Normal Numbers

Introductory question:

Which numbers in $[0, 1]$ are truly random?

Random in base 10 would mean:

$$x = 0.x_1 x_2 x_3 x_4 \dots \qquad x_i \in \{0, 1, 2, \dots, 9\}$$

and the **frequency** of every digits is $\frac{1}{10}$:

$$\lim_{n\to\infty} \frac{1}{n} \#\{1 \le i \le n : x_i = a\} = \frac{1}{10} \text{ of each } a \in \{0, 1, 2, \dots, 9\}.$$

In fact, the **frequency** of every block $d_1 \dots d_k$ of k digits is 10^{-k}:

$$\lim_{n\to\infty} \frac{1}{n} \#\{1 \le i \le n : x_i x_{i+1} \dots x_{i+k-1} = d_1 d_2 \dots d_k\} = 10^{-k}.$$

Normal Numbers

In more generality, a number is called normal in base $b \geq 2$ if the **frequency** of every block $d_1 \dots d_k \in \{0, \dots, , b-1\}^k$ is b^{-k}:

$$\lim_{n\to\infty} \frac{1}{n} \#\{1 \leq i \leq n : x_i x_{i+1} \dots x_{i+k-1} = d_1 d_2 \dots d_k\} = 10^{-k}.$$

Emile Borel proved in 1909 that Lebesgue-a.e., is normal w.r.t. every base $b \geq 2$, based on an intricate use of the (now called) Borel-Cantelli Lemma.

But it is not trivial to find any normal number, even in base 10.

Exercise 1.1: Every rational number $x \in [0,1] \cap \mathbb{Q}$ is not normal w.r.t. any base.

Question: Is $\pi - 3$ normal? Or $e - 2$? Or $\sqrt{2} - 1$?

Normal Numbers

The best known example of a normal number is Champernowne's Number:

$x = 0.1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 22\ldots$

but it doesn't look random at all! Similar normal numbers can be obtained by concatenating the primes $0.2\ 3\ 5\ 7\ 11\ 13\ldots$ (Copeland & Erdös, 1946) or the squares $0.1\ 4\ 9\ 16\ 25\ 36\ldots$ (Besicovich, 1953).

A general result (well outside the scope of this lecture):

Theorem: For every polynomial p with real coefficients so that $p(\mathbb{R}^+) \subset \mathbb{R}^+$,

$$0.[p(1)]\ [p(2)]\ [p(3)]\ [p(4)]\ [p(5)]\ [p(6)]\ [p(7)]\ [p(8)]\ \ldots$$

is a normal number in base 10. ($[x]$ is the integer part of x.)

Dynamical Systems

Let (X, T) be dynamical system:

- X is the space, usually compact metric,
- T is the evulation rule, usually a continous (or at least piecewise continuous) map from X to itself, which we iterate. That is, we study the orbit of points $x \in X$:

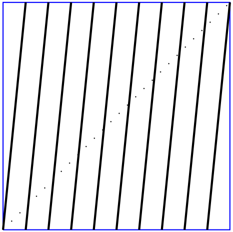
$$\mathrm{orb}(x) = x, T(x), \underbrace{T \circ T(x)}_{T^2(x)}, \underbrace{T \circ T \circ T(x)}_{T^3(x)}, \dots$$

A point x is periodic if $T^p(x) = x$ for some $p \geq 1$ (the minimal such p is called the period. If $p = 1$, then x is a fixed point. If x is not periodic, but $T^n(x)$ is periodic, it is called eventually periodic.

Orbits can be very chaotic; even the slightest computation or round-off error can blow up rapidly. Therefore orbits are in general very difficult to computed with useful accuracy. In Ergodic Theory we study the average behaviour of orbits.

Dynamical Systems

Example:



$T : [0, 1) \to [0, 1)$
$T(x) = 10x \bmod 1$
On $\mathbb{S}^1$, T is continuous!

The fixed points are $x = k/9$, $k = 0, \ldots, 8$. The points $y = k/10$, $k = 0, \ldots, 9$ are prefixed.

