Class notes for Nonlinear ODEs and dynamical systems
Hamiltonian dynamics

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Hamiltonian Dynamics

In general coordinates $(p, q) \in \mathbb{R}^{2n}$, q = position, p = momentum (usually), the Hamiltonian equations are:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = X_H \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix} = \underbrace{\begin{pmatrix} \nabla_p H \\ -\nabla_q H \end{pmatrix}}_{\text{in vector notation}}$$

for the Hamiltonian function $H : \mathbb{R}^{2n} \to \mathbb{R}$. Throughout we assume that H is C^2-smooth. The H is preserved along orbits:

$$\dot{H} = \nabla_q H \bullet \dot{q} + \nabla_p H \bullet \dot{q} = \nabla_q H \bullet \nabla_q H + \nabla_p H \bullet (-\nabla_q H) = 0.$$

(Here $\bullet$ stands for the usual dot-product.) Usually H represent the energy, but other preserved quantities (e.g. momentum or angular momentum) are possible too.

Theorem 1. *The Hamiltonian flow ϕ_H^t preserves volume in $\mathbb{R}^{2n}$.*

This "volume" is sometimes called **Liouville measure** in this context.

Proof. Let $\Omega \subset \mathbb{R}^{2n}$ be some region in $\mathbb{R}^2$, and $V(t) = \mathrm{Vol}(\phi_H^t(\Omega))$ its volume after flowing for time t. In other words

$$V(t) = \int_{\phi_H^t(\Omega)} d\mathrm{Vol}.$$

A standard result in multivariate calculus (see Theorem 14 Schmeiser's Notes) says that

$$\dot{V}(t) = \int_{\phi_H^t(\Omega)} \mathrm{div} X_H d\mathrm{Vol},$$

where the **divergence** is defined as

$$\begin{aligned} \mathrm{div} X_H &= \sum_{i=1}^n \frac{\partial}{\partial q_i} (X_H)_{p_i} + \sum_{i=1}^n \frac{\partial}{\partial p_i} (X_H)_{p_i} \\ &= \sum_{i=1}^n \frac{\partial}{\partial q_i} (\frac{\partial}{\partial p_i} H) + \sum_{i=1}^n \frac{\partial}{\partial p_i} (-\frac{\partial}{\partial q_i} H) = 0. \end{aligned}$$

Therefore $\dot{V}(t) = 0$ and $V(t)$ is constant as claimed. $\square$

Lagrangian Dynamics

In general coordinates $(v, q) \in \mathbb{R}^{2n}$, q = position, $v = \dot{q}$ = velocity (usually), the Lagrangian is $L : \mathbb{R}^2 \to \mathbb{R}$. We assume that L is C^2-smooth.

The task is to find an extremum of the action of the Lagrangian over all C^1-paths, i.e., minimize

$$I(\{(\dot{q}(t), q(t)) : t_1 \leq t \leq t_2\}) = \int_{t_1}^{t_2} L(\dot{q}(t), q(t))\ dt,$$

where $q(t_1), q(t_2)$ are both fixed.

Figure 1: The path $q(t)$ and its perturbation $b(t) = \varepsilon r(t)$

For a perturbation path $\{r(t) : t_1 \leq t \leq t_2\}$ with $r(t_1) = r(t_2) = 0$ (see Figure 1), compute the Gateaux derivative:

$$\begin{aligned}
\frac{d}{d\varepsilon} I(q + \varepsilon r)|_{\varepsilon=0} &= \int_{t_1}^{t_2} \nabla_q L(\dot{q} + \varepsilon\dot{r}, q + \varepsilon r) \bullet r + \underbrace{\nabla_v L(\dot{q} + \varepsilon\dot{r}, q + \varepsilon r) \bullet \dot{r}}_{\text{integrate by parts}}\ dt|_{\varepsilon=0} \\
&= \int_{t_1}^{t_2} \left(\nabla_q L(\dot{q} + \varepsilon\dot{r}, q + \varepsilon r) - \frac{d}{dt} \nabla_v L(\dot{q} + \varepsilon\dot{r}, q + \varepsilon r) \right) \bullet r\ dt|_{\varepsilon=0} \\
&\quad + [\nabla_v L(\dot{q} + \varepsilon\dot{r}, q + \varepsilon r) \bullet r]_{t_1}^{t_2}|_{\varepsilon=0} \\
&= \int_{t_1}^{t_2} \left(\nabla_q L(\dot{q}, q) - \frac{d}{dt} \nabla_v L(\dot{q}, q) \right) \bullet r\ dt.
\end{aligned}$$

The constant terms in the penultimate line above disappear because $r(t_1) = r(t_2) = 0$. In order to have $\frac{d}{d\varepsilon} I(q + \varepsilon r)|_{\varepsilon=0} = 0$ over **all** paths r, we need

$$\nabla_q L(\dot{q}, q) = \frac{d}{dt} \nabla_v L(\dot{q}, q). \tag{1}$$

These are the **Euler-Lagrange equations** (one for every coordinate of q).

