Spectral Theory for Dynamical Systems

Spectral Theory of a dynamical system refers to the properties of eigenvalues and eigenfunctions of the Koopman operator of a dynamical system.

We recall: Given $(X, \mathcal{B}, \mu, T)$, we can take the space of complex-valued square-integrable observables $L^2(\mu)$. This is a Hilbert space, equipped with inner product

$$\langle f, g \rangle = \int_X f(x) \cdot \overline{g(x)} \, d\mu.$$

The Koopman operator is defined as

$$U_T : L^2(\mu) \to L^2(\mu), \qquad U_T f = f \circ T.$$

The Koopman Operator

By T-invariance of μ, it is a unitary operator (it preserves the inner product). Indeed

$$\begin{aligned} \langle U_T f, U_T g \rangle &= \int_X f \circ T(x) \cdot \overline{g \circ T(x)} \, d\mu \\ &= \int_X (f \cdot \overline{g}) \circ T(x) \, d\mu = \int_X f \cdot \overline{g} \, d\mu = \langle f, g \rangle, \end{aligned}$$

and therefore $U_T^* U_T = U_T U_T^* = I$.

The Koopman Operator

Theorem: The eigenvalues of U_T form a multiplicative group subgroup of the unit circle. Eigenfunctions to different eigenvalues are orthogonal, and if μ is ergodic, then the eigenfunctions have constant modulus and each eigenspace is one-dimensional.

Examples: A transformation is weakly mixing if 1 is the only eigenvalue (and it has multiplicity 1, so constant functions are the only eigenfunctions). This is not so easy to see from our earlier definition of weak mixing, but we will not give the proof for now.

For rational circle rotations $(\mathbb{S}^1, \mathcal{B}, Leb, R_{p/q})$ the eigenvalues are $\{e^{2\pi i r/q} : r \in \{0, 1, \ldots, q-1\}\}$, and each eigenvalue has infinite multiplicity. Note: no ergodicity!

For irrational circle rotations $(\mathbb{S}^1, \mathcal{B}, Leb, R_\alpha)$ the eigenvalues are $\{e^{2\pi i n\alpha} : n \in \mathbb{Z}\}$, and each eigenfunction of $e^{2\pi i n\alpha}$ is $x \mapsto e^{2\pi i n x}$, up to a multiplicative constant.

The Koopman Operator

Proof: If λ is the eigenvalue of eigenfunction v, then

$$\langle v, v\rangle = \langle U_T v, U_T v\rangle = \langle \lambda v, \lambda v\rangle = |\lambda|^2 \langle v, v\rangle,$$

so λ lies on the unit circle. Assuming that λ, μ are eigenvalues with eigenfunctions v and w respectively, we have

$$U_T(vw) = (vw) \circ T = (v \circ T) \cdot (w \circ T) = U_T v \cdot U_T w = \lambda\mu(vw),$$

so $\lambda\mu$ is an eigenvalue. Also

$$U_T(\bar{v}) = \bar{v} \circ T = \overline{v \circ T} = \overline{U_T v} = \bar{\lambda}\bar{v} = \lambda^{-1}\bar{v},$$

so the eigenvalues form a multiplicative group of the unit circle. If v and w are eigenfunctions to different eigenvalues λ and μ, then

$$\langle v, w\rangle = \langle U_T v, U_T w\rangle = \langle \lambda v, \mu w\rangle = \lambda\bar{\mu}\langle v, w\rangle,$$

and this can only be true of $\langle v, w\rangle = 0$.

The Koopman Operator

Assume now that μ is ergodic, so the only eigenvectors of eigenvalue 1 are constant μ-a.e. If v is the eigenfunction of eigenvalue λ, then $|v|$ is an eigenfunction of eigenvalue $|\lambda| = 1$, so $|v|$ is constant; we can scale $|v| = 1$. If w is another eigenfunction of λ, scaled so that $|w| = 1$ and independent of v, then v/w is an eigenfunction of 1, so $v = w$ μ-a.e.

The Koopman Operator

Lemma If (Y, S, ν) is a measure-theoretical factor of (X, T, μ) (with factor map π and $\nu = \mu \circ \pi^{-1}$), then every eigenvalue of (Y, S, ν) is also an eigenvalues of (X, T, μ).