Take $x = \sqrt{2} - 1$. Is $T^{1000}(x) > \frac{1}{2}$ or $< \frac{1}{2}$? For this you need to know digit x_{1001} of x!

Dynamical Systems

Exercise 1.2: Prove that $T^{1000} \neq \frac{1}{2}$ for $x = \sqrt{2} - 1$.

Exercise 1.3: Prove: $x \in [0, 1] \cap \mathbb{Q}$ if and only if x is (eventually) periodic.

Exercise 1.4: Show that Chapernowne's number x has are current orbit: $x \in \overline{\mathrm{orb}(T(x))}$.

Exercise 1.5: Show that a normal number in base 10 has a dense orbit under $T(x) = 10x \bmod 1$.

Invariant measures

Definition: A measure is T-invariant if $\mu(T^{-1}(A)) = \mu(A)$ for every set A in the algebra of μ-measurable sets.

In the example $T(x) = 10x \bmod 1$, Lebesgue measure is T-invariant, because if $A = (a, b) \subset [0, 1]$,

$$b - a = \mathrm{Leb}(A) = 10 * \frac{b - a}{10} = \mathrm{Leb}(T^{-1}(A)).$$

Since the Borel sets are generated by the open sets, $\mathrm{Leb}(B) = \mathrm{Leb}(T^{-1}(B))$ for every Borel set. Also $\mathrm{Leb}(N) = \mathrm{Leb}(T^{-1}(N))$ for every null-set N, so Leb is T-invariant.

Dynamical Systems

To simplify our lives a bit, we switch to the doubling map:

$$T : \mathbb{S}^1 \to \mathbb{S}^1, \qquad T(x) = 2x.$$

Invariant measures are:

- Lebesgue measure;
- The Dirac measure δ_0 at the fixed point 0;
- Equidistributions of periodic orbits, e.g. $\frac{1}{2}(\delta_{1/3} + \delta_{2/3})$;
- A measure μ is called atomic if there is $x \in X$ such that $\mu(\{x\}) > 0$.
- Bernoulli measures are T-invariant.
- Convex combinations $\alpha\mu + (1-\alpha)\nu$ of invariant measures are invariant.
- Many many more...

Birkhoff's Ergodic Theorem

Birkhoff's Ergodic Theorem formalizes a frequent observation in physics:

Space Average = Time Average (for typical points).

This is expressed in the Birkhoff's Ergodic Theorem:

Theorem: Let μ be a probability measure and $\psi \in L^1(\mu)$. Then the ergodic average

$$\psi^*(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi \circ T^i(x)$$

exists μ-a.e., and ψ^* is T-invariant, i.e., $\psi^* \circ T = \psi^*$ μ-a.e.

If in addition μ is ergodic then

$$\psi^* = \int_X \psi \, d\mu \qquad \mu\text{-a.e.}$$

Recall that μ-a.e. means: all points $x \in X$, except for a null-set.

Birkhoff's Ergodic Theorem

For $T(x) = 10x \bmod 1$ and $\mu = \mathrm{Leb}$, we can apply Birkhoff's Theorem. We didn't define ergodic yet, but Lebesgue measure is indeed ergodic in this case.

Take $\psi = 1_{[0,\frac{1}{10})}$ (then $\psi \in L^1(\mu)$. Hence, for Lebesgue-a.e. x:

$$\begin{aligned} \frac{1}{10} &= \int_0^1 \psi \, d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi \circ T^k(x) \\ &= \lim_{n\to\infty} \frac{1}{n} \#\{1 \le k \le n : x_k = 0\}. \end{aligned}$$

Birkhoff's Ergodic Theorem

By varying the map ψ, we can get the frequency of the other digits, and the frequency of blocks of digits.

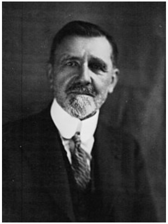


Figure: Emile Borel (1871-1956) and George Birkhoff (1884–1944).

Exercise 1.6: Complete the proof of Borel's result, using Birkhoff's Ergodic Theorem.