepsilon > 0$ the set $\mathrm{K}(\varepsilon) = \{n \in \omega : \rho(x_n, x_0) \geq \varepsilon\}$ has natural density zero.

• The following definition was introduced by [Fast, CM, 1951], [Steinhaus, CM, 1951] and before [Zygmund, 1936], all independently, with different names (for sequences of real numbers)

- **Definition 3.** A sequence (x_n) in (X, ρ) is said to be statistically convergent to $x_0 \in X$ if for arbitrary $\varepsilon > 0$ the set $\mathrm{K}(\varepsilon) = \{n \in \omega : \rho(x_n, x_0) \geq \varepsilon\}$ has natural density zero.
- [Salat, MS, 1980] A sequence (x_n) of real numbers is statistically convergent to ξ if and only if there exist a set $M = \{m_1 < m_2 < ...\} \subset \omega$ such that $d(M) = 1$ and $\lim_{k \to \infty} x_{m_k} = \xi$.

• The following definition was introduced by [Fast, CM, 1951], [Steinhaus, CM, 1951] and before [Zygmund, 1936], all independently, with different names (for sequences of real numbers)

- **Definition 3.** A sequence (x_n) in (X, ρ) is said to be statistically convergent to $x_0 \in X$ if for arbitrary $\varepsilon > 0$ the set $\mathrm{K}(\varepsilon) = \{n \in \omega : \rho(x_n, x_0) \geq \varepsilon\}$ has natural density zero.
- [Salat, MS, 1980] A sequence (x_n) of real numbers is statistically convergent to ξ if and only if there exist a set $M = \{m_1 < m_2 < ...\} \subset \omega$ such that $d(M) = 1$ and $\lim\limits_{k\to\infty} x_{m_k} = \xi$.
- $\star$ This particular property of statistical convergence make it ”non-trivial” yet ”not too wild” and this is the reason why it has been used to extend several classical results and present new characterizations of existing concepts.

Definition (Dikranjan, Das, Bose, FM, 2020)

For a sequence of integers (a_n) the subgroup

$$t^s_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \text{ statistically in } \mathbb{T}\}$$

of $\mathbb{T}$ is called a statistically characterized (shortly, an s-characterized) (by (a_n)) subgroup of $\mathbb{T}$.

Definition (Dikranjan, Das, Bose, FM, 2020)

For a sequence of integers (a_n) the subgroup

$$t^s_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \ \text{statistically in} \ \mathbb{T}\}$$

of $\mathbb{T}$ is called a statistically characterized (shortly, an s-characterized) (by (a_n)) subgroup of $\mathbb{T}$.

- One reason behind this new approach

Definition (Dikranjan, Das, Bose, FM, 2020)

For a sequence of integers (a_n) the subgroup

$$t^s_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \text{ statistically in } \mathbb{T}\}$$

of $\mathbb{T}$ is called a statistically characterized (shortly, an s-characterized) (by (a_n)) subgroup of $\mathbb{T}$.

• One reason behind this new approach
⋆ Even if the correspondence $(a_n) \mapsto t_{(a_n)}(\mathbb{T})$ is monotone decreasing (with respect to inclusion), in many cases (as in the classical examples) the subgroup $t_{(a_n)}(\mathbb{T})$ is rather small, even if the sequence (a_n) is not too dense (in the above example, it is a geometric progression, so has exponential growth). This suggests that asking $a_n x \to 0$ is maybe somewhat too restrictive.

Theorem

For any sequence of integers (a_n), $t^s_{(a_n)}(\mathbb{T})$ is a $F_{\sigma\delta}$ (hence, Borel) subgroup of $\mathbb{T}$ containing $t_{(a_n)}(\mathbb{T})$.

- In general the subgroup $t^s_{(a_n)}(\mathbb{T})$ may not be complete with respect to the usual norm $\|.\|$ prevalent in $\mathbb{T}$
- Let $\delta : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ be defined as follows. For any $x, y \in \mathbb{T}$, let

$$\delta(x, y) = \sup_{n \in \mathbb{N}} \{\|x - y\|, \|a_n(x - y)\|\}.$$

  The subgroup $t^s_{(a_n)}(\mathbb{T})$ is closed in $\mathbb{T}$ with respected to this new metric.

