Invariant Measures

Definition: A measure μ for a dynamical system (X, T) is T-invariant if

$$\mu(A) = \mu(T^{-1}(A))$$

for every measurable set $A \subset X$.

Lemma 1: Let $T : X \to X$ be a continuous map on a compact space X. Then μ is T-invariant if and only if

$$\int_X f \, d\mu = \int_X f \circ T \, d\mu$$

for every $f \in C(X)$. (Here $C(X)$ is the space of all **continuous** functions on X, equipped with the norm $\| \ \|_\infty$.)

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Proof of Lemma 1:

Assume that μ is T-invariant and $A \in \mathcal{B}$, the σ-algebra of μ-measurable sets. Then

$$\int_X 1_A \circ T \, d\mu = \mu(T^{-1}A) = \mu(A) = \int_X 1_A \, d\mu.$$

A similar expression works for linear combinations of indicator sets. These linear combinations of indicator sets lie dense in $C(X)$, so the result carries over to $C(X)$.

(NB: We used here that continuous function of a compact space are bounded).

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For the other direction: for every closed set A and $\varepsilon > 0$, we can find a function $f \in C(X)$ such that $f \equiv 1$ on A and $\int |f - 1_A| \, d\mu < \varepsilon$ as well as $\int |f - 1_A| \circ T \, d\mu < \varepsilon$. Then

$$\begin{aligned} |\mu(T^{-1}A) - \mu(A)| &= \left| \int 1_A \circ T \, d\mu - \int 1_A \, d\mu \right| \\ &= \left| \int (f - 1_A) \circ T \, d\mu - \int (f - 1_A) \, d\mu \right| \\ &\leq 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, $\mu(T^{-1}A) = \mu(A)$. The closed sets generate the Borel σ-algebra, so also $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{B}$.

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We have seen examples that Lebesgue measure is T-invariant for the doubling map and also for rotations

$$R_\alpha(x) = x + \alpha \bmod 1$$

on $\mathbb{S}^1$. It is reasonable to ask:

Does every dynamical system have an invariant measure?

Theorem of Krylov-Bogol'ubov: If $T : X \to X$ is a continuous map on a nonempty compact metric space X, then the set of invariant probability measures $\mathcal{M}(X, T) \neq \varnothing$.

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Example: To show that compactness is an essential assumption in the Theorem of Krylov-Bogol'ubov, consider

$$T : (0,1) \to (0,1), \qquad T(x) = x^2.$$

$0, 1 \notin (0,1)$, so the Dirac measures δ_0 and δ_1 are not allowed.

Suppose that μ is T-invariant, and assume that $A \subset (0,1)$ is a compact set with $\mu(A) > 0$. Then there is $n_1 \in \mathbb{N}$ such that $A_1 := T^{-n_1}(A)$ is disjoint from A. Hence $\mu(A_1) = \mu(A) > 0$.

Proceed by induction: $n_i \in \mathbb{N}$ is such that $A_i := T^{-n_i}(A_{i-1})$ is disjoint from A_{i-1}. Then $\mu(A) = \mu(A_i) > 0$ for all $i \in \mathbb{N}$ and all A_i are pairwise disjoint. Hence

$$\mu((0,1)) \geq \sum_i \mu(A_i) = \infty,$$

so μ is not a probability measure.

Exercise 2.1: Show that continuity is an essential assumption in the Theorem of Krylov-Bogol'ubov.

Proof of the Theorem of Krylov-Bogol'ubov:

Let ν be any probability measure (invariant or not) and define Cesaro means:

$$\nu_n(A) = \frac{1}{n}\sum_{j=0}^{n-1} \nu(T^{-j}A).$$

These are all probability measures.

The collection of probability measures on a compact metric space is known to be compact in the weak* topology.

This means that there is limit probability measure μ and a subsequence $(n_i)_{i\in\mathbb{N}}$ such that for every continuous function $\psi : X \to \mathbb{R}$:

$$\int_X \psi \; d\nu_{n_i} \to \int \psi \; d\mu \text{ as } i \to \infty.$$

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On a metric space, we can, for any $\varepsilon > 0$ and closed set A, find a continuous function $\psi_A : X \to [0,1]$ such that $\psi_A(x) = 1$ if $x \in A$ and

$$\mu(A) \leq \int_X \psi_A d\mu \leq \mu(A) + \varepsilon$$

and

$$\mu(T^{-1}A) \leq \int_X \psi_A \circ T \, d\mu \leq \mu(T^{-1}A) + \varepsilon.$$

Here we use outer regularity of the measure μ:

$$\mu(A) = \inf\{\mu(G) : G \supset A \text{ is open}\}.$$

We take $G \supset A$ so small that $\mu(G) - \mu(A) < \varepsilon$ and make sure that $\psi_A = 0$ for all $x \notin G$. Note that it is important that A is closed, because if there exists $a \in \partial A \setminus A$, then the above property fails for $\mu = \delta_a$.

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Now by Lemma 1 and the definition of μ

$$\begin{aligned}
|\mu(T^{-1}(A)) &- \mu(A)| \leq \left| \int \psi_A \circ T \, d\mu - \int \psi_A \, d\mu \right| + \varepsilon \\
&= \lim_{i\to\infty} \left| \int \psi_A \circ T \, d\nu_{n_i} - \int \psi_A \, d\nu_{n_i} \right| + \varepsilon \\
&= \lim_{i\to\infty} \frac{1}{n_i} \left| \sum_{j=0}^{n_i-1} \left(\int \psi_A \circ T^{j+1} \, d\nu - \int \psi_A \circ T^j \, d\nu \right) \right| + \varepsilon \\
&\leq \lim_{i\to\infty} \frac{1}{n_i} \left| \int \psi_A \circ T^{n_i} \, d\nu - \int \psi_A \, d\nu \right| + \varepsilon \\
&\leq \lim_{i\to\infty} \frac{2}{n_i} \|\psi_A\|_\infty + \varepsilon = \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\mu(T^{-1}(A)) = \mu(A)$. The closed sets generate Borel sets, so $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$.