In particular, the spectrum of (Y, S, ν) is contained in the spectrum of (X, T, μ), and isomorphic systems have the same eigenvalues and spectrum.

Proof: Let g be an eigenvalue of (Y, S, ν), with eigenvalue $e^{2\pi i\alpha}$. Then $f := g \circ \pi$ is an eigenvector of (X, T, μ), because

$$f \circ T = g \circ \pi \circ T = g \circ S \circ \pi = e^{2\pi i\alpha} g \circ \pi = e^{2\pi i\alpha} f \ \ \mu\text{-a.e.}$$

Hence f is an eigenfunction of (X, T, μ) with the same eigenvalue $e^{2\pi i\alpha}$.

Spectral Measures

Given a non-negative measure $\nu \in \mathcal{M}(\mathbb{T})$ on the circle, the Fourier coefficients of ν are defined as

$$\hat{\nu}(n) = \int_{\mathbb{T}} z^n \, d\nu = \int_0^1 e^{2\pi i n x} d\nu(e^{2\pi i x}).$$

For every sequence $(z_j)_{j\in\mathbb{N}} \subset \mathbb{C}$ and $N \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{j,k=1}^{N} z_j \bar{z}_k \hat{\nu}(j-k) &= \sum_{j,k=1}^{N} \int_0^1 z_j e^{2\pi i j x} \, \overline{z_k e^{2\pi i k x}} \, d\nu \\ &= \int_0^1 \sum_{j=1}^{N} z_j e^{2\pi i j x} \overline{\sum_{k=1}^{N} z_k e^{2\pi i k x}} \, d\nu \\ &= \int_0^1 \| \sum_{j=1}^{N} z_j e^{2\pi i j x} \|^2 \, d\nu \geq 0. \end{aligned}$$

Spectral Measures

This property of $(\hat{\nu}(n))_{n\in\mathbb{Z}}$ is called positive definiteness.

Conversely, the **Bochner-Herglotz Theorem** states that for every positive definite sequence $(a_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$, there is a unique non-negative measure $\nu \in \mathcal{M}(\mathbb{T})$ such that $\hat{\nu}(n) = a_n$ for each n, and $\nu(\mathbb{T}) = \sqrt{\sum_n |a_n|^2}$.

Spectral Measures

Let $(X, \mathcal{B}, \mu, T)$ be an invertible dynamical system. Given a function $f \in L^2(\mu)$, the sequence $a_n := \langle U_T^n f, f\rangle = \int_X f \circ T^n \overline{f}\, d\mu$ is positive definite because

$$\begin{aligned}\sum_{j,k=1}^{N} z_j \bar{z}_k a_{j-k} &= \sum_{j,k=1}^{N} z_j \bar{z}_k \langle U_T^{j-k} f, f\rangle = \sum_{j,k=1}^{N} \langle z_j U_T^j f, z_k U_T^k f\rangle \\ &= \left\langle \sum_{j=1}^{N} z_j U_T^j f, \sum_{k=1}^{N} z_k U_T^k f \right\rangle = \| \sum_{j=1}^{N} z_j U_T^j f\|^2 \geq 0.\end{aligned}$$

Therefore the **Bochner-Herglotz Theorem** associates a non-negative measure $\nu_f \in \mathcal{M}(\mathbb{T})$ to f, which is called the spectral measure of f.

Spectral Measures

Remarks

- If U is an invertible unitary operator, then

$$\begin{aligned}\hat{\nu}_f(-n) &= \langle U^{-n}f, f\rangle = \langle U^{-n}f, U^{-n}U^n f\rangle \\ &= \langle f, U^n f\rangle = \overline{\langle U^n f, f\rangle} \\ &= \overline{\hat{\nu}_f(n)},\end{aligned}$$

  for every $n \in \mathbb{N}$. Therefore it makes sense to define $\hat{\nu}_f(-n) := \overline{\hat{\nu}_f(n)}$ also for non-invertible unitary operators. Most of the theory remains valid.