Theorem

For any sequence of integers (a_n), $t^s_{(a_n)}(\mathbb{T})$ is a $F_{\sigma\delta}$ (hence, Borel) subgroup of $\mathbb{T}$ containing $t_{(a_n)}(\mathbb{T})$.

- In general the subgroup $t^s_{(a_n)}(\mathbb{T})$ may not be complete with respect to the usual norm $\|.\|$ prevalent in $\mathbb{T}$
- Let $\delta : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ be defined as follows. For any $x, y \in \mathbb{T}$, let

$$\delta(x, y) = \sup_{n \in \mathbb{N}} \{\|x - y\|, \|a_n(x - y)\|\}.$$

  The subgroup $t^s_{(a_n)}(\mathbb{T})$ is closed in $\mathbb{T}$ with respected to this new metric.

Corollary

There is a finer topology on the subgroup $t^s_{(a_n)}(\mathbb{T})$ which is completely metrizable.

- [Srivastava] Let X be a Polish space. Then for every Borel set B in X there is a finer Polish topology τ_B on X such that B is closed in X with respect to τ_B.

- [Srivastava] Let X be a Polish space. Then for every Borel set B in X there is a finer Polish topology τ_B on X such that B is closed in X with respect to τ_B.
- $\mathbb{T}$ is Polish with respect to usual topology, and $t^s_{(a_n)}(\mathbb{T}) = B$ (say) is Borel in $\mathbb{T}$. There is a finer Polish topology τ_B on $\mathbb{T}$ such that B is closed in X with respect to τ_B and so $t^s_{(a_n)}(\mathbb{T})$ is Polishable.

- [Srivastava] Let X be a Polish space. Then for every Borel set B in X there is a finer Polish topology τ_B on X such that B is closed in X with respect to τ_B.
- $\mathbb{T}$ is Polish with respect to usual topology, and $t^s_{(a_n)}(\mathbb{T}) = B$ (say) is Borel in $\mathbb{T}$. There is a finer Polish topology τ_B on $\mathbb{T}$ such that B is closed in X with respect to τ_B and so $t^s_{(a_n)}(\mathbb{T})$ is Polishable.

Theorem

Let (a_n) be an arithmetic sequence. Then $|t^s_{(a_n)}(\mathbb{T})| = \mathfrak{c}$.

Theorem

For any arithmetic sequence (a_n), $t^s_{(a_n)}(\mathbb{T}) \neq t_{(a_n)}(\mathbb{T})$.

Lemma (see Dikranjan, Impieri, CA, 2014)

For any arithmetic sequence (a_n) and $x \in \mathbb{T}$, we can build a unique sequence of integers (c_n), where $0 \leq c_n < q_n$, such that

$$x = \sum_{n=1}^{\infty} \frac{c_n}{a_n} \tag{2}$$

and $c_n < q_n - 1$ for infinitely many n.

Lemma (see Dikranjan, Impieri, CA, 2014)

For any arithmetic sequence (a_n) and $x \in \mathbb{T}$, we can build a unique sequence of integers (c_n), where $0 \leq c_n < q_n$, such that

$$x = \sum_{n=1}^{\infty} \frac{c_n}{a_n} \qquad (2)$$

and $c_n < q_n - 1$ for infinitely many n.

• For $x \in \mathbb{T}$ with canonical representation (2), we define $supp(x) = \{n \in \mathbb{N} : c_n \neq 0\}$ and $supp_q(x) = \{n \in \mathbb{N} : c_n = q_n - 1\}$. Clearly $supp_q(x) \subseteq supp(x)$.

- A typical example for the sequence (2^n): Choose $x \in \mathbb{T}$ with

$$supp_{(2^n)}(x) = \bigcup_{n=1}^{\infty} [(2n)^2, (2n+1)^2]. \tag{3}$$

- We can show that $x \in t^s_{(2^n)}(\mathbb{T})$.

