Ergodicity

Definition: A measure μ for a dynamical system (X, T) is ergodic if

$$\mu(A) = 0 \text{ or } \mu(A^c) = 0$$

for every measurable $A \subset X$ such that $T^{-1}(A) = A \bmod \mu$.

- Here $T^{-1}(A) = A \bmod \mu$ means that the mass of the symmetric difference $\mu(T^{-1}(A) \triangle A) = 0$.
- If μ is a probability measure, the we can write $\mu(A) = 0$ or 1 for every measurable $A \subset X$ such that $T^{-1}(A) = A \bmod \mu$. As stated, the definition also applies to infinite measures.
- Usually ergodicity is stated for invariant measures, but the definition wroks for non-invariant measures too.

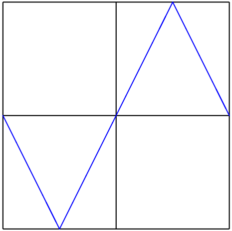
Ergodicity

Ergodicity means that the space X doesn't fall apart in two separate parts.

Example 1: T is the doubling map on $X = \mathbb{S}^1$ and $\mu = \frac{1}{2}(\mathrm{Leb} + \delta_0)$. This measure is not ergodic, because $A = \{0\}$ and $A^c = \mathbb{S}^1 \setminus \{0\}$ are both invariant mod μ, but $\mu(A) = \mu(A^c) = \frac{1}{2}$.

Example 2: $X = [0, 1]$ and

$$T(x) = \begin{cases} \frac{1}{2} - 2x & \text{if } x \in [0, \frac{1}{4}]; \\ 2x - \frac{1}{2} & \text{if } x \in [\frac{1}{4}, \frac{3}{4}]; \\ \frac{5}{2} - 2x & \text{if } x \in [0, \frac{1}{4}]. \end{cases}$$



Lebesgue measure T-invariant but not ergodic, because $A = [0, \frac{1}{2}]$ and $A^c = (\frac{1}{2}, 1]$ are both invariant mod μ, but $\mu(A) = \mu(A^c) = \frac{1}{2}$.

Ergodicity

Proposition: Let μ be an invariant measure for (X, T). Then μ is ergodic if and only if the only T-invariant functions $\psi \in L^1(\mu)$ *i.e.,* $\psi \circ T = \psi$ μ-a.e., are constant μ-a.e.

Proof: $\Rightarrow$ Let $\psi : X \to \mathbb{R}$ be T-invariant μ-a.e., but not constant. Thus there exists $a \in \mathbb{R}$ such that

$$A := \psi^{-1}((-\infty, a]) \quad \text{and} \quad A^c = \psi^{-1}((a, \infty))$$

both have positive measure. By T-invariance, $T^{-1}A = A \pmod{\mu}$, and we have a contradiction to ergodicity.

$\Leftarrow$ Let A be a set of positive measure such that $T^{-1}A = A$. Let $\psi = 1_A$ be its indicator function; it is T-invariant because A is T-invariant. By assumption, ψ is constant μ-a.e., but as $\psi(x) = 0$ for $x \in A^c$, it follows that $\mu(A^c) = 0$.

Proving Ergodicity

Proving that a measure is ergodic is not always simple. We illustrate two different proofs, applicable to different systems.

Lemma 1: Lebesgue measure ergodic for the doubling map $T : \mathbb{S}^1 \to \mathbb{S}^1$, $x \mapsto 2x \bmod 1$.

Proof: Suppose by contradiction that A and B are disjoint T-invariant sets, both of positive measure.

Lebesgue measure has the property that if $\mathrm{Leb}(A) > 0$, then Leb-a.e. $x \in A$ is a density point, which means that

$$\lim_{\varepsilon \to 0} \sup_{x \in J, \mathrm{diam}(J) < \varepsilon} \frac{\mathrm{Leb}(A \cap J)}{\mathrm{Leb}(J)} = 1.$$

That is, the closer to $x \in A$, the more points belong to A, relatively.

