Dynamical Systems and Nonlinear ODEs – PS 25100.1
Exercises   Summer Semester 2021

Exercise 1 *Given are the one-parameter quadratic families* $f_c : z \mapsto z^2 + c$ *and* $Q_a : x \to ax(1-x)$.
a) *Show that for* $c \in [-2, \frac{1}{4}]$ *there is an* $a \in [1,4]$ *such that* f_c *is conjugate to* Q_a.
b) *Find the parameter regions in* $\mathbb{R}$ *where* f_c *has a stable fixed point resp. stable period* 2 *point.*
c) *Name the bifurcations that take place at parameters* $c = \frac{1}{4}, c = -\frac{3}{4}, c = -\frac{5}{4}$.

Exercise 2 *Consider the initial value problem*

$$\dot{x} = f(x) := x^3 - 4x + c, \quad x(0) = x_0,$$

for some parameter $c \in \mathbb{R}$.
a) *First take* $c = 0$. *Draw the phase portrait and determine whether the stationary points are stable/unstable.*
b) *Still for* $c = 0$, *solve the ODE (separation of variables, partial fractions).*
c) *As* c *varies over* $\mathbb{R}$, *at what values of* c *do which types of bifurcation occur?*

Exercise 3 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a* C^2 *map with fixed point* x_0 *such that* $f'(x_0) = 1$. *Show (by example) that* x_0 *can be stable or unstable, but also prove that it cannot be exponentially stable.*

Exercise 4 *The map* $T : [0,1] \to [0,1]$, $T(x) = \min\{2x, 2(1-x)\}$ *is called the tent-map (or full tent-map, because it is onto* $[0,1]$*).*
a) *Compute the multiplier of each periodic point. Compute the Lyapunov exponents of arbitrary points. Which points* $x \in [0,1]$ *do not have a Lyapunov exponent?*
b) *Let* $Q : [0,1] \to [0,1]$ *and* $\psi : [0,1] \to [0,1]$ *be defined by* $Q(x) = 4x(1-x)$ *and* $\psi : [0,1] \to [0,1]$ *and* $\psi(x) = \frac{1}{2}(1 - \cos \pi x)$. *Show that* $Q \circ \psi = \psi \circ T$.
c) *Conclude that every* n*-periodic point* $p \neq 0$ *of* Q *has multiplier* $|(Q^n)'(p)| = 2^n$. *Why doesn't this argument apply also to* $p = 0$*?*
d) *What is the Lyapunov exponent of points* $x \in [0,1]$ *w.r.t.* Q*? Is this Lyapunov exponent defined for all* x*?*

Exercise 5 *Consider the two ODEs on* $\mathbb{R}$:

$$\dot{x} = -x \qquad \text{and} \qquad \dot{y} = -2y.$$

a) *Show that the corresponding flows, say* $\varphi^t(x)$ *and* $\psi^t(y)$ *are conjugate, i.e., find a homeomoprhism such that* $\varphi^t(h(x)) = h(\psi^t(x))$. *Is your solution* h *a diffeomorphism? Is it unique*
b) *A function* $h : \mathbb{R} \to \mathbb{R}$ *is called* **Hölder continuous** *with exponent* $\alpha \in (0,1]$ *if there is a constant* K *such that*

$$\sup_{x \neq y} \frac{|h(x) - h(y)|}{|x-y|^\alpha} \leq K.$$

(So Hölder continuous with exponent $\alpha = 1$ *is the same as Lipschitz continuous.) Show that* $h(x) = |x|^\alpha$ *is indeed Hölder continuous with exponent* $\alpha \in (0,1]$. *Check that your solution in*

a) is a Hölder conjugacy, i.e., both h and h^{-1} are Hölder continuous ina neighbourhood of 0.
c) *Consider two ODEs*

$$\dot{x} = f(x) \qquad \textit{and} \qquad \dot{y} = g(y).$$

with $f(0) = g(0) = 0$ and $f'(0) < g'(0) < 0$. Prove that their flows can be Hölder conjugate, but not with an exponent $> g'(0)/f'(0)$.

Exercise 6 *Consider the circle map $f_c : \mathbb{S}^1 \to \mathbb{S}^1$, $x \mapsto 2x + c \pmod 1$.*
a) Show that this map is chaotic in the sense of Devaney.
b) A pair of points (x, y) is called Li-Yorke if

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0 \qquad \textit{and} \qquad \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0.$$

Show that $f_c : \mathbb{S}^1 \to \mathbb{S}^1$ has a Li-Yorke pair. Find a set $\{x, y, z\}$ such that every two of them form a Li-Yorke pair. (Such set is called a **scrambeled set**. *A map if Li-Yorke chaotic if there exists an uncountable scrambeled set.)*