- A typical example for the sequence (2^n): Choose $x \in \mathbb{T}$ with

$$supp_{(2^n)}(x) = \bigcup_{n=1}^{\infty} [(2n)^2, (2n+1)^2]. \tag{3}$$

- We can show that $x \in t^s_{(2^n)}(\mathbb{T})$.
- $x \notin t_{(2^n)}(\mathbb{T}) = \mathcal{Z}(2^\infty)$ because $x \in \mathcal{Z}(2^\infty)$ precisely when $supp(x)$ is finite [see Dikranjan, Impieri, CA, 2014].

- A typical example for the sequence (2^n): Choose $x \in \mathbb{T}$ with

$$supp_{(2^n)}(x) = \bigcup_{n=1}^{\infty} [(2n)^2, (2n+1)^2]. \tag{3}$$

- We can show that $x \in t^s_{(2^n)}(\mathbb{T})$.
- $x \notin t_{(2^n)}(\mathbb{T}) = \mathcal{Z}(2^\infty)$ because $x \in \mathcal{Z}(2^\infty)$ precisely when $supp(x)$ is finite [see Dikranjan, Impieri, CA, 2014].
- the element $x \in \mathbb{T}$ above can be replaced by a more generally defined element of $\mathbb{T}$ by taking the support from $\mathbb{I}$ where
-
$$\mathbb{I} = \{\bigcup_{n=1}^{\infty} B_n : B_n = [b_n, d_n], b_{n+1} > d_n + 1 \ \forall n; \}$$

  and

$$\lim_{n\to\infty} |d_n - b_n| = \infty = \lim_{n\to\infty} |b_{n+1} - d_n|.$$

• The notion of natural density can be further extended as follows [Balcerzak,Das, Filipczak, Swaczina, AMH, 2015].

• The notion of natural density can be further extended as follows [Balcerzak,Das, Filipczak, Swaczina, AMH, 2015].
• Let $g : \omega \to [0, \infty)$ be a function with $\lim_{n\to\infty} g(n) = \infty$. The upper density of weight g was defined by the formula

$$\overline{d}_g(A) = \limsup_{n\to\infty} \frac{|A \cap [1, n]|}{g(n)}$$

for $A \subset \omega$.

• The notion of natural density can be further extended as follows [Balcerzak,Das, Filipczak, Swaczina, AMH, 2015].
• Let $g : \omega \to [0, \infty)$ be a function with $\lim_{n\to\infty} g(n) = \infty$. The upper density of weight g was defined by the formula

$$\overline{d}_g(A) = \limsup_{n\to\infty} \frac{|A \cap [1, n]|}{g(n)}$$

for $A \subset \omega$.
• The family $\mathcal{I}_g = \{A \subset \omega : \overline{d}_g(A) = 0\}$ forms an ideal. It has been observed that $\omega \in \mathcal{I}_g$ iff. $\frac{n}{g(n)} \to 0$. So we additionally assume that $n/g(n) \nrightarrow 0$ so that $\omega \notin \mathcal{I}_g$ and it can be proved that $\mathcal{I}_g$ is a proper admissible P-ideal of ω. The collection of all such functions g satisfying the above mentioned properties will be denoted by G.

- The modulus functions are defined as functions $f : [0, \infty) \to [0, \infty)$ which satisfy the following properties.

(i) $f(x) = 0 \Leftrightarrow x = 0$

(ii) $f(x + y) \leq f(x) + f(y)$ for all $x, y \in (0, \infty)$ [Triangle inequality]

(iii) f is non-decreasing

(iv) f is right continuous at 0.

• The modulus functions are defined as functions $f : [0, \infty) \to [0, \infty)$ which satisfy the following properties.

(i) $f(x) = 0 \Leftrightarrow x = 0$

(ii) $f(x + y) \leq f(x) + f(y)$ for all $x, y \in (0, \infty)$ [Triangle inequality]

(iii) f is non-decreasing

(iv) f is right continuous at 0.

