Toral Automorphisms

In this lecture we study hyperbolic toral automorphisms $T_A : \mathbb{T}^d \to T^d$ on the d-dimensional torus $\mathbb{T}^d = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, which are basically a linear map given by a matrix A, taken (mod 1) to fit on the torus.

The best know example is Arnol'd Catmap:

$$T_C(x,y) = C\begin{pmatrix} x \\ y \end{pmatrix} \pmod 1 \qquad \text{for the matrix } C = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

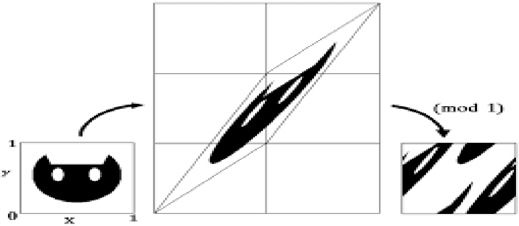

Figure: Arnol'd Catmap.

Toral Automorphisms

The name catmap comes solely from the fact that Vladimir Arnold used this picture of a cat's head in his book to illustrate the map.

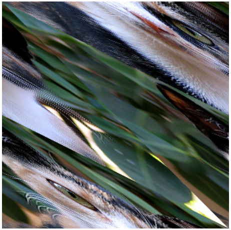

Figure: Catmap taken from Jason Davies' page, https://www.jasondavies.com/catmap/ (check iterate 348)

Although this map is extremely chaotic (unpredictable dynamics) and the cat is distorted beyond recognition, on the web you see simulations where the cat returns.

Exercise 10.1: Show that every rational point $x \in \mathbb{T}^2$ is periodic under T_C. Explain why this implies that the cat returns.

Toral Automorphisms

Definition: A toral automorphism $T : \mathbb{T}^d \to \mathbb{T}^d$ is an invertible linear map on the (d-dimensional) torus $\mathbb{T}^d$. Each such T is of the form $T_A(x) = Ax \pmod 1$, where the matrix A satisfies:

- A is an integer matrix with $\det(A) = \pm 1$;
- If the eigenvalues of A are not on the unit circle, then the toral automorphism is called hyperbolic.

For example, the eigenvalues of C are $\lambda_\pm = (3 \pm \sqrt{5})/2$, and the corresponding eigenspaces $E_\pm$ are spanned $(-1, (\sqrt{5}+1)/2)^t$ and $(1, (\sqrt{5}-1)/2)^t$. These are orthogonal (naturally, since C is symmetric), and have irrational slopes, so they wrap densely around the torus.

Toral Automorphisms

Properties of T_A are:

- To avoid degenerate examples including $A = Id$, we assume that A is primitive, i.e., A^n is strictly positive for some $n \geq 1$.
- A preserves the integer lattice $\mathbb{Z}^d$, so T_A is well-defined and continuous.
- $\det(A) = \pm 1$, so Lebesgue measure m is preserved (both by A and T_A). Also A and T_A are invertible, and A^{-1} is still an integer matrix (so T_A^{-1} is well-defined and continuous too).
- T_A fixes the origin, so δ_0 is an invariant measure, too. There are in fact many invariant measures (the Choquet simplex is Poulsen!).

Markov Partitions

Somewhat easier to treat than the catmap is T_A for $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, which is an orientation reversing matrix with $A^2 = C$.
The map T_A has a Markov partition, that is a partition $\{R_i\}_{i=1}^N$ for sets such that

1. The R_i have disjoint interiors and $\cup_i R_i = \mathbb{T}^d$;
2. If $T_A(R_i) \cap R_j \neq \emptyset$, then $T_A(R_i)$ stretches across R_j in the unstable direction (i.e., the direction spanned by the unstable eigenspaces of A).
3. If $T_A^{-1}(R_i) \cap R_j \neq \emptyset$, then $T_A^{-1}(R_i)$ stretches across R_j in the stable direction (i.e., the direction spanned by the stable eigenspaces of A).

Markov Partitions

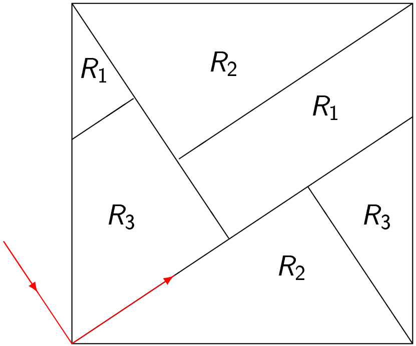

Figure: The Markov partition for T_A. The arrows indicate the stable and unstable directions at $(0, 0)$.