The Legendre Transforms

Let f : domain of $f = U \subset \mathbb{R}^n \to \mathbb{R}$ be a C^2-smooth strictly convex function. This means that

$$D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

is a positive definite matrix: $(D^2 f)v \bullet v > 0$ unless $v = 0$.

Define the **Legendre transform**

$$f^*(v) = \sup_{v \in U} p \bullet v - f(v). \tag{2}$$

The supremum $f^*(p)$ corresponds to the infimum of $-f^*(p)$, and this is the intersection of the vertical axis with lines of slope p intersecting the graph of f, see Figure 2. This infimum is achieved when this line of slope p is tangent to the graph.

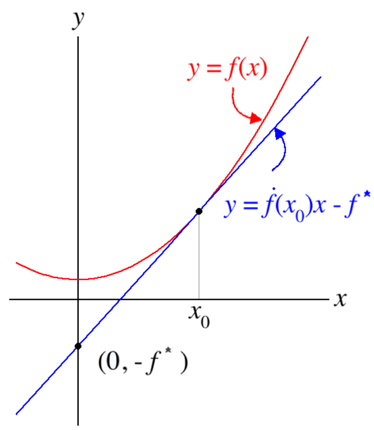

Figure 2: Illustration of the Legendre transform with $x = v$, $p = \dot{f}(v)$ and $x_0 = v^*$.

Let $v^*(p)$ be the value where the supremum is assumed (so $(v^*(p), f(v^*(p)))$ is the point of tangency). Then

$$p = \nabla_v f(v^*(p)), \tag{3}$$

$$f^*(p) = p \bullet v^*(p) - f(v^*(p)). \qquad \text{(by inserting (3) in (2))} \tag{4}$$

Theorem 2. *Let f be strictly convex C^2 function. Then*

- *f^* is a strictly convex C^2 function;*
- *$f^{**} = f$.*

Proof. Compute

$$\begin{aligned} \nabla_p f^*(p) \underbrace{=}_{\text{by (4)}} & \nabla_p \left(p \bullet v^*(p) \right) - f(v^*(p))) \\ = & \; v^*(p) + \underbrace{p \bullet \nabla_p v^*(p)}_{p = \nabla_v f(v^*(p)) \text{ by (3)}} - \nabla_v f(v^*(p)) \bullet \nabla_p v^*(p) \\ = & \; v^*(p). \end{aligned}$$

so (3) holds with f^* instead of f and the roles of v and p swapped. But then (4) follows as well. So $f^{**} = f$.

In fact, we also have $\nabla_p f^*(p) = v$, $\nabla_v f(v) = p$, so

$$\nabla_v f = (\nabla_p f^*)^{inv} \tag{5}$$

Now f is C^2 strictly convex
$\Leftrightarrow$ the Jacobian matrix $D^2 f = J(\nabla f)$ is positive definite
$\Leftrightarrow$ $J(\nabla f)^{-1}$ is positive definite
$\Leftrightarrow$ (by (5)) $J(\nabla f^*) = D^2 f^*$ is positive definite
$\Leftrightarrow$ f^* is C^2 strictly convex. $\square$

Theorem 3. *Assume that (for each q fixed) the Hamiltonian is the Legendre transform of the Lagrangian, so by* (3) *and* (4)

$$p = \nabla_v L(v^*, q) \qquad \text{and} \qquad H(p,q) = L^*(v,q) = p \bullet v^*(p) - L(v^*(p)), q).$$

Then the Euler-Lagrange equations are equivalent to the Hamiltonian equations.

Proof. We assumed $H = L^*$ so by the previous theorem also $L = H^*$. Using (3) with $f = H$ and v and p swapped, we find the first Hamiltonian equation:

$$\dot{q} = v = \nabla_p H.$$

Also, by (4)

$$\begin{aligned}
\nabla_q H &= \nabla_q \left(p \bullet v^*(p,q)) - L(v^*(p,q)), q)\right) \\
&= p \bullet \nabla_q v^*(p,q) - \nabla_v L \bullet \nabla_q v^* - \nabla_q L \\
&\underbrace{=}_{(3)\ \text{with} f=L} p \bullet \nabla_q v^* - p \bullet \nabla_q v^* - \nabla_q L \\
&= -\nabla_q L \\
&\underbrace{=}_{\text{by } (1)} -\frac{d}{dt} \nabla_v L \\
&\underbrace{=}_{(3)\ \text{with } f=L} -\frac{d}{dt} p = -\dot{p}.
\end{aligned}$$

This is the second Hamiltonian equation. □

The example of the pendulum

Newton's equation of motion for the pendulum (see Figure 3) is:

$$\ddot{q} + \frac{g}{\ell} \sin q = 0, \tag{6}$$

for the angle q.