Spectral Measures

Remarks

- For $U = U_T$, the Koopman operator of an invertible dynamical system $(X, \mathcal{B}, \mu, T)$, the Fourier coefficients $\hat{\nu}_f(n) = \langle U_T^n f, f \rangle = \int_X f \circ T^n \, \bar{f} \, d\mu$ are the autocorrelation coefficients of the observable $f \in L^2(\mu)$.

  If μ is mixing, then $\hat{\nu}_f(n) \to 0$ for every $f \in L^2(\mu)$ with $\int_X f \, d\mu = 0$.

  In fact, the correlation coefficients $\langle U_T^n f, g \rangle = \int_X f \circ T^n \, \bar{g} \, d\mu$ of two observables $f, g \in L^2(\mu)$ are the Fourier coefficients of a complex measure $\sigma_{f,g}$; this is an application of a somewhat more general version of the Bochner-Herglotz Theorem.

Spectral Measures

Suppose the unitary operator U acts on a the Hilbert space $\mathbb{H}$. We can decompose $\mathbb{H}$ into subspaces that are the linear spans of U-orbits of well-chosen functions in $\mathbb{H}$:

Theorem: Let U be an invertible unitary operator acting on a separable Hilbert space $\mathbb{H}$. Then there is a (possibly finite) sequence of functions $h_j \in \mathbb{H}$ such that

$$\begin{gathered}\mathbb{H} = \oplus_j \overline{\mathrm{Span}(U^n h_j : n \in \mathbb{Z})} \\ \text{and if } j \neq k, \text{ then} \\ \overline{\mathrm{Span}(U^n h_j : n \in \mathbb{Z})} \perp \overline{\mathrm{Span}(U^n h_k : n \in \mathbb{Z})}.\end{gathered} \tag{1}$$

The corresponding spectral measures satisfies

$$\nu_{h_1} \gg \nu_{h_2} \gg \nu_{h_3} \gg \dots$$

Moreover, if (h'_j) satisfy (1), then $\nu_{h_j} \sim \nu_{h'_j}$ for each j.

Spectral Measures

Definition: The spectral measure ν_{h_1} of the leading function h_1 in (1) is called the maximal spectral type. If $U = U_T$ is the Koopman operator of an invertible dynamical system, then we call ν_{h_1} the spectral measure of T and we will denote it as ν_T.

Example: If f is an eigenfunction of U_T to eigenvalue λ scaled so that $\|f\|_2 = 1$, then $\nu_f = \delta_\lambda$ is the Dirac measure at the eigenvalue. Indeed,

$$\hat{\delta}_\lambda(n) = \int_{\mathbb{T}} z^n d\delta_\lambda = \lambda^n = \langle \lambda^n f, f\rangle = \langle U_T^n f, f\rangle.$$

For each eigenfunction f, $\mathrm{Span}(U_T^n f : n \in \mathbb{Z}) =: \mathrm{Span}(f)$ is only a one-dimensional subspace. However, closure of the span of all eigenvalues $\overline{\mathrm{Span}(f : U_T f = \lambda f)}$, called the Kronecker factor, can be as large as the whole Hilbert space $L^2(\mu)$.

Pure Point Spectrum

The spectral measure of T decomposes as

$$\nu_T = \nu_{pp} + \nu_{ac} + \nu_{sing}$$

where

- ν_{pp} is the discrete or pure point part of ν_T. It is an at most countable linear combination of Dirac measures, namely at every eigenvalue, so in particular at $\lambda = 1$. For weak mixing transformations $\nu_{pp} = c\delta_0$ for some $c \in (0, 1]$.
- ν_{ac} is absolutely continuous w.r.t. Lebesgue measure.
- ν_{sing} is non-atomic but singular w.r.t. Lebesgue measure.

Then parts $\nu_{ac} + \nu_{sing} = \nu_{cont}$ together are called the continuous part of the spectral measure.

Pure Point Spectrum

Definition: A measure-preserving dynamical system $(X, \mathcal{B}, \mu, T)$ is said to have a pure point spectrum (also called discrete spectrum if the collection of eigenfunctions of the Koopman operator U_T spans $L^2(\mu)$. That is: the Kronecker factor is $L^2(\mu)$.