• The moduler simple density function [Bose, Das, Kwela, IM, 2018] is defined by

$$\underline{d}_g^f(A) = \liminf_{n\to\infty} \frac{f(|A \cap [1, n]|)}{f(g(n))} \quad \text{and} \quad \overline{d}_g^f(A) = \limsup_{n\to\infty} \frac{f(|A \cap [1, n]|)}{f(g(n))}.$$

If $\underline{d}_g^f(A) = \overline{d}_g^f(A)$, we say that $d_g^f(A)$ exists.
$\mathcal{Z}_g(f) = \{A \subset \mathbb{N} : d_g^f(A) = 0\}$ denotes the corresponding ideal.

Some basic observations [Bose, Das, Kwela, IM, 2018]:

(a) For a modulus function f and $g \in \mathbb{G}$, the ideal $\mathcal{Z}_g(f)$ is a P-ideal. In fact $\mathcal{Z}_g(f)$ is equal to $Exh(\varphi)$, where φ is a lower semicontinuos submeasure on ω given by

$$\varphi(A) = \sup_{n \in \omega} \frac{f(|A \cap [0, n-1])}{f(g(n))}$$

for $A \subset \omega$.

(b) Let f be an unbounded modulus function. Then $\mathcal{Z}_g(f)$ is a density ideal for every $g \in \mathbb{G}$ (an ideal $\mathcal{I}$ on ω is a density ideal in the sense of Farah if $\mathcal{I} = Exh(\varphi)$ where $\varphi := \sup_{i \in \omega} \mu_i$ and μ_i are measures with pairwise disjoint supports being finite subsets of ω).

(c) For any unbounded modulus function f and $g \in \mathbb{G}$, the ideal $\mathcal{Z}_g(f)$ is not F_σ.

- Further observations [Das, Ghosh, IM, 2021]:

(a) For any unbounded modulus function f and for any $g \in \mathbb{G}$, the ideal $\mathcal{Z}_g(f)$ is tall or dense.

(b) Let f be an unbounded modulus function. If $g_1, g_2 \in \mathbb{G}$ are such that $\frac{f(n)}{f(g_2(n))} \geq a > 0$ and $\frac{f(g_2(n))}{f(g_1(n))} \to \infty$ then $\mathcal{Z}_{g_1}(f) \subsetneq \mathcal{Z}_{g_2}(f)$.

(c) For any two unbounded modulus functions f_1, f_2, there exists a family $\mathbb{G}_0 \subseteq \mathbb{G}$ of cardinality $\mathfrak{c}$ such that $\mathcal{Z}_{g_1}(f_i)$, $\mathcal{Z}_{g_2}(f_j)$ are incomparable for $i, j \in \{1, 2\}$ and any two distinct $g_1, g_2 \in \mathbb{G}_0$.

Definition

For a sequence of integers (a_n) the subgroup

$$t^{f,g}_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \ f^g\text{-statistically in } \ \mathbb{T}\} \tag{4}$$

of $\mathbb{T}$ is called an f^g-statistically characterized (shortly, an f^g-characterized) (by (a_n)) subgroup of $\mathbb{T}$.

Definition

For a sequence of integers (a_n) the subgroup

$$t^{f,g}_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \ f^g\text{-statistically in } \mathbb{T}\} \tag{4}$$

of $\mathbb{T}$ is called an f^g-statistically characterized (shortly, an f^g-characterized) (by (a_n)) subgroup of $\mathbb{T}$.

- Let us define

$$\mathbb{I}^f_g = \left\{ \bigcup_{r=1}^{\infty} B_r : B_r = [n_{(2r-1)}, n_{(2r)}], \text{ for some } A = \{n_r\}_{r\in\mathbb{N}} \subset \mathbb{N} \text{ with } d^f_g(\tag{5}$$

Definition

For a sequence of integers (a_n) the subgroup

$$t^{f,g}_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \ f^g\text{-statistically in } \mathbb{T}\} \tag{4}$$

of $\mathbb{T}$ is called an f^g-statistically characterized (shortly, an f^g-characterized) (by (a_n)) subgroup of $\mathbb{T}$.