Figure 3: The pendulum with gravitational force along the rod $F = -mg \sin q$

In the Hamiltonian setting, we set $p = m\ell^2 \dot{q}$,

$$E_{kin} = \frac{m(\ell \dot{q})^2}{2} = \frac{p^2}{2m\ell^2}, \qquad E_{pot} = -mg\ell \cos q,$$

and Hamiltonian

$$H(p,q) = E_{kin} + E_{pot} = \frac{p^2}{2m\ell^2} - mg\ell \cos q.$$

Then

$$\begin{cases} \frac{\partial H}{\partial p} = \frac{p}{m\ell^2} = \dot{q} \\ -\frac{\partial H}{\partial q} = -mg\ell \sin q \qquad \underbrace{=}_{\text{Newton's eq. of motion}} \qquad \ell^2 m \ddot{q} = \dot{p}. \end{cases}$$

From Lagrangian point of view with $v = \dot{q}$:

$$\text{Lagrangian } = L(v,q) = E_{kin} - E_{pot} = \frac{m\ell^2 v^2}{2} + mg\ell\cos q.$$

The Euler-Lagrangian equation gives

$$\begin{aligned} & 0 = \nabla_q L - \frac{d}{dt}\nabla_v L = -mg\ell\sin q - \frac{d}{dt} m\ell^2 \dot{q} \\ \Leftrightarrow \quad & 0 = \frac{g}{\ell}\sin q + \ddot{q}, \end{aligned}$$

which is Newton's equation of motion.

Now we compare the Hamiltonian and Lagrangian using the Legendre transform. Start with $v = \dot{q}$ and $L(v,q) = \frac{m\ell^2 v^2}{2} + mg\ell\cos q$. Take the Legendre transform to get the Hamiltonian:

$$H(p,q) = \sup_{v\in\mathbb{R}} p \bullet v - \left(\frac{m\ell^2 v^2}{2m} + mg\ell\cos q\right).$$

Set the derivative $\frac{d}{dv} = 0$: $p - m\ell^2 v = 0$, so $v = \frac{p}{m\ell^2}$. (Thus $p = m\ell^2 v = m\ell\dot{q}$ – this somewhat strange form of the momentum is a result of Legendre transforming a somewhat strange form of velocity, which is really an angular velocity of physical dimension [1/sec]). Insert

$$\begin{aligned} H(p,q) &= \frac{p^2}{m\ell^2} - \frac{m\ell^2 p^2}{2m^2\ell^4} - mg\ell\cos q \\ &= \frac{p^2}{2m\ell^2} - mg\ell\cos q \\ &= E_{kin} + E_{pot}. \end{aligned}$$

Relativistic pendulum

In order to illustrate that the relation between Lagangian and Hamiltonian can be more complicated than just flipping the sign of the potential energy, we consider the pendulum within the framework of Einstein's relativistic mechanics.

$$\begin{cases} E_{kin} = m(v)c^2, \\ E_{pot} = E_{pot}(q). \end{cases} \qquad m(v) = m_0\sqrt{1 + \tfrac{|v|^2}{c^2}}.$$

Here c is the speed of light, v the velocity and m_0 the rest-mass of the particle. Note that $v = \dot{q}$ is a vector, and $|v| = \sqrt{\sum_i v_i^2}$.

The Lagrangian is defined as $L(v,q) = E_{kin} - E_{pot}$, and therefore

$$\frac{\partial}{\partial v_i} L(v,q) = \frac{m_0 c^2}{2\sqrt{1 + \frac{|v|^2}{c^2}}} \frac{2v_i}{c^2} = \frac{m_0 v_i c}{\sqrt{c^2 + |v|^2}}.$$

Therefore the Euler-Lagrange equations become

$$\nabla_q E_{pot}(q) = \frac{d}{dt}\frac{m_0 c\dot{q}}{\sqrt{c^2 + |\dot{q}|^2}}.$$

We find the Hamiltonian via the Legendre transform:

$$H(p,q) = L^*(v,q) = \sup_{v\in\mathbb{R}^n} v \bullet p - \left(m_0 c^2 \sqrt{1 + \frac{|v|^2}{c^2}} - E_{pot}(q) \right). \tag{7}$$

Solve $\nabla_v L(v,q) = 0$:

$$p = \frac{m_0 c v}{\sqrt{c^2 + |v|^2}} \qquad \Rightarrow \qquad |v|^2 = \frac{c^2 |p|^2}{m_0^2 c^2 - |p|^2}.$$

Insert in (7)

$$\begin{aligned} H(p,q) &= \frac{c|p|^2}{\sqrt{m_0^2 c^2 - |p|^2}} - m_0 c^2 \sqrt{1 + \frac{|p|^2}{m_0^2 c^2 - |p|^2}} + E_{pot}(q) \\ &= -c\sqrt{m_0^2 c^2 - |p|^2} + E_{pot}(q). \end{aligned}$$

Noether's Theorem and First Integrals

Not only energy can supply the Hamiltonian; there are often other preserved (additional) quantities, e.g. momentum, angular momentum. Such constants of motion are called **first integrals**. These can often be related to a **continuous symmetry** in the system. This is the content of Noether's Theorem.