Equivalently, the spectral measure $\nu_T = \nu_{pp}$ is a countable linear combination of Dirac measures.

Pure Point Spectrum

We quote (without proof) two structure theorems due to Halmos & von Neumann, that illustrate the use of pure point spectrum transformations.

Theorem Two measure-preserving dynamical systems with pure point spectra are isomorphic if and only if their eigenvalues are the same.

Theorem An ergodic probability measure preserving system (X, T, μ) on compact metric space has pure point spectrum if and only if it is isomorphic to a rotation on a compact metrizable Abelian group G with Haar measure μ_G, so there is $g_0 \in G$ such that $Tx = \phi^{-1}(\phi(x) + g_0)$, where $\phi : X \to G$ is the isomorphism.

Pure Point Spectrum

We give some examples to illustrate the second theorem. Assume that (X, T, μ) has pure point spectrum.

Example: Let α be irrational suppose that the set of eigenvalues of U_T is $\{e^{2\pi i n\alpha} : n \in \mathbb{Z}\}$.
Then the system is isomorphic to $(\mathbb{S}^1, R_\alpha, Leb)$, with eigenfunctions

$$v_n = e^{2\pi i n x}$$

These are the usual Fourier modes, and they span $L^2(\mathbb{S}^1, Leb)$.

Pure Point Spectrum

Example: Let α and β two irrationals that are rationally independent, i.e., $x\alpha + y\beta + z = 0$ for $x, y, z \in \mathbb{Q}$ implies $x = y = z = 0$.
Then the "group" rotation

$$R_{\alpha,\beta} : \mathbb{T}^2 \to \mathbb{T}^2, \quad (x, y) \mapsto (x + \alpha, y + \beta).$$

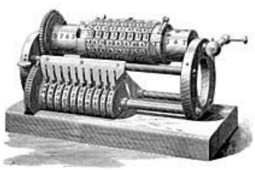
has spectrum $\{e^{2\pi i(m\alpha+n\beta)} : m, n \in \mathbb{Z}\}$. The eigenfunctions are

$$v_{m,n} = e^{2\pi i(mx+ny)}.$$

These are the two-dimensional Fourier modes, and they span $L^2(\mathbb{T}^2, Leb)$.

The dyadic odometer

The following example, called dyadic odometer or dyadic adding machine has spectrum $\{e^{2\pi i m 2^{-n}} : m, n \in \mathbb{N}\}$.
It is a map $a : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ defined by "add-and-carry".



$$
\begin{array}{rcl}
x & = & 01110001101101010100\ldots \\
+1 & = & 10000000000000000000\ldots \\
\hline
a(x) & = & 11110001101101010100\ldots \\
+1 & = & 10000000000000000000\ldots \\
\hline
a^2(x) & = & 00001001101101010100\ldots
\end{array}
$$

The dyadic odometer

Let us write the map a down as a computer algorithm:

$$
\begin{array}{ll}
c := 1 \quad ; & k := 1 \\
\text{Repeat} & \\
 & s := x_k + c; \\
 & \text{If } s \geq 2 \text{ then } c := 1 \text{ else } c := 0 \\
 & x_k := s \bmod 2 \;;\; k := k + 1 \\
\text{Until} & c = 0
\end{array}
$$

Check that this algorithm indeed gives

$$a(11111111111\ldots) = 00000000000\ldots$$

The dyadic odometer

Let us also generalize this to show that the odometer is a topological group under addition.

The addition $z = x + y$ of two sequences $x, y \in X$ with add-and-carry goes according to the algorithm:

$$
\begin{array}{ll}
c := 0 \quad ; & k := 1 \\
\text{Repeat} & \text{for all } k \in \mathbb{N} \\
 & s := x_k + y_k + c; \\
 & \text{If } s \geq 2 \text{ then } c := 1 \text{ else } c := 0 \\
 & z_k := s \bmod 2 \ ; \ k := k + 1
\end{array}
$$

One can check that this is continuous in x and y.

The dyadic odometer

As an interval map, the odometer has the form

$$a(x) = x - (1 - 3 \cdot 2^{1-n}) \qquad \text{if } x \in [1 - 2^{1-n}, 1 - 2^{-n}),\ \ n \geq 1,$$

It preserves Lebesgue measure.