In fact, every hyperbolic toral automorphism has a Markov partition, but in general they are fiendishly difficult to find explicitly.

Markov Partitions

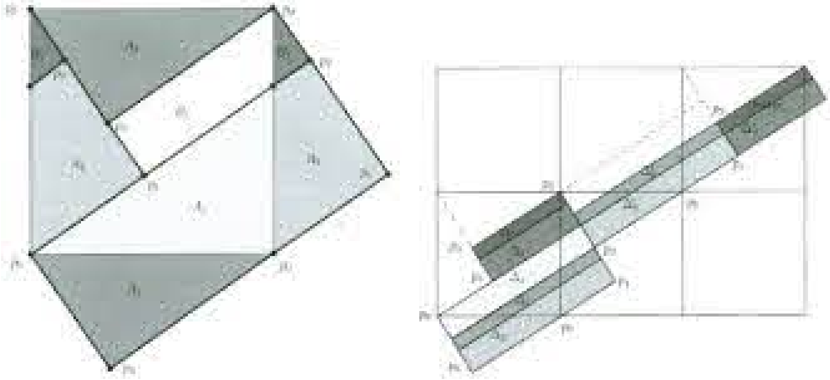

Figure: The Markov partition of Arnol'd catmap.

Markov Partitions

The corresponding transition matrix is

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ where } B_{ij} = \begin{cases} 1 & \text{if } T_A(R_i) \cap R_j \neq \emptyset \\ 0 & \text{if } T_A(R_i) \cap R_j = \emptyset. \end{cases}$$

The characteristic polynomial of B is

$$\begin{aligned} \det(B - \lambda I) &= -\lambda^3 + 2\lambda + 1 \\ &= -(\lambda + 1)(\lambda^2 - \lambda - 1) \\ &= -(\lambda + 1)\det(A - \lambda I). \end{aligned}$$

Note that B has the eigenvalues of A (no coincidence!), together with $\lambda = -1$.

Exercise 10:2 Find the transition matrix of the (same) Markov partition for the catmap T_C.

Markov Partitions

The transition matrix B generates a subshift of finite type:

$$\Sigma_B = \{(x_i)_{i\in\mathbb{Z}} : x_i \in \{1,2,3\}, B_{x_i x_{i+1}} = 1 \ \forall\ i \in \mathbb{Z}\},$$

equipped with the left-shift σ. That is, Σ_B contains only sequences in which each $x_i x_{i+1}$ indicate transitions from Markov partition elements that are allowed by the map T_A.
It can be shown that $(\mathbb{T}^d, \mathcal{B}, T_A, Leb)$ is isomorphic to the shift space $(\Sigma_B, \mathcal{C}, \sigma, \mu)$ where

$$\mu([x_k x_{k+1} \dots x_n]) = m_{x_k} \Pi_{x_k x_{k+1}} \Pi_{x_{k+1} x_{k+2}} \dots \Pi_{x_{n-1} x_n},$$

for $m_i = Leb(R_i)$, $i = 1, \dots, d\}$, and weighted transition matrix Π where

$$\Pi_{ij} = \frac{Leb(T_B(R_i) \cap R_j)}{Leb(R_i)}$$ is the relative mass that T_A transports from R_i to R_j.

The σ-algebra $\mathcal{C}$ is generated by the allowed cylinder sets.

Lebesgue is ergodic and mixing

Theorem: For every hyperbolic toral automorphism based on a primitive matrix A, Lebesgue measure is ergodic and mixing.

Proof: We only give the proof for dimension 2. The higher dimensional case goes similarly. Consider the Fourier modes (also called characters)

$$\chi_{(m,n)} : \mathbb{T}^2 \to \mathbb{C}, \qquad \chi_{(m,n)}(x, y) = e^{2\pi i(mx+ny)}.$$

These form an orthogonal system (w.r.t. $\langle \varphi, \psi \rangle = \int \varphi \overline{\psi} \, d\lambda$), spanning $L^2(\lambda)$ for Lebesgue measure λ.

Lebesgue is ergodic

We have for the Koopman operator

$$\begin{aligned} U_{T_A}\chi_{(m,n)}(x,y) &= \chi_{(m,n)}\circ T_A(x,y) \\ &= e^{2\pi i (am+cn)x+(bm+dn)y)} \\ &= \chi_{A^t(m,n)}(x,y). \end{aligned}$$

In other words, U_{T_A} maps the character with index (m,n) to the character with index $A^t(m,n)$, where A^t is the transpose matrix.