Exercise 7 *Consider the map $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x/2$. Show that this map is C^1 structurally stable, but not C^0 structurally stable.*

Exercise 8 *Consider the map*

$$f : [0,1] \to [0,1], \quad x \mapsto \begin{cases} \frac{x}{1-x} & \textit{if } x \in [0, \frac{1}{2}]; \\ \frac{2x-1}{x} & \textit{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

a) Show that every $x \in \mathbb{Q} \cap (0, 1]$ is eventually mapped to 1.
b) Show that x and $f(x)$ have the same Lyapunov exponent. Find the Lyapunov exponent $\lambda(x)$ of the fixed points and the period 2 points of f.
c) For which $\Lambda \in \mathbb{R}$ do you think there are points $x \in [0, 1]$ such that its Lyapunov exponent $\lambda(x) = \Lambda$? Does every point x have a well-defined Lyapunov exponent?

Exercise 9 *Suppose $f, g : [0, 1] \to [0, 1]$ are two C^1 maps that are conjugate via $h : [0, 1] \to [0, 1]$, i.e., $h \circ f = g \circ h$.*
a) Show that f is chaotic in the sense of Devaney if and only if g is chaotic in the sense of Devaney.
b) Assume in addition that h is a C^1 diffeomorphism. Show that if p is periodic for f, then $q := h(p)$ is periodic for g, with the same period and **multiplier**.
c) For general (i.e., not necessarily periodic) points, do x and $y = h(x)$ have the same Lyapunov exponent?

Exercise 10 *a) Given is a general Lotka-Voterra equation:*

$$\begin{cases} \dot{x} = (A - By)x, \\ \dot{y} = (Cx - D)y, \end{cases} \qquad A, B, C, D > 0.$$

Find changes of coordinates that bring this equation into the form

$$\begin{cases} \dot{x} = (1-y)x, \\ \dot{y} = \alpha(x-1)y, \end{cases} \qquad \alpha > 0.$$

b) Consider the following variation of the Lotka Volterra equations:

$$\begin{cases} \dot{x} = (1-y-\lambda(x-1))x, \\ \dot{y} = \alpha(x-1+\lambda(1-y))y, \end{cases} \qquad 1 \geq \alpha > \lambda > 0.$$

Find the stationary points and their type. Use a Lyapunov function if linearization at the stationary point is not sufficient to draw a conclusion.

Exercise 11 *Consider the standard Van der Pol equation:*

$$\ddot{x} + x = \varepsilon(1-x^2)\dot{x}, \qquad \varepsilon > 0. \tag{1}$$

a) *Write this system as a first order ODE in* $\mathbb{R}^2$*, and then write the first order ODE in polar coordinates.*

b) *Assume that there is a periodic solution* $R(\phi)$*. Argue that by "averaging over* ϕ*", this solution should satisfy*

$$\dot{R} = \frac{-\varepsilon R}{8}(R^2-4), \text{ with some initial condition } R(0) = R_0.$$

and show that its solution is $R(t) = \frac{2}{\sqrt{1+(4/R_0^2-1)e^{-\varepsilon t}}}$.

c) *Analyse what happens in* (1) *if* $\varepsilon < 0$*: compare this case with the case* $\varepsilon > 0$.

Exercise 12 *Let* $\mathcal{A}$ *be a finite alphabet and* $\Sigma = \mathcal{A}^{\mathbb{N}}$*, equipped with product topology.*

a) *Show that* Σ *is a Cantor set, i.e., it is compact, totally disconnected (*$\forall x, y \in \Sigma\ \exists U, V \subset \Sigma$ *open,* $x \in U, y \in V, U \cap V = \emptyset, U \cup V = \Sigma$*) and without isolated points.*

b) *Show that the metric*

$$d_\Sigma(x,y) = \begin{cases} 2^{-\max\{k\ :\ x_i = y_i\ \forall |i| < k\}} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

induces the product topology.

c) *Another metric is*

$$d'_\Sigma(x,y) = \begin{cases} \frac{1}{1+\max\{k\ :\ x_i = y_i\ \forall |i| < k\}} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Two metrics d *and* d' *are* **equivalent** *if*

$$\exists C > 0\ \forall x, y \quad \frac{1}{C} d(x,y) \leq d'(x,y) \leq C d(x,y). \tag{2}$$

Show that d_Σ *and* d'_Σ *are not equivalent in the sense of* (2)*, but that the identity map* $x \in (\Sigma, d_\Sigma) \mapsto x \in (\Sigma, d'_\Sigma)$ *is a homeomorphism. Conclude that* d_Σ *and* d'_Σ *induce the same topology.*

