Bernoulli - Mixing - Ergodic - Recurrent

In this lecture we study the relation between roperties of measure preserving systems, most of which we introduced before:
A measure preserving system $(X, \mathcal{B}, \mu; T)$ is

- Bernoulli if it is isomorphic to a two-sided Bernoulli shift.
- strong mixing if for all $A, B \in \mathcal{B}$

$$\mu(T^{-n}(A) \cap B) - \mu(A)\mu(B) \to 0.$$

- weak mixing if for all $A, B \in \mathcal{B}$ the average

$$\frac{1}{n}\sum_{i=0}^{n-1} |\mu(T^{-i}(A) \cap B) - \mu(A)\mu(B)| \to 0.$$

- ergodic if $T^{-1}(A) = A \bmod \mu$ implies $\mu(A) = 0$ or $\mu(A^c) = 0$.
- recurrent if for all $A \in \mathcal{B}$ with $\mu(A) > 0$ there is $n \geq 1$ such that $\mu(T^n(A) \cap A) > 0$.

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First an alternative way of stating ergodicity:

Lemma: A probability preserving dynamical system $(X, \mathcal{B}, T, \mu)$ is ergodic if and only if

$$\frac{1}{n}\sum_{i=0}^{n-1} \mu(T^{-i}(A) \cap B) - \mu(A)\mu(B) \to 0 \text{ as } n \to \infty,$$

for all $A, B \in \mathcal{B}$. (Compared to weak mixing, note the absence of absolute value bars.)

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Proof: Assume that T is ergodic, so by Birkhoff's Ergodic Theorem $\frac{1}{n}\sum_{i=0}^{n-1} 1_A \circ T^i(x) \to \mu(A)$ μ-a.e. Multiplying by 1_B gives

$$\frac{1}{n}\sum_{i=0}^{n-1} 1_A \circ T^i(x) 1_B(x) \to \mu(A) 1_B(x) \quad \mu\text{-a.e.}$$

Integrating over x (using the Dominated Convergence Theorem to swap limit and integral), gives

$$\lim_n \frac{1}{n}\sum_{i=0}^{n-1} \int_X 1_A \circ T^i(x) 1_B(x)\, d\mu = \mu(A)\mu(B).$$

Conversely, assume that $A = T^{-1}A$ and take $B = A$. Then we obtain $\mu(A) = \frac{1}{n}\sum_{i=0}^{n-1} \mu(A \cap T^{-i}(A)) \to \mu(A)^2$, hence $\mu(A) \in \{0, 1\}$.

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Theorem We have the implications:

Bernoulli $\Rightarrow$ *mixing* $\Rightarrow$ *weak mixing* $\Rightarrow$ *ergodic* $\Rightarrow$ *recurrent.*

None of the reverse implications holds.

Example We know already that irrational rotations $R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ are ergodic (even uniquely ergodic). Let us show R_α is not mixing:

Take an interval A of length $\frac{1}{4}$. There are infinitely many n such that $R_\alpha^{-n}(A) \cap A = \emptyset$, so

$$\liminf_n \mu(R^{-n}(A) \cap A) = 0 \neq (\frac{1}{4})^2.$$

Circle rotations are not weak mixing either.

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Proof Bernoulli $\Rightarrow$ mixing holds for any pair of cylinder sets $Z = Z_{[a,b]}$, $Z' = Z_{[c,d]}$ because $\mu(\sigma^{-n}(Z) \cap Z') = \mu(Z)\mu(Z')$ for $n > d - a$. The property carries over to all measurable sets by the Kolmogorov Extension Theorem.

Mixing $\Rightarrow$ weak mixing is immediate from the definition.

Weak mixing $\Rightarrow$ ergodic is immediate from the characterization of ergodicity in the previous lemma.

Ergodic $\Rightarrow$ recurrent. If $B \in \mathcal{B}$ has positive measure, then $A := \cup_{i \in \mathbb{N}} T^{-i}(B)$ is T-invariant up to a set of measure 0, see the Poincaré Recurrence Theorem. By ergodicity, $\mu(A) = 1$, and thus μ-a.e. $x \in B$ returns to B.

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An example of a recurrent but not ergodic transformation is the identity map $T : [0,1] \to [0,1]$ with $\mu =$ Lebesgue.

There are standard examples to show that

- weak mixing $\not\Rightarrow$ mixing. The first counter-example in the literature is Chácon's cutting and stacking example.
- mixing $\not\Rightarrow$ Bernoulli.

But I will not cover these examples in class. (See the notes for Chacon's cutting and stacking example.)