Invariant Measures

Take x and y density points of A and B respectively. Let $J_x \ni x$ and $J_y \ni y$ be dyadic intervals of length 2^{-n} where $n \in \mathbb{N}$ is so large that

$$\frac{\mathrm{Leb}(J_x \cap A)}{\mathrm{Leb}(J_x)} > \frac{2}{3} \quad \text{and} \quad \frac{\mathrm{Leb}(J_y \cap B)}{\mathrm{Leb}(J_y)} > \frac{2}{3}.$$

By linearity and T-invariance of A and B also:

$$\frac{\mathrm{Leb}(T^n(J_x \cap A))}{\mathrm{Leb}(T^n(J_x)))} > \frac{2}{3} \quad \text{and} \quad \frac{\mathrm{Leb}((T^n(J_y \cap B))}{\mathrm{Leb}(T^n(J_y)))} > \frac{2}{3}.$$

But $T^n(J_x) = T^n(J_y) = \mathbb{S}^1$. Therefore

$$\mathrm{Leb}(A) > \frac{2}{3} \quad \text{and} \quad \mathrm{Leb}(B) > \frac{2}{3}.$$

This contradicts that A and B are disjoint. This concludes this proof by contradiction.

Proving Ergodicity

Lemma 2: Let $\alpha \in \mathbb{R}$ be irrational. Lebesgue measure ergodic for the doubling map $R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$, $x \mapsto x + \alpha \bmod 1$.

Exercise 3.7: Show that Lebesgue measure is not ergodic if $\alpha \in \mathbb{Q}$.

Proof of Lemma 2: We show that every T-invariant function $\psi \in L^2$ must be constant. Indeed, write

$$\psi(x) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x}$$

as a Fourier series. The T-invariance implies

$$\psi \circ T(x) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n (x+\alpha)} = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n \alpha} e^{2\pi i n x} = \psi(x)$$

so $a_n e^{2\pi i n \alpha} = a_n$ for all $n \in \mathbb{Z}$. Since $\alpha \notin \mathbb{Q}$, we have $a_n = 0$ for all $n \neq 0$, so $\psi(x) \equiv a_0$ is indeed constant. Finally, L^2 is dense in L^1, so the same conclusion holds for $\psi \in L^1$.

Circle Rotations

Another Lebesgue preserving map that we will frequently use as example is the circle rotation:

$$R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1, \qquad R_\alpha(x) = x + \alpha \bmod 1.$$

- Lebesgue measure is always R_α-invariant, regardless what α is.
- If $\alpha \in \mathbb{Q}$, then every orbit is periodic. Hence for every $x \in \mathbb{S}^1$, there is an atomic R_α-measure such that $\mu(\{x\}) > 0$.
- If $\alpha \notin \mathbb{Q}$, then every orbit is dense in $\mathbb{S}^1$.

Exercise 3.2: If $\alpha \notin \mathbb{Q}$ and μ is an R_α-invariant atomic measure, show that μ is an infinite measure: $\mu(\mathbb{S}^1) = \infty$. Are all infinite R_α-invariant measure atomic?

More on Irrational Rotations

The rotation $R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ is defined as $R_\alpha(x) = x + \alpha \pmod 1$.
Let α be irrational.

Theorem (Poincaré): Every orbit is dense in $\mathbb{S}^1$, and for every interval J and every $x \in \mathbb{S}^1$, the visit frequency

$$v(J) := \lim_{n\to\infty} \frac{1}{n} \#\{0 \le i < n : R_\alpha^i(x) \in J\} = |J|.$$

Figure: Henri Poicaré (1854-1912): pioneer of dynamical systems.

More on Irrational Rotations

Proof: As $\alpha \notin \mathbb{Q}$, then x cannot be periodic, so its orbit is infinite. Let $\varepsilon > 0$. Since $\mathbb{S}^1$ is compact, there must be $m < n$ such that

$$0 < \delta := d(R_\alpha^m(x), R_\alpha^n(x)) < \varepsilon.$$

Since R_α is an isometry,

$$|R_\alpha^{k(n-m)}(x) - R_\alpha^{(k+1)(n-m)}(x)| = \delta$$

for every $k \in \mathbb{Z}$, and

$$\{R_\alpha^{k(n-m)}(x) : k \in \mathbb{Z}\}$$

is a collection of points such that every two neighbours are exactly δ apart. Since $\varepsilon > \delta$ is arbitrary, this shows that $\mathrm{orb}(x)$ is dense.