Assume that φ is a T_A-invariant L^2-function. Write it as Fourier series:

$$\varphi(x,y) = \sum_{m,n\in\mathbb{Z}} \varphi_{(m,n)}\chi_{(m,n)}(x,y),$$

where the Fourier coefficients $\varphi_{m,n}\to 0$ as $|m|+|n|\to\infty$.

Lebesgue is ergodic

By T_A-invariance, we have

$$\varphi(x,y) = \varphi \circ T_A(x,y) = \sum_{m,n\in\mathbb{Z}} \varphi_{(m,n)} \chi_{A^t(m,n)}(x,y),$$

and hence $\varphi_{(m,n)} = \varphi_{A^t(m,n)}$ for all m,n. For $(m,n) = (0,0)$ this is not a problem, but this only produces constant functions.

If $(m,n) \neq (0,0)$, then the A^t-orbit of (m,n), so infinitely many equal Fourier coefficients

$$\varphi_{(m,n)} = \varphi_{A^t(m,n)} = \varphi_{(A^t)^2(m,n)} = \varphi_{(A^t)^3(m,n)} = \varphi_{(A^t)^4(m,n)} \cdots$$

As the Fourier coefficients converge to zero as $|m| + |n| \to \infty$, they all must be equal to zero, and hence φ is a constant function. This proves ergodicity.

Lebesgue is mixing

For the proof of mixing, we need a lemma, which we give without proof.

Lemma: A transformation (X, T, μ) is mixing if and only if for all φ, ψ in a complete orthogonal system spanning $L^2(\mu)$, we have

$$\int_X \varphi \circ T^N(x)\overline{\psi(x)}\, d\mu \to \int_X \varphi(x)\, d\mu \cdot \int_X \overline{\psi(x)}\, d\mu$$

as $N \to \infty$.

Lebesgue is mixing

To use this lemma on $\varphi = \chi_{(m,n)}$ and $\psi = \chi_{(k,l)}$, we compute

$$\int_X \chi_{(m,n)} \circ T_A^N(x)\overline{\chi_{(k,l)}(x)}\, d\lambda = \int_X \chi_{(A^t)^N(m,n)}\overline{\chi_{(k,l)}(x)}\, d\lambda.$$

- If $(m,n) = (0,0)$, then $(A^t)^N(m,n) = (0,0) = (m,n)$ for all N. Hence, the integral is non-zero only if $(k,l) = (0,0)$, but then

$$\int_X \chi_{(0,0)} \circ T_A^N(x)\overline{\chi_{(0,0)}(x)}\, d\lambda = 1 = \int_X \chi_{(0,0)}\, d\lambda \int_X \overline{\chi_{(0,0)}}\, d\lambda.$$

  If $(k,l) \neq (0,0)$, then

$$\int_X \chi_{(0,0)} \circ T_A^N(x)\overline{\chi_{(k,l)}(x)}\, d\lambda = 0 = \int_X \chi_{(0,0)}\, d\lambda \int_X \overline{\chi_{(0,0)}(x)}\, d\lambda.$$

Lebesgue is mixing

Repeat from previous slide:

$$\int_X \chi_{(m,n)} \circ T_A^N(x)\overline{\chi_{(k,l)}(x)}\, d\lambda = \int_X \chi_{(A^t)^N(m,n)}\overline{\chi_{(k,l)}(x)}\, d\lambda.$$

- If $(m,n) \neq (0,0)$, then, regardless what (k,l) is, there is N such that $(A^t)^M(m,n) \neq (k,l)$ for all $M \geq N$. Therefore

$$\int_X \chi_{(m,n)} \circ T^M(x)\overline{\chi_{(k,l)}(x)}\, d\lambda = 0 = \int_X \chi_{(m,n)}\, d\lambda \int_X \overline{\chi_{(k,l)}}\, d\lambda.$$

The lemma therefore guarantees mixing.

Lebesgue is mixing

Repeat from previous slide:

$$\int_X \chi_{(m,n)} \circ T_A^N(x)\overline{\chi_{(k,l)}(x)}\,d\lambda = \int_X \chi_{(A^t)^N(m,n)}\overline{\chi_{(k,l)}(x)}\,d\lambda.$$

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  The lemma therefore guarantees mixing.

Exercise: Where is hyperbolicity used in this proof. Is every non-hyperbolic toral automorphism ergodic and/or mixing?