- Let us define

$$\mathbb{I}^f_g = \left\{ \bigcup_{r=1}^{\infty} B_r : B_r = [n_{(2r-1)}, n_{(2r)}], \text{ for some } A = \{n_r\}_{r\in\mathbb{N}} \subset \mathbb{N} \text{ with } d^f_g(\tag{5}$$

- $|\mathbb{I}^f_g| = |\mathcal{Z}_g(f)| = \mathfrak{c}$.

Definition

For a sequence of integers (a_n) the subgroup

$$t^{f,g}_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \ f^g\text{-statistically in } \mathbb{T}\} \tag{4}$$

of $\mathbb{T}$ is called an f^g-statistically characterized (shortly, an f^g-characterized) (by (a_n)) subgroup of $\mathbb{T}$.

• Let us define

$$\mathbb{I}^f_g = \left\{ \bigcup_{r=1}^{\infty} B_r : B_r = [n_{(2r-1)}, n_{(2r)}], \text{ for some } A = \{n_r\}_{r \in \mathbb{N}} \subset \mathbb{N} \text{ with } d^f_g(\right. \tag{5}$$

• $|\mathbb{I}^f_g| = |\mathcal{Z}_g(f)| = \mathfrak{c}$.

• Let (a_n) be an arithmetic sequence and let $x \in \mathbb{T}$ be such that $supp(x) \in \mathbb{I}^f_g$ and $c_n = q_n - 1$ for all $n \in supp(x)$. Then $x \in t^{f,g}_{(a_n)}(\mathbb{T})$.

• Let (a_n) be an arithmetic sequence and $x \in \mathbb{T}$. If $d_g^f(supp(x)) = 0$, then $x \in t_{(a_n)}^{f,g}(\mathbb{T})$.

• Let (a_n) be an arithmetic sequence and $x \in \mathbb{T}$. If $d_g^f(supp(x)) = 0$, then $x \in t_{(a_n)}^{f,g}(\mathbb{T})$.

Theorem

For any arithmetic sequence (a_n), We have $|t_{(a_n)}^{f,g}(\mathbb{T})| = \mathfrak{c}$.

• Let (a_n) be an arithmetic sequence and $x \in \mathbb{T}$. If $d_g^f(supp(x)) = 0$, then $x \in t_{(a_n)}^{f,g}(\mathbb{T})$.

Theorem

For any arithmetic sequence (a_n), We have $|t_{(a_n)}^{f,g}(\mathbb{T})| = \mathfrak{c}$.

Theorem

$t_{(a_n)}^{f,g}(\mathbb{T}) \neq t_{(a_n)}(\mathbb{T})$ for any arithmetic sequence (a_n).

Theorem

For the unbounded modulus function $f(x) = \log(1 + x)$ *and for any* $g \in \mathbb{G}$, $t^{f,g}_{(a_n)}(\mathbb{T}) \neq t^{\alpha}_{(a_n)}(\mathbb{T})$ *and* $t^{f,g}_{(a_n)}(\mathbb{T}) \neq t^{s}_{(a_n)}(\mathbb{T})$.

Theorem

For any unbounded modulus function f, *there exists* $\mathfrak{c}$ *many* $g \in \mathbb{G}$ *such that* $t^{f}_{(a_n)}(\mathbb{T}) \subsetneq t^{f,g}_{(a_n)}(\mathbb{T})$.

Theorem

For the unbounded modulus function $f(x) = \log(1 + x)$ *and for any* $g \in \mathbb{G}$, $t^{f,g}_{(a_n)}(\mathbb{T}) \neq t^{\alpha}_{(a_n)}(\mathbb{T})$ *and* $t^{f,g}_{(a_n)}(\mathbb{T}) \neq t^{s}_{(a_n)}(\mathbb{T})$.

Theorem

For any unbounded modulus function f*, there exists* $\mathfrak{c}$ *many* $g \in \mathbb{G}$ *such that* $t^{f}_{(a_n)}(\mathbb{T}) \subsetneq t^{f,g}_{(a_n)}(\mathbb{T})$.