Exercise 13 *Let $T : \mathbb{S}^1 \to \mathbb{S}^1$, $x \mapsto 2x \bmod 1$ be the doubling map. Take $a \in [0, \frac{1}{4}]$ and let $J_0 = (a, a + \frac{1}{2})$ and $J_1 = \mathbb{S}^1 \setminus J_0$ represent a partition of the circle $\mathbb{S}^1$. Let us call this partition* **generating** *if every two points $x, y \in \mathbb{S}^1$ whose orbits do not contain a or $a + \frac{1}{2}$, have distinct symbolic itineraries: $\boldsymbol{i}(x) \neq \boldsymbol{i}(y)$.*
a) *Show that for $a = 0$, the partition $\{J_0, J_1\}$ is generating.*
b) *Let $S(x) = 1 - x$. Show that $T \circ S = S \circ T$. Use this to show that for $a = \frac{1}{4}$, the partition is $\{J_0, J_1\}$ not generating. In fact, $\boldsymbol{i} : \mathbb{S}^1 \to \{0, 1\}^{\mathbb{N}}$ is two-to-one.*
c) *For which $a \in [0, \frac{1}{4}]$ is the partition $\{J_0, J_1\}$ generating?*

Exercise 14 *Let A be an $N \times N$ transition matrix, and (Σ_A, σ) is the corresponding subshift of finite type.*
a) *Prove that trace(A^n) gives the number of periodic sequences $s \in \Sigma_A$ of period n (although this need not be the minimal period).*
b) *Assume that there is $m \geq 1$ such that A^m has only positive entries. Show that (Σ_A, σ) is chaotic in the sense of Devaney.*
c) *Show that (Σ_A, σ) is chaotic in the sense of Li-Yorke.*

Exercise 15 *Let $R_\alpha : S^1 \to \mathbb{S}^1$, $x \mapsto x + \alpha \bmod 1$ be a circle rotation.*
a) Show that (i) $\alpha \in \mathbb{Q}$ if and only if every point is periodic, and $\alpha \notin \mathbb{Q}$ if and only if every point has a dense orbit.
b) Compute the Lyapunov exponent of every point.
c) If $\alpha \neq \beta \bmod 1$, show that R_α and R_β are not conjugate.

Exercise 16 *In this example, we make the Denjoy example of a circle homeomorphism without dense orbits more concrete. Let $R_\alpha : \mathbb{S}^1 \to S^1$ be a circle rotation with irrational α. Let $R_\alpha^n(0)$ for $n \in \mathbb{Z}$. Let $I_n = [a_n, b_n]$ be intervals of length $|I_n| = \frac{1}{1+n^2}$.*
a) *Define*

$$\psi_n : [a_n, b_n] \to [a_{n+1}, b_{n+1}, \qquad x \mapsto a_{n+1} + \int_{a_n}^{x} 1 + 6\frac{|I_{n+1}| - |I_n|}{|I_n|}(b_n - t)(t - a_n)\, dt.$$

Show that $\psi_n : I_n \to I_{n+1}$ is a C^2 diffeomorphism. In particular, show that ψ' is bounded with $\psi_n'(a_n) = \psi_n'(b_n) = 1$. Also compute that $\psi''(\frac{a_n+b_n}{2}) = 0$.
b) *We construct a sequence of maps $(f_N)_{N\geq 0}$ as follows. To create f_0, replace 0 with an interval I_0 and map $f_0(x) = R_\alpha(0)$ for every $x \in I_n$, and $f_0(x) = R_\alpha(x)$ for every $x \notin I_0$.*

Once f_{N-1} is contructed, construct f_N by replacing $R_\alpha^N(0)$ by an interval I_N and replacing $R_\alpha^{-N}(0)$ interval I_{-N}. Also define f_N on I_{N-1} as ψ_{N-1} and on I_{-N} as ψ_{-N} and on I_N as constant $R_\alpha^{N+1}(0)$. Show that f_N is a C^1 map.
c) *Let $f = \lim_N f_N$. Show that it is a C^1 diffeomorphism. Is it C^2?*

Exercise 17 *The harmonic oscillator with damping is given by the ODEs*

$$\ddot{x} + r\dot{x} + \omega^2 x = 0. \qquad\qquad r > 0$$

Depending on the size of the dampint parameter r, there is moderate damping, overdamping (when the solution is no longer oscillitory) and critical damping in between. Find the critical damping parameter $r = r_c$, and find the solution of ODE at critical damping.