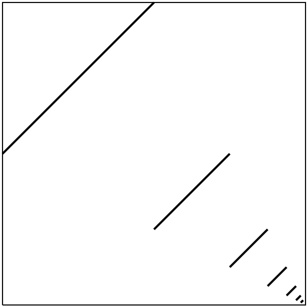


Figure: The dyadic odometer represented as a map a on the interval (von Neumann-Kakutani map).

The dyadic odometer

The map a permutes the n-cylinders cyclically, so if we define $v_{m,n} : \{0,1\}^{\mathbb{N}} \to \mathbb{C}$ as

$$v_{m,n}(x) = e^{2\pi imk/2^n} \text{ if } x \in [a_1 \dots a_n], \; k = \sum_{j=1}^{n} a_j 2^{j-1},$$

then

$$v_{m,n} \circ a = e^{2\pi im/2^n} v_{m,n}.$$

This shows that $e^{2\pi im/2^n}$ are indeed all eigenvalues.

Exercise: Verify that these eigenfunctions form a orthonormal system. Show that the dyadic adding machine is uniquely ergodic.

The dyadic odometer

Theorem The dyadic odometer has pure point spectrum.

Proof: Let (X, a) be the odometer, with a-invariant measure μ. The n-cylinder $Z_{[0^n]} = [0 \dots 0]$ (n zeros) is periodic with period 2^n under the map a. Abbreviate $Z_j^n = a^j(Z_{[0^n]})$. Now

$$\lambda_m^n := e^{2\pi i m/2^n} \text{ for } 0 \leq m < 2^n$$

is an eigenvalue, because we can construct a corresponding eigenfunction $v_{m,n}$ of the Koopman operator as $v_{m,n}|_{Z_j^n} = e^{-2\pi i j m/2^n}$. In particular,

$(v_{m,n})_{n \in \mathbb{N}, 0 \leq m < 2^n}$ *forms an orthogonal system,*

and it is also easy to check that the $L^2(\mu)$-norms $\|v_{m,n}\|_2 = 1$ for all $n \in \mathbb{N}, 0 \leq m < 2^n$.

The dyadic odometer

Proof continued: To show that it is a complete orthonormal system, i.e.,

$$\mathrm{Span}(\{v_{m,n} : n \in \mathbb{N}, 0 \leq m < 2^n\}) \text{ is dense in } L^2(\mu),$$

it suffices to show that if $g \in L^2(\mu)$ is such that $\int_X g \, \overline{v_{m,n}} \, d\mu = 0$ for all $n \in \mathbb{N}, 0 \leq m < 2^n$, then $g \equiv 0$ μ-a.e. Since $C(X)$ is dense in $L^2(\mu)$, we can assume that g is continuous.

Assume that there is a cylinder set Z_j^n such that $\int_{Z_j^n} g \, d\mu \neq 0$. Let

$$w_\varepsilon = \varepsilon + (1 - \varepsilon) e^{-2\pi i j/2^n} v_{1,n},$$

The dyadic odometer

Proof continued: Then $w_\varepsilon|_{Z_j^n} = 1$ and $|w_\varepsilon(x)| \leq 1 - \varepsilon^2 < 1$ for $x \notin Z_j^n$ for $\varepsilon > 0$ sufficiently small. Clearly w_ε is a linear combination of eigenfunctions, so

$$\int_X g\, \overline{w_\varepsilon}\, d\mu = 0.$$

The algebraic power w_ε^r is a linear combination of eigenfunctions too, and hence also

$$\int_X g\, \overline{w_\varepsilon^r}\, d\mu = 0.$$

But since $|w_\varepsilon^r(x)| < (1-\varepsilon^2)^r \to 0$ for each $x \notin Z_j^n$, we get $\lim_{r\to\infty} \int_X g\, \overline{w_\varepsilon^r}\, d\mu = \int_{Z_j^n} g\, d\mu \neq 0$. This contradiction shows that $(v_{m,n})_{n\in\mathbb{N},0\leq m<2^n}$ is indeed a complete orthonormal system,