Let $Q(s,q)$ be a continuous family of motions in $\mathbb{R}^n$. For instance, rotations in $\mathbb{R}^2$ over an angle s:

$$Q : (s,q) \mapsto \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} q_1 & q_2 \end{pmatrix}.$$

Formally, $Q(0,q) = q$ and $Q(s+t,q) = Q(s,Q(t,q))$ for all $q \in \mathbb{R}^n$ and $s,t \in \mathbb{R}$. Therefore we can describe $Q(s,q)$ as the solution of a flow of a vector field, say $f : \mathbb{R}^n \to \mathbb{R}^n$:

$$\begin{cases} \frac{\partial}{\partial s} Q(s,q) = f(Q(s,q)), \\ Q(0,q) = q. \end{cases} \tag{8}$$

The corresponding action on tangent vectors $v = \dot{q} \in T_q\mathbb{R}^n$ (which is isomorphic to $\mathbb{R}^n$) is

$$v \mapsto D_q Q(s,q) v,$$

with s-derivative at $s = 0$:

$$\frac{\partial}{\partial s} D_q Q(s,q)|_{s=0} = D_q \frac{\partial}{\partial s} Q(s,q)|_{s=0} = D_q f(Q(s,q))|_{s=0} = D_q f(q). \tag{9}$$

Definition 4. *A vector field f (with $\frac{\partial}{\partial s}(Q(s,q)) = f(Q(s,q))$) generates a* **symmetry** *of a Lagrangian system if*

$$L(v,q) = L(D_q Q(s,q) v, Q(s,q)) \tag{10}$$

for all $s \in \mathbb{R}$ and $q \in \mathbb{R}^n$, $v \in T_q\mathbb{R}^n$. That is, L remains constant over actions of the symmetry.

Theorem 5 (Noether's Theorem)**.** *If f generates a symmetry, then*

$$\mathcal{I}(v,q) := \nabla_v L(v,q) \bullet f(q)$$

is a first integral.

Proof. Differentiate (10) with respect to s at $s=0$. Because L is constant over the action of $Q(s,q)$, we obtain

$$\begin{aligned} 0 &= \frac{d}{ds} L(D_q Q(s,q)v, Q(s,q))|_{s=0} \\ &= \nabla_v L \bullet \frac{d}{ds} D_q Q(s,q)v + \nabla_q L \bullet \underbrace{\frac{d}{ds} Q(s,q)}_{=f(q)} |_{s=0} \end{aligned} \tag{11}$$

Hence

$$\begin{aligned} \dot{\mathcal{I}}(v,q) &= \frac{d}{dt}\nabla_v L(v,q) \bullet f + \nabla_v L(v,q) \bullet \underbrace{\left(\nabla_q f \bullet \underbrace{\frac{d}{dt} q}_{v}\right)}_{\frac{d}{ds} D_q Q(s,q)v|_{s=0} \text{ by } (9)} \\ &\underbrace{=}_{\text{by } (11)} \frac{d}{dt}\nabla_v L \bullet f(q) - \nabla_q L \bullet f(q) \\ &\underbrace{=}_{\text{by } (1)} \left(\frac{d}{dt}\nabla_v L - \nabla_q L\right) \bullet f(q) = 0. \end{aligned}$$

Hence $\mathcal{I}$ is constant as claimed. □

Examples: Assume that $q \in \mathbb{R}^n$ and $L(v,q) = E_{kin}(v) - E_{pot}(q)$ (with $E_{kin} = \frac{mv^2}{2}$) does **not** depend on the coordinate q_j. Take $Q(s,q) = q + se_j$, so $Q(0,q) = q$ and the corresponding vector field is

$$f(q) = e_j \qquad \text{the } j\text{-th basis vector in } \mathbb{R}^n.$$

Then

$$\begin{aligned} \mathcal{I}(v,q) &= \nabla_v L(v,q) \bullet f(q) \\ &= \nabla_v (E_{kin}(v) - E_{pot}(q)) \bullet f(q) \\ &= mv \bullet e_j = mv_j = p_j \end{aligned}$$

is the j-th component of the momentum. That is: the j-th component of the momentum is preserved.

For rotational symmetries $q \mapsto Q(s,q) = A(s)q$ for a family of skew-symmetric matrices, e.g., $A(s) = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}$, the preservation of

$$\mathcal{I}(v,q) = mv^T A(0)q.$$

leads to the preservation of (a component of) the angular momentum.

Definition 6. *When a Hamiltonian system in $\mathbb{R}^{2n}$ has n first integrals $\mathcal{I}_j$, $j = 1, \ldots, n$, then the system is called* **integrable**. *This doesn't mean that you can solve the system explicitly (in general), but that trajectories are confined to the intersection of level sets*

$$\bigcap_{j=1}^{n} \{(v, q) : \mathcal{I}_j(v, q) = a_j\}.$$

When these level sets are compact, they are almost surely (i.e., for almost all values a_j) n-dimensional tori.

Figure 4: Parallel flow on the torus and on nested tori.

As an example, note that constant parallel motion

$$\dot{x} = \vec{C}$$

on a d-dimensional torus is dense in a $d - i$-dimensional torus if the d coordinates of $\vec{C}$ have i rational relations among them. For instance, if $\vec{C} = (c_1, c_2)^T$, then the motion is periodic if $ac_1 = bc_2$ for some $(a, b) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, and dense fills the 2-dimensional torus if not.