Exercise 18 *The harmonic oscillator with* **parametric driving** *is given by the non-autonomous ODEs*

$$\ddot{x} + r(t)\dot{x} + \omega^2(t)x = 0.$$

a) Show that you can eliminate the linear term using the change of coordinates $q(t) = e^{\frac{1}{2}\int^t r(s)\,ds}x(t)$*. The result should be*

$$\ddot{q} + \Omega^2(t)q = 0,$$

for $\Omega^2(t) = \omega^2(t) - \frac{1}{2}\dot{r}(t) - \frac{1}{4}r^2(t)$.

b) Assume now that $r(t)$ *and* $\omega^2(t)$ *are functions that oscillate mildly with the same frequency around some fixed value. That is*

$$r(t) = \omega_0(b + O(\varepsilon)) \qquad\qquad \omega^2(t) = \omega_0^2(1 + O(\varepsilon))$$

where the $O(\varepsilon)$ *stand for oscillating functions of fixed frequency* ω_1 *and small amplitude* $\approx \varepsilon$*. Show that this reduces the ODE to*

$$\ddot{q} + \omega_0^2(1 - \frac{b^2}{4})(1 + \varepsilon f(t))q = 0,$$

where f *is periodic with frequency* $2\omega_2$ *for some* ω_2.

c) Assume $f(t) = f_0 \sin 2\omega_2 t$*. Use the change of coordinates* $q(t) = A(t)\cos(\omega_2 t) + B(t)\sin(\omega_2 t)$ *to come to an ODEs*

$$\begin{cases} 2\omega_2 \dot{A} = \frac{f_0}{2}\omega_0^2 A - \left(\omega_2^2 - \omega_0^2\right) B, \\ 2\omega_2 \dot{B} = -\frac{f_0}{2}\omega_0^2 B + \left(\omega_2^2 - \omega_0^2\right) A. \end{cases}$$

d) Approximate the solutions of this latter ODE using the Ansatz $A(t) = p(t)\cos\theta(t)$ *and* $B(t) = p(t)\sin\theta(t)$*. This should lead to*

$$\begin{cases} \dot{p} = p_{max}\cos(2\theta(t))\,p(t) \\ \dot{\theta} = -p_{\max}\left(\sin 2\theta - \sin 2\theta_{eq}\right) \end{cases} \qquad \begin{aligned} p_{\max} &= \tfrac{f_0\omega_0^2}{4\omega_2} \\ \sin 2\theta_{eq} &= \tfrac{2(\omega_2^2 - \omega_0^2)}{f_0\omega_0^2} \end{aligned}$$

e) The equation for $\theta(t)$ *is independent of* $p(t)$*, and is close to a linear equation. Its solution decays exponentially fast to the constant solution* $\theta(t) \equiv \theta_{eq}$*. Use this solution to solve the equation for* $p(t)$.

f) What conclusion can you draw for the original variable $x(t) = q(t)e^{-\frac{1}{2}\int^t r(s)\,ds}$*? Specifically, is the equilibrium solution* $x(t) \equiv 0$ *stable?*

Exercise 19 *Show that if the Hamiltonian* $H = E_{kin}(p) + E_{pot}(q)$ *and* $E_{kin} = \frac{p^2}{2m}$*, then the Lagrangian is* $L = E_{kin}(p) - E_{pot}(q)$.

Exercise 20 *Assume that* X_H *is a Hamiltonian vector field in* $\mathbb{R}^2$*:*

- *Show that equilibria of* X_H *can only be centers or saddles.*
- *Which bifurcations (of the ones we treated in class) can occur in a family of Hamiltonian vector fields?*
- *Find a family of Hamiltonians* $H_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ *such that at* $\varepsilon = 0$*, a saddle becomes a center.*

Exercise 21 *A Lagrangian system in* $\mathbb{R}^3$ *has the Lagrangian*

$$L(v,q) = \frac{v_1^2 + v_2^2 + v_3^2}{2} - \frac{q_1^2 + q_2^2 + q_3^3}{2}.$$

Use Noether's Theorem to find first integrals. Is the system integrable?

Exercise 22 *We have a Hamiltonian system in coordinates* $(x,y) \in \mathbb{R}^2$ *where the Hamiltonian has the form*

$$H(x,y) = \frac{y^2}{2} + V(x), \qquad V \text{ is } C^2\text{-smooth},$$

and assume that $V(x) = V(-x)$ *has* $V''(0) > 0$*. This means that* $(0,0)$ *is*

(a) *Show that* $(0,0)$ *is a center, with periodic motion around it.*

(b) *Let* $T(a)$ *be the period of the orbit starting at* $(a,0)$*. Show that*

$$T(a) = \int_0^a \frac{4}{\sqrt{2(V(a) - V(x))}} dx.$$

Hint: Integrate $T(a) = \int_{t_1}^{t_2}$ *a quarter of the periodic orbit and invert* $t = t(x)$ *(instead of* $x = x(t)$*) to rewrite the integral.*

- *Show that* $T(a) = 2\pi$ *is constant for* $V(x) = \frac{x^2}{2}$ *(harmonic oscillator).*
- *Show that* $T(a)$ *is increasing if* $V(x) = -\cos x$ *(pendulum), and find* $\lim_{a \searrow 0} T(a)$ *and* $\lim_{a \nearrow 0} T(a)$*.*