Definition 1 *A map f is C^r-***structurally stable** *if there is $\varepsilon > 0$ such that every map g with $\|f-g\|_{C^r} < \varepsilon$ is conjugate to f. Here $\|h\|_{C^r}$ is the C^r-norm, i.e., $\|h\|_{C^r} = \max_{i=0,\dots,r} \sup_x |h^{(i)}|$, where $h^{(i)}$ denotes the i-th derivative.*

Theorem 2 *The map $Q_a(x) = ax(1-x)$ is C^2-structurally stable for $a \in (0,1)$.*

Proof. Compute $Q_a'(x) = a(1-2x)$ and $Q_a''(x) = -2a$. Fix $a \in (0,1)$. If $\|Q_a - g\|_{C^2} < \varepsilon$, then we can write $g = Q_a + \varepsilon h$ where $h = h_\varepsilon$ and $\|h\|_{C^2} < 1$. We need to find a homeomorphism $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi \circ Q_a = g \circ \psi$.

Clearly Q_a has a fixed point at 0. Set $F(x,\varepsilon) = Q_a(x) + \varepsilon h(x) - x$. Then $\frac{\partial F}{\partial x}(0,0) = a-1 \neq 0$, so the Implicit Function Theorem tells us that there is a neighbourhood $U \ni 0$ and a continuous function $p : U \to \mathbb{R}$ with $p(0) = 0$, such that $F(p(\varepsilon),\varepsilon) = g(p(\varepsilon)) - p(\varepsilon) = 0$. That is, $p = p(\varepsilon)$ is a fixed point of g. Set $\psi(0) = p(\varepsilon)$.

Also assume that $\varepsilon < \min\{a, 1-a\}$ is so small that $g'(p(\varepsilon)) = a(1-2p(\varepsilon)) + \varepsilon h'(p(\varepsilon)) < 1$. Therefore, the fixed point $p(\varepsilon)$ of g is attracting.

Next note that $g''(x) = -2a + \varepsilon h''(x) < 0$ (because $\sup_x |h''(x)| < 1$). Therefore f is strictly convex, and it has therefore a unique global maximum $m = m(\varepsilon)$ with $m(0) = \frac{1}{2}$. Set $\psi(\frac{1}{2}) = m(\varepsilon)$.

Since $g'(x)$ is decreasing on (p,m) and $0 < g'(p) < 1$, we have $p < g(m) < m$. We call $(g(m), m]$ a **fundamental domain** for g.

Note that for $\varepsilon = 0$, so $g = Q_a$, $Q_a(m) = Q_a(\frac{1}{2}) = \frac{a}{4}$. Set $\psi(\frac{a}{4}) = g(m)$ and extend $\psi : [\frac{a}{4}, \frac{1}{2}] \to [g(m), m]$ in some homeomorphic (e.g. affine) way.

For every $x \in (0, \frac{1}{2}]$, there is a unique $y \in (\frac{a}{4}, \frac{1}{2}]$ and $n \geq 0$ such that $Q_a^n(y) = x$. Set $\psi(x) = g^m(\psi(y))$. This defines ψ continuously on $[0, \frac{1}{2}]$ so that $\psi \circ Q_a = g \circ \psi$.

Since $m > p$ and $g'(x) < 0$, there is a unique $q = q(\varepsilon) > m$ such that $g(q) = p$. Clearly $q(0) = 1$, and $g : [m,q] \to [g(m), p]$ is a strictly decreasing homeomorphism. For $x \in (\frac{a}{4}, 1]$, let $\psi(x)$ the unique point in $(m,q]$ such that $\psi(Q_a(x)) = g(\psi(x))$. We now have $\psi : [0,1] \to [p,q]$ continuous (and in fact strictly increasing) such that $\psi \circ Q_a = g \circ \psi$.

Using similar arguments, we can extend ψ to $(-\infty, 0) \cup (1, \infty)$. $\square$