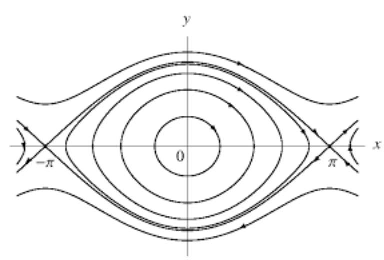

Figure 5: The phase portrait of the pendulum.

A second example is the pendulum, with $n = 1$ and $H = E_{kin} + E_{pot}$ a first integral. Therefore this system is integrable. In Figure 5, if we take $x = q \mod 2\pi \in [-\pi, \pi]$, then all orbits are compact in the cylinder. All the orbits are topological circles, except the two equilibria at $(0, 0)$ and $(\pm\pi, 0)$.

0.1 The n-body problem

The 2-body problem (Kepler's Problem) is about the motio of two particles (e.g. the Sun and the Earth) moving in ech other's gravitational field in $\mathbb{R}^3$, and no other forces around. Kepler, as well known in science history, gave three laws to describe the motion of the planets:

1. The orbit of a planet is an ellipse with the Sun at one of the two foci.

2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time:

3. The square of the orbital period T of a planet is directly proportional to the cube of the semi-major axis a of its orbit:
$$T^2 \sim a^3.$$

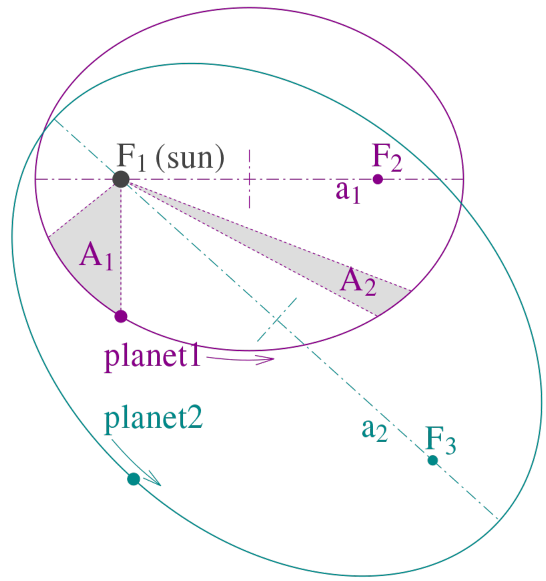

Figure 6: Illustration of Kepler's Laws.

Here the dimension of the problem is $n = 6$ (3 space coordinates $\vec{q}_S$ of the Sun, 3 space coordinates $\vec{q}_E$ of the Earth), and there are (more than) 6 first integrals:

- the momentum vector $p^* = \vec{p}_S + \vec{p}_E$ (3 components), where $\vec{p}_S = m_S \frac{d}{dt}\vec{q}_S$ and $\vec{p}_E = m_E \frac{d}{dt}\vec{q}_E$;
- the angular momentum vector $\ell^* = \vec{p}_S \times \vec{q}_S + \vec{p}_E \times \vec{q}_E$. (This is equivalent to Kepler's second law).

Therefore all bounded motions are confined to a 6-dimensional torus (integrable).

There are, however, more first integrals, each of them reducing the dimension in which orbits

- From the Galilei transformations $(\vec{v}, \vec{q}) \mapsto (\vec{v} - \vec{v}_0, \vec{q} - t\vec{v}_0)$ (which allows us to put $p^* = 0$).
- Energy $E_{kin} + E_{pot}$.
- The Laplace-Runge-Lenz vector $p^* \times \ell^* - \alpha \frac{\vec{q}}{|\vec{q}|}$, where the constant α is depends on the masses of Sun and Earth an don the gravitational constant G in Newton law of attraction:

$$F_{grav} = G\frac{m_S m_E}{r^2}. \qquad G \approx 6.67410^{-11} \frac{\mathrm{m}^3}{\mathrm{kg.sec}^2}. \tag{12}$$

  This first integral exists precisely because the gravitation force is inverse square proportional to the mutual distance between Sun and Earth.

This reduces the orbits to 1-dimensional tori, i.e., ellipses, as Kepler's first law states (or to hyperbolas or parabolas if the orbit is not compact). In fact, also Kepler's third law is a consequence of the preservation of the Laplace-Runge-Lenz vector (so again of the precise form of the Newtonian gravitation).

Let us try to derive the elliptical motion (i.e., solve Kepler's problem) with a bit of physical insight:

$$\begin{cases} m_S\vec{q}_S + m_E\vec{q}_E = 0 & \text{the center of mass fixed at the origin;} \\ p^* = m_S\vec{v}_S + m_E\vec{v}_E = 0 & \text{preservation of momentum;} \\ \ell^* = \vec{p}_S \times \vec{q}_S + \vec{p}_E \times \vec{q}_E = c\vec{e}_3 & \text{preservation of angular momentum,} \\ & \text{we choose coordinates to keep orbits in } x, y\text{-plane.} \end{cases}$$

Use polar coordinates for

$$\vec{r} := \vec{q}_S - \vec{q}_E = r\cos\phi\ \vec{e}_1 + r\sin\phi\ \vec{e}_2. \qquad\qquad r = |\vec{r}|.$$