More on Irrational Rotations

Let $J_\delta^0 = [R_\alpha^m(x), R_\alpha^n(x))$ and $J_\delta^k = R_\alpha^{k(n-m)}(J_\delta)$. Then for $K = \lfloor 1/\delta \rfloor$, $\{J_\delta^k\}_{k=0}^K$ is a cover $\mathbb{S}^1$ of adjacent intervals, each of length δ, and $R_\alpha^{j(n-m)}$ is an isometry from J_δ^i to J_δ^{i+j}. Therefore the visit frequencies

$$\underline{v}_k = \liminf_n \frac{1}{n} \#\{0 \leq i < n : R_\alpha^i(x) \in J_\delta^k\}$$

are all the same for $0 \leq k \leq K$, and together they add up to at most $1 + \frac{1}{K}$. This shows for example that

$$\frac{1}{K+1} \leq \underline{v}_k \leq \overline{v}_k := \limsup_n \frac{1}{n} \#\{0 \leq i < n : R_\alpha^i(x) \in J_\delta^k\} \leq \frac{1}{K},$$

and these inequalities are independent of the point x.

More on Irrational Rotations

Now an arbitrary interval J can be covered by $\lfloor |J|/\delta \rfloor + 2$ such adjacent J_δ^k, so

$$v(J) \leq \left(\frac{|J|}{\delta} + 2\right) \frac{1}{K} \leq (|J|(K+1) + 2) \frac{1}{K} \leq |J| + \frac{3}{K}.$$

A similar computation gives $v(J) \geq |J| - \frac{3}{K}$.

Taking $\varepsilon \to 0$ (hence $\delta \to 0$ and $K \to \infty$), we find that the limit $v(J)$ indeed exists, and is equal to $|J|$. This concludes the proof.

More on Irrational Rotations

Example: Consider the first digits of the powers of 2.

1	16	256	4096	
2	32	512	8192	
4	64	1024	16384	etc.
8	128	2048	32768	

Exercise 3.3: Does 9 ever appear as first digit?

Exercise 3.4: Does 2 appear infinitely often?

Exercise 3.5: With which frequency does 1 appear?

More on Irrational Rotations

Hint for Solution: Define $h : \mathbb{R}^+ \to \mathbb{S}^1$ as $h(x) = \log_{10} x \bmod 1$. Then

$$h(2x) = h(x) + \log_{10} 2.$$

so the following diagram commutes:

$$\begin{tikzcd} \mathbb{R}^+ \arrow[r, "\times 2"] \arrow[d, "h"] & \mathbb{R}^+ \arrow[d, "h"] \\ \mathbb{S}^1 \arrow[r, "+\log_{10} 2"] & \mathbb{S}^1 \end{tikzcd}$$

and note also that $\log_{10} 2 \neq \mathbb{Q}$.

The first digit of 2^n is $b \in \{1, \ldots, 9\}$ if and only if

$$n \log_{10} 2 \in [\log_{10} b, \log_{10} b + 1).$$

Unique ergodicity

Definition: A system (X, T) is called uniquely ergodic if there is exactly one T-invariant **probability** measure.

Irrational rotations R_α are uniquely ergodic, with Lebesgue as unique R_α-invariant probability measure.

Exercise 3.6: Show that the unique invariant meaure of a uniquely ergodic system is necessarily ergodic.

Unique ergodicity

Oxtoby's Theorem: Let X be a compact space and $T : X \to X$ continuous. A transformation (X, T) is uniquely ergodic if and only if, for every continuous function ψ and every point $x \in X$, the Birkhoff averages

$$\frac{1}{n}\sum_{i=0}^{n-1} \psi \circ T^i(x)$$

converge uniformly to a constant function.

Unique ergodicity

Proof: If μ and ν were two different ergodic measures, then we can find a continuous function $f : X \to \mathbb{R}$ such that $\int f d\mu \neq \int f d\nu$. Using Birkhoff's Ergodic Theorem for both measures (with their own typical points x and y), we see that

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) = \int f d\mu \neq \int f d\nu = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(y),$$

so there is not even convergence to a constant function.

Unique ergodicity

Conversely, we know by the Ergodic Theorem that $\lim_n \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k(x) = \int f d\mu$ is constant μ-a.e. But if the convergence is not uniform, then there is a sequence $(y_i) \subset X$ and $(n_i) \subset \mathbb{N}$, such that

$$\lim_i \frac{1}{n_i} \sum_{k=0}^{n_i-1} f \circ T^k(y_i) \neq \int_X f \, d\mu.$$

Define probability measures $\nu_i := \frac{1}{n_i}\sum_{k=0}^{n_i-1} \delta_{T^k(x_i)}$. This sequence (ν_i) has a weak accumulation points ν which is shown to be T-invariant measures in the same way as in the proof of Krylov-Bogol'ubov Theorem. But $\nu \neq \mu$ because $\int f \, d\nu \neq \int f \, d\mu$. Hence (X, T) cannot be uniquely ergodic.