Theorem

For any unbounded modulus function f*, if* $g_1, g_2 \in \mathbb{G}$ *are such that* $\frac{f(n)}{f(g_2(n))} \geq a > 0$ *and* $\frac{f(g_2(n))}{f(g_1(n))} \to \infty$*, then* $t^{f,g_1}_{(a_n)}(\mathbb{T}) \subsetneq t^{f,g_2}_{(a_n)}(\mathbb{T})$.

- An interesting observation [Das, Bose, PMH, 2021]

Theorem

Let $\alpha_1, \alpha_2 \in (0,1]$ *with* $\alpha_1 < \alpha_2$. *Then* $t^{\alpha_1}_{(2^n)}(\mathbb{T}) \subsetneq t^{\alpha_2}_{(2^n)}(\mathbb{T})$.

Theorem

For $\alpha \in (0,1]$, $|t^{\alpha}_{(2^n)}(\mathbb{T})| = \mathfrak{c}$.

Theorem

Both the set differences $\bigcap_{\alpha\in(0,1)} t^{\alpha}_{(2^n)}(\mathbb{T}) \setminus t_{(2^n)}(\mathbb{T})$ *and* $t^{s}_{(2^n)}(\mathbb{T}) \setminus \bigcup_{\alpha\in(0,1)} t^{\alpha}_{(2^n)}(\mathbb{T})$ *are non-empty.*

[1] J. Arbault, Sur l'ensemble de convergence absolute d'une série trigonométrique., Bull. Soc. Math. Fr. 80 (1952), 253–317.
[2] D. Armacost, The structure of locally compact abelian groups, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker Inc. New York 68 (1981).
[3] G. Babieri, D. Dikranjan, C, Milan, H. Weber, Answer to Raczkowski's quests on converging sequences of integers, Topol. Appl. 132(1) (2003), 89–101.
[4] G. Babieri, D. Dikranjan, C, Milan, H. Weber, Convergent sequences in precompact group topologies, Appl. Gen. Topology **6**, n. 2 (2005) 149–169.
[5] G. Babieri, D. Dikranjan, C, Milan, H. Weber, Topological torsion related to some recursive sequences of integers, Math. Nachr. 281(7) (2008), 930–950.
[6] M. Balcerzak, P. Das, M. Filipczak, J. Swaczyna, Generalized kinds of density and the associated ideals, Acta Math. Hungar, **147 (1)** (2015), 97 - 115.

A. Bíró, Characterizations of groups generated by Kronecker sets, J. de Th. des Nom. de Bordeaux 19(3) (2007), 567–582.
[8] S. Bhunia, P. Das and S. K. Pal, Restricting statistical convergence, Acta Math. Hungar., **134 (1-2)** (2012), 153-161.
[9] K. Bose, P. Das, A. Kwela, Generating new ideals using weighted density via modulus functions, Indag. Math., 29 (2018) 1196–1209.
[10] K. Bose. P. Das, W. He, Generating subgroups of the circle using statistical convergence of order α, Acta Math. Hungar., **162 (2)** (2020), 633 - 646.
[11] H. Cartan, Théorie des filtres, C. R. Acad. Sci. Paris, **205** (1937), 595–598.
[12] P.Das, K. Bose, Existence of an uncountable tower of Borel subgroups between the Prüfer group and the s-characterized group, Periodica Math. Hungar., 84 (1) (2022), 47 - 55.
[13] P. Das, A. Ghosh, Generating subgroups of the circle using a generalized class of density functions, Indag. Math., 32 (3) (2021), 598 - 618.

D. Dikranjan, P. Das, K. Bose, Statistically characterized subgroups of the circle, Fund. Math., 249 (2020), 185 - 209.
[15] H. Fast, Sur la convergence statistique, Colloq. Math., **2** (1951), 241–244.
[16] J. A. Fridy, On statistical convergence, Analysis, 5 (1985), 301–313.
[17] T. Šalǎt, On statistically convergent sequences of real numbers, Math. Slovaca, **30**, (1980), 139–150.
[18] I.J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly, **66**, (1959), 361–375.
[19] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. **2** (1951), 73–74.

THANK YOU FOR YOUR ATTENTION