Since the origin is the center of mass, we have

$$\vec{q}_S = \frac{m_E}{m_E + m_S}\vec{r}, \qquad \vec{q}_E = -\frac{m_S}{m_E + m_S}\vec{r}.$$

Now we compute the angular momenta of Sun and Earth explicitly:

$$\begin{aligned} \ell_S &= m_S\frac{m_E}{m_S + m_E}\begin{pmatrix} \frac{d}{dt}(r\cos\phi) \\ \frac{d}{dt}(r\sin\phi) \\ 0 \end{pmatrix} \times \frac{m_E}{m_S + m_E}\begin{pmatrix} r\cos\phi \\ r\sin\phi \\ 0 \end{pmatrix} \\ &= m_S\left(\frac{m_E}{m_S + m_E}\right)^2\begin{pmatrix} \dot{r}\cos\phi - r\dot{\phi}\sin\phi \\ \dot{r}\sin\phi + r\dot{\phi}\sin\phi \\ 0 \end{pmatrix} \times \begin{pmatrix} r\cos\phi \\ r\sin\phi \\ 0 \end{pmatrix} \\ &= m_S\left(\frac{m_E}{m_S + m_E}\right)^2\begin{pmatrix} 0 \\ 0 \\ r^2\dot{\phi} \end{pmatrix}. \end{aligned}$$

Similarly

$$\ell_E = m_E\left(\frac{m_S}{m_S + m_E}\right)^2\begin{pmatrix} 0 \\ 0 \\ r^2\dot{\phi} \end{pmatrix}.$$

Thus

$$|\ell^*| := |\ell_S| + |\ell_E| = m^* r^2\dot{\phi}, \qquad\qquad m^* = \frac{m_S m_E}{m_S + m_E}. \tag{13}$$

Next we compute the kinetic energies of Sun and Earth explicitly:

$$\begin{aligned} E_{kin,S} &= \frac{m_S|\vec{v}_S|^2}{2} = \frac{m_S}{2}\left|\begin{pmatrix} \frac{d}{dt}(r\cos\phi) \\ \frac{d}{dt}(r\sin\phi) \\ 0 \end{pmatrix}\right|^2 \\ &= \frac{m_S}{2}\left(\frac{m_E}{m_S + m_E}\right)^2(\dot{r}^2 + r^2\dot{\phi}^2), \end{aligned}$$

and similarly

$$E_{kin,E} = \frac{m_E}{2}\left(\frac{m_S}{m_S + m_E}\right)^2(\dot{r}^2 + r^2\dot{\phi}^2).$$

Their combined potential energies are

$$E_{pot} = -\frac{m_S m_E G}{r}.$$

Adding all the energies together:

$$E = \frac{m^*}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{m_S m_E G}{r}. \tag{14}$$

Solve r as function of ϕ, using (13) (i.e., $|\ell^*| = m^* r^2 \dot{\phi}$):

$$\dot{r} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\dot{\phi} = -\frac{dr}{d\phi}\frac{|\ell^*|}{m^* r^2}.$$

Inserted in (14):

$$E = \frac{|\ell^*|^2}{2m^*}\left(\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{r^2}\right) - \frac{m_S m_E G}{r}.$$

We can simplify this differential equation by taking $r = \frac{1}{\rho}$, so $\frac{dr}{d\phi} = -\frac{1}{\rho^2}\frac{d\rho}{d\phi}$. This gives

$$E = \frac{|\ell^*|^2}{2m^*}\left(\left(\frac{d\rho}{d\phi}\right)^2 + \rho^2\right) - m_S m_E G\ \rho.$$

Next split off the square:

$$\left(\frac{d\rho}{d\phi}\right)^2 + (\rho - A)^2 = B^2, \qquad \begin{cases} A = \frac{m^* m_S m_E G}{|\ell^*|^2}, \\ B = A^2 + \frac{2m^* E}{|\ell^*|}. \end{cases}$$

Introduce a new variable ψ by setting $\rho = A + B\cos\psi$:

$$B^2 \sin^2\psi \left(\frac{d\psi}{d\phi}\right)^2 + B^2\cos^2\psi = B^2 \qquad \Rightarrow \qquad \left(\frac{d\psi}{d\phi}\right)^2 = 1.$$

We choose the positive solution $\frac{d\psi}{d\phi} = 1$ (the negative solution will result in the same shape of orbit, traversed in the opposite direction). Therefore

$$\psi = \psi(\phi) = \phi - \phi_0, \quad \rho(\phi) = A + B\cos(\phi - \phi_0),$$

and

$$r(\phi) = \frac{A^{-1}}{1 + \varepsilon\cos(\phi - \phi_0)} \qquad \text{with eccentricity } \varepsilon = \frac{B}{A} = \sqrt{1 + \frac{2E|\ell^*|}{m^*(m_S m_E)^2 G}}.$$

In Cartesian coordinates $x = r(\phi)\cos(\phi - \phi_0)$, $y = r(\phi)\sin(\phi - \phi_0)$, this is

$$\begin{aligned}
(1-\varepsilon^2)x^2 + y^2 &= r^2\left((1-\varepsilon^2)\cos^2(\phi-\phi_0) + \sin^2(\phi-\phi_0)\right) \\
&= \frac{A^{-2}}{(1+\varepsilon\cos(\phi-\phi_0))^2}\left(1 - \varepsilon^2\cos^2(\phi-\phi_0)\right) \\
&= \frac{1}{A^2}\frac{1-\varepsilon\cos(\phi-\phi_0)}{1+\varepsilon\cos(\phi-\phi_0)} \\
&= \frac{1}{A^2}\frac{1+\varepsilon\cos(\phi-\phi_0) - 2\varepsilon\cos(\phi-\phi_0)}{(1+\varepsilon\cos(\phi-\phi_0))} = \frac{1}{A^2} - \frac{2\varepsilon x}{A}.
\end{aligned}$$

which gives

$$(1-\varepsilon^2)x^2 + \frac{2\varepsilon x}{A} + y^2 = A^2.$$

Finally, for the eccentricity we have:

$$\varepsilon \begin{cases} < 1 & \text{if } E < 0 \text{ and we have an ellipse,} \\ = 1 & \text{if } E = 0 \text{ and we have a parabola,} \\ > 1 & \text{if } E > 0 \text{ and we have a hyperbola.} \end{cases}$$

Figure 7: Paris mathematicians with an L, in alphabetical and chronological order.

The mathematical explanation (in first place due to Newton) for Kepler's Laws was a big triumph of the use of mathematics to understand the world. In the wake of the Enlightening (Aufklärung) philosophical movement of the late 17th century, it gave ground for an optimistic view on determinism, i.e., the view that in nature everything is predictable. This was most famously expressed by Laplace:

> *We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect*[1] *which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.* [2]

However, the three body problem, and more generally the eight body problem (i.e., the Solar system, Neptune was discovered in 1848, making it a nine body problem, Pluto was only discovered in 1930) stubbornly refused to be integrable, at least all the efforts of 19th century mathematics went in vain. As such, it remained an open question whether the planets would circle around the Sun for ever, or actually have unstable orbits, maybe swinging off to outer space, or colliding with one another at some point in the future. The Swedish mathematician Mittag-Leffler persuaded King Oscar II to arrange an international prize competition in mathematics honouring the king's 60th birthday in 1884. This competition was won by the French

[1]also called: Laplace's demon

[2]Pierre Simon Laplace, A Philosophical Essay on Probabilities

mathematician Henri Poincaré with a lengthy essay[3] which can be seen as the beginning of "Dynamical Systems", and in which he showed that predicting motions as complicated as the three body problem can, in general and to any degree of accuracy and in spite of "determinism", not be accomplished. The reason is the occurrence of sensitive dependence on initial conditions and of horseshoes, but the words for these notions emerge much later.

Modern computer simulations confirm that there is no way of saying whether, for instance, the Earth and Mars will collide within the next 50 million years or not. Computers also found some very unexpected periodic orbits (choreographies) for three or more bodies, see Figure 8.

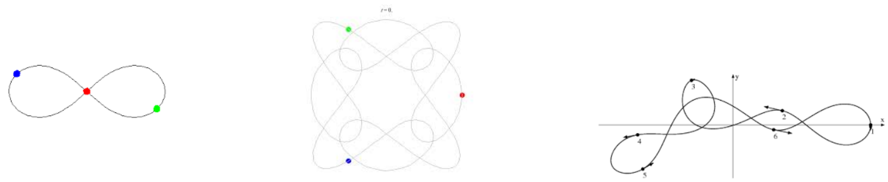

Figure 8: Some possible periodic orbits for three (and even six) bodies.

Exercise 7. *Show that if the Hamiltonian* $H = E_{kin}(p) + E_{pot}(q)$ *and* $E_{kin} = \frac{p^2}{2m}$*, then the Lagrangian is* $L = E_{kin}(p) - E_{pot}(q)$*.*

Exercise 8. *Given is the differential equation*

$$\dot{x} = C, \qquad x \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2, \; C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \mathbb{R}^2.$$

Show that:

- *the return map* $F : \mathbb{S}^1 \to S^1$ *of the flow* φ^t *to the circle* $\mathbb{S}^1 = \{x \in \mathbb{T}^2 : x_1 = 0\}$ *is a rotation over* c_1/c_2*.*
- *hence all orbits of* F *are dense if and only if* $c_1/c_2 \notin \mathbb{Q}$*.*
- *hence all orbit of* φ^t *are dense if and only if* $c_1/c_2 \notin \mathbb{Q}$*.*

What does this say about the Poincaré-Bendixson theorem on the torus $\mathbb{T}^2$*?*

Exercise 9. *Assume that* X_H *is a Hamiltonian vector field in* $\mathbb{R}^2$*:*

- *Show that equilibria of* X_H *can only be centers or saddles.*
- *Which bifurcations (of the ones we treated in class) can occur in a family of Hamiltonian vector fields?*
- *Find a family of Hamiltonians* $H_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ *such that at* $\varepsilon = 0$*, a saddle becomes a center.*

Exercise 10. *A Lagrangian system in* $\mathbb{R}^3$ *has the Lagrangian*

$$L(v, q) = \frac{v_1^2 + v_2^2 + v_3^2}{2} - \frac{q_1^2 + q_2^2 + q_3^3}{2}.$$

Use Noether's Theorem to find first integrals. Is the system integrable?

[3]Poincaré discovered a mistake in the first version of his essay, corrected it, and had to spend all his prize money to recall the wrong first version that was already in print.

Exercise 11. *We have a Hamiltonian system in coordinates* $(x, y) \in \mathbb{R}^2$ *where the Hamiltonian has the form*

$$H(x,y) = \frac{y^2}{2} + V(x), \qquad V \text{ is } C^2\text{-smooth,}$$

and assume that $V(x) = V(-x)$ *has* $V''(0) > 0$. *This means that* $(0,0)$ *is*

(a) *Show that* $(0,0)$ *is a center, with periodic motion around it.*

(b) *Let* $T(a)$ *be the period of the orbit starting at* $(a, 0)$. *Show that*

$$T(a) = \int_0^a \frac{4}{\sqrt{2(V(a) - V(x))}} dx.$$

Hint: Integrate $T(a) = \int_{t_1}^{t_2}$ *a quarter of the periodic orbit and invert* $t = t(x)$ *(instead of* $x = x(t)$*) to rewrite the integral.*

- *Show that* $T(a) = 2\pi$ *is constant for* $V(x) = \frac{x^2}{2}$ *(harmonic oscillator).*
- *Show that* $T(a)$ *is increasing if* $V(x) = -\cos x$ *(pendulum), and find* $\lim_{a \searrow 0} T(a)$ *and* $\lim_{a \nearrow 0} T(a)$.

Solution to Exercise 9: By Liouville's Theorem, the Hamiltonian flow preserves area, so there cannot be sinks or saddles (because locally these contract resp. expand area). As such, fold bifurctions and Hopf bifurcations cannot occur, neither pitchfork bifurcations. However, for the Hamiltonian $H(x,y) = \frac{y^2}{2} + \varepsilon x^2 + x^4$, the center at $(0,0)$ turns into saddle as ε becomes negative. The eigenvalues of this equilibrium, which are two purely imaginary complex conjugates, first decrease to 0, and then become real (one postive, the other negative).

Solution to Exercise 11: (a) Equilibria at $(x,0)$ where $V'(x) = 0$, and $DX_H(x,0) = \begin{pmatrix} 0 & 1 \\ -V''(x) & 0 \end{pmatrix}$. Since $V''(0) > 0$, the eigenvalues are $\lambda_\pm = \pm\sqrt{V''(0)}i$, so purely imaginary, and therefore 0 is a center.

(b) Let t_1 and t_2 be the points in time where the orbit through $(a,0)$ hits the positive vertical resp. positive horizontal axis. We have, by change of coordinates $x = x(t)$.

$$T(a) = 4\int_{t_1}^{t_2} dt = \int_0^a \frac{dt}{dx}\, dx = \int_0^a \frac{1}{y}\, dx.$$

We compute y from $H = \frac{y^2}{2} + V(x)$, so $y = \sqrt{2(H - V(x))}$. But $H(x,y) = H(y,0) = V(a)$ because it is a constant of motion. Therefore

$$T(a) = 4\int_0^a \frac{1}{\sqrt{2(V(a) - V(0))}}\, dx.$$

(c) Take $V(x) = \frac{x^2}{2}$. Then, using $x = a\sin\psi$,

$$T(a) = 4\int_0^a \frac{1}{\sqrt{a^2 - x^2}}\, dx = 4\int_0^{\frac{\pi}{2}} \frac{a\sin\psi}{\sqrt{a^2(1 - \sin\psi^2)}}\, d\psi = 4\int_0^{\frac{\pi}{2}} d\psi = 2\pi.$$

(d) For $V(x) = -\cos x$ we have $T(a) = 4\int_0^a \frac{1}{\sqrt{2(\cos(x) - \cos(a))}}\, dx$. This integral cannot be solved with an explicit expression, ut for $0 \le x \le a$ both small, we can use the Taylor expansion:

$$\lim_{a\to 0} T(a) = \lim_{a\to 0} 4\int_0^a \frac{1}{\sqrt{2(1 - \frac{x^2}{2} - (1 - \frac{a^2}{2}))}}\, dx = \lim_{a\to 0} 4\int_0^a \frac{1}{\sqrt{a^2 - x^2}}\, dx = 2\pi$$

as in (c). For $a \to \pi/2$, we simply plug in the value:

$$\lim_{a\to\frac{\pi}{2}} T(a) = 4\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\cos x}}\, dx = 4\int_0^{\frac{\pi}{2}} \frac{\cos x}{(\cos x)^{3/2}}\, dx.$$

Change coordinates $u = \sin x$ and then $v = 1 - u$:

$$\lim_{a\to\frac{\pi}{2}} T(a) = 4\int_0^1 \frac{du}{(1-u^2)^{3/2}} = 4\int_0^1 v^{-3/2}(2-v)^{-3/2} dv \ge \sqrt{2}\int_0^1 v^{-3/2} dv